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# The Generalized Linear Sampling Method for limited aperture measurements

Lorenzo Audibert\* and Houssem Haddar †

**Abstract.** We extend the so-called Generalized Linear Sampling Method (GLSM) to the case of limited aperture data at a fixed frequency. In this case the factorization of the sampling operator does not obey the symmetry required in the justification of the GLSM introduced in Audibert-Haddar [Inverse Problems, 2014]. We propose a new formulation by adding an extra penalty term that asymptotically correct the non symmetry of the GLSM original penalty term. The analysis of the new formulation is first presented in an abstract framework. We then show how to apply our setting to the scalar problem with far field measurements or near field measurements on a limited aperture. We finally validate the method through some numerical tests in two dimensions and for far field measurements.

**Key words.** Inverse scattering problems, Linear Sampling Method, Generalized Linear Sampling Method, Factorization Method, Qualitative methods

**AMS subject classifications.** 35R60, 35R30, 65M32

**1. Introduction.** This work is concerned with the design of so-called sampling methods [7, 6, 8, 13, 4] for inverse scattering problems where one would like to determine the shape of extended targets from fixed frequency multi-static data. More precisely we extend and analyze the recently introduced Generalized Linear Sampling Method [3] (GLSM) to limited aperture data. The GLSM framework developed in [3] provides an exact characterization of the target shape in terms of the so-called far field operator (at fixed frequency and for full aperture). This characterization is based on two factorizations of the far field operator. The first one is used to justify the Linear Sampling Method (LSM) and the second one is at the heart of the Factorization Method (FM). Considering general limited aperture data break the symmetry of the second factorization and prevent the application of the results of [3] or [13] on the FM. The characterization of the GLSM is based on constructing nearby solution to the far field equation as minimizing sequences of a special cost functional. In this cost functional the symmetric factorization is important to ensure that the regularization term has suitable properties. In this article we propose a modification of the regularization term and analyze this modification in order to prove exact characterization even for non symmetric factorization.

The main idea behind our method is that without symmetric factorization it is not possible to control directly the norm of the Herglotz wave that approximately solves the far field equation. However we have access to a term that is close to this quantity and we can bound the error we made, therefore controlling the norm of the associated Herglotz wave. Due to this splitting the control is coarser and therefore it reflects the fact that this situation is less favorable for imaging. The fact that the regularization involves compact operators or the case of noisy operators are covered using the idea already proposed in [3]. However the interesting property of strong convergence of the minimizing sequence of the cost functional demonstrated

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in [2] could not be simply extended. The second main contribution of this article is to add a regularization term to lower the hypothesis of [2] on the regularization term. This new results extend the validity of the results of [2] and enable an extension to non symmetric factorization. In order to introduce those ideas we choose to present the case of scalar inverse scattering from inhomogeneous inclusions for limited aperture far field measurements. We also indicate how the method can be easily extended to near field data.

On the numerical side we introduce a second order method to minimize the cost functional, this method prove to be more efficient than the one use in [3]. The superiority of our indicator function is demonstrated for symmetric factorization. The theory does not say how to choose the regularization parameter for symmetric factorization, the method does not seem to be very sensitive and an heuristic choice give good result. For non symmetric factorization this choice is by far more important and we propose three heuristics to set this parameter.

The article is organized as follows. In Section 2 a model problem is introduced to motivate the GLSM for non symmetric factorization. Theoretical extension for the symmetric factorization is given in section 3.1 and the case of non-symmetric factorization is treated in section 3.2. Section 4 provides an example of application by completely treating the model problem introduce in section 2. Section 5 show how nearfield data easily fit into the theory developed in Section 3. The last section (Section 6) is devoted to numerical algorithms issued from section 4 along with validating numerical results and discussion on the difference between symmetric and non-symmetric cases.

**2. A model problem for limited aperture data.** We choose to present our method for the simple model of inverse time harmonic scattering problem from inhomogeneous targets. For a wave number  $k > 0$ , the total field solve the following scalar wave equation:

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

with  $d = 2$  or  $3$  and with  $n \in L^\infty(\mathbb{R}^d)$  denoting the refractive index such that the support of  $n - 1$  is included inside  $\bar{D}$  with  $D$  a bounded domain with Lipschitz boundary and connected complement and such that  $\Im(n) \geq 0$ .

We consider the cases where the total field is generated by incident plane waves,  $u^i(\theta, x) := e^{ikx \cdot \theta}$  with  $x \in \mathbb{R}^d$  and  $\theta \in \Gamma_s$  ( $\Gamma_s \subset \mathbb{S}^{d-1}$  the unit sphere) and we denote by  $u^s$  the scattered field defined by

$$u^s(\theta, \cdot) = u - u^i(\theta, \cdot) \text{ in } \mathbb{R}^d,$$

which is assumed to be satisfying the Sommerfeld radiation condition,

$$\lim_{r \rightarrow \infty} \int_{|x|=r} \left| \frac{\partial u^s}{\partial r} - ik u^s \right|^2 ds = 0.$$

The data for the inverse problem is formed by noisy measurements of the so called far field pattern  $u^\infty(\theta, \hat{x})$  defined by

$$u^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} (u^\infty(\theta, \hat{x}) + O(1/|x|))$$

65 as  $|x| \rightarrow \infty$  for all  $(\theta, \hat{x}) \in \Gamma_s \times \Gamma_m$ , where  $\Gamma_m$  is a subset of  $\mathbb{S}^{d-1}$  possibly different from  $\Gamma_s$ .  
 66 The goal is to be able to reconstruct  $D$  from these measurements (without knowing  $n$ ). We  
 67 introduce the far field operator  $F : L^2(\Gamma_s) \rightarrow L^2(\Gamma_m)$ , defined by

$$Fg(\hat{x}) := \int_{\Gamma_s} u^\infty(\theta, \hat{x})g(\theta)ds(\theta), \quad \hat{x} \in \Gamma_m.$$

68 Let us define, for  $\psi \in L^2(D)$ , the unique function  $w \in H_{\text{loc}}^1(\mathbb{R}^d)$  satisfying

$$69 \quad (1) \quad \begin{cases} \Delta w + nk^2w = -k^2(n-1)\psi \text{ in } \mathbb{R}^d, \\ \lim_{r \rightarrow \infty} \int_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds = 0. \end{cases}$$

By linearity of the forward scattering problem,  $Fg$  is nothing but the far field pattern of  $w$  solution of (1) with  $\psi = v_g$  in  $D$ , where

$$v_g(x) := \int_{\Gamma_s} e^{ikx \cdot \theta} g(\theta)ds(\theta), \quad g \in L^2(\Gamma_s), \quad x \in \mathbb{R}^d.$$

70 Now consider the (compact) operator  $H_s : L^2(\Gamma_s) \rightarrow L^2(D)$  defined by

$$71 \quad (2) \quad H_s g := v_g|_D,$$

72 and the (compact) operator  $G_m : \overline{\mathcal{R}(H_s)} \subset L^2(D) \rightarrow L^2(\Gamma_m)$  defined by

$$73 \quad (3) \quad G_m \psi := w^\infty|_{\Gamma_m}$$

where  $w^\infty$  is the far field of  $w \in H_{\text{loc}}^1(\mathbb{R}^d)$  solution of (1) and where  $\overline{\mathcal{R}(H_s)}$  denotes the closure of the range of  $H_s$  in  $L^2(D)$ . Then clearly

$$F = G_m H_s$$

74 One can still decompose  $F$  to get the second factorisation of the far field operator. More  
 75 precisely, for the case under consideration, since the far field pattern of  $w$  has the following  
 76 expression ([4])

$$77 \quad w^\infty(\hat{x}) = - \int_D e^{-iky \cdot \hat{x}} (1-n)k^2(\psi(y) + w(y))dy,$$

one simply has  $G_m = H_m^* T \psi$ , where  $H_m^* : L^2(D) \rightarrow L^2(\Gamma_m)$  is the adjoint of the operator  $H_m$  (defined similarly to  $H_s$  but with  $\Gamma_s$  replaced by  $\Gamma_m$ ) and whose expression is given by

$$H_m^* \varphi(\hat{x}) := \int_D e^{-iky \cdot \hat{x}} \varphi(y)dy, \quad \varphi \in L^2(D), \quad \hat{x} \in \Gamma_m,$$

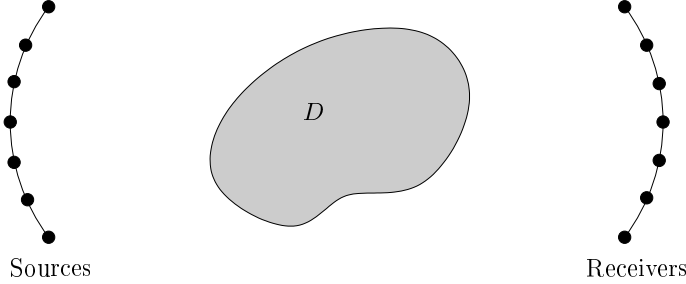
78 and where the operator  $T : L^2(D) \rightarrow L^2(D)$  is defined by

$$79 \quad (4) \quad T\psi := -k^2(1-n)(\psi + w),$$

80 with  $w \in H_{\text{loc}}^1(\mathbb{R}^d)$  being the solution of (1). Finally we end up with

$$81 \quad (5) \quad F = H_m^* T H_s.$$

82 This factorization is called "non symmetric" in the cases  $H_m \neq H_s$  which correspond to  
 83  $\Gamma_s \neq \Gamma_m$ . The GLSM as formulated in [3] applies to the "symmetric" cases, i.e.  $\Gamma_s = \Gamma_m$ .  
 84 Physically the latter correspond with sources and receivers on symmetric opposite sides of the  
 85 target (as shown in Figure 1).



**Figure 1.** Sources-receivers configurations that correspond with symmetric factorizations of the far field operator.

86 Our focus in the following is to extend the GLSM to non symmetric factorizations of the  
 87 measurement operator.

**3. Theoretical foundation of the GLSM for limited aperture.** In this section we shall give the theoretical foundation of the extension of the GLSM to non symmetric factorizations. We will adopt an abstract framework that can be applied to other settings than the one presented in the previous section (See for instance Section 5 where the case of near field data is considered). As pointed above, the "symmetry" in the factorization of the far field operator is of primary importance in the GLSM framework of [3] where the following cost functional (for noise-free data) was introduced:

$$J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \alpha^{1-\eta} |\langle Fg - \phi, g \rangle| + \|Fg - \phi\|^2$$

88 with  $B$  being an operator constructed from  $F$  and that has a "symmetric" factorization. The  
 89 latter seems to be hard to ensure in general when  $F$  itself has not a "symmetric" factorization.  
 90 In some special cases this can be done as for heterogeneous backgrounds [10] or special settings  
 91 of the near field data [5]. However, in the case of limited aperture presented above with  
 92  $\Gamma_m \neq \Gamma_s$ , this type of construction seems to be impossible to achieve. This is why we shall  
 93 consider in the following only the case  $B = F$ .

94 As has been pointed out in [2], for the case  $B = F$ , one cannot guarantee in general the  
 95 strong convergence of Herglotz waves associated with the minimizing sequences of  $J_\alpha(\phi; g)$   
 96 (when the sampling point is inside  $D$ ). Since this convergence is an important property for  
 97 some imaging algorithms (as in [2] for the case of differential measurements), we shall first  
 98 modify the setting of GLSM so that one obtain this convergence result even in the case  $B = F$ .  
 99 The idea is to add an extra (carefully chosen) penalty term that is inspired from difficulties  
 100 encountered in establishing the over mentioned convergence result in the classical setting of  
 101 GLSM.

### 3.1. A new formulation of the GLSM for symmetric factorizations.

**3.1.1. Analysis of the noise free case.** We denote by  $X$  and  $Y$  two (complex) reflexive Banach spaces with duals  $X^*$  and  $Y^*$  respectively and shall denote by  $\langle \cdot, \cdot \rangle$  a duality product that refers to  $\langle X^*, X \rangle$  or  $\langle Y^*, Y \rangle$  duality. We consider the linear operator  $F : X \rightarrow X^*$ . Moreover we shall assume that the following factorization holds

$$(6) \quad F = H^*TH$$

where the operators  $H : X \rightarrow Y$  and  $T : Y \rightarrow Y^*$  are bounded. We denote by  $G : \overline{\mathcal{R}(H)} \subset Y \rightarrow X^*$  the linear operator  $H^*T$  restricted to  $\overline{\mathcal{R}(H)}$ .

Let  $\alpha > 0$  be a given parameter and  $\phi \in X^*$ . The new GLSM (for noise free measurements) is based on considering minimizing sequences of the functional  $J_\alpha(\phi; \cdot) : X \rightarrow \mathbb{R}$

$$(7) \quad J_\alpha(\phi; g) := \alpha |\langle Fg, g \rangle| + \alpha^{1-\eta} |\langle Fg - \phi, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X,$$

where  $\eta \in ]0, 1]$  is a fixed parameter. Following [3], we first observe that

$$(8) \quad j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g) \rightarrow 0 \text{ as } \alpha \rightarrow 0.$$

for all  $\phi \in X^*$  if one assumes that  $F$  has dense range. Indeed in this case, for a given  $\varepsilon > 0$  there exists  $g_\varepsilon$  such that  $\|Fg_\varepsilon - \phi\| < \frac{\varepsilon}{2}$ . Then one can choose  $\alpha_0(\varepsilon)$  such for all  $\alpha \leq \alpha_0(\varepsilon)$ ,  $\alpha |\langle Fg_\varepsilon, g_\varepsilon \rangle| + \alpha^{1-\eta} |\langle Fg_\varepsilon - \phi, g_\varepsilon \rangle| < \frac{\varepsilon}{2}$  so that  $j_\alpha(\phi) < \varepsilon$ , which proves (8). One then can prove the following characterization of the range of  $G$  in terms of  $F$ :

**Theorem 1.** *We assume that  $H$  is compact,  $G$  is injective and  $F$  is injective with dense range. We also assume that  $T$  satisfies the coercivity property*

$$(9) \quad |\langle Th, h \rangle| > \mu \|h\|^2 \quad \forall h \in \mathcal{R}(H),$$

where  $\mu > 0$  is a constant independent of  $h$ . Consider for  $\alpha > 0$  and  $\phi \in X^*$ ,  $g_\alpha \in X$  such that

$$(10) \quad J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha)$$

where  $\frac{p(\alpha)}{\alpha}$  is bounded with respect to  $\alpha$ . Then

$$\phi \in \mathcal{R}(G) \quad \text{iff} \quad \lim_{\alpha \rightarrow 0} |\langle Fg_\alpha, g_\alpha \rangle| < \infty.$$

In the case  $\phi = G\varphi$ , the sequence  $Hg_\alpha$  converges strongly to  $\varphi$  in  $Y$  as  $\alpha$  goes to zero.

*Proof.* Assume that  $\phi \in \mathcal{R}(G)$  and let  $\varphi \in \overline{\mathcal{R}(H)}$  such that  $G\varphi = \phi$ . For  $\alpha > 0$  one can choose  $g_0 \in X$  such that  $\|Hg_0 - \varphi\|^2 < \alpha^2$ . Then by continuity of  $G$ ,  $\|Fg_0 - \phi\|^2 < \|G\|^2 \alpha^2$ . On the other hand the continuity of  $T$  implies

$$|\langle Fg_0, g_0 \rangle| = |\langle THg_0, Hg_0 \rangle| \leq \|T\| \|Hg_0\|^2 < 2 \|T\| (\alpha^2 + \|\varphi\|^2)$$

and

$$|\langle Fg_0 - \phi, g_0 \rangle| = |\langle T(Hg_0 - \varphi), Hg_0 \rangle| \leq \|T\| \|Hg_0 - \varphi\| \|Hg_0\| < 2 \|T\| \alpha (\alpha + \|\varphi\|).$$

From the definitions of  $j_\alpha(\phi)$  and  $g_\alpha$  we have

$$\alpha|\langle Fg_0, g_0 \rangle| + \alpha^{1-\eta}|\langle Fg_0 - \phi, g_0 \rangle| + \|Fg_0 - \phi\|^2 > j_\alpha(\phi) > J_\alpha(\phi, g_\alpha) - p(\alpha).$$

126 We then deduce from the definition of  $J_\alpha$ , the fact that  $\eta \in ]0, 1]$  and previous inequalities

$$127 \quad (11) \quad \alpha|\langle Fg_\alpha, g_\alpha \rangle| \leq J_\alpha(\phi, g_\alpha) \leq p(\alpha) + 2\alpha \|T\| (\alpha^2 + \|\varphi\|^2) + \alpha^2 \|G\|^2 + 2 \|T\| \alpha^{2-\eta}(\alpha + \|\varphi\|).$$

Therefore  $\limsup_{\alpha \rightarrow 0} |\langle Fg_\alpha, g_\alpha \rangle| < \infty$ . The coercivity property of  $T$  implies that  $\|Hg_\alpha\|^2$  is

bounded. From (8) and (10) and the injectivity of  $G$  we infer that the only possible weak limit of (any subsequence of)  $Hg_\alpha$  is  $\varphi$ . Thus the whole sequence  $Hg_\alpha$  weakly converges to  $\varphi$  in  $Y$ . On the other hand we have that:

$$\begin{aligned} \|Hg_\alpha - \varphi\|^2 &\leq |\langle T(Hg_\alpha - \varphi), Hg_\alpha - \varphi \rangle| \\ &\leq |\langle T(Hg_\alpha - \varphi), Hg_\alpha \rangle| + |\langle T(Hg_\alpha - \varphi), \varphi \rangle| \\ &\leq |\langle Fg_\alpha - \phi, g_\alpha \rangle| + |\langle T(Hg_\alpha - \varphi), \varphi \rangle| \end{aligned}$$

The last term goes to zero due to the weak convergence of  $Hg_\alpha$ . The first term goes to zero since the second inequality in (11) implies in particular that  $|\langle Fg_\alpha - \phi, g_\alpha \rangle| \leq \alpha^\eta$ . Therefore we conclude that  $Hg_\alpha$  strongly converges to  $\varphi$  and consequently

$$\lim_{\alpha \rightarrow 0} |\langle Fg_\alpha, g_\alpha \rangle| = |\langle T\varphi, \varphi \rangle|.$$

128 We now consider the case  $\phi \notin \mathcal{R}(G)$ . Assume that  $\liminf_{\alpha \rightarrow 0} |\langle Fg_\alpha, g_\alpha \rangle| < \infty$ . Then, (for some

129 extracted subsequence  $g_\alpha$ )  $|\langle Fg_\alpha, g_\alpha \rangle| < A$  for some constant  $A$  independent of  $\alpha \rightarrow 0$ . The

130 coercivity of  $T$  implies that  $\|Hg_\alpha\|$  is also bounded and therefore one can assume that, up to

131 an extracted subsequence,  $Hg_\alpha$  weakly converges to some  $\varphi \in \overline{\mathcal{R}(H)}$ . Since  $G$  is compact,

132 we obtain that  $GHg_\alpha$  strongly converges to  $G\varphi$  as  $\alpha \rightarrow 0$ . On the other hand, (8) and the

133 definition of  $J_\alpha(\phi, g_\alpha)$  imply that  $\|Fg_\alpha - \phi\| \leq J_\alpha(\phi, g_\alpha) \leq j_\alpha(\phi) + C\alpha \rightarrow 0$  as  $\alpha \rightarrow 0$ . Since

134  $Fg_\alpha = GHg_\alpha$  we obtain that  $G\varphi = \phi$  which is a contradiction. We then conclude that if

135  $\phi \notin \mathcal{R}(G)$  then  $\lim_{\alpha \rightarrow 0} |\langle Fg_\alpha, g_\alpha \rangle| = \infty$ . ■

136 **Remark 1.** *The extension proposed in Theorem 1 requires indeed less assumptions to ensure*  
 137 *strong convergence than the one proposed in [2] for the case of symmetric factorizations.*  
 138 *However the result from [2] is still interesting for practical applications (when applicable)*  
 139 *since it uses a convex cost functional which is easier to minimize numerically.*

**3.1.2. Analysis for the case of noisy measurements.** Let  $F^\delta : X \rightarrow X^*$  be the operator associated with noisy far field measurements such that

$$\|F^\delta - F\| \leq \delta$$

140 for some  $\delta > 0$ . We assume that the operators  $F^\delta$  and  $F$  are compact. Again let  $\eta \in ]0, 1]$  be

141 a fixed parameter. We define for  $\alpha > 0$  and  $\phi \in X^*$  the regularized functional

$$142 \quad (12) \quad J_\alpha^\delta(\phi; g) := \alpha|\langle F^\delta g, g \rangle| + \alpha^{1-\eta}|\langle F^\delta g - \phi, g \rangle| + \alpha^{1-\eta}\delta \|g\|^2 + \|F^\delta g - \phi\|^2$$

143 for  $g \in X$ . This functional has a minimizer

$$144 \quad (13) \quad g_\alpha^\delta := \arg \min_{g \in X} J_\alpha^\delta(\phi; g)$$

145 and we also have, assuming that  $F$  has a dense range,

$$146 \quad (14) \quad \lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} J_\alpha^\delta(\phi; g_\alpha^\delta) = 0.$$

The latter can be proved exactly the same way as in [3, Lemma 4] or Lemma 5 below and is based on the estimate

$$J_\alpha^\delta(\phi; g) \leq J_\alpha(\phi; g) + (\alpha\delta + \alpha^{1-\eta}\delta + \alpha^{1-\eta}\delta + \delta^2) \|g\|^2$$

147 and (8). We now state and prove the main result of this section.

148 **Theorem 2.** *Assume that the hypothesis of Theorem 1 hold true. Let  $g_\alpha^\delta$  be the minimizer*  
 149 *of  $J_\alpha^\delta(\phi; \cdot)$  for  $\alpha > 0$ ,  $\delta > 0$  and  $\phi \in X^*$ . Then*

$$150 \bullet \phi \in \mathcal{R}(G) \text{ implies } \lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \delta \alpha^{-\eta} \|g_\alpha^\delta\|^2 \right) < \infty.$$

$$151 \bullet \phi \notin \mathcal{R}(G) \text{ implies } \lim_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \delta \alpha^{-\eta} \|g_\alpha^\delta\|^2 \right) = \infty.$$

Moreover, when  $\phi \in \mathcal{R}(G)$  we also have

$$\lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \delta \|g_\alpha^\delta\|^2 = 0.$$

152 *If  $G\varphi = \phi$ , then there exists  $\delta_0(\alpha)$  such that for all  $\delta(\alpha) \leq \delta_0(\alpha)$ ,  $Hg_\alpha^{\delta(\alpha)}$  converges strongly*  
 153 *to  $\varphi$  as  $\alpha$  goes to zero.*

*Proof.* The proof follows the lines of the proof of Theorem 1. Assume that  $\phi = G(\varphi)$  for some  $\varphi \in \overline{\mathcal{R}(H)}$ . We consider  $g_0$  (that depends on  $\alpha$  but is independent from  $\delta$ ) such that  $\|Hg_0 - \varphi\|^2 < \alpha^2$ . Choosing  $\delta$  sufficiently small such that

$$(\alpha\delta + \alpha^{1-\eta}\delta + \alpha^{1-\eta}\delta + \delta^2) \|g_0\|^2 \leq \alpha$$

154 we get

$$155 \quad (15) \quad J_\alpha^\delta(\phi; g_\alpha^\delta) \leq J_\alpha^\delta(\phi; g_0) \leq J_\alpha(\phi; g_0) + \alpha.$$

Consequently, following the same arguments as for the second inequality in (11), we arrive at

$$\alpha \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \alpha^{-\eta} \delta \|g_\alpha^\delta\|^2 \right) \leq J_\alpha^\delta(\phi; g_\alpha^\delta) \leq C\alpha,$$

for sufficiently small  $\alpha$  with  $C$  a constant independent from  $\alpha$ . This proves  $\limsup_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \alpha^{-\eta} \delta \|F^\delta g_\alpha^\delta\|^2 \right) < \infty$ . We also have, as a consequence of the inequalities above, that

$$\delta \|g_\alpha^\delta\|^2 \leq C\alpha^\eta,$$



which proves  $\limsup_{\alpha \rightarrow 0} \lim_{\delta \rightarrow 0} \delta \|g_\alpha^\delta\|^2 = 0$ . We also have

$$|\langle F^\delta g_\alpha^\delta - \phi, g_\alpha^\delta \rangle| \leq C\alpha^\eta$$

which proves, with the estimate on  $\delta \|g_\alpha^\delta\|^2$  given above, that

$$\lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} |\langle F g_\alpha^\delta - \phi, g_\alpha^\delta \rangle| = 0.$$

156 We then get that  $\lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| < \infty$  and can conclude as in the proof of Theorem  
 157 1 that  $Hg_\alpha^{\delta(\alpha)}$  converges strongly to  $\varphi$  as  $\alpha$  goes to zero for  $\delta(\alpha)$  sufficiently small. This also  
 158 proves that  $\lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \delta \alpha^{-\eta} \|g_\alpha^\delta\|^2 \right) < \infty$ .

159 Now assume that  $\phi \notin \mathcal{R}(G)$  and  $\liminf_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} \left( |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \alpha^{-\eta} \delta \|g_\alpha^\delta\|^2 \right)$  is finite. The  
 160 coercivity of  $T$  and  $\alpha < 1$  implies that

$$161 \quad \mu \|Hg_{\alpha(\delta)}^\delta\|^2 \leq |\langle F g_\alpha^\delta, g_\alpha^\delta \rangle| \leq |\langle F^\delta g_\alpha^\delta, g_\alpha^\delta \rangle| + \alpha^{-\eta} \delta \|g_\alpha^\delta\|^2. \quad \blacksquare$$

162 Therefore  $\liminf_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} \|Hg_\alpha^\delta\|^2$  is also finite. This means the existence of a subsequence  
 163  $(\alpha', \delta(\alpha'))$  such that  $\alpha' \rightarrow 0$  and  $\delta(\alpha') \rightarrow 0$  as  $\alpha' \rightarrow 0$  and  $\|Hg_{\alpha'}^{\delta(\alpha')}\|^2$  is bounded independently  
 164 from  $\alpha'$ . One can also choose  $\delta(\alpha')$  such that  $\delta(\alpha') \leq \alpha'^{1-\eta}$ . On the other hand Equation  
 165 (14) indicates that one can choose this subsequence such that  $J_{\alpha'}^{\delta(\alpha')}(g_{\alpha'}^{\delta(\alpha')}) \rightarrow 0$  as  $\alpha' \rightarrow 0$   
 166 and therefore  $\|F^\delta g_{\alpha'}^{\delta(\alpha')} - \phi\| \rightarrow 0$  as  $\alpha' \rightarrow 0$  and  $\alpha'^{1-\eta} \delta(\alpha') \|g_{\alpha'}^{\delta(\alpha')}\|^2 \rightarrow 0$  as  $\alpha' \rightarrow 0$ . By a  
 167 triangular inequality and  $\delta(\alpha') \leq \alpha'^{1-\eta}$  we then deduce that  $\|Fg_{\alpha'}^{\delta(\alpha')} - \phi\| \rightarrow 0$  as  $\alpha' \rightarrow 0$ .  
 168 The compactness of  $G$  implies that a subsequence of  $GHg_{\alpha'}^{\delta(\alpha')}$  converges for some  $G\varphi$  in  $X^*$ .  
 169 The uniqueness of the limit implies that  $G\varphi = \phi$ , which is a contradiction.

170 **3.2. The GLSM for non symmetric factorizations.** In this section we shall extend GLSM  
 171 formalism presented in the previous section to the case of non symmetric factorisations. The  
 172 general framework is given by the following assumptions. We shall denote by  $X_1$ ,  $X_2$  and  
 173  $Y$  three (complex) reflexive Banach spaces with duals  $X_1^*$ ,  $X_2^*$  and  $Y^*$  respectively and shall  
 174 denote by  $\langle \cdot, \cdot \rangle$  a duality product that refers to  $\langle X_1^*, X_1 \rangle$ ,  $\langle X_2^*, X_2 \rangle$  or  $\langle Y^*, Y \rangle$  duality. We  
 175 also set  $X := X_1 \times X_2$ .

176 We consider a linear operator  $F : X_2 \rightarrow X_1^*$  that is assumed to be bounded and has the  
 177 following factorization

$$178 \quad (16) \quad F = U^*TV$$

179 where the operators  $V : X_2 \rightarrow Y$ ,  $T : Y \rightarrow Y^*$  and  $U : X_1 \rightarrow Y$  are bounded. We set  
 180  $G : \overline{\mathcal{R}_Y(V)} \subset Y \rightarrow X_1^*$  the restriction of  $U^*T$  to  $\overline{\mathcal{R}_Y(V)}$  where  $\overline{\mathcal{R}_Y(V)}$  is the closure of the

181 range of  $V$  in  $Y$ . We shall assume in addition the existence of a space  $\hat{Y}$  such that  $U$  and  $V$   
 182 can be extended to bounded operators  $V : X_2 \rightarrow \hat{Y}$  and  $U : X_1 \rightarrow \hat{Y}$  such that

$$183 \quad (17) \quad \|Vg_2 + Ug_1\|_Y \leq \|Vg_2 + Ug_1\|_{\hat{Y}}, \quad \forall (g_1, g_2) \in X.$$

184 We finally assume that

$$185 \quad (18) \quad \overline{\mathcal{R}_Y(V)} = \overline{\mathcal{R}_Y(U)} \quad \text{and} \quad \overline{\mathcal{R}_{\hat{Y}}(V)} = \overline{\mathcal{R}_{\hat{Y}}(U)}.$$

A typical example is the case of limited aperture presented above with  $X_2 = L^2(\Gamma_s)$ ,  $X_1 = L^2(\Gamma_m)$ ,  $Y = L^2(D)$  and  $\hat{Y} = L^2(\Sigma)$  with  $\Sigma$  being any domain such that  $D \subset \Sigma$ . The domain  $\Sigma$  is assumed to be known a priori (which can coincide with the whole probed domain) and therefore the operators  $V : X_2 \rightarrow \hat{Y}$  and  $U : X_1 \rightarrow \hat{Y}$  are also known a priori. In the case of limited aperture presented above these operators are defined by

$$Vg(x) = \int_{\Gamma_s} e^{ikx \cdot \theta} g(\theta) ds(\theta) \quad \text{and} \quad Ug(x) = \int_{\Gamma_m} e^{ikx \cdot \theta} g(\theta) ds(\theta), \quad x \in \Sigma.$$

186 **3.2.1. Analysis of the noise free case.** Let  $\alpha > 0$  be a given parameter and  $\phi \in X_1^*$ . We  
 187 redefine the functional  $J_\alpha$  as  $J_\alpha(\phi; \cdot) : X = X_1 \times X_2 \rightarrow \mathbb{R}$

$$188 \quad (19) \quad J_\alpha(\phi; g) := \alpha |\langle Fg_2, g_1 \rangle| + \alpha^{1-\eta} \|Vg_2 - Ug_1\|_{\hat{Y}}^2 + \alpha^{1-\eta} |\langle Fg_2 - \phi, g_1 \rangle| + \|Fg_2 - \phi\|^2$$

189 for all  $g = (g_1, g_2) \in X$  where  $\eta \in ]0, 1[$  is again a fixed parameter. We also set

$$190 \quad (20) \quad j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g).$$

Indeed the role of the extra term  $\|Vg_2 - Ug_1\|$  is to formally ensure  $Vg_2 \simeq Ug_1$  which cannot be done exactly since the ranges of the operators  $V$  and  $U$  are different in general. We then observe that the penalty term is of the form

$$\langle Fg_2, g_1 \rangle = \langle TVg_2, Ug_1 \rangle \simeq \langle TVg_2, Vg_2 \rangle$$

191 and therefore formally behaves as in the case of symmetric factorizations. The goal of the  
 192 following analysis is to show that this is indeed asymptotically the case as  $\alpha \rightarrow 0$ . We first  
 193 prove that with the additional penalty term, the inf still goes to 0 as  $\alpha \rightarrow 0$  which guarantee  
 194 that we can construct nearby solutions of the  $Fg \simeq \phi$ .

195 **Lemma 3.** *Assume that  $F$  has dense range. Then for all  $\phi \in X_1^*$ ,  $j_\alpha(\phi) \rightarrow 0$  as  $\alpha \rightarrow 0$ .*

196 **Proof.** Since  $F$  has dense range, for a given  $\varepsilon > 0$  there exists  $g_2^\varepsilon$  such that

$$197 \quad (21) \quad \|Fg_2^\varepsilon - \phi\| \leq \varepsilon/3.$$

198 Using (18) and (17) we can choose  $g_1^\varepsilon$  such that:

$$199 \quad (22) \quad \|Vg_2^\varepsilon - Ug_1^\varepsilon\|_{\hat{Y}}^2 < \|Vg_2^\varepsilon - Ug_1^\varepsilon\|_{\hat{Y}}^2 < \varepsilon/3$$

200 One then can choose  $\alpha$  small enough such that

$$201 \quad (23) \quad \alpha |\langle Fg_2^\varepsilon, g_1^\varepsilon \rangle| + \alpha^{1-\eta} |\langle Fg_2^\varepsilon - \phi, g_1^\varepsilon \rangle| \leq \varepsilon/3.$$

Together with inequalities (21) and (22) the latter inequality implies

$$j_\alpha(\phi) \leq J_\alpha(\phi; g^\varepsilon) \leq \varepsilon$$

202 for sufficiently small  $\alpha$  where  $g^\varepsilon := (g_1^\varepsilon, g_2^\varepsilon)$ . ■

203 We now can state and prove the main theorem of this section that provides a characterization  
204 of the range of  $G$  in terms of  $F$  and  $U$  and  $V$  as operators with values in  $\hat{Y}$ .

205 **Theorem 4.** *We assume that  $G : \overline{\mathcal{R}_Y(V)} \subset Y \rightarrow X_1^*$  is injective and that  $F$  has dense  
206 range. We also assume that  $T$  satisfies the coercivity property*

$$207 \quad (24) \quad |\langle T\varphi, \varphi \rangle| > \mu \|\varphi\|^2 \quad \forall \varphi \in \overline{\mathcal{R}(U)} = \overline{\mathcal{R}(V)},$$

208 where  $\mu > 0$  is a constant independent of  $\varphi$ . Let  $p(\alpha)$  be a given function such that  $\frac{p(\alpha)}{\alpha} = O(1)$   
209 and consider for  $\alpha > 0$  and  $\phi \in X_1^*$ ,  $g^\alpha = (g_1^\alpha, g_2^\alpha) \in X$  such that

$$210 \quad (25) \quad J_\alpha(\phi; g^\alpha) \leq j_\alpha(\phi) + p(\alpha).$$

211 Then we have the following:

212 •  $\phi \in \mathcal{R}(G)$  implies  $\limsup_{\alpha \rightarrow 0} \left( |\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_{\hat{Y}}^2 \right) < \infty$ .

213 •  $\phi \notin \mathcal{R}(G)$  implies  $\lim_{\alpha \rightarrow 0} \left( |\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_{\hat{Y}}^2 \right) = \infty$

214 In the case  $\phi = G\varphi$ , the two sequences  $Vg_2^\alpha$  and  $Ug_1^\alpha$  strongly converge to  $\varphi$  in  $Y$ .

215 **Proof.** The proof follows roughly the same steps and ideas as the proof for the case of  
216 symmetric factorizations. We start with the case  $\phi \in \mathcal{R}(G)$ . We consider  $\varphi \in \overline{\mathcal{R}_Y(V)}$  such  
217 that  $G\varphi = \phi$  and  $h_2^\alpha \in X_2$  such that  $\|Vh_2^\alpha - \varphi\|_{\hat{Y}}^2 \leq \alpha^2$ . According to (18) and (17), there  
218 exists  $h_1^\alpha \in X_1$  such that:

$$219 \quad (26) \quad \|Vh_2^\alpha - Uh_1^\alpha\|_{\hat{Y}}^2 < \|Vh_2^\alpha - Uh_1^\alpha\|_{\hat{Y}}^2 < \alpha^\eta.$$

220 We also have

$$221 \quad (27) \quad \begin{aligned} |\langle Fh_2^\alpha, h_1^\alpha \rangle| &= |\langle TVh_2^\alpha, Uh_1^\alpha \rangle| \\ &\leq |\langle TVh_2^\alpha, Vh_2^\alpha \rangle| + |\langle TVh_2^\alpha, Uh_1^\alpha - Vh_2^\alpha \rangle| \\ &\leq \|T\| \|Vh_2^\alpha\|_{\hat{Y}}^2 + \|T\| \|Vh_2^\alpha\|_{\hat{Y}} \sqrt{\alpha^\eta} \end{aligned}$$

and

$$|\langle Fh_2^\alpha - \phi, h_1^\alpha \rangle| = |\langle T(Vh_2^\alpha - \varphi), Uh_1^\alpha \rangle| \leq \|T\| \|Vh_2^\alpha - \varphi\| \|Uh_1^\alpha\| < 2 \|T\| \alpha(\alpha + \|\varphi\| + \sqrt{\alpha^\eta}).$$

The two previous inequalities and the definitions  $g^\alpha$  and  $j_\alpha(\phi)$  lead to

$$\alpha(|\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_{\hat{Y}}^2 + \alpha^{-\eta} |\langle Fh_2^\alpha - \phi, h_1^\alpha \rangle|) \leq j_\alpha(\phi) + p(\alpha) \leq C\alpha,$$

222 where  $C$  is bounded independently of  $\alpha$ . This implies in particular that  $(|\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_Y^2)$   
 223 remains bounded as  $\alpha \rightarrow 0$ . We also get

$$224 \quad (28) \quad \|Vg_2^\alpha - Ug_1^\alpha\|_Y^2 \leq C\alpha^\eta$$

225 and

$$226 \quad (29) \quad |\langle Fg_2^\alpha - \phi, g_1^\alpha \rangle| \leq C\alpha^\eta. \quad \blacksquare$$

We shall prove now the convergence of  $Vg_2^\alpha$  strongly converges to  $\varphi$  in  $Y$  where  $G\varphi = \phi$ . The coercivity of  $T$  implies

$$\mu \|Vg_2^\alpha\|_Y^2 \leq |\langle TVg_2^\alpha, Vg_2^\alpha \rangle| \leq |\langle TVg_2^\alpha, Vg_2^\alpha \rangle| + |\langle TVg_2^\alpha, Ug_1^\alpha - Vg_2^\alpha \rangle| + |\langle TVg_2^\alpha, Ug_1^\alpha - Vg_2^\alpha \rangle|$$

On the one hand

$$|\langle TVg_2^\alpha, Vg_2^\alpha \rangle| + |\langle TVg_2^\alpha, Ug_1^\alpha - Vg_2^\alpha \rangle| = |\langle Fg_2^\alpha, g_1^\alpha \rangle| \leq C$$

and on the other hand

$$|\langle TVg_2^\alpha, Ug_1^\alpha - Vg_2^\alpha \rangle| \leq \|T\| \|Vg_2^\alpha\|_Y \|Vg_2^\alpha - Ug_1^\alpha\|_Y \leq \|T\| C\alpha^\eta \|Vg_2^\alpha\|_Y$$

These inequalities show that  $\|Vg_2^\alpha\|_Y$  is bounded. Second, from Lemma 3 and (25) and the injectivity of  $G$  we infer that the only possible weak limit of (any subsequence of)  $Vg_2^\alpha$  in  $Y$  is  $\varphi$ . Thus the whole sequence  $Vg_2^\alpha$  weakly converges to  $\varphi$  in  $Y$ . Following the idea of proof of Theorem 1 we use the formula:

$$|\langle T(Vg_2^\alpha - \varphi), Vg_2^\alpha - \varphi \rangle| \leq |\langle T(Vg_2^\alpha - \varphi), \varphi \rangle| + \underbrace{|\langle T(Vg_2^\alpha - \varphi), Vg_2^\alpha - Ug_1^\alpha \rangle|}_{\leq \|T\|(\|Vg_2^\alpha\| + \|\varphi\|)\|Vg_2^\alpha - Ug_1^\alpha\|_Y} + |\langle Fg_2^\alpha - \phi, g_1^\alpha \rangle|$$

227 The first term on the right hand side goes to zero thanks to the weak convergence, the second  
 228 term goes to zero thanks to (28) and the third term goes to zero thanks to (29). The coercivity  
 229 property of  $T$  implies that  $Vg_2^\alpha$  converges strongly to  $\varphi$  in  $Y$ . The strong convergence of  $Ug_1^\alpha$   
 230 to  $\varphi$  in  $Y$  is a direct consequence of (28).

231 We now consider the case  $\phi \notin \mathcal{R}(G)$  and assume that  $\liminf_{\alpha \rightarrow 0} (|\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_Y^2) < \infty$ .  
 232 Then, (for some extracted subsequence  $g^\alpha$ )  $|\langle Fg_2^\alpha, g_1^\alpha \rangle| + \alpha^{-\eta} \|Vg_2^\alpha - Ug_1^\alpha\|_Y^2 \leq A$  for some  
 233  $A$  independent of  $\alpha$  as  $\alpha$  goes to 0. Using the same reasoning as in the first part of the theorem  
 234 this implies that  $\|Vg_2^\alpha\|_Y$  is bounded. We then obtain a contradiction exactly in the same  
 235 way as in the proof of the second part of Theorem 1.

**3.2.2. Analysis of the case of perturbed operators.** We now consider the case of noisy data and/or non exact models. The noise in the data is modelled with an operator  $F^\delta$  such that

$$\|F^\delta - F\| \leq \delta$$

for some  $\delta > 0$ . We can also assume error in the "model" by considering perturbed operators  $U^\delta, V^\delta$

$$\|U^\delta - U\| \leq \delta \quad \text{and} \quad \|V^\delta - V\| \leq \delta.$$

236 The noisy operators  $F^\delta$ ,  $U^\delta$  and  $V^\delta$  are assumed to be compact. We introduce the counterpart  
 237 of (19) in the noisy case (for a constant  $\eta \in ]0, 1[$ ) as

$$238 \quad (30) \quad J_\alpha^\delta(\phi; g) := \alpha \left| \langle F^\delta g_2, g_1 \rangle \right| + \delta \alpha^{1-\eta} (\|g_1\|_{X_1}^2 + \|g_2\|_{X_2}^2) + \alpha^{1-\eta} |\langle F^\delta g_2 - \phi, g_1 \rangle| \\ + \alpha^{1-\eta} \|V^\delta g_2 - U^\delta g_1\|_{\tilde{Y}}^2 + \|F^\delta g_2 - \phi\|_{X_1^*}^2$$

for  $g = (g_1, g_2) \in X$ . We can also treat the case of noisy incorrect knowledge of  $\phi$  by assume that one would consider  $\phi^\delta \in X$  such that

$$\|\phi^\delta - \phi\| \leq \delta.$$

239 The analysis of the noisy case will then be mainly based on the following simple estimate

$$240 \quad (31) \quad J_\alpha^\delta(\phi^\delta; g) \leq J_\alpha(\phi; g) + n(\delta, \alpha, g),$$

241 where

$$242 \quad (32) \quad n(\delta, \alpha, g) := \delta(\alpha + \alpha^{1-\eta})(\|g_1\|_{X_1}^2 + \|g_2\|_{X_2}^2) \\ + \delta^2 \left( \|g_2\|_{X_2}^2 + \alpha^{1-\eta}(\|g_1\|_{X_1}^2 + \|g_2\|_{X_2}^2) + 1 \right).$$

**Lemma 5.** *For for all  $\alpha, \delta > 0$  the functional  $J_\alpha^\delta(\phi^\delta; \cdot)$  has a minimizer  $g^{\alpha, \delta}$ . Assume in addition that  $F$  has dense range. Then we have*

$$\lim_{\alpha \rightarrow 0} \lim_{\delta \rightarrow 0} J_\alpha^\delta(\phi^\delta; g^{\alpha, \delta}) = 0.$$

*Proof.* The existence of a minimizer is clear: for a fixed  $\alpha > 0$ ,  $\delta > 0$  and  $\phi^\delta$ , any minimizing sequence  $g^n$  of  $J_\alpha^\delta(\phi^\delta; \cdot)$  is bounded and therefore there exists a weakly convergent subsequence to some  $g^{\alpha, \delta}$ . The lower semi-continuity of the norm with respect to the weak convergence and the compactness property of the operators then imply:

$$J_\alpha^\delta(\phi^\delta; g^{\alpha, \delta}) \leq \liminf_{n \rightarrow +\infty} J_\alpha^\delta(\phi^\delta; g^n) \leq \inf_g J_\alpha^\delta(\phi^\delta; g)$$

which proves that  $g^{\alpha, \delta}$  is a minimizer of  $J_\alpha^\delta(\phi; \cdot)$ . Let  $\epsilon > 0$  be given. We consider  $g^\epsilon$  as introduced in the proof of Lemma 3 and choose  $\delta$  sufficiently small ( $\delta \leq \delta_0(\alpha, \epsilon)$ ) such that

$$n(\delta, \alpha, g^\epsilon) \leq \epsilon.$$

We then deduce from (31) and the definition of  $g^{\alpha, \delta}$  that

$$J_\alpha^\delta(\phi^\delta; g^{\alpha, \delta}) \leq J_\alpha(\phi; g^\epsilon) + \epsilon$$

and conclude as in Lemma 3 that

$$J_\alpha^\delta(\phi; g^{\alpha, \delta}) \leq 2\epsilon$$

243 for sufficiently small  $\alpha$ , which proves the second claim of the lemma. ■

We now can prove the following asymptotic characterization of the range of  $G$  (as  $\delta 0$ ). In order to shorten the notaion we define

$$R(g, \alpha, \delta) := |\langle F^\delta g_2, g_1 \rangle| + \delta \alpha^{-\eta} (\|g_1\|_{X_1}^2 + \|g_2\|_{X_2}^2) + \alpha^{-\eta} \left\| V^\delta g_2 - U^\delta g_1 \right\|_{\hat{Y}}^2.$$

244

245 **Theorem 6.** *Assume that the hypothesis of Theorem 4 hold true. Let  $g^{\alpha, \delta} = (g_1^{\alpha, \delta}, g_2^{\alpha, \delta})$  be*  
 246 *the minimizer of  $J_\alpha^\delta(\phi^\delta; \cdot)$ . Then:*

247•  $\phi \in \mathcal{R}(G)$  implies  $\limsup_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} R(g^{\alpha, \delta}, \alpha, \delta) < \infty$ .

248•  $\phi \notin \mathcal{R}(G)$  implies  $\liminf_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} R(g^{\alpha, \delta}, \alpha, \delta) = \infty$ .

249 Moreover, if  $G\varphi = \phi$ , then we also have

$$250 \quad (33) \quad \limsup_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \delta \left( \|g_1^{\alpha, \delta}\|_{X_1}^2 + \|g_2^{\alpha, \delta}\|_{X_2}^2 \right) = 0$$

251 and there exists  $\delta_0(\alpha)$  such that for all  $\delta(\alpha) \leq \delta_0(\alpha)$ ,  $Vg_2^{\alpha, \delta(\alpha)}$  and  $Ug_1^{\alpha, \delta(\alpha)}$  converge strongly  
 252 to  $\varphi$  in  $Y$  as  $\alpha$  goes to zero.

*Proof.* Consider first the case  $\phi \in \mathcal{R}(G)$ . We shall make use of same function  $h^\alpha = (h_1^\alpha, h_2^\alpha)$  as in the first part of the proof of Theorem 4 (that only depends on  $\alpha$ ). If we choose  $\delta(\alpha)$  such that :

$$n(\delta(\alpha), \alpha, h^\alpha) \leq \alpha$$

(where  $n$  is defined in (32)) then we get (as in first part of the proof of Theorem 4)

$$J_\alpha^\delta(\phi; g^{\alpha, \delta}) \leq C\alpha + \alpha.$$

253 Consequently

$$254 \quad (34) \quad R(g^{\alpha, \delta}, \alpha, \delta) \leq C$$

which proves the first assertion of the theorem. We also get, as a consequence of the inequalities above, that

$$\delta \left( \|g_1^{\alpha, \delta}\|_{X_1}^2 + \|g_2^{\alpha, \delta}\|_{X_2}^2 \right) \leq C\alpha^\eta$$

255 which proves

$$256 \quad (35) \quad \lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \delta \left( \|g_1^{\alpha, \delta}\|_{X_1}^2 + \|g_2^{\alpha, \delta}\|_{X_2}^2 \right) = 0.$$

257 For the same reasons, since  $\eta > 0$  we have

$$258 \quad (36) \quad \lim_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \left\| V^\delta g_2^{\alpha, \delta} - U^\delta g_1^{\alpha, \delta} \right\|_{\hat{Y}}^2 = 0.$$

Now choose  $\delta_0(\alpha)$  small enough such that,  $\limsup_{\alpha \rightarrow 0} n(\delta_0(\alpha), \alpha, h^\alpha) = 0$ , consider  $\delta(\alpha) \leq \delta_0(\alpha)$  and denote by  $\tilde{g}^\alpha := g^{\alpha, \delta(\alpha)}$ . Then, from (34) and (36) we clearly obtain that the quantity

$\langle TV^{\delta(\alpha)}\tilde{g}_2^\alpha, V^{\delta(\alpha)}\tilde{g}_2^\alpha \rangle$  is bounded. To conclude as in the proof of Theorem 4 that  $V\tilde{g}_2^\alpha$  and  $U\tilde{g}_1^\alpha$  converge strongly to  $\varphi$  in  $Y$  as  $\alpha$  goes to zero, one just need to remark that

$$\|V\tilde{g}_2^\alpha - U\tilde{g}_1^\alpha\|_Y^2 \leq \|V^{\delta(\alpha)}\tilde{g}_2^\alpha - U^{\delta(\alpha)}\tilde{g}_1^\alpha\|_Y^2 + \delta(\alpha)^2 \|\tilde{g}_1^\alpha\|_{X_1}^2 + \delta(\alpha)^2 \|\tilde{g}_2^\alpha\|_{X_2}^2 \rightarrow 0$$

as  $\alpha \rightarrow 0$  and

$$|\langle F\tilde{g}_2^\alpha - \phi, \tilde{g}_1^\alpha \rangle| \leq |\langle F^{\delta(\alpha)}\tilde{g}_2^\alpha - \phi, \tilde{g}_1^\alpha \rangle| + \delta(\|\tilde{g}_1^\alpha\|_{X_1}^2 + \|\tilde{g}_2^\alpha\|_{X_2}^2) \rightarrow 0.$$

259 as  $\alpha \rightarrow 0$ .

260 Consider now the case  $\phi \notin \mathcal{R}(G)$  and assume that  $\liminf_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} R(g^{\alpha, \delta}, \alpha, \delta)$  is finite.

261 Then, from

$$262 \quad (37) \quad \left| \langle Fg_2^{\alpha, \delta}, g_1^{\alpha, \delta} \rangle \right| \leq \left| \langle F^\delta g_2^{\alpha, \delta}, g_1^{\alpha, \delta} \rangle \right| + \frac{\delta}{2} \|g_1^{\alpha, \delta}\|_{X_1}^2 + \frac{\delta}{2} \|g_2^{\alpha, \delta}\|_{X_2}^2$$

263 we deduce that  $\left| \langle Fg_2^{\alpha, \delta}, g_1^{\alpha, \delta} \rangle \right|$  is bounded for some subsequence  $\delta(\alpha)$ . One also get that

264  $\|Vg_2^{\alpha, \delta} - Ug_1^{\alpha, \delta}\|_Y^2$  is bounded for the same sequence  $\delta(\alpha)$  meaning that, similarly to the

265 second part of the proof of Theorem 4, the sequence  $\|Vg_2^{\alpha, \delta(\alpha)}\|_Y$  is bounded as  $\alpha \rightarrow 0$ . We

266 then can obtain a contradiction exactly the same way as in the proof of Theorem 4.  $\blacksquare$

267 **4. Application to inverse scattering.** The purpose of this section is to apply the result  
268 of section 3 to limited aperture data (described in section 2). This will be possible if  $D \subset \Sigma$   
269 where  $\Sigma$  is some bounded known domain. We then define  $\hat{Y}$  from section 3 to be  $L^2(\Sigma)$  and  
270 set  $V = H_s$  and  $U = H_m$ .

271 The basis of the GLSM is the characterization of the obstacle  $D$  in term of the range of  
272  $G_m$ . This characterization is based on the solvability of the following interior transmission  
273 problem for  $u, v \in L^2(D)$  such that  $u - v \in H^2(D)$ ,

$$274 \quad (38) \quad \begin{cases} \Delta u + k^2 nu = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ (u - v) = f & \text{on } \partial D, \\ \frac{\partial}{\partial \nu}(u - v) = g & \text{on } \partial D, \end{cases}$$

275 for a given  $f \in H^{\frac{3}{2}}(\partial D)$  and  $g \in H^{\frac{1}{2}}(\partial D)$ . We should make the following assumption

276 **Hypothesis 1.** We assume that  $k^2 \in \mathbb{R}_+$  is such that for all  $f \in H^{\frac{3}{2}}(\partial D)$  and  $g \in H^{\frac{1}{2}}(\partial D)$   
277 problem (38) has a unique solution in  $(u, v) \in L^2(D) \times L^2(D)$  and  $u - v \in H^2(D)$ .

We recall that it is known [15] that if  $n - 1$  positive definite or negative definite in a neighborhood of  $\partial D$ , Hypothesis 1 is verified for all  $k \in \mathbb{R}$  except a countable set without finite accumulation point.

Defining

$$\phi_z(\hat{x}) := e^{-ik\hat{x} \cdot z} \text{ for } \hat{x} \in \Gamma_m$$

278 we have:

279 **Theorem 7.** Under Hypothesis 1,  $\phi_z \in \mathcal{R}(G_m)$  (for  $G_m$  defined in (3)) if and only if  $z \in D$ .

280 **Lemma 8.**  $\overline{\mathcal{R}(H_s)} = \{v \in L^2(D) \text{ s.t. } \Delta v + k^2 v = 0 \text{ in } D\} = \overline{\mathcal{R}(H_m)}$

281 The proof of this theorem is rather straightforward using the result of Lemma 8 (see [13]) and  
 282 the fact that  $\phi_z$  is the far field of  $\Phi(\cdot, z)$ , the fundamental solution of the Helmholtz equation  
 283 satisfying the Sommerfeld radiation condition.

284 The central additional theorem needed in order to apply the theory developed in Section  
 285 3 is the following coercivity property of the operator  $T$ . This theorem holds true under the  
 286 following assumptions:

287 **Hypothesis 2.** We assume that  $n \in L^\infty(\mathbb{R}^d)$  with  $\Im(n) \geq 0$  and there exist constants  $n_0, \alpha >$   
 288  $0$  such that  $1 - \Re(n) + \alpha \Im(n) \geq 0$  for a.e.  $x$  in a neighborhood of  $\partial D$  or  $\Re(n) - 1 + \alpha \Im(n) \geq 0$   
 289 for a.e.  $x$  in a neighborhood of  $\partial D$ .

290 The following lemma has been proved in [1].

291 **Lemma 9.** Assume that Hypothesis 2 holds and that  $k^2$  is not a transmission eigenvalue.  
 292 Then the operator  $T$  satisfies the coercivity property (9).

293 Let  $C > 0$  be a given constant (independent of  $\alpha$ ) and consider  $\alpha > 0$  and  $z \in \mathbb{R}^d$ ,  
 294  $g^{z,\alpha} = (g_1^{z,\alpha}, g_2^{z,\alpha}) \in L^2(\Gamma_m) \times L^2(\Gamma_s)$  such that :

$$\begin{aligned} J_\alpha(\phi_z, g^{z,\alpha}) &= \alpha |\langle F g_2^{z,\alpha}, g_1^{z,\alpha} \rangle| + \alpha^{1-\eta} \|H_s g_2^{z,\alpha} - H_m g_1^{z,\alpha}\|_{L^2(\Sigma)}^2 \\ &+ \alpha^{1-\eta} |\langle F g_2^{z,\alpha} - \phi_z, g_1^{z,\alpha} \rangle| + \|F g_2^{z,\alpha} - \phi_z\|^2 \\ &\leq j_\alpha(\phi_z) + C\alpha, \end{aligned}$$

where  $\eta \in ]0, 1[$  and

$$j_\alpha(\phi_z) = \inf_{g \in L^2(\Gamma_m) \times L^2(\Gamma_s)} J_\alpha(\phi_z, g).$$

295 Combining the results of Theorems 4 and 7 we obtain the following theorem:

296 **Theorem 10.** Assume that Hypotheses 1 and 2 hold. Then  $z \in D$  if and only if  $\limsup_{\alpha \rightarrow 0} |\langle F g_2^{z,\alpha}, g_1^{z,\alpha} \rangle| +$

297  $\alpha^{-\eta} \|H_s g_2^{z,\alpha} - H_m g_1^{z,\alpha}\|_{L^2(\Sigma)}^2 < \infty$ .

298 Moreover, if  $z \in D$  then the sequence of Herglotz wave functions associated to  $g^{z,\alpha}$  converges  
 299 strongly to the solution  $v$  of (38) with  $(f, g) = (\Phi_z, \frac{\partial \Phi_z}{\partial \nu})$  as  $\alpha$  goes to zero.

For applications, it is important to rather use the criterion provided in Theorem 6. Con-  
 sider  $F^\delta : L^2(\Gamma_s) \rightarrow L^2(\Gamma_m)$  a compact operator such that:

$$\|F^\delta - F\| \leq \delta.$$

Then consider for  $\alpha > 0$  and  $\phi \in L^2(\Gamma_m)$  the functional  $J_\alpha^\delta(\phi, \cdot) : L^2(\Gamma_s) \times L^2(\Gamma_m) \rightarrow \mathbb{R}$ ,

$$\begin{aligned} J_\alpha^\delta(\phi_z, g) &= \alpha |\langle F^\delta g_2, g_1 \rangle| + \alpha^{1-\eta} \|H_s g_2 - H_m g_1\|_{L^2(\Sigma)}^2 + \alpha^{1-\eta} \delta \|g\|^2 \\ &+ \alpha^{1-\eta} |\langle F^\delta g_2 - \phi_z, g_1 \rangle| + \|F^\delta g_2 - \phi_z\|^2 \end{aligned}$$

300 where  $\eta \in ]0, 1[$ . Then as a direct consequences of Theorem 6 we obtain the following charac-  
 301 terization of  $D$ ,



**Theorem 11.** *Assume that the hypothesis of Theorem 10 hold true. For  $z \in \mathbb{R}^d$  denote by  $g^{z,\alpha,\delta}$  the minimizer of  $J_\alpha^\delta(\phi, \cdot)$  over  $L^2(\Gamma_s) \times L^2(\Gamma_m)$ . Then  $z \in D$  if and only if*

$$\limsup_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \left| \left\langle F^\delta g_2^{z,\alpha,\delta}, g_1^{z,\alpha,\delta} \right\rangle \right| + \alpha^{-\eta} \left\| H_s g_2^{z,\alpha,\delta} - H_m g_1^{z,\alpha,\delta} \right\|_{L^2(\Sigma)}^2 + \alpha^{-\eta} \delta \left\| g^{z,\alpha,\delta} \right\|^2 < \infty.$$

302 *If  $z \in D$ , there exists  $\delta_0(\alpha)$  such that for all  $\delta(\alpha) \leq \delta_0(\alpha)$ ,  $Hg^{z,\alpha,\delta(\alpha)}$  converges strongly*  
 303 *to the solution  $v$  of (38) with  $(f, g) = (\Phi_z, \frac{\partial \Phi_z}{\partial \nu})$  as  $\alpha$  goes to zero.*

304 **5. Extension to near field data.** We concentrated in the previous sections on incident  
 305 plane waves and far field measurement and raise the problem of "non symmetric factorization"  
 306 in the case of limited apertures. We here show how the theory of Section 3 can be applied to  
 307 other configurations of non symmetric factorization. This is the case for instance of near field  
 308 data that we shall present in this section.

309 The total field is generated by point sources and the scattered field is recorded on a surface  
 310 of  $\mathbb{R}^d$  (usually where the point source lies). If we denote by  $\partial\Omega$  the surface where the sources  
 311 lie, we consider an incident field  $u^i(y, x) := \Phi(y, x)$  with  $x \in \mathbb{R}^d$  and  $y \in \partial\Omega$ . We introduce  
 312  $N : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$  defined by

$$313 \quad (39) \quad Ng := \int_{\partial\Omega} u^s(y, x)g(y)ds(y), \quad g \in H^{-\frac{1}{2}}(\partial\Omega), \quad x \in H^{\frac{1}{2}}(\partial\Omega),$$

314 where  $u^s(y, \cdot = w)$  solution of (1) with an incident field  $\psi = u^i(y, \cdot)$ . We introduce the  
 315 compact operator  $S : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow L^2(D)$  (which plays the role of  $H_s$ ) defined by

$$316 \quad (40) \quad Sg := \int_{\partial\Omega} \Phi(y, x)g(y)ds(y), \quad g \in H^{-\frac{1}{2}}(\partial\Omega), \quad x \in D$$

and the (compact) operator  $G : \overline{\mathcal{R}(S)} \subset L^2(D) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$  defined by

$$G\psi := w|_{\partial\Omega},$$

317 where  $\overline{\mathcal{R}(S)}$  denotes the closure of the range of  $S$  in  $L^2(D)$  and  $w$  is defined as in (1). Then  
 318 clearly

$$319 \quad (41) \quad N = GS.$$

In the case under consideration, since the scattered field has the following expression :

$$w(x) = - \int_D \Phi(y, x)(1-n)k^2(\psi(y) + w(y))dy,$$

one simply has  $G = \bar{S}^*T\psi$  where  $\bar{S}^* : L^2(D) \rightarrow L^2(\partial\Omega)$  is the conjugate of the adjoint of  $S$   
 given by:

$$\bar{S}^* \varphi(x) = \int_D \Phi(y, x)\varphi(y)dy, \quad x \in \Gamma,$$

320 and  $T$  is defined by (4). Finally we get

$$321 \quad (42) \quad N = \bar{S}^*TS.$$

322 As for the limited aperture case this factorization is "non symmetric".

**5.1. Point sources and point measurements on the same surface.** The case where the point sources and the measurements are on the same surface can be solved without relying on the theory developed in Section 3.2. At the cost of computing an operator  $C$  (introduced in the following) such that :

$$B = CF = H^*TH,$$

one can rely on the theory of Section 3.1 or on the theory proposed in [3]. In [13] an inf-criterion is proposed to tackle the case of near field full aperture, through the use of the corresponding far field operator. We refer to [12] for similar ideas using near field measurement. We propose to adapt this idea to the setting of the GLSM and to revisit its analysis to avoid the use of the corresponding far field operator. To do so we need to introduce the following operator, which is closely connected to  $S$  and a technical lemma.

$$(43) \quad S_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega), S_{\partial\Omega}(f)(x) = \int_{\partial\Omega} \Phi(x, y)f(y)ds(y), \quad x \in \partial\Omega$$

**Lemma 12.** *If  $k^2$  is not a Dirichlet eigenvalue of the Laplace operator in  $\Omega$ , we have that:*

$$S_{\partial\Omega}^* S_{\partial\Omega}^{-1} \bar{S}^* = S^*.$$

*Proof.* If  $k^2$  is not a Dirichlet eigenvalue of the Laplace operator in  $\Omega$ , for  $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$  we have that  $\bar{S}S_{\partial\Omega}^{-1,*}S_{\partial\Omega}\varphi$  and  $S\varphi$  solves the Helmholtz equation in  $\Omega$ . Straightforward calculations provide that  $\bar{S}_{\partial\Omega}S_{\partial\Omega}^{-1,*}S_{\partial\Omega}^*\varphi = \bar{S}_{\partial\Omega}\varphi$  therefore the two solutions share the same boundary values on  $\partial\Omega$ . By taking the adjoint we conclude the proof. ■

Using (42) we arrived at

$$B = S_{\partial\Omega}^* S_{\partial\Omega}^{-1} N = S^* T S$$

From this factorization one can either use the framework developed in [3] or the factorization method developed in [13]. One can also apply the results from Section 3.1 by substituting  $\langle Fg - \phi, g \rangle$  with  $\langle C(Fg - \phi), g \rangle$ .

**5.2. Point sources and measurements lying on different surfaces.** One can consider a limited aperture nearfield measurement by considering that the point sources are located on  $\Gamma_s \subset \partial\Omega$  and the measurements are done on  $\Gamma_m \subset \partial\Omega$  and assume that  $\Gamma_s$  and  $\Gamma_m$  are analytic surfaces. In this case similarly to the far field case we obtain a factorization :

$$(44) \quad N = \bar{S}_m^* T S_s,$$

where  $S_m$  and  $S_s$  are defined similarly to  $S_{\partial\Omega}$  with  $\partial\Omega$  replaced by  $\Gamma_m$  and  $\Gamma_s$  respectively. Similarly to Section 2, we define the compact operator  $G$  from  $\overline{\mathcal{R}(S_s)}$  to  $H^{\frac{1}{2}}(\Gamma_m)$  by  $G := \bar{S}_m^* T$ . As for the far field case we have the following result which is proven in [13],

**Lemma 13.** *If Hypothesis 1 is verified,  $\Phi_z \in \mathcal{R}(G)$  if and only if  $z \in D$ .*

**Lemma 14.** *If  $k$  is not a dirichlet eigenvalue of  $\Omega$  we have that  $S_s$  and  $S_m$  are dense in  $\{v \in L^2(D) \text{ s.t. } \Delta v + v = 0 \text{ in } D\}$*

348 As already pointed out the operator  $T$  is not changed by the type of incident wave and  
 349 measurement therefore it keeps the coercivity property given in section 4.

350 The two previous lemmas, the coercivity of  $T$  and (44) are all the required ingredients to  
 351 apply the framework of section 3.2 with  $V = S_s$ ,  $U = \bar{S}_m$  and  $F = N$ . We therefore obtain  
 352 the following corollary for the GLSM with nearfield measurements.

353 **Corollary 15.** *Assume that Hypotheses 1 and 2 hold and that  $D \subset \Sigma$ . Then  $z \in D$  if and*  
 354 *only if*

355•  $|\langle N g_2^{z,\alpha}, g_1^{z,\alpha} \rangle| + \alpha^{-\eta} \|S_s g_2^{z,\alpha} - \bar{S}_m g_1^{z,\alpha}\|_{L^2(\Sigma)}^2$  remains bounded for  $g_1^{z,\alpha}$  and  $g_2^{z,\alpha}$  defined as in  
 356 Section 3 with  $\phi = \phi_z$

357•  $R(g^{z,\alpha,\delta}, \alpha, \delta)$  (defined in Theorem 6) remains bounded for  $g^{z,\alpha,\delta}$  defined as in Section 3 with  
 358  $\phi = \phi_z$ ,  $V = S_s$  and  $U = \bar{S}_m$ .

359 Moreover we have that one can extract a subsequence from the sequence of herglotz wave  
 360 functions associated to  $g^{z,\alpha}$  (resp.  $g^{z,\alpha,\delta}$ ) which will converge strongly to the solution  $v$  of  
 361 (38) with  $(f, g) = (\Phi_z, \frac{\partial \Phi_z}{\partial \nu})$  as  $\alpha$  goes to zero (resp. as  $\alpha$  and  $\delta$  go to zero for  $\delta \leq \delta_0$ ).

362 **6. Numerical Algorithm and results.** In order to fix the ideas, we shall restrict ourselves  
 363 in a two dimensional setting with far field measurement. We identify  $\mathbb{S}^1$  with the interval  
 364  $[0, 2\pi[$ . In order to collect the data of the inverse problem we solve numerically (1) for  $N$   
 365 incident fields using the surface integral equation forward solver available in [11]. The discrete  
 366 version of  $F$  is then the matrix  $F_N$ . We add some noise to the data to build a noisy far  
 367 field matrix  $F_N^\delta$  where  $(F_N^\delta)_{j,k} = (F_N)_{j,k}(1 + \sigma N_{ij})$  for  $\sigma > 0$  and  $N_{ij}$  an uniform complex  
 368 random variable in  $[-1, 1]^2$ . We denote  $\Phi_{z,N} \in \mathbb{C}^N$ , the vector defined by  $\Phi_{z,N}(j) = \phi_z(\frac{2\pi j}{N})$   
 369 for  $0 \leq j \leq N-1$ . In all our experiments we take  $\eta = 0$  as we do not find a significant  
 370 influence for this parameter.

**6.1. Symmetric case.** First we will look at the result given when  $\Gamma_m = \Gamma_s$ . This setting  
 could be seen as a reference image as it does not introduce any new regularization term based  
 on a priori knowledge on  $D$  (the choice of  $\Sigma$ ). Moreover it can be formulated [3] as a convex  
 functional if one introduces  $F_\#^\delta = |\Re(F^\delta)| + |\Im(F^\delta)|$ , we introduce:

$$g_\#^{z,\alpha,\delta} = \arg \min_{g \in \mathbb{C}^N} \alpha \left\| (F_\#^\delta)^{\frac{1}{2}} g \right\|^2 + \alpha^{1-\eta} \delta \|g\|^2 + \left\| F^\delta g - \phi_z \right\|^2$$

This minimization is solved using the normal equation:

$$g_\#^{z,\alpha,\delta} = (\alpha F_\# + \alpha^{1-\eta} \delta Id + F^{\delta,*} F^\delta)^{-1} F^{\delta,*} \phi_z$$

And finally we use the following indicator function to retrieve the  $D$

$$\mathcal{I}_\#(z) = \frac{1}{\left\| (F_\#^\delta)^{\frac{1}{2}} g_\#^{z,\alpha,\delta} \right\|^2 + \alpha^{-\eta} \delta \left\| g_\#^{z,\alpha,\delta} \right\|^2}$$

To compare with setting where  $\Gamma_m \neq \Gamma_s$  we also introduced :

$$g^{z,\alpha,\delta} = \arg \min_{g \in \mathbb{C}^N} \alpha |\langle F^\delta g, g \rangle| + \alpha^{1-\eta} \delta \|g\|^2 + \alpha^{1-\eta} |\langle F^\delta g - \phi_z, g \rangle| + \left\| F^\delta g - \phi_z \right\|^2$$

and consider the following indicator function:

$$\mathcal{I}(z) = \frac{1}{|\langle F^\delta g^{z,\alpha,\delta}, g^{z,\alpha,\delta} \rangle| + \alpha^{-\eta} \delta \|g^{z,\alpha,\delta}\|^2}.$$

371 Computing  $g^{z,\alpha,\delta}$  is much more challenging as the functional is non convex nor differentiable  
 372 in general. In [3], a first order gradient method is used. We here improve the efficiency of this  
 373 scheme by using a second order method. We give the formula of the gradient and the hessian  
 374 explicitly in the more general case where  $\Gamma_m \neq \Gamma_s$ . The iteration are initialized by using the  
 375 original LSM [4] with Tikhonov regularization :

$$g_0^{z,\beta,\delta} = \arg \min_{g \in \mathbb{C}^N} \beta \|g\|^2 + \|F^\delta g - \phi_z\|^2$$

376 where we choose  $\beta$  such that  $\delta \|g_0^{z,\beta,\delta}\| = \|F_0^{z,\beta,\delta} - \phi_z\|$ . From this choice of  $\beta$  we set  $\alpha = \frac{\beta}{\|F_\# \|}$   
 377 or  $\frac{\beta}{\|F\|}$ .

378 We consider two examples one with two ellipses and one with a kite shape obstacle both  
 379 penetrable obstacle with index of refraction of 0.2. The axis are labelled as multiple of the  
 380 wavelength  $\lambda = 2\pi/k$ . We consider three apertures :  $[\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  $[7\pi/8, 9\pi/8[$   
 381 with a noise  $\delta = 1\%$ . In figure 2, we show the results of  $\mathcal{I}_\#$  and  $\mathcal{I}$ .

382 **6.2. NonSymetric case.** We consider the case where  $\Gamma_m \neq \Gamma_s$ . In this case we have to  
 383 define  $g^{z,\alpha,\delta}$  as the minimizer of a (non convex nor differentiable) cost functional,

$$g^{z,\alpha,\delta} = \arg \min_{g \in \mathbb{C}^N \times \mathbb{C}^N} \alpha |\langle F^\delta g_2, g_1 \rangle| + \alpha^{1-\eta} \delta \|g\|^2 + \alpha^{1-\eta} |\langle F^\delta g_2 - \phi_z, g_1 \rangle| \\ + \alpha^{1-\eta} \|H_s g_2 - H_m g_1\|^2 + \|F^\delta g_2 - \phi_z\|^2,$$

384 and we introduced the indicator function:

$$\mathcal{I}(z) = \frac{1}{|\langle F^\delta g_2^{z,\alpha,\delta}, g_1^{z,\alpha,\delta} \rangle| + \alpha^{-\eta} \delta \|g^{z,\alpha,\delta}\|^2 + \alpha^{-\eta} \|H_s g_2^{z,\alpha,\delta} - H_m g_1^{z,\alpha,\delta}\|^2}.$$

385 To minimize the cost functional we will rely on a second order descent method. We will  
 386 choose the starting point of the descent,  $g_0$ , as

$$g_{0,2}^{z,\beta_2,\delta} = \arg \min_{g \in \mathbb{C}^N} \beta_2 \|g\|^2 + \|F^\delta g - \phi_z\|^2$$

387 (45) 
$$g_{0,1}^{z,\beta_1,\delta} = \arg \min_{g \in \mathbb{C}^N} \beta_1 \|g\|^2 + \|H_m g - H_s g_{0,2}^{z,\beta_2,\delta}\|^2$$

388 where we choose  $\beta_2$  such that  $\delta \|g_{0,2}^{z,\beta_2,\delta}\| = \|F_{0,2}^{z,\beta_2,\delta} - \phi_z\|$  and  $\beta_1$  such that  $\|g_{0,1}^{z,\beta_1,\delta}\| =$   
 389  $\|g_{0,2}^{z,\beta_2,\delta}\|$ . This second choice is purely arbitrary, our purpose in setting  $\beta_1$  is to avoid,  $g_{0,1}^{z,\beta_1,\delta}$   
 390 to have large norm which would dominate numerically all other quantities.

391 The minimization of  $J_\alpha^\delta$  causes numerical problem. Indeed first numerically  $H_m$  is a  
 392 compact operator and even if it is not important for the theory we are implicitly inverting it  
 393 by minimizing  $J_\alpha^\delta$  therefore we have to be careful on the balance between the terms  $\|g_1\|^2$  and  
 394  $\|H_m g_1 - H_s g_2\|$ . This is even more important as  $H_s g_2$  is not in the range of  $H_m$ . Since the  
 395 theory does not give a strategy to set  $\alpha$ , we proposed and tested three strategies that give  
 396 similar results. Those strategies are based on the idea (we also use to pick an initial guess  $g_0$ )  
 397 that  $\|g_1^{z,\beta_1,\delta}\|$  and  $\|g_2^{z,\beta_1,\delta}\|$  should have the same order of magnitude.

First one should remark that we have used the same parameter,  $\eta$  in front of all the terms but it could have been chosen with a different value for each term (as long as it stays between 0 and 1 for the theory). Using different  $\alpha$  instead of  $\eta$  to keep simple notation we introduce:

$$J_\alpha^\delta(g_1, g_2) = \alpha |\langle F^\delta g_2, g_1 \rangle| + \alpha_1 \delta \|g_1\|^2 + \alpha_2 \delta \|g_2\|^2 + \alpha^{1-\eta} |\langle F^\delta g_2 - \phi_z, g_1 \rangle| \\ + \alpha_3 \|H_s g_2 - H_m g_1\|^2 + \|F^\delta g_2 - \phi_z\|^2$$

398 We have actually increase the number of parameter in order to get some freedom to balance  
 399 the term involving  $g_1$  and  $H_m g_1$ . To set  $\alpha$  we use again our heuristic:  $\alpha = \frac{\beta_2}{\|F^\delta\|}$ . We propose  
 400 to choose  $\alpha_1 = \alpha_2 = \alpha$  and  $\alpha_3 = \alpha\delta/\beta_1$  and therefore keep the regularizing power used to  
 401 find the initial guess. The parameters set, we used a newton method to minimize  $J_\alpha^\delta$ .

402 A second solution we have experienced is to alternatively minimize  $J_\alpha^\delta$  as a function of  $g_2$   
 403 with  $\alpha_3 = \alpha_1 = 0$  and to minimize the same Tikhonov functional (45) we used to find the  
 404 initial guess  $g_{0,1}$ . This will impose  $\|g_1^{z,\beta_1,\delta}\| = \|g_2^{z,\beta_2,\delta}\|$  and limit the number of parameters  
 405 to set, however it is not a scheme that is cover by the theory.

406 A third solution closely related to our heuristic for symmetric factorization, we have set  
 407  $\alpha_3$  to 1 and  $\alpha_1 = \alpha_2 = \alpha$  where  $\alpha$  is chosen to be equal to  $\max(\beta_1, \beta_2)/\|F^\delta\|$ .

408 All those three methods give similar result. In the following we will show only the results  
 409 of the first method. In order to perform the Newton method we need to compute the gradient  
 410 and the Hessian which we explicit in the following for the original cost functional, both gradient  
 411 and Hessian can be easily derive from those formulas. If  $\cdot$  is the dot product without conjugate,  
 412  $^t$  the transposition and by  $^*$  the classical transpose-conjugate, we can rewrite  $J_\alpha^\delta(\phi, \cdot)$  :

$$\alpha |g^* \cdot (F^\delta g)| + \alpha^{1-\eta} |g^* \cdot (F^\delta g - \phi)| + \delta \alpha^{1-\eta} \|F^\delta\| \|g^* \cdot g + \alpha^{1-\eta} (H g)^* \cdot (H g) + (F^\delta g - \phi)^* \cdot (F^\delta g - \phi)\|$$

where we use the matrix:

$$F^\delta = \begin{bmatrix} 0 & F_N^\delta \\ 0 & 0 \end{bmatrix} \text{ and } H = [ H_m \quad -H_s ] \text{ and } g = \begin{bmatrix} g_2 \\ g_1 \end{bmatrix} \text{ and } \phi = \begin{bmatrix} \Phi_z \\ 0 \end{bmatrix}$$

413 Using this notation we can compute following the framework of [14] the gradient

$$\delta \alpha^{1-\eta} \|F^\delta\| \|g + F^{\delta*} (F^\delta g - \phi) + \alpha^{1-\eta} H^* H g\| + \alpha \frac{\overline{g^* \cdot (F^\delta g)} F^\delta g + (g^* \cdot (F^\delta g)) F^{\delta*} g}{|g^* \cdot (F^\delta g)|} \\ + \alpha^{1-\eta} \frac{(g^* \cdot (F^\delta g - \phi)) (F^\delta g - \phi) + (g^* \cdot (F^\delta g - \phi)) F^{\delta*} g}{|g^* \cdot (F^\delta g - \phi)|}$$

414 and the Hessian,

$$\begin{aligned}
 & \delta\alpha^{1-\eta} \left\| F^\delta \right\| \left\| Id + F^{\delta*} F^\delta + \alpha^{1-\eta} H^* H + \alpha \frac{\overline{g^* \cdot (F^\delta g)} F^\delta + (g^* \cdot (F^\delta g)) F^{\delta*} + F^\delta g g^* F^{\delta*} + F^{\delta*} g g^* F^\delta}{|g^* \cdot (F^\delta g)|} \right. \\
 & - \alpha \frac{\overline{(g^* \cdot (F^\delta g)) F^\delta g + (g^* \cdot (F^\delta g)) F^{\delta*} g} \overline{(g^* \cdot (F^\delta g)) F^\delta g + (g^* \cdot (F^\delta g)) F^{\delta*} g}^*}{2|g^* \cdot (F^\delta g)|^{\frac{3}{2}}} \\
 & + \alpha^{1-\eta} \left( \frac{\overline{(g^* \cdot (F^\delta g - \phi)) F^\delta} + (g^* \cdot (F^\delta g - \phi)) F^{\delta*} + (F^\delta g - \phi)(g^* F^{\delta*} - \phi^*) + F^{\delta*} g g^* F^\delta}{|g^* \cdot (F^\delta g - \phi)|} \right. \\
 & \left. - \frac{\overline{((g^* \cdot (F^\delta g - \phi))(F^\delta g - \phi) + (g^* \cdot (F^\delta g - \phi)) F^{\delta*} g)} \overline{((g^* \cdot (F^\delta g - \phi))(F^\delta g - \phi) + (g^* \cdot (F^\delta g - \phi)) F^{\delta*} g)}^*}{2|g^* \cdot (F^\delta g - \phi)|^{\frac{3}{2}}} \right).
 \end{aligned}$$

416 We apply those techniques to the case of back scattering data which is when  $\Gamma_m = -\Gamma_s$ , for  
 417 apertures  $\Gamma_s = [\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  $[7\pi/8, 9\pi/8[$ . The result are shown in figure 3 for  
 418 the kite example for a domain  $\Sigma$  which occupies the whole image and the smallest rectangle  
 419 that contains  $D$ . We also consider the case of  $\Gamma_s$  being either  $[\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  
 420  $[7\pi/8, 9\pi/8[$  and  $\Gamma_m$  being either  $[0, \pi[$ ,  $[\pi/4, 3\pi/4[$  and  $[3\pi/8, 5\pi/8[$ . The results are shown  
 421 in figure 4, again for a kite example and an original setting of sources and measurements. On  
 422 those simulation the size of  $\Sigma$  has no clear impact therefore we will only show simulation for  
 423 the large grid.

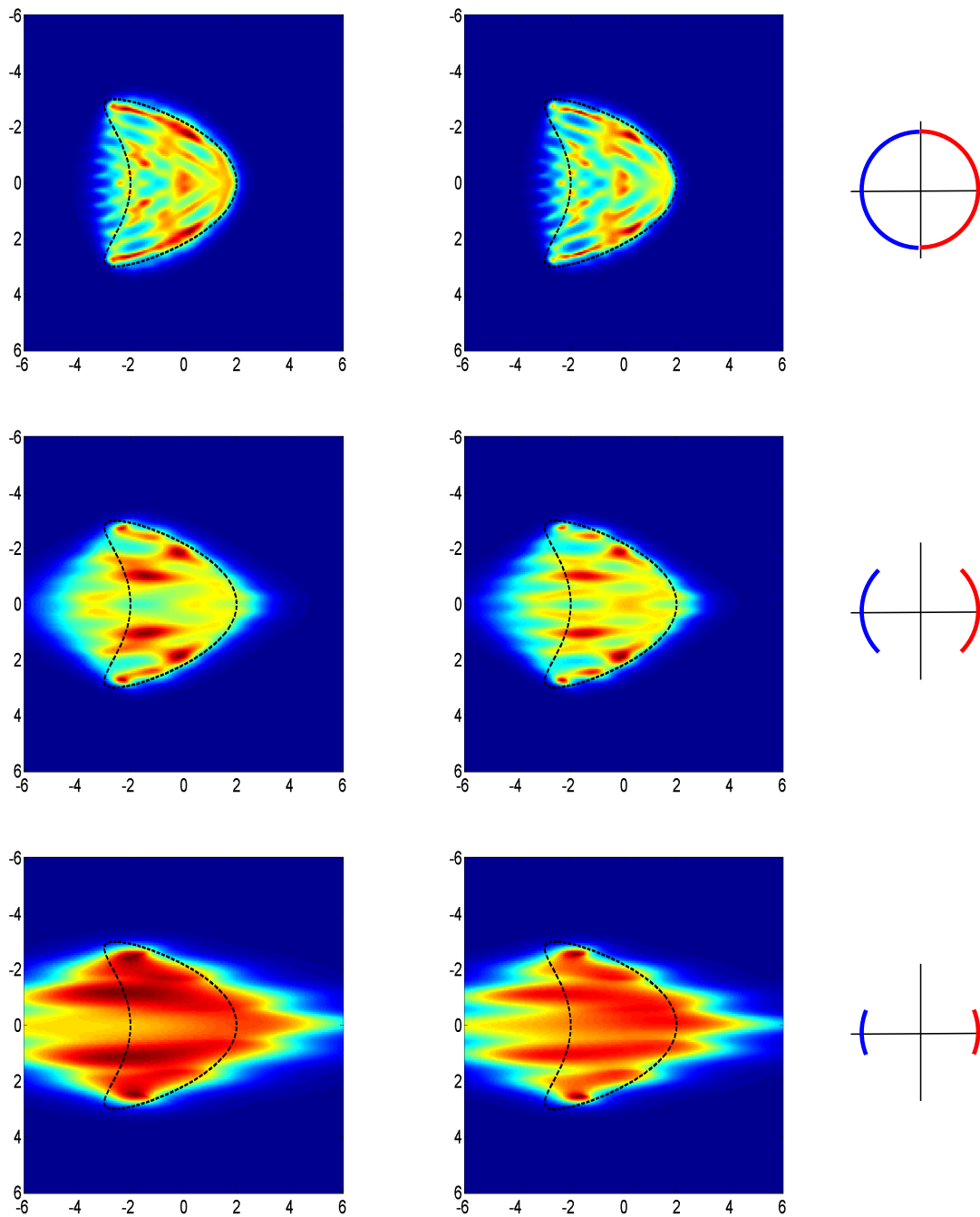
424 Figures 5 and 6 consider backscattering data from aperture of the same size as previously,  
 425 but rotated around the obstacle. We see the strong dependency with the mean direction of the  
 426 aperture. The fact that the results are coherent with the aperture we consider lets us think  
 427 that non symmetric aperture is intrinsically worst than symmetric one. Connected to that  
 428 subject in [9] they study invisibility for a finite number of incident direction and demonstrate  
 429 that imposing invisibility in symmetric direction is equivalent to impose invisibility in all  
 430 direction. Meaning that there is more information inside symmetric-factorization like far field  
 431 operator than any other setting of sources and measurements.

432

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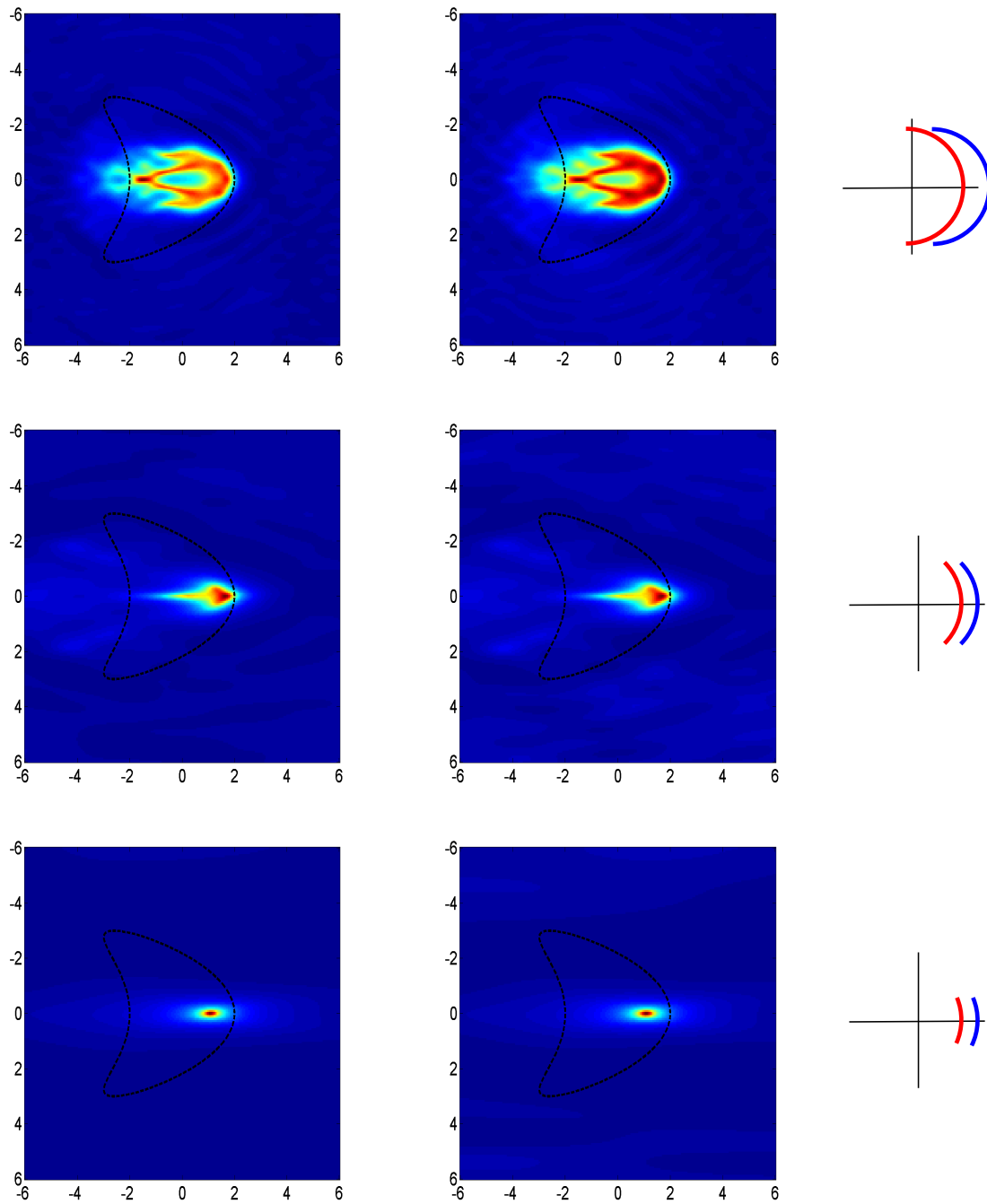
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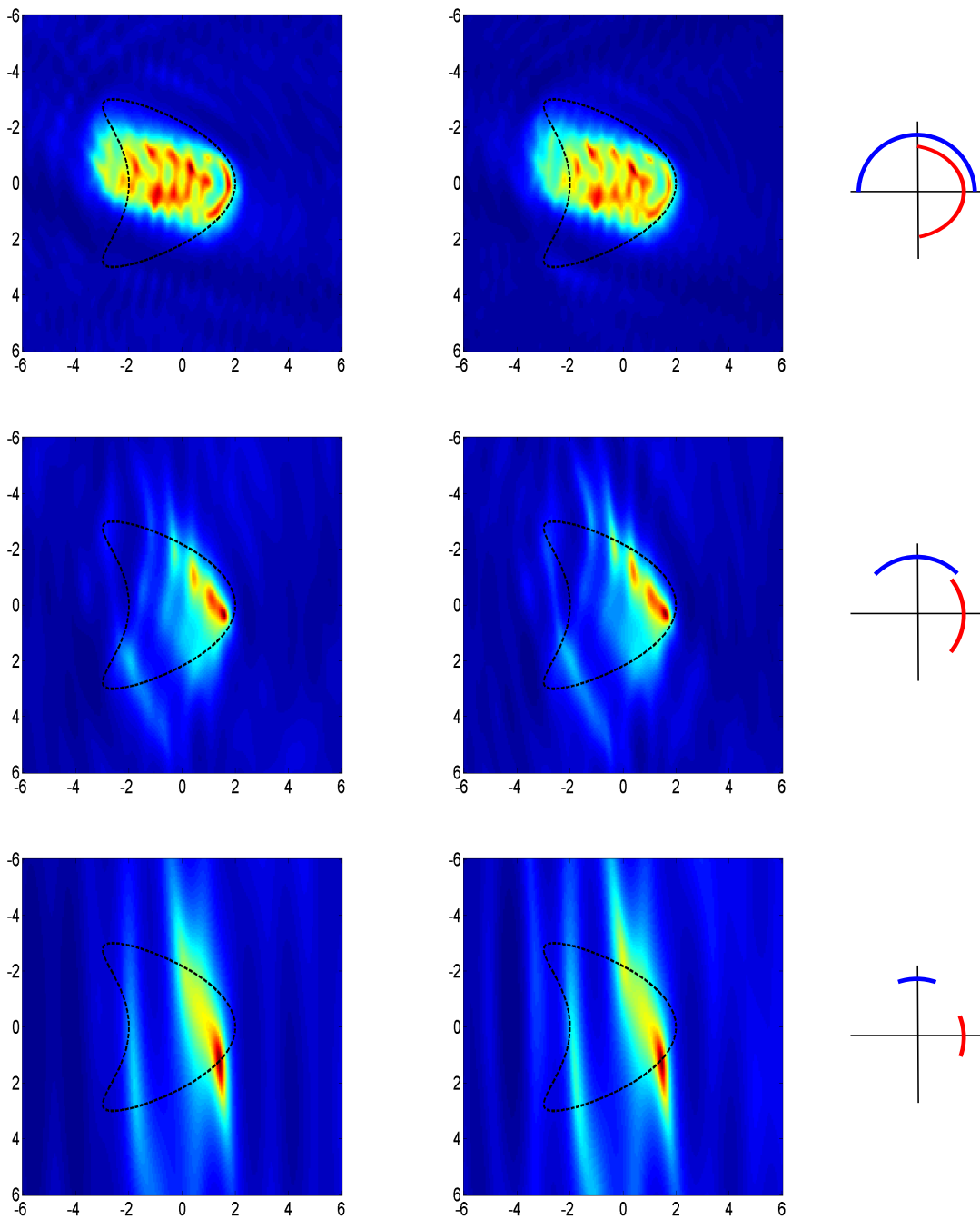


**Figure 2.** On the left  $\mathcal{I}_\#$  and on the right  $\mathcal{I}$ . From up to down the aperture is :  $[\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  $[7\pi/8, 9\pi/8[$  (as depicted on the right column).

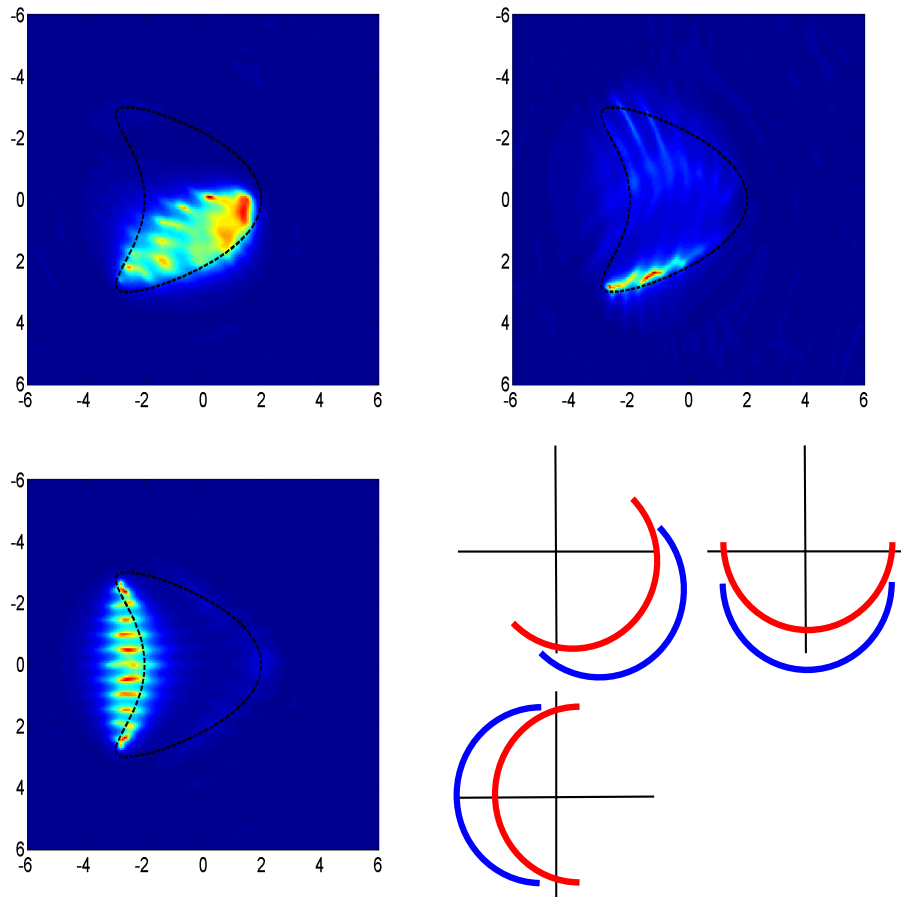




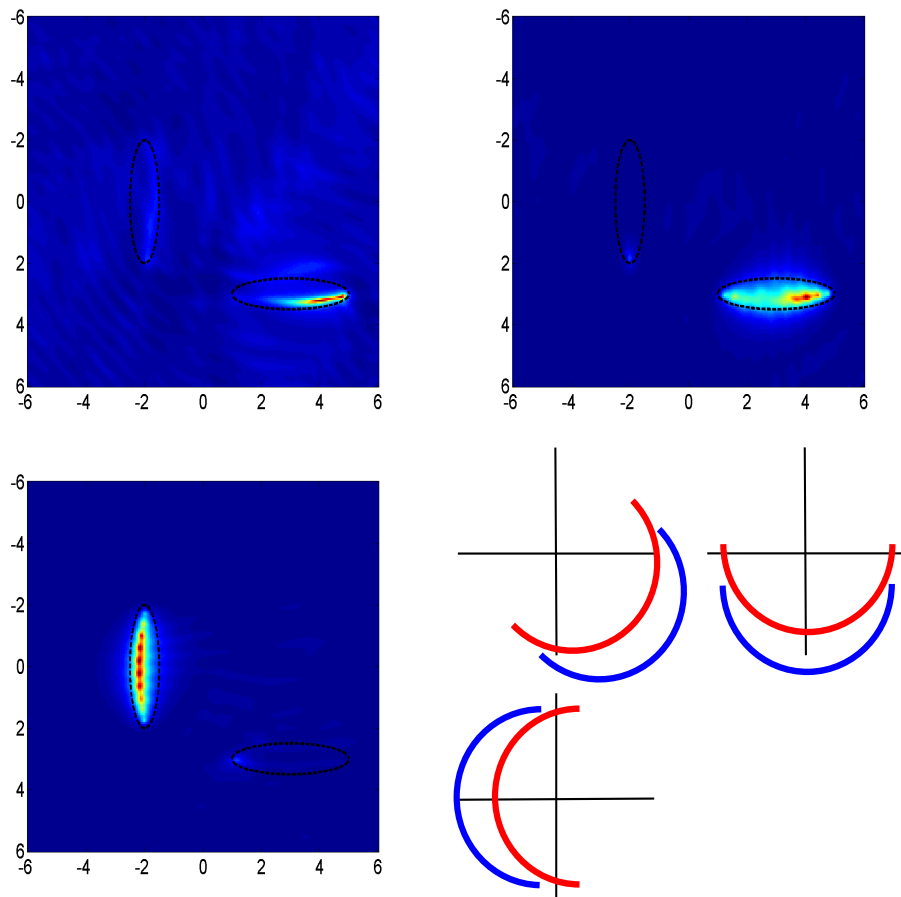
**Figure 3.**  $\mathcal{I}$  computed on the left with a large  $\Sigma$  and with on the right with a small one. From up to down the apertures are :  $\Gamma_s = [\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  $[7\pi/8, 9\pi/8[$  (as depicted in the right column).



**Figure 4.**  $\mathcal{I}$  computed on the left with a large  $\Sigma$  and with on the right with a small one. From up to down the apertures are  $:\Gamma_s = [\pi/2, 3\pi/2[$ ,  $[3\pi/4, 5\pi/4[$  and  $[7\pi/8, 9\pi/8[$  and  $\Gamma_m = [0, \pi[$ ,  $[\pi/4, 3\pi/4[$  and  $[3\pi/8, 5\pi/8[$  (as depicted in the right column).



**Figure 5.**  $\mathcal{I}$  computed with  $\Sigma$  equals the full grid. From left to right and up to down the aperture are :  $\Gamma_s[3\pi/4, 7\pi/4[$ ,  $[\pi, 2\pi[$  and  $[-\pi/2, \pi/2[$  and  $\Gamma_m = \Gamma_s + \pi$  (the sensor setting are depicted following the same order in the last image).



**Figure 6.**  $\mathcal{I}$  computed with  $\Sigma$  equals the full grid. From left to right and up to down the aperture are :  $\Gamma_s = [3\pi/4, 7\pi/4[$ ,  $[\pi, 2\pi[$  and  $[-\pi/2, \pi/2[$  and  $\Gamma_m = \Gamma_s + \pi$  (the sensor setting are depicted following the same order in the last image).