

Hölder-type inequalities and their applications to concentration and correlation bounds

Christos Pelekis* Jan Ramon† Yuyi Wang‡

November 29, 2016

Abstract

Let $Y_v, v \in V$, be real-valued random variables having a dependency graph $G = (V, E)$. We show that

$$\mathbb{E} \left[\prod_{v \in V} Y_v \right] \leq \prod_{v \in V} \left\{ \mathbb{E} \left[Y_v^{\frac{\chi_b}{b}} \right] \right\}^{\frac{b}{\chi_b}},$$

where χ_b is the b -fold chromatic number of G . This inequality may be seen as a dependency-graph analogue of a generalised Hölder inequality, due to Helmut Finner. Additionally, we provide applications of the aforementioned Hölder-type inequalities to concentration and correlation bounds for sums of weakly dependent random variables whose dependencies can be described in terms of graphs or hypergraphs.

Keywords: fractional chromatic number; Finner's inequality; Janson's inequality; dependency graph; hypergraphs

1 Introduction and related work

The main purpose of this article is to illustrate that certain Hölder-type inequalities can be employed in order to obtain concentration and correlation bounds for sums of weakly dependent random variables whose dependencies are described in terms of graphs, or hypergraphs. Before being more precise, let us begin with some notation and definitions that will be fixed throughout the text.

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001, Belgium,
Email: pelekis.chr@gmail.com

†Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001, Belgium,
Email: Jan.Ramon@cs.kuleuven.be

‡Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001, Belgium,
Email: yuyiwang920@gmail.com

A *hypergraph* \mathcal{H} is a pair (V, \mathcal{E}) where V is a finite set and \mathcal{E} is a family of subsets of V . The set V is called the *vertex set* of \mathcal{H} and the set \mathcal{E} is called the *edge set* of \mathcal{H} ; the elements of \mathcal{E} are called *hyperedges* or just edges. The cardinality of the vertex set will be denoted by $|V|$ and the cardinality of the edge set by $|\mathcal{E}|$. A hypergraph is called *k-uniform* if every edge from \mathcal{E} has cardinality k . A 2-uniform hypergraph is a *graph*. The *degree* of a vertex $v \in V$ is defined as the number of edges that contain v . A hypergraph will be called *d-regular* if every vertex has degree d . A subset $V' \subseteq V$ is called *independent* if it does not contain any edge from \mathcal{E} . A *fractional matching* of a hypergraph, $\mathcal{H} = (V, \mathcal{E})$, is a function $\phi : \mathcal{E} \rightarrow [0, 1]$ such that $\sum_{e: v \in e} \phi(e) \leq 1$, for all vertices $v \in V$. The *fractional matching number* of \mathcal{H} , denoted $\nu^*(\mathcal{H})$, is defined as $\max_{\phi} \sum_{e \in \mathcal{E}} \phi(e)$ where the maximum runs over all fractional matchings of \mathcal{H} . The fractional chromatic number of a graph G is defined in the following way. A *b-fold colouring* of G is an assignment of sets of size b to the vertices of the graph in such a way that adjacent vertices have disjoint sets. A graph is *(a : b)-colourable* if it has a b -fold colouring using a different colours. The least a for which the graph is $(a : b)$ -colourable is the *b-fold chromatic number* of the graph, denoted $\chi_b(G)$. The *fractional chromatic number* of a graph G is defined as $\chi^*(G) = \inf_b \frac{\chi_b(G)}{b}$. Here and later, $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ will denote probability and expectation, respectively.

Let us also recall Hölder's inequality. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let A be a finite set and let $Y_a, a \in A$, be random variables from Ω into \mathbb{R} . Suppose that $w_a, a \in A$ are non-negative weights such that $\sum_{a \in A} w_a \leq 1$ and each Y_a has finite $\frac{1}{w_a}$ -moment, i.e., $\mathbb{E} \left[Y_a^{1/w_a} \right] < +\infty$, for all $a \in A$. Hölder's inequality asserts that

$$\mathbb{E} \left[\prod_{a \in A} Y_a \right] \leq \prod_{a \in A} \mathbb{E} \left[Y_a^{1/w_a} \right]^{w_a}.$$

This is a classic result (see [2]). In this article we shall be interested in applications of Hölder-type inequalities to concentration and correlation bounds for sums of weakly dependent random variables. We focus on two particular types of dependencies between random variables. The first one is described in terms of a hypergraph.

Definition 1 (hypergraph-correlated random variables). *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Suppose that $\{Y_e\}_{e \in \mathcal{E}}$ is a collection of real-valued random variables, indexed by the edge set of \mathcal{H} , which satisfy the following: there exist independent random variables $\{X_v\}_{v \in V}$ indexed by the vertex set V such that, for every edge $e \in \mathcal{E}$, we have $Y_e = f_e(X_v; v \in e)$ is a function that depends only on the random variables X_v with $v \in e$. We will refer to the aforementioned random variables $\{Y_e\}_{e \in \mathcal{E}}$ as \mathcal{H} -correlated, or simply as hypergraph-correlated, when there is no confusion about the underlying hypergraph.*

As an example of hypergraph-correlated random variables the reader may think of the indicators of the triangles in an Erdős-Rényi random graph model $\mathcal{G}(n, p)$. Recall that

such a model generates a random graph, G , on n labelled vertices by joining pairs of vertices, independently, with probability $p \in (0, 1)$. To see that the indicators of the triangles in G are hypergraph-correlated, let \mathcal{H}_G be the hypergraph whose vertex set, \mathcal{V} , has an element for every (potential) edge in G and whose edge set, \mathcal{E} , has a hyperedge for every triplet of edges in G that form (potential) triangles. Let $X_v, v \in \mathcal{V}$, be independent Bernoulli $\text{Ber}(p)$ random variables. For every triangle T in G , let E_T be the corresponding hyperedge of \mathcal{H}_G . Now notice that the indicator of each triangle, T , in G can be written as a product $\prod_{v \in E_T} X_v$. More examples of hypergraph-correlated random variables can be found in Section 4. Hypergraph-correlated random variables are encountered in the theory of random graphs and machine learning (see [9, 11, 17, 19] and references therein); a quintessential question that motivates their study asks for upper bounds on the probability that a random graph $G \in \mathcal{G}(n, p)$ contains more triangles than expected. Notice that the indicators of the triangles in $G \in \mathcal{G}(n, p)$ are not mutually independent and therefore the standard Chernoff-Hoeffding bounds do not apply. Another type of "dependency structure" that plays a key role in probabilistic combinatorics and related areas involves the notion of a dependency graph (see also [1, 12]).

Definition 2 (Graph-dependent random variables). *A dependency graph for the random variables $\{Y_v\}_{v \in V}$, indexed by a finite set V , is any loopless graph, $G = (V, E)$, whose vertex set V is the index set of the random variables and whose edge set is such that if $V' \subseteq V$ and $v_i \in V$ is not incident to any vertex of V' , then Y_{v_i} is mutually independent of the random variables $Y_{v'}$ for which $v' \in V'$. We will refer to random variables $\{Y_v\}_{v \in V}$ having a dependency graph G as G -dependent or as graph-dependent.*

A difference between the notions of graph-dependent and hypergraph-correlated random variables is illustrated in Example 2.2 below. An example of graph-dependent random variables is the indicators of the triangles in a random graph $G \in \mathcal{G}(n, p)$. Let us remark that the previous notions of weakly dependent random variables arise in the study of the, so-called, Lovász Local Lemma. In this setting one is dealing with a (finite) number of events in some probability space, and is interested in lower bounds on the probability that none of the events will occur. The events need not be independent and their indicators are assumed to be graph-dependent, or hypergraph-correlated (see [6, 16]). In this article we focus on upper bounds on the probability that none of the events will occur.

Notice that if $\{Y_e\}_{e \in \mathcal{E}}$ are hypergraph-correlated random variables then they *induce* a dependency graph whose vertex set is \mathcal{E} and with edges joining any two sets $e, e' \in \mathcal{E}$ such that $e \cap e' \neq \emptyset$. Hence a set of hypergraph-correlated random variables is graph-dependent. The reader may wonder whether the converse holds true. We will show, using a particular generalisation of Hölder's inequality, that this is not the case (see Example 2.2 below) and so the aforementioned notions of dependencies between random variables are not equivalent.

In the present paper we provide a Hölder-type inequality for graph-dependent random variables. Our result can be seen as a dependency-graph analogue of the following theorem, due to Helmut Finner.

Theorem 1.1 (Finner [7]). *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let $\{Y_e\}_{e \in \mathcal{E}}$ be \mathcal{H} -correlated random variables. If $\phi : \mathcal{E} \rightarrow [0, 1]$ is a fractional matching of \mathcal{H} then*

$$\mathbb{E} \left[\prod_{e \in \mathcal{E}} Y_e \right] \leq \prod_{e \in \mathcal{E}} \left\{ \mathbb{E} \left[Y_e^{1/\phi(e)} \right] \right\}^{\phi(e)}.$$

Notice that, by applying the previous result to the random variables $Z_e = Y_e^{\phi(e)}$, one concludes $\mathbb{E} \left[\prod_{e \in \mathcal{E}} Y_e^{\phi(e)} \right] \leq \prod_{e \in \mathcal{E}} \left\{ \mathbb{E} [Y_e] \right\}^{\phi(e)}$. See [7] for a proof of Theorem 1.1 which is based on Fubini's theorem and Hölder's inequality. Alternatively, see [17] for a proof that uses the concavity of the weighted geometric mean and Jensen's inequality. In other words, Theorem 1.1 provides a Hölder-type inequality for hypergraph-correlated random variables which is formalised in terms of a fractional matching of the underlying hypergraph.

In this article we illustrate how certain Hölder-type inequalities yield concentration and correlation bounds for sums of hypergraph-correlated random variables as well as for sums of graph-dependent random variables. In that regard, our contribution is two-fold. On the one hand, we show that a particular graph-dependent analogue of Theorem 1.1 yields a new proof of the following result, due to Svante Janson, which provides an estimate on the probability that the sum of graph-dependent random variables is significantly larger than its mean.

Theorem 1.2 (Janson [11]). *Let $\{Y_v\}_{v \in V}$ be $[0, 1]$ -valued random variables having a dependency graph $G = (V, E)$. Set $q := \frac{1}{|V|} \mathbb{E} [\sum_v Y_v]$. If $t = n(q + \varepsilon)$ for some $\varepsilon > 0$, then*

$$\mathbb{P} \left[\sum_v Y_v \geq t \right] \leq \exp \left(-\frac{2\varepsilon^2 |V|}{\chi^*} \right),$$

where $\chi^* = \chi^*(G)$ is the fractional chromatic number of G .

See [11, Theorem 2.1] for a proof of this result which is based on breaking up the sum into a particular linear combination of sums of independent random variables. On the other hand, we show that Finner's inequality applies to the following problem, that is interesting on its own.

Problem 1.3. *Fix a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and let \mathbb{I} be a random subset of V formed by including vertex $v \in V$ in \mathbb{I} with probability $p_v \in (0, 1)$, independently of other vertices. What is an upper bound on the probability that \mathbb{I} is independent?*

Here and later, given a set of parameters in $(0, 1)$, say $\mathbf{p} = \{p_v\}_{v \in V}$, indexed by the vertex set of a hypergraph \mathcal{H} , we will denote by $\pi(\mathbf{p}, \mathcal{H})$ the probability that \mathbb{I} is independent. Let us remark that Problem 1.3 has attracted the attention of several authors and appears to be related to a variety of topics (see [4, 8, 13, 15, 19] and references therein). A particular line of research is motivated by question about independent sets and subgraph counting in random graphs. In this context, Problem 1.3 has been considered by Janson et al. [13], Krivelevich et al. [15] and Wolfowitz [19]. It is observed in [15] that when \mathcal{H} is k -uniform, d -regular and $p_v = p$ for all $v \in V$, an exponential estimate on $\pi(\mathbf{p}, \mathcal{H})$ can be obtained using the so-called Janson's correlation inequality. Additionally, it is shown in [15, Section 5] that under certain "mild additional assumptions" the bound provided by Janson's inequality can be improved to

$$\pi(\mathbf{p}, \mathcal{H}) \leq \exp \left(-\Omega \left(\frac{p|\mathcal{E}|}{(1-p)kd} \right) \right).$$

See [15] for a precise formulation of the "mild additional assumptions" and a proof of this result which is based on a martingale-type concentration inequality.

The remaining part of our paper is organised as follows. In the Section 2 we collect our main results. In particular, we provide a Hölder-type inequality for graph-dependent random variables and we apply this result in order to deduce upper bounds on the probability that a sum of graph-dependent random variables is significantly larger than its mean. We also show that Finner's inequality yields an upper bound on $\pi(\mathbf{p}, \mathcal{H})$. The proofs of our main results are deferred to Section 3. Finally, in Section 4, we illustrate that Finner's inequality can be seen as an alternative to the, well-known, Janson's correlation inequality. Our paper ends with Section 5 in which we briefly sketch how one can apply similar ideas to yet another class of weakly dependent random variables.

2 Main results

In this section we collect our main results. Our first result provides an analogue of Theorem 1.1 for random variables having a dependency graph G . The corresponding Hölder-type inequality is formalised in terms of the b -fold chromatic number of G and reads as follows.

Theorem 2.1. *Let $\{Y_v\}_{v \in V}$ be real-valued random variables having a dependency graph $G = (V, E)$. Then, for every b -fold colouring of G using $\chi_b := \chi_b(G)$ colours, we have*

$$\mathbb{E} \left[\prod_{v \in V} Y_v \right] \leq \prod_{v \in V} \left\{ \mathbb{E} \left[Y_v^{\frac{\chi_b}{b}} \right] \right\}^{\frac{b}{\chi_b}}.$$

We prove Theorem 2.1 in Section 3. The proof employs the concavity of the weighted geometric mean and the definition of b -fold chromatic number. Recall that Finner's inequality applies to hypergraph-correlated random variables and that such an ensemble of random variables induces a dependency graph. However, as is shown in the following example, there exist graph-dependent random variables which are not hypergraph-correlated and therefore Theorem 2.1 provides a Hölder-type inequality can be applied to more general ensembles of "weakly dependent" random variables.

Example 2.2 (R.v.'s that are graph-dependent but not hypergraph-correlated). *Let G be a cycle-graph on 5 vertices $\{v_1, \dots, v_5\}$ such that v_i is incident to v_{i+1} , for $i \in \{1, 2, 3, 4\}$ and v_5 is incident to v_1 . Let $Y = (Y_1, \dots, Y_5)$ be a vector of Bernoulli 0/1 random variables whose distribution is defined as follows. The vector Y takes the value $(0, 0, 0, 0, 0)$ with probability $\frac{1}{2}(2-p)(1-p)^2$, the value $(1, 1, 1, 1, 1)$ with probability $\frac{p^2+p^3}{2}$, the values*

$$(0, 0, 0, 1, 1), (0, 0, 1, 1, 0), (0, 1, 1, 0, 0), (1, 1, 0, 0, 0), (1, 0, 0, 0, 1)$$

with probability $\frac{p(1-p)^2}{2}$, the values

$$(0, 0, 1, 1, 1), (0, 1, 1, 1, 0), (1, 1, 1, 0, 0), (1, 1, 0, 0, 1), (1, 0, 0, 1, 1)$$

with probability $\frac{p^2-p^3}{2}$ and the remaining values with probability 0. Elementary, though quite tedious, calculations show that $\mathbb{E}[Y_j] = p$, for $j = 1, \dots, 5$ and that G is a dependency graph for $\{Y_j\}_{j=1}^5$. Now assume that $\{Y_j\}_{j=1}^5$ are \mathcal{H} -correlated, for some hypergraph $\mathcal{H} = (V, \mathcal{E})$. Notice that the cardinality of \mathcal{E} must be equal to 5. If $e_i \in \mathcal{E}$ is the edge corresponding to the random variable Y_i , $i = 1, \dots, 5$, then the fact that $\{Y_i\}_{i=1}^5$ have G as a dependency graph implies that $e_i \cap e_{(i+2) \bmod 5} = \emptyset$, for $i \in \{1, 2, 3, 4, 5\}$. This means that the fractional matching number of \mathcal{H} is at least 2.5 and therefore Theorem 1.1 implies that $\mathbb{P}[Y = (1, 1, 1, 1, 1)] \leq p^{2.5}$. However, the arithmetic geometric means inequality implies $\frac{p^2+p^3}{2} > p^{2.5}$ and therefore the random variables $\{Y_j\}_{j=1}^5$ cannot be hypergraph-correlated.

We also show that Theorem 2.1, combined with standard techniques based on exponential moments, yields, on the one hand, a new proof of Theorem 1.2 and, on the other hand, the following Bennett-type inequality for graph-dependent random variables.

Theorem 2.3. *Let $\{Y_v\}_{v \in V}$ be random variables having a dependency graph $G = (V, E)$. For every $v \in V$ let $\sigma_v^2 := \text{Var}(Y_v)$ and assume further that $Y_v \leq 1$ and $\mathbb{E}[Y_v] = 0$. Set $S = \sum_v \sigma_v^2$ and fix $t > 0$. Then*

$$\mathbb{P} \left[\sum_v Y_v \geq t \right] \leq \exp \left(-\frac{S}{\chi^*(G)} \psi \left(\frac{t}{S} \right) \right),$$

where $\psi(x) = (1+x) \ln(1+x) - x$.

Let us remark that Theorem 2.3 improves on [11, Theorem 2.3] which contains the same bound but with $\psi\left(\frac{4t}{5S}\right)$ instead of our $\psi\left(\frac{t}{S}\right)$. Note that we assume a one-sided bound on each Y_v . The proof of Theorem 2.3 is based on Theorem 2.1 and can be found in Section 3. Notice also that Theorems 2.1 and 2.3 are concerned with graph-dependent random variables. Our final result concerns hypergraph-correlated indicators. In Section 3 we show that Finner's inequality yields the following upper bound on the probability that a random subset of the vertex set of a hypergraph is independent.

Theorem 2.4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $\mathbf{p} = \{p_v\}_{v \in V}$ be a set of parameters from $(0, 1)$. Then*

$$\pi(\mathbf{p}, \mathcal{H}) \leq \prod_e \left(1 - \prod_{v \in e} p_v\right)^{\phi(e)},$$

where $\phi : \mathcal{E} \rightarrow [0, 1]$ is a fractional matching of \mathcal{H} . In particular, if the hypergraph \mathcal{H} is k -uniform, d -regular and $p_v = p$, for all $v \in V$, then

$$\pi(\mathbf{p}, \mathcal{H}) \leq \left(1 - p^k\right)^{\nu^*(\mathcal{H})} \leq \exp\left(-p^k \frac{|\mathcal{E}|}{d}\right),$$

where $\nu^*(\mathcal{H})$ is the fractional matching number of \mathcal{H} .

Notice that the first bound in the second statement of Theorem 2.4 has a *monotonicity property*, in the sense that if \mathcal{H}_1 is a superhypergraph of \mathcal{H}_2 then $(1 - p^k)^{\nu^*(\mathcal{H}_1)} \leq (1 - p^k)^{\nu^*(\mathcal{H}_2)}$. In Section 4 we illustrate that Theorem 2.4 may be seen as an alternative to Janson's correlation inequality.

Let us end this section by noting that Theorem 1.1, combined with standard exponential-moment ideas, yields concentration inequalities for sums of hypergraph-correlated random variables. This has been reported in prior work and so we only provide the statement without proof. In [17] one can find a proof of the following result.

Theorem 2.5 (Ramon et al. [17]). *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and assume that $\{Y_e\}_{e \in \mathcal{E}}$ are \mathcal{H} -correlated random variables. Assume further that $Y_e \in [0, 1]$, for all $e \in \mathcal{E}$, and that $\mathbb{E}[Y_e] = p_e$, for some $p_e \in (0, 1)$. Let $\phi : \mathcal{E} \rightarrow [0, 1]$ be a fractional matching of \mathcal{H} and set $\Phi = \sum_e \phi(e)$, $p = \frac{1}{|\mathcal{E}|} \sum_e \mathbb{E}[Y_e]$. If t is a real number from the interval $(\Phi p, \Phi)$ such that $t = \Phi(p + \varepsilon)$, then*

$$\mathbb{P}\left[\sum_{e \in \mathcal{E}} \phi(e) Y_e \geq t\right] \leq \exp(-2\Phi\varepsilon^2).$$

In particular, if d is the maximum degree of \mathcal{H} and $\phi(e) = \frac{1}{d}$, for all $e \in \mathcal{E}$, then Theorem

2.5 yields the bound

$$\mathbb{P} \left[\sum_e Y_e \geq t \right] \leq \exp \left(-2 \frac{|\mathcal{E}|}{d} \varepsilon^2 \right), \text{ for } t = |\mathcal{E}|(p + \varepsilon).$$

This inequality has also been obtained in Gavinsky et al. [9] using entropy ideas. As is mentioned in the introduction, the indicators, say $\{I_T\}_T$, of the triangles in an Erdős-Rényi random graph are both graph-dependent and hypergraph-correlated; their dependency graph is the graph which is induced from the random variables $\{I_T\}_T$ when they are viewed as hypergraph-correlated. This means that both Theorem 1.2 and Theorem 2.5 provide an upper bound on the probability that $G \in \mathcal{G}(n, p)$ contains more triangles than expected. We invite the reader to verify that Theorem 2.5 provides a better bound. An intuitive explanation as to why Theorem 2.5 gives a better bound is that, while considering the induced dependency-graph of a set of hypergraph-correlated random variables, one loses information. This behaviour has also been reported in the context of Lovász Local Lemma (see Kolipaka et al. [14]).

3 Proofs

In this section we prove our main results. We begin with Theorem 2.1, whose proof is based on the concavity of the weighted geometric mean.

Lemma 3.1. *Let $\beta = (\beta_1, \dots, \beta_k)$ be a vector of non-negative real numbers such that $\sum_{i=1}^k \beta_i = 1$. Then the function $g : \mathbb{R}^k \rightarrow \mathbb{R}$ defined by $g(t) = \prod_{i=1}^k t_i^{\beta_i}$ is concave.*

Proof. This is easily verified by showing that the Hessian matrix is positive definite. See [5], or [17] for details. \square

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We show that

$$\mathbb{E} \left[\prod_{v \in V} \{Y_v\}^{\frac{b}{\chi_b}} \right] \leq \prod_{v \in V} \{\mathbb{E} [Y_v]\}^{\frac{b}{\chi_b}}.$$

The theorem follows by applying this inequality to the random variables $Z_v = Y_v^{\frac{\chi_b}{b}}$. For every colour $i = 1, \dots, \chi_b$ let I_i be the set consisting of the vertices that are coloured with

colour i . Note that each I_i is an independent subset of V and every vertex $v \in V$ appears in exactly b independent sets I_i . Therefore,

$$\mathbb{E} \left[\prod_{v \in V} \{Y_v\}^{\frac{b}{\chi_b}} \right] = \mathbb{E} \left[\prod_{i=1}^{\chi_b} \prod_{v \in I_i} \{Y_v\}^{\frac{1}{\chi_b}} \right] = \mathbb{E} \left[\prod_{i=1}^{\chi_b} \left\{ \prod_{v \in I_i} Y_v \right\}^{\frac{1}{\chi_b}} \right].$$

Lemma 3.1 and Jensen's inequality combined with the observation that the random variables $\{Y_v\}_{v \in I_i}$ are mutually independent yield

$$\mathbb{E} \left[\prod_{i=1}^{\chi_b} \left\{ \prod_{v \in I_i} Y_v \right\}^{\frac{1}{\chi_b}} \right] \leq \prod_{i=1}^{\chi_b} \left\{ \mathbb{E} \left[\prod_{v \in I_i} Y_v \right] \right\}^{\frac{1}{\chi_b}} = \prod_{i=1}^{\chi_b} \prod_{v \in I_i} \{\mathbb{E}[Y_v]\}^{\frac{1}{\chi_b}}.$$

Now, using again the fact that each vertex v appears in exactly b sets I_i , we conclude

$$\prod_{i=1}^{\chi_b} \prod_{v \in I_i} \{\mathbb{E}[Y_v]\}^{\frac{1}{\chi_b}} = \prod_{v \in V} \{\mathbb{E}[Y_v]\}^{\frac{b}{\chi_b}}$$

and the result follows. □

Recall the following, well-known, result whose proof is included for the sake of completeness.

Lemma 3.2. *Let X be a random variable that takes values on the interval $[0, 1]$. Suppose that $\mathbb{E}[X] = p$, for some $p \in (0, 1)$, and let B be a Bernoulli 0/1 random variable such that $\mathbb{E}[B] = p$. If $f : [0, 1] \rightarrow \mathbb{R}$ is a convex function, then $\mathbb{E}[f(X)] \leq \mathbb{E}[f(B)]$.*

Proof. Given an outcome from the random variable X , define the random variable B_X that takes the values 0 and 1 with probability $1 - X$ and X , respectively. It is easy to see that $\mathbb{E}[B_X] = p$ and so B_X has the same distribution as B . Now Jensen's inequality implies

$$\mathbb{E}[f(X)] = \mathbb{E}[f(\mathbb{E}[B_X|X])] \leq \mathbb{E}[f(B_X|X)] = \mathbb{E}[f(B_X)],$$

as required. □

We now proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $h > 0$ and let $q_v = \mathbb{E}[Y_v]$, for $v \in V$. Using Markov's inequality and Theorem 2.1 we estimate

$$\begin{aligned} \mathbb{P}\left[\sum_{v \in V} Y_v \geq t\right] &\leq e^{-ht} \mathbb{E}\left[e^{h \sum_{v \in V} Y_v}\right] \\ &= e^{-ht} \mathbb{E}\left[\prod_{v \in V} e^{hY_v}\right] \\ &\leq e^{-ht} \prod_{v \in V} \left\{\mathbb{E}\left[\exp\left(\frac{\chi_b}{b} h Y_v\right)\right]\right\}^{\frac{b}{\chi_b}} \end{aligned}$$

For $v \in V$ let B_v be a Bernoulli 0/1 random variable of mean q_v . Lemma 3.2 implies

$$\begin{aligned} e^{-ht} \prod_{v \in V} \left\{\mathbb{E}\left[\exp\left(\frac{\chi_b}{b} h Y_v\right)\right]\right\}^{\frac{b}{\chi_b}} &\leq e^{-ht} \prod_{v \in V} \left\{\mathbb{E}\left[\exp\left(\frac{\chi_b}{b} h B_v\right)\right]\right\}^{\frac{b}{\chi_b}} \\ &= e^{-ht} \prod_{v \in V} \left\{(1 - q_v) + q_v e^{\frac{\chi_b}{b} h}\right\}^{\frac{b}{\chi_b}}. \end{aligned}$$

Using the weighted arithmetic-geometric means inequality we conclude

$$\begin{aligned} e^{-ht} \prod_{v \in V} \left\{(1 - q_v) + q_v e^{\frac{\chi_b}{b} h}\right\}^{\frac{b}{\chi_b}} &\leq e^{-ht} \left\{\sum_{v \in V} \frac{1}{|V|} \left((1 - q_v) + q_v e^{\frac{\chi_b}{b} h}\right)\right\}^{\frac{b}{\chi_b} |V|} \\ &= e^{-ht} \left\{1 - q + q e^{\frac{\chi_b}{b} h}\right\}^{\frac{b}{\chi_b} |V|}. \end{aligned}$$

If we minimise the last expression with respect to $h > 0$ we get that h must satisfy $e^{\frac{\chi_b}{b} h} = \frac{t(1-q)}{q(|V|-t)}$ and therefore, since $t = |V|(q + \varepsilon)$, we conclude

$$\mathbb{P}\left[\sum_{v \in V} Y_v \geq t\right] \leq \left\{\left(\frac{q}{q + \varepsilon}\right)^{q + \varepsilon} \left(\frac{1 - q}{1 - (q + \varepsilon)}\right)^{1 - (q + \varepsilon)}\right\}^{\frac{b}{\chi_b} |V|} = e^{-\frac{b}{\chi_b} |V| D(q + \varepsilon || q)},$$

where $D(q + \varepsilon || q)$ is the Kullback-Leibler distance between $q + \varepsilon$ and q . Finally, using the standard estimate $D(q + \varepsilon || q) \geq 2\varepsilon^2$, we deduce

$$\mathbb{P}\left[\sum_{v \in V} Y_v \geq t\right] \leq e^{-\frac{b}{\chi_b} |V| 2\varepsilon^2}$$

and the result follows upon minimising the last expression with respect to b . \square

The proof of Theorem 2.3 is similar.

Proof of Theorem 2.3. Fix $h > 0$ to be determined later. As in the proof of Theorem 1.2, Markov's inequality and Theorem 2.1 yield

$$\mathbb{P} \left[\sum_v Y_v \geq t \right] \leq e^{-ht} \prod_{v \in V} \left\{ \mathbb{E} \left[\exp \left(\frac{\chi_b}{b} h Y_v \right) \right] \right\}^{\frac{b}{\chi_b}}.$$

Using an inequality proved in [11] (see inequality (3.7), page 240), we have

$$\mathbb{E} \left[\exp \left(\frac{\chi_b}{b} h Y_v \right) \right] \leq \exp \left(\sigma_v^2 g \left(\frac{\chi_b}{b} h \right) \right),$$

where $g(a) := e^a - 1 - a$. Summarising, we have shown

$$\mathbb{P} \left[\sum_v Y_v \geq t \right] \leq \exp \left(-ht + \left(\frac{b}{\chi_b} e^{h\chi_b/b} - \frac{b}{\chi_b} - h \right) S \right).$$

Now choose $h = \frac{b}{\chi_b} \cdot \ln \left(1 + \frac{t}{S} \right)$ to deduce

$$\begin{aligned} \mathbb{P} \left[\sum_v Y_v \geq t \right] &\leq \exp \left(\frac{b}{\chi_b} t - S \frac{b}{\chi_b} \left(1 + \frac{t}{S} \right) \ln \left(1 + \frac{t}{S} \right) \right) \\ &= \exp \left(-S \frac{b}{\chi_b} \psi \left(\frac{t}{S} \right) \right). \end{aligned}$$

The result follows upon minimising the last expression with respect to b . □

We end this section with the proof of Theorem 2.4.

Proof of Theorem 2.4. Let $X_v, v \in V$, be indicators of the event $v \in \mathbb{I}$. For each $e \in \mathcal{E}$ set $Y_e = \prod_{v \in e} B_v$. Clearly, the random variables $\{Y_e\}_{e \in \mathcal{E}}$ are \mathcal{H} -correlated. Now look at the probability $\mathbb{P}[\sum_e Y_e = 0]$. Notice that if all Y_e are equal to zero, then every edge $e \in \mathcal{E}$ contains a vertex, v , such that $B_v = 0$ and vice versa. This implies that if $\sum_e Y_e = 0$ then the set of vertices, v , for which $B_v = 1$ is an independent subset of V and vice versa. Therefore

$$\mathbb{P} \left[\sum_{e \in \mathcal{E}} Y_e = 0 \right] = \mathbb{E} \left[\prod_{e \in \mathcal{E}} (1 - Y_e) \right] = \pi(\mathbf{p}, \mathcal{H}).$$

From Theorem 1.1 we deduce

$$\mathbb{E} \left[\prod_{e \in \mathcal{E}} (1 - Y_e) \right] \leq \mathbb{E} \left[\prod_{e \in \mathcal{E}} (1 - Y_e)^{\phi(e)} \right] \leq \prod_{e \in \mathcal{E}} (\mathbb{E} [1 - Y_e])^{\phi(e)}$$

and the first statement follows. To prove the second statement notice that $\mathbb{E}[Y_e] = p^k$, for all $e \in \mathcal{E}$, and therefore

$$\mathbb{E} \left[\prod_{e \in \mathcal{E}} (1 - Y_e) \right] \leq (1 - p^k)^{\sum_e \phi(e)}.$$

The result follows by maximising the exponent on the right hand side over all fractional matchings of \mathcal{H} . \square

In the next section we provide several applications of Theorem 2.4 to the theory of random graphs.

4 Applications

In this section we discuss comparisons between Finner's and Janson's inequality. Janson's inequality (see Janson [10] and Janson et al. [12, Chapter 2]) is a well known result that provides upper estimates on the probability that a sum of dependent indicators is equal to zero. It is described in terms of the dependency graph corresponding to the indicators. More precisely, let $\{B_v\}_{v \in V}$ be indicators having a dependency graph G . Set $\mu = \mathbb{E}[\sum_v B_v]$ and $\Delta = \sum_{e=\{u,v\} \in G} \mathbb{E}[B_u B_v]$. Janson's inequality asserts that

$$\mathbb{P} \left[\sum_v B_v = 0 \right] \leq \min \left\{ e^{-\mu + \Delta}, \exp \left(\frac{\Delta}{1 - \max_v \mathbb{E}[B_v]} \right) \prod_v (1 - \mathbb{E}[B_v]) \right\}.$$

This inequality has been proven to be very useful in the study of the Erdős-Rényi random graph model $\mathcal{G}(n, p)$. For $G \in \mathcal{G}(n, p)$ let us denote by T_G the number of triangles in G . A typical application of Janson's inequality provides the estimate

$$\mathbb{P}[G \in \mathcal{G}(n, p) \text{ is triangle-free}] \leq (1 - p^3)^{\binom{n}{3}} \cdot \exp \left(\frac{\Delta}{2(1 - p^3)} \right),$$

where $\Delta = 6 \binom{n}{4} p^5$. In this section we juxtapose the previous bound with the bound provided by Finner's inequality.

Proposition 4.1. *Let $G \in \mathcal{G}(n, p)$ be an Erdős-Rényi random graph and denote by T_G the number of triangles in G . Then*

$$\mathbb{P}[T_G = 0] \leq (1 - p^3)^{\frac{1}{n-2} \binom{n}{3}}.$$

Proof. We apply Theorem 2.4. Define a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ as follows. Let $v_i, i = 1, \dots, \binom{n}{2}$, be an enumeration of all (potential) edges in G and consider $v_1, \dots, v_{\binom{n}{2}}$ as

the vertex set \mathcal{V} of the hypergraph \mathcal{H} . Let $B_{v_i}, i = 1, \dots, \binom{n}{2}$ be independent Bernoulli $\text{Ber}(p)$ random variables, corresponding to the edges of G , and let $E_i, i = 1 \dots, \binom{n}{3}$, be an enumeration of all triplets of edges in G that form (potential) triangles in G . Define $\mathcal{E} = \{E_1, \dots, E_{\binom{n}{3}}\}$ to be the edge set of \mathcal{H} and let Z_i be the indicator of triangle E_i ; thus $T_G = \sum_i Z_i$. Now form a subset \mathbb{I} of \mathcal{V} by picking each vertex, independently, with probability p . Then the probability that \mathbb{I} is independent equals $\mathbb{P}[T_G = 0]$ and, in order to apply Theorem 2.4, we have to find a fractional matching of \mathcal{H} . Since every vertex of \mathcal{H} belongs to $n - 2$ edges in $\mathcal{E} = \{E_1, \dots, E_{\binom{n}{3}}\}$, we obtain a fractional matching, $\phi(\cdot)$ of \mathcal{H} by setting $\phi(E_i) = \frac{1}{n-2}$, for $i = 1, \dots, \binom{n}{3}$. The result follows. \square

Notice that the bound obtained from Janson's inequality is smaller than the bound given by Proposition 4.1 for values of p that are close to 0, but the bound of Proposition 4.1 does better for large values of p . Similar estimates can be obtained for the probability that a graph $G \in \mathcal{G}(n, p)$ contains no k -clique, for $k \geq 3$. The details are left to the reader.

We now proceed with one more application of Finner's inequality. Let $G \in \mathcal{G}(n, p)$ be a random graph on n labelled vertices. Fix two vertices, say u and v . What is an upper bound on the probability that there is no path of length k between u and v ?

A path of length k is a sequence of edges $\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-2}, v_{k-1}\}, \{v_{k-1}, v_k\}$ such that $v_i \neq v_j$, for $i \neq j$. We assume $k \geq 3$, otherwise the problem is easy. Let us remark that it may be cumbersome to apply Janson's inequality to the previous question. By contrast, Theorem 2.4 reduces the problem to a counting one. More explicitly, let $\{P_i\}_i$ be an enumeration of all (potential) paths of length k between u and v . Clearly, there are $\binom{n-2}{k-1} \cdot (k-1)!$ such paths. Define the hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ as follows. The vertices of \mathcal{H} correspond to the (potential) edges of G and the edges of \mathcal{H} correspond to the sets of edges in G which form a path of length k between u and v . Hence the probability that there is no path of length k between u and v equals $\pi(p, \mathcal{H})$. In order to apply Theorem 2.4 we have to find a fractional matching of \mathcal{H} and, in order to do so, it is enough to find an upper bound on the maximum degree of \mathcal{H} . To this end, fix an edge, $e = \{x, y\}$, in G . In case one of the vertices x or y is equal to either u or v , then there are $\binom{n-3}{k-2} \cdot (k-2)!$ paths of length k from u to v that pass through edge e . If none of the vertices x, y is equal to u or v , then we count the paths as follows. We first create a path, P_{k-2} , of length $k-2$ from u to v that does not pass through any of the points x, y and then we place the edge $e = \{x, y\}$ in one of $k-2$ available edges in the path P_{k-2} . Since there are two ways of placing the edge e in each slot of P_{k-2} it follows that the number of paths from u to v that go through edge e is equal to $2(k-2) \cdot \binom{n-4}{k-3} \cdot (k-3)!$. If $k \leq (n-1)/2$ then the later quantity is smaller than $\binom{n-3}{k-2} \cdot (k-2)!$, otherwise it is larger than $\binom{n-3}{k-2} \cdot (k-2)!$. Therefore, if $k \leq (n-1)/2$, the fractional matching number of \mathcal{H} is at least $\frac{\binom{n-2}{k-1} \cdot (k-1)!}{\binom{n-3}{k-2} \cdot (k-2)!} = n-2$. If $k > (n-1)/2$ then the fractional matching number of \mathcal{H} is at least $\frac{\binom{n-2}{k-1} \cdot (k-1)!}{2 \binom{n-3}{k-2} \cdot (k-2)!}$. We have thus proven the following.

Proposition 4.2. Let $G \in \mathcal{G}(n, p)$. Fix two vertices u, v in G and a positive integer $k \geq 3$. Let E be the event "there is no path of length k between u and v ". If $k \leq (n - 1)/2$ then

$$\mathbb{P}[E] \leq (1 - p^k)^{n-2}.$$

If $k > (n - 1)/2$, then

$$\mathbb{P}[E] \leq (1 - p^k)^{\frac{(n-2)(n-3)}{2(k-2)}}.$$

We end this section with an estimate on the probability that a $G \in \mathcal{G}(n, p)$ contains no vertex of fixed degree.

Proposition 4.3. Let $G \in \mathcal{G}(n, p)$ and fix a positive integer $d \in \{0, 1, \dots, n - 1\}$. Then the probability that there is no vertex in G whose degree equals d is less than or equal to

$$\left(1 - \binom{n-1}{d} p^d (1-p)^{n-1-d}\right)^{\frac{n}{2}}.$$

Proof. This is yet another application of Theorem 2.4, so we sketch it. Let v_1, \dots, v_n be an enumeration of the vertices of G . Let the hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be defined as follows. The vertex set \mathcal{V} corresponds to the (potential) edges of G . The edge set $\mathcal{E} = \{E_1, \dots, E_n\}$ corresponds to the vertices of G . That is, for $i = 1, \dots, n$ the edge E_i contains those $u \in \mathcal{V}$ for which the corresponding edges of G are incident to vertex v_i . The result follows from the fact that $|\mathcal{E}| = n$ and the maximum degree of \mathcal{H} is equal to 2. \square

5 Remarks

As mentioned in Janson [11], there exist collections of weakly dependent random variables that *do not* have a dependency graph. The dependencies between such collections of random variables can occasionally be described using an independence system. Recall that an *independence system* is a pair $\mathcal{A} = (V, I)$ where V is a finite set and I is a collection of subsets of V (called the *independent sets*) with the following properties (see [3]):

- The empty set is independent, i.e., $\emptyset \in I$. (Alternatively, at least one subset of V is independent, i.e., $I \neq \emptyset$.)
- Every subset of an independent set is independent, i.e., for each $A' \subset A \subset \mathcal{A}$, if $A \in I$ then $A' \in I$. This is sometimes called the *hereditary property*.

Given a set of random variables $\{Y_v\}_{v \in V}$, we say that their joint distribution is *described with an independence system*, say $\mathcal{A} = (V, I)$, if for every $A \in I$ the random variables $\{Y_a\}_{a \in A}$ are mutually independent. Let us remark that this definition includes the case of k -wise independent random variables (see [1, Chapter 16], or [18]). Notice that if $\{Y_v\}_{v \in V}$ are random variables whose joint distribution is described with an independence system $\mathcal{A} = (V, I)$ then $\{v\} \in I$, for all $v \in V$. It is easy to see that if the random variables $\{Y_v\}_{v \in V}$ have a dependency graph then their joint distribution is described with an independence system. However, the converse need not be true (see [11]). In a similar way as in Section 1, one may define the fractional chromatic number of an independence system as follows. A b -fold colouring of an independence system $\mathcal{A} = (V, I)$ is a function $\lambda : I \rightarrow \mathbb{Z}_+$ such that $\sum_{A: v \in A} \lambda(A) = b$, for all $v \in V$. The b -fold chromatic number of \mathcal{A} is defined as $\chi_b(\mathcal{A}) := \inf_{\lambda} \sum_{A \in I} \lambda(A)$, where the infimum is over all b -fold colourings, $\lambda(\cdot)$, of $\mathcal{A} = (V, I)$. Finally, the fractional chromatic number of \mathcal{A} is $\chi^*(\mathcal{A}) := \inf_b \frac{\chi_b(\mathcal{A})}{b}$.

With these concepts by hand, one can prove a corresponding Hölder-type inequality using a similar argument as in Theorem 2.1. As a consequence one can obtain tail bounds similar to Theorem 1.2 and Theorem 2.3, the only difference being that the fractional chromatic number of the dependency graph, $\chi^*(G)$, is replaced with the fractional chromatic number of the independence system, $\chi^*(\mathcal{A})$. We leave the details to the reader.

Acknowledgements The authors are supported by ERC Starting Grant 240186 "MiGraNT, Mining Graphs and Networks: a Theory-based approach". We are grateful to an anonymous referee for several suggestions that improved the presentation of the paper.

References

- [1] N. Alon and J. Spencer. *The Probabilistic Method*. J. Wiley and Sons, New York, 3rd edition, 2008.
- [2] V. I. Bogachev. *Measure Theory*, volume I. Springer, 2007.
- [3] J. A. Bondy and U. S. R. Murty. *Graph Theory*. Graduate Texts in Mathematics 244, Springer, 2008.
- [4] R. Boppana and J. Spencer. A useful elementary correlation inequality. *Journal of Combinatorial Theory, Series A*, 50:305–307, 1982.
- [5] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge UK, 2004.

- [6] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *In A. Hajnal, R. Rado, and V. T. Sós, eds. Infinite and Finite Sets (to Paul Erdős on his 60th birthday)*, pages 609–627. North-Holland, 1975.
- [7] H. Finner. A generalization of Hölder’s inequality and some probability inequalities. *The Annals of Probability*, 20(4):1893–1901, 1992.
- [8] D. Galvin and Y. Zhao. The number of independent sets in a graph with small maximum degree. *Graphs and Combinatorics*, 27:177–186, 2011.
- [9] D. Gavinsky, S. Lovett, M. Saks, S. Srinivasan, and A. Tail. Bound for Read-k Families of Functions. *Random Structures & Algorithms*, 47:99–108, 2015.
- [10] S. Janson. New versions of Suen’s correlation inequality. *Random Structures & Algorithms*, 13:467–483, 1998.
- [11] S. Janson. Large Deviations for Sums of Partly Dependent Random Variables. *Random Structures & Algorithms*, 24:234–248, 2004.
- [12] S. Janson, T. Luczak, and A. Ruciński. *Random Graphs*. Wiley, New York, 2000.
- [13] S. Janson, L. Warnke, and The Lower Tail: Poisson Approximation Revisited. *Random Structures & Algorithms*. 2015.
- [14] K. Kolipaka and M. Szegedy. Moser and Tardos meet Lovász. In *STOC ’11 Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 235–244, 2011.
- [15] M. Krivelevich, B. Sudakov, V. H. Vu, and N. C. Wormald. On the probability of independent sets in random graphs. *Random Structures & Algorithms*, 22:1–14, 2003.
- [16] R. A. Moser and G. Tardos. A constructive proof of the general Lovász local lemma. *Journal of the ACM*, 57(2), 2010.
- [17] J. Ramon, Y. Wang, and Z. Guo. Learning from networked examples. *Journal of Machine Learning Research (to appear) arXiv:1405.2600v2*, 2015.
- [18] J. P. Schmidt, A. Siegel, and A. Srinivasan. Chernoff-Hoeffding bounds for applications with limited independence. *SIAM Journal on Discrete Mathematics*, 8(2):223–250, 1995.
- [19] G. Wolfowitz. A concentration result with application to subgraph count. *Random Structures & Algorithms*, 40:254–267, 2012.