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# HARMONIC FACTORIZATION AND RECONSTRUCTION OF THE ELASTICITY TENSOR

M. OLIVE, B. KOLEV, B. DESMORAT, AND R. DESMORAT

ABSTRACT. In this paper, we propose a factorization of a fourth-order harmonic tensor into second-order tensors. We obtain moreover explicit equivariant reconstruction formulas, using second-order covariants, for transverse isotropic and orthotropic harmonic fourth-order tensors, and for trigonal and tetragonal harmonic fourth-order tensors up to a cubic fourth order covariant remainder.

## 1. INTRODUCTION

Interest in coordinate-free representations formulas for linear anisotropic elastic materials has been an active research area, starting by the formulation of elastic energy functionals in the framework of finite strains [49, 43, 46, 24] and having a key role in the classification of linear elastic or piezoelectric materials [16, 18]. It has also been of main importance in the Continuum Mechanics representation of cracked/damaged media [11, 29, 12, 40].

One underlying difficulty is that several Elasticity tensors may represent the same linear anisotropic elastic material in different orientations. More precisely, any change of orientation of a material specified by some rotation  $g \in \text{SO}(3, \mathbb{R})$  defines a new Elasticity tensor  $\bar{\mathbf{E}}$  deduced from the previous one by some *group action*

$$\mathbf{E} \mapsto \bar{\mathbf{E}} = g \star \mathbf{E}, \quad \bar{E}_{ijkl} = g_{ip}g_{jq}g_{kr}g_{ls}E_{pqrs}$$

where Einstein convention on repeated indices is used. As the material is rotated the tensor  $\mathbf{E}$  moves on its *orbit* in the space  $\mathbb{E}a$  of Elasticity tensors. Thus, a linear elastic material is represented by the *orbit* of an Elasticity tensor under this group action, and any intrinsic parameter is necessary an *invariant* of this group action.

In a seminal paper, Boehler–Kirilov–Onat [8] emphasized the fundamental role played by polynomial *invariants* of the Elasticity tensor and which have been used by Auffray–Kolev–Petitot [2] to classify the orbits of the elasticity tensor. The old problem of finding a basis of polynomial invariants for the fourth order Elasticity tensor was finally solved by Olive in 2014 [38] after several attempts [8, 5, 44, 55, 41]. A *minimal integrity basis of 297 invariants* for this space was definitively obtained in [1]. Note that in 2D, this integrity basis is composed by only 6 invariants [53, 54], which shows the incredible complexity of 3D linear elasticity compared to 2D case. In 3D, the problem

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cannot be reduced as finding an integrity basis of second-order tensor-valued functions.

Two main tools have been used to solve this extremely difficult computational problem.

The first one is the so-called *harmonic decomposition* which corresponds to the splitting of the Elasticity tensor into *irreducible pieces* [42, 45] which was first achieved by Backus in 1970 [3]. Such a decomposition is useful to compute the *material symmetry classes* [16, 17, 18]. As mentioned, it has been used also in the study of effective elastic properties of cracked media [29, 26, 40, 9, 13] and – without explicit reference to it – in the homogenization techniques of laminated composites [52, 51, 37]. Note that in the later case, the polar decomposition method for 2D media has been used [53, 50], while its equivalence with 2D harmonic decomposition has been shown in [14].

The second one is a purely mathematical tool, *binary forms*, which was the cornerstone of Invariant Theory in the nineteenth century [10, 20, 21, 22, 23] and is connected to spinors. This tool has been brought first to the knowledge of the mechanical community by Backus [3], and then by Boehler–Kirilov–Onat [8] but its deep power did not seem to have been widely considered so far. It happens to be extremely useful for solving problems in tensor analysis for higher order tensors in 3D. For instance, this tool was used by Auffray–Olive [39] to produce a minimal integrity basis for a traceless and totally symmetric third order tensor, which appears in piezoelectricity [56] and second-gradient strain elasticity theory [36]. Note, furthermore, that applications of binary forms are not just bounded to questions in continuum mechanics but are also related to other fields such as quantum computation [34] and cryptography [32].

The harmonic decomposition of the Elasticity tensor produces a quintuple

$$(\alpha, \beta, \mathbf{a}', \mathbf{b}', \mathbf{H})$$

where  $\alpha, \beta$ , are scalar invariants,  $\mathbf{a}', \mathbf{b}'$  are second order traceless tensors and  $\mathbf{H}$  is a *fourth order harmonic tensor*. The scalar and second order tensor components are linearly related to the Voigt and the dilatation tensors, the two independent traces of the Elasticity tensor. The irreducible fourth order component remains problematic in the study the Elasticity tensor.

In the present work, we try to go further in the representation of the harmonic fourth order component  $\mathbf{H}$ . We propose to introduce a secondary decomposition of this tensor, which was first discovered by Sylvester [48] and now known as *Maxwell multipoles* (see also [3, 4]), through a binary operation called *harmonic product* (a commutative and associative product directly inherited from the product of binary forms). This decomposition aims ideally to reduce  $\mathbf{H}$  to a family of vectors, the Maxwell multipoles, or, as detailed next, to reduce  $\mathbf{H}$  to a family of second order tensors. In 2D, the factorization of the harmonic component  $\mathbf{H}$  by means of second order tensors was already established in [14], it results from the decomposition of Elasticity tensor by Verchery’s polar method [53]. Besides, this factorization has the good taste to be *equivariant*, which means that it commutes with the action of the rotation group.

However, the solution in 3D of such a decomposition – *i.e.* the four Maxwell multipoles or in an equivalent manner the two second order tensors that factorize  $\mathbf{H}$  – is far from being unique and constructive and is thus of poor value in practice. Moreover, there are no global *equivariant section* of such mappings. We concentrate therefore on producing an *equivariant explicit solution* of the problem that we call a *reconstruction* and which are limited to specific symmetry classes. This process uses the concept of *covariants*, which generalizes the idea of invariants but which are tensors (rather than scalars) depending in an equivariant manner of the original tensor. For obvious reasons, no such reconstruction can be globally defined. As it is well known [33], only triclinic, monoclinic, orthotropic, transversely isotropic and isotropic classes have a chance to be reconstructed using second-order covariants.

In this paper, we obtain for the first time explicit reconstruction formulas in the orthotropic and the transversely isotropic cases, using *polynomial* second-order harmonic covariants and rational invariants. Moreover, to overpass the geometrical constraints in the trigonal and in the tetragonal cases, we establish for these symmetry classes reconstruction formulas by means of second order covariants up to a single *cubic fourth order covariant remainder*.

The considered reconstruction problem is closely related to the so-called *isotropic extension* of anisotropic constitutive functions via structural tensors developed by Boehler [6] and Liu [33] independently (see also [7, 35]). In the case of a *linear constitutive law*, an isotropic extension is just an *equivariant reconstruction limited to a given symmetry class* of the constitutive tensor. Furthermore, our approach is more constructive since we are able to give *explicit equivariant formulas* in which the “structural tensors” are polynomial covariant tensors of the constitutive tensor.

**Organization of the paper.** In Section 2, we recall basic materials on tensorial representations of the orthogonal group, recalling the link between totally symmetric tensor and homogeneous polynomials and the harmonic decomposition. In Section 3, we introduce the *harmonic product* between harmonic polynomials and formulate an harmonic factorisation theorem which proof is postponed in Appendix B. The general reconstruction problem is formulated in Section 4, where a geometric obstruction for an equivariant reconstruction of fourth-order tensors, by means of second order polynomial covariants is explained. Explicit reconstructions formulas, using rational invariants and second order polynomial covariants are obtained in Section 5 for transversely isotropic and orthotropic tensors. In Section 6, we propose similarly equivariant reconstructions for tetragonal and trigonal fourth order harmonic tensors up to a cubic covariant remainder.

**Notations.** The following spaces are involved:

- $\mathbb{E}la$ : the space of *Elasticity tensors*;
- $\mathbb{T}^n(\mathbb{R}^3)$ : the space of  $n$ -th order tensors on  $\mathbb{R}^3$ ;
- $\mathbb{S}^n(\mathbb{R}^3)$ : the space of  $n$ -th order *totally symmetric tensors* on  $\mathbb{R}^3$ ;
- $\mathbb{H}^n(\mathbb{R}^3)$ : the space of  $n$ -th order *harmonic tensors* on  $\mathbb{R}^3$ ;
- $\mathbb{S}^n(\mathbb{R}^3)$ : the space of homogeneous polynomials of degree  $n$  on  $\mathbb{R}^3$ ;

- $H^n(\mathbb{R}^3)$ : the space of *harmonic polynomials* of degree  $n$  on  $\mathbb{R}^3$ ;
- $S^n(\mathbb{C}^2)$ : the space of binary forms of degree  $n$  on  $\mathbb{C}^2$ ;
- $S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$ : the space of binary forms of degree  $2n$  which correspond to real harmonic tensors of degree  $n$ .

In addition, we adopt the following conventions:

- $x, y, z, x_i$ : coordinates on  $\mathbb{R}^3$ ;
- $u, v$ : coordinates on  $\mathbb{C}^2$ ;
- $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{n}, \mathbf{e}_i$ : vectors of  $\mathbb{R}^3$ ;
- $\boldsymbol{\xi}$ : a vector in  $\mathbb{C}^2$ ;
- $\mathbf{a}, \mathbf{b}, \mathbf{d}_k, \mathbf{s}, \mathbf{h}, \mathbf{h}_k$ : second-order tensors;
- $\mathbf{d}$ : second-order dilatation tensor;
- $\mathbf{v}$ : second-order Voigt tensor;
- $\mathbf{1}$ : second order unit tensor;
- $\mathbf{E}$ : elasticity tensor;
- $\mathbf{I}$ : fourth-order unit tensor;
- $\mathbf{T}, \mathbf{S}$ : generic tensors;
- $\mathbf{H}$ : fourth-order harmonic tensor;
- $p$ : polynomial on  $\mathbb{R}^3$ ;
- $h$ : harmonic polynomial on  $\mathbb{R}^3$ ;
- $\mathbf{f}, \mathbf{g}, \mathbf{w}$ : binary forms;
- $g$ : element of  $\text{SO}(3, \mathbb{R})$ ;
- $\cdot$ : the scalar product;
- $\otimes$ : the tensor product;
- $\odot, \otimes_{(4)}$ : the symmetric tensor product;
- $*$ : the harmonic product;
- $\star$ : the action;
- $\|\mathbf{T}\| = \sqrt{\mathbf{T} \cdot \mathbf{T}}$ : the Euclidean norm;
- $(\cdot)^t$ : the transpose;
- $(\cdot)'$ : the deviatoric part;
- $(\cdot)_0$ : the harmonic projection;
- $(\cdot), \bar{\lambda}$ : the complex conjugate.

## 2. TENSORIAL REPRESENTATIONS OF THE ROTATION GROUP

In this section, we recall classical facts about tensorial representations of the rotation group  $\text{SO}(3, \mathbb{R})$ . More details on the subject can be found in [47] or [19]. We consider thus tensors on the Euclidean space  $\mathbb{R}^3$  and, thanks to the Euclidean product, we do not have to distinguish between *upper* and *lower* indices. Therefore, a  $n$ -th order tensor may always be considered as a  $n$ -linear mapping

$$\mathbf{T} : \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

Let  $\mathbb{T}^n(\mathbb{R}^3)$  be the space of  $n$ -th order tensors. The rotation group  $\text{SO}(3, \mathbb{R})$  acts on  $\mathbb{T}^n(\mathbb{R}^3)$ , by the rule

$$(g \star \mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) := \mathbf{T}(g^{-1} \cdot \mathbf{x}_1, \dots, g^{-1} \cdot \mathbf{x}_n),$$

where  $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$  and  $g \in \text{SO}(3, \mathbb{R})$ .

Usually, we are not interested in the whole tensor space  $\mathbb{T}^n(\mathbb{R}^3)$  but rather in a subspace  $\mathbb{V}$  defined by some particular *index symmetries* and *stable* (or

invariant) under the action of the rotation group  $\text{SO}(3, \mathbb{R})$ , which means that

$$g \star \mathbf{T} \in \mathbb{V}, \quad \forall \mathbf{T} \in \mathbb{V}, \quad \forall g \in \text{SO}(3, \mathbb{R}).$$

*Example 2.1.* The permutation group  $\mathfrak{S}_n$  acts on  $\mathbb{T}^n(\mathbb{R}^3)$  by the rule

$$(\sigma \star \mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) := \mathbf{T}(\mathbf{x}_{\sigma^{-1}(1)}, \dots, \mathbf{x}_{\sigma^{-1}(n)}), \quad \sigma \in \mathfrak{S}_n,$$

and this action commutes with the action of  $\text{SO}(3, \mathbb{R})$ . The space of *totally symmetric tensors* of order  $n$ , denoted by  $\mathbb{S}^n(\mathbb{R}^3)$ , is the space of tensors  $\mathbf{T}$  which are invariant under this action, in other words, such that

$$(\sigma \star \mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n), \quad \forall \sigma \in \mathfrak{S}_n.$$

*Example 2.2.* The space of *alternate tensors* of order  $n$ , denoted by  $\Lambda^n(\mathbb{R}^3)$ , is the space of tensors  $\mathbf{T}$  such that

$$(\sigma \star \mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) = \epsilon(\sigma) \mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n), \quad \forall \sigma \in \mathfrak{S}_n,$$

where  $\epsilon(\sigma)$  is the signature of the permutation  $\sigma$ .

**2.1. Harmonic tensors.** The representation  $\mathbb{V}$  is *irreducible* if the only stable subspaces are  $\{0\}$  and  $\mathbb{V}$ . The irreducible representations of the rotation group  $\text{SO}(3, \mathbb{R})$  are known (up to isomorphism) [19]. Explicit models for these irreducible representations are given by *harmonic tensor spaces* which are described below.

Contracting two indices  $i, j$  on a totally symmetric tensor  $\mathbf{T}$  does not depend on the particular choice of the pair  $i, j$ . Thus, we can refer to this contraction without any reference to a particular choice of indices. We will denote this contraction as  $\text{tr } \mathbf{T}$ , which is a totally symmetric tensor of order  $n - 2$  and is called the *trace* of  $\mathbf{T}$ . Iterating the process leads to

$$\text{tr}^k \mathbf{T} = \text{tr}(\text{tr}(\dots(\text{tr } \mathbf{T})))$$

which is a totally symmetric tensor of order  $n - 2k$ .

**Definition 2.3.** A *harmonic tensor* of order  $n$  is a totally symmetric tensor  $\mathbf{T} \in \mathbb{S}^n(\mathbb{R}^3)$  such that  $\text{tr } \mathbf{T} = 0$ . The space of harmonic tensors of order  $n$  will be denoted by  $\mathbb{H}^n(\mathbb{R}^3)$ . It is a sub-vector space of  $\mathbb{S}^n(\mathbb{R}^3)$  of dimension  $2n + 1$ .

The sub-space  $\mathbb{H}^n(\mathbb{R}^3)$  of  $\mathbb{S}^n(\mathbb{R}^3)$  is invariant under the action of  $\text{SO}(3, \mathbb{R})$  and is irreducible (see for instance [19]). Moreover, every finite dimensional irreducible representation of  $\text{SO}(3, \mathbb{R})$  is isomorphic to some  $\mathbb{H}^n(\mathbb{R}^3)$ . As a special case of the Peter–Weil theorem [47], every  $\text{SO}(3, \mathbb{R})$ -representation  $\mathbb{V}$  splits into a direct sum of *harmonic tensor spaces*  $\mathbb{H}^n(\mathbb{R}^3)$ , and such a direct sum is called the *harmonic decomposition* of  $\mathbb{V}$ .

*Example 2.4.* The harmonic decomposition of a second order symmetric tensor  $\mathbf{s}$  corresponds to its decomposition  $\mathbf{s}' + \frac{1}{3}(\text{tr } \mathbf{s}) \mathbf{1}$  into its deviatoric and spheric parts

$$\mathbb{S}^2(\mathbb{R}^3) = \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^0(\mathbb{R}^3).$$

There is a well-known correspondence between totally symmetric tensors of order  $n$  and homogeneous polynomial of degree  $n$  on  $\mathbb{R}^3$ , which extends the well-known correspondence between a symmetric bilinear form and a

quadratic form via polarization. Indeed, to each symmetric tensor  $\mathbf{T} \in \mathbb{S}^n(\mathbb{R}^3)$  corresponds a homogeneous polynomial of degree  $n$  given by

$$\mathfrak{p}(\mathbf{x}) := \mathbf{T}(\mathbf{x}, \dots, \mathbf{x}), \quad \mathbf{x} := (x, y, z) \in \mathbb{R}^3.$$

This correspondence defines a linear isomorphism  $\phi$  between the tensor space  $\mathbb{S}^n(\mathbb{R}^3)$  and the polynomial space  $\mathbb{S}^n(\mathbb{R}^3)$  of homogeneous polynomials of degree  $n$  on  $\mathbb{R}^3$ . The linear isomorphism  $\phi$  satisfies moreover

$$\phi(g \star \mathbf{T}) = g \star \phi(\mathbf{T}), \quad g \in \mathrm{SO}(3, \mathbb{R}),$$

it is said to be an *equivariant* mapping.

*Remark 2.5.* The rotation group  $\mathrm{SO}(3, \mathbb{R})$  acts on the polynomial space  $\mathbb{S}^n(\mathbb{R}^3)$ , by the rule

$$(g \star \mathfrak{p})(\mathbf{x}) := \mathfrak{p}(g^{-1} \cdot \mathbf{x}), \quad g \in \mathrm{SO}(3, \mathbb{R}).$$

The inverse  $\mathbf{T} = \phi^{-1}(\mathfrak{p})$  can be recovered by *polarization*. More precisely, the expression  $\mathfrak{p}(t_1 \mathbf{x}_1 + \dots + t_n \mathbf{x}_n)$  is a homogeneous polynomial in the variables  $t_1, \dots, t_n$  and we get

$$\mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{1}{n!} \left. \frac{\partial^n}{\partial t_1 \dots \partial t_n} \right|_{t_1 = \dots = t_n = 0} \mathfrak{p}(t_1 \mathbf{x}_1 + \dots + t_n \mathbf{x}_n).$$

*Remark 2.6.* Note that the *Laplacian operator* of  $\mathfrak{p} = \phi(\mathbf{T})$  writes as

$$\Delta \phi(\mathbf{T}) = n(n-1) \phi(\mathrm{tr} \mathbf{T}).$$

Thus, *totally symmetric tensors with vanishing trace* correspond *via*  $\phi$  to *harmonic polynomials* (polynomials with vanishing Laplacian). This justifies the denomination of *harmonic tensors* for elements of  $\mathbb{H}^n(\mathbb{R}^3)$ . More generally, for any non-negative integer  $k$ , we get

$$(2.1) \quad \Delta^k \phi(\mathbf{T}) = \frac{n!}{(n-2k)!} \phi(\mathrm{tr}^k \mathbf{T}).$$

The equivariant isomorphism  $\phi$  sends  $\mathbb{H}^n(\mathbb{R}^3)$  to  $\mathbb{H}^n(\mathbb{R}^3)$ , the space of homogeneous harmonic polynomials of degree  $n$ . Thus, the spaces  $\mathbb{H}^n(\mathbb{R}^3)$  provide alternative models for *irreducible representations* of the rotation group  $\mathrm{SO}(3, \mathbb{R})$ . A third model of these irreducible representations is given by the spaces of *binary forms*  $\mathbb{S}_{\mathbb{R}}^{2n}(\mathbb{C}^2)$ , whose construction is detailed in Appendix A.

**2.2. Harmonic decomposition of totally symmetric tensors.** The harmonic decomposition of a homogeneous polynomial of degree  $n$  is described by the following proposition.

**Proposition 2.7.** *Every homogeneous polynomial  $\mathfrak{p} \in \mathbb{S}^n(\mathbb{R}^3)$  can be decomposed uniquely as*

$$(2.2) \quad \mathfrak{p} = \mathfrak{h}_0 + \mathfrak{q} \mathfrak{h}_1 + \dots + \mathfrak{q}^r \mathfrak{h}_r,$$

where  $\mathfrak{q}(x, y, z) = x^2 + y^2 + z^2$ ,  $r = [n/2]$  – with  $[\cdot]$  integer part – and  $\mathfrak{h}_k$  is a harmonic polynomial of degree  $n - 2k$ .

The harmonic polynomial  $(p)_0 := h_0$  is the orthogonal projection of  $p$  onto  $\mathbb{H}^n(\mathbb{R}^3)$  (for the induced Euclidean structure of  $\mathbb{R}^3$ ). Its explicit expression can be obtained recursively as follows (see [1]). Let first define

$$(2.3) \quad h_r = \begin{cases} \frac{1}{(2r+1)!} \Delta^r p, & \text{if } n \text{ is even;} \\ \frac{3! \times (r+1)}{(2r+3)!} \Delta^r p, & \text{if } n \text{ is odd.} \end{cases}$$

Then  $h_k$  is obtained inductively by the formula

$$(2.4) \quad h_k = \mu(k) \Delta^k \left( p - \sum_{j=k+1}^r q^j h_j \right)$$

with

$$(2.5) \quad \mu(k) := \frac{(2n-4k+1)!(n-k)!}{(2n-2k+1)!k!(n-2k)!},$$

which leads to  $h_0$  after  $r$  iterations.

Introducing the symmetric tensor product

$$\mathbf{T}_1 \odot \mathbf{T}_2 := \frac{1}{(n_1+n_2)!} \sum_{\sigma \in \mathfrak{S}_{n_1+n_2}} \sigma \star (\mathbf{T}_1 \otimes \mathbf{T}_2),$$

for  $\mathbf{T}_1 \in \mathbb{S}^{n_1}(\mathbb{R}^3)$  and  $\mathbf{T}_2 \in \mathbb{S}^{n_2}(\mathbb{R}^3)$ , we can recast proposition 2.7 in tensorial form:

$$\mathbb{S}^n(\mathbb{R}^3) = \mathbb{H}^n(\mathbb{R}^3) \oplus \mathbb{H}^{n-2}(\mathbb{R}^3) \oplus \dots \oplus \mathbb{H}^{n-2r}(\mathbb{R}^3)$$

where  $r = [n/2]$ . More precisely, every totally symmetric tensor  $\mathbf{T} \in \mathbb{S}^n(\mathbb{R}^3)$  of order  $n$  can be decomposed uniquely as

$$\mathbf{T} = \mathbf{H}_0 + \mathbf{1} \odot \mathbf{H}_1 + \dots + \mathbf{1}^{\odot r-1} \odot \mathbf{H}_{r-1} + \mathbf{1}^{\odot r} \odot \mathbf{H}_r,$$

where  $\mathbf{H}_k$  is a harmonic tensor of degree  $n-2k$  and  $\mathbf{1}^{\odot k}$  means the symmetrized tensorial product of  $k$  copies of  $\mathbf{1}$ . The harmonic part  $(\mathbf{T})_0 := \mathbf{H}_0$  is the orthogonal projection of  $\mathbf{T}$  onto  $\mathbb{H}^n(\mathbb{R}^3)$ . Using (2.1), the tensorial form of (2.3) reads

$$\mathbf{H}_r = \begin{cases} \frac{1}{2r+1} \text{tr}^r \mathbf{T}, & \text{if } n \text{ is even;} \\ \frac{3}{2r+3} \text{tr}^r \mathbf{T}, & \text{if } n \text{ is odd.} \end{cases}$$

which is a scalar for even  $n$  or a vector for odd  $n$ . Finally, the tensorial form of (2.4) is given by:

$$\mathbf{H}_k = \phi^{-1}(h_k) = \mu(k) \frac{n!}{(n-2k)!} \text{tr}^k \left[ \mathbf{T} - \sum_{j=k+1}^r \mathbf{1}^{\odot j} \odot \mathbf{H}_j \right]$$

Note that in this equation,  $\text{tr}^k \left[ \mathbf{T} - \sum_{j=k+1}^r \mathbf{1}^{\odot j} \odot \mathbf{H}_j \right]$  is the orthogonal projection of  $\text{tr}^k \mathbf{T}$  on  $\mathbb{H}^{n-2k}(\mathbb{R}^3)$ .



**2.3. Harmonic decomposition of the elasticity tensor.** Harmonic decomposition of the space  $\mathbb{E}la$  of Elasticity tensors was first obtained by Backus [3] and is given by

$$\mathbb{E}la = \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^4(\mathbb{R}^3).$$

More precisely, an elasticity tensor  $\mathbf{E}$  admits the following harmonic decomposition [3]:

$$(2.6) \quad \mathbf{E} = \alpha \mathbf{1} \otimes_{(4)} \mathbf{1} + \beta \mathbf{1} \otimes_{(2,2)} \mathbf{1} + \mathbf{1} \otimes_{(4)} \mathbf{a}' + \mathbf{1} \otimes_{(2,2)} \mathbf{b}' + \mathbf{H}.$$

In this formula,  $\mathbf{1}$  is the Euclidean canonical bilinear 2-form represented by the components  $(\delta_{ij})$  in any orthonormal basis and the Young-symmetrized tensor products  $\otimes_{(4)}$  and  $\otimes_{(2,2)}$ , between symmetric second-order tensors  $\mathbf{a}, \mathbf{b}$ , are defined as follows:

$$(\mathbf{a} \otimes_{(4)} \mathbf{b})_{ijkl} = \frac{1}{6} (a_{ij}b_{kl} + b_{ij}a_{kl} + a_{ik}b_{jl} + b_{ik}a_{jl} + a_{il}b_{jk} + b_{il}a_{jk}),$$

where  $\otimes_{(4)}$  is the totally symmetric tensor product also denoted by  $\odot$ , and

$$(\mathbf{a} \otimes_{(2,2)} \mathbf{b})_{ijkl} = \frac{1}{6} (2a_{ij}b_{kl} + 2b_{ij}a_{kl} - a_{ik}b_{jl} - a_{il}b_{jk} - b_{ik}a_{jl} - b_{il}a_{jk}).$$

In the harmonic decomposition (2.6),  $\alpha, \beta$  are scalars,  $\mathbf{a}', \mathbf{b}'$  are second order harmonic tensors (also known as symmetric deviatoric tensors) and  $\mathbf{H}$  is a fourth-order harmonic tensor.

*Remark 2.8.* In this decomposition,  $\alpha, \beta$  and  $\mathbf{a}', \mathbf{b}'$  are related to the *dilatation tensor*  $\mathbf{d} := \text{tr}_{12} \mathbf{E}$  and the *Voigt tensor*  $\mathbf{v} := \text{tr}_{13} \mathbf{E}$  by the formulas

$$\alpha = \frac{1}{15} (\text{tr} \mathbf{d} + 2 \text{tr} \mathbf{v}) \quad \beta = \frac{1}{6} (\text{tr} \mathbf{d} - \text{tr} \mathbf{v})$$

and

$$\mathbf{a}' = \frac{2}{7} (\mathbf{d}' + 2\mathbf{v}') \quad \mathbf{b}' = 2 (\mathbf{d}' - \mathbf{v}')$$

where  $\mathbf{d}' := \mathbf{d} - \frac{1}{3} \text{tr}(\mathbf{d}) \mathbf{1}$  and  $\mathbf{v}' := \mathbf{v} - \frac{1}{3} \text{tr}(\mathbf{v}) \mathbf{1}$  are the deviatoric parts of  $\mathbf{d}$  and  $\mathbf{v}$  respectively.

### 3. HARMONIC FACTORIZATION

In this section, we introduce a product between harmonic tensors (which is associative and commutative). Then, we show that every harmonic tensor  $\mathbf{H} \in \mathbb{H}^n(\mathbb{R}^3)$  can be factorized as a harmonic product of lower order tensors.

**3.1. Harmonic product.** The product of two harmonic polynomials is not harmonic in general. However, we define the following binary operation on the space of harmonic polynomials:

$$h_1 * h_2 := (h_1 h_2)_0.$$

This product is commutative:

$$h_1 * h_2 = h_2 * h_1,$$

and is moreover associative, which means that

$$(h_1 * h_2) * h_3 = h_1 * (h_2 * h_3).$$

The fact that this binary operation is associative is essential in practice. It means that the final result does not depend on the way we have chosen to “put the brackets” to compute it. We can therefore write

$$\mathbf{h}_1 * \mathbf{h}_2 * \mathbf{h}_3 := (\mathbf{h}_1 * \mathbf{h}_2) * \mathbf{h}_3 = \mathbf{h}_1 * (\mathbf{h}_2 * \mathbf{h}_3),$$

and omit the brackets. Endowed with this product, the vector space

$$\mathbb{H}(\mathbb{R}^3) := \bigoplus_{n \geq 0} \mathbb{H}^n(\mathbb{R}^3)$$

of harmonic polynomials becomes a *commutative algebra*. The unity (for the multiplicative operation  $*$ ) is represented by the constant harmonic polynomial 1.

*Remark 3.1.* Using the equivariant isomorphism  $\phi$  between  $\mathbb{H}^n(\mathbb{R}^3)$  and  $\mathbb{H}^n(\mathbb{R}^3)$ , this algebraic structure is inherited on harmonic tensors. More precisely, given two harmonic tensors  $\mathbf{H}_1$  and  $\mathbf{H}_2$  of order  $n_1$  and  $n_2$  respectively, we define a harmonic tensor  $\mathbf{H}_1 * \mathbf{H}_2$  of order  $n_1 + n_2$  by the following formula:

$$\mathbf{H}_1 * \mathbf{H}_2 := \phi^{-1}(\phi(\mathbf{H}_1) * \phi(\mathbf{H}_2)) = (\mathbf{H}_1 \odot \mathbf{H}_2)_0.$$

*Example 3.2* (Harmonic product of two vectors). For two vectors  $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{H}^1(\mathbb{R}^3)$ , we get

$$\mathbf{v}_1 * \mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_1 \otimes \mathbf{v}_2 + \mathbf{v}_2 \otimes \mathbf{v}_1) - \frac{1}{3}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{1}.$$

*Example 3.3* (Harmonic product of two second order harmonic tensors). For two second order harmonic tensors  $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H}^2(\mathbb{R}^3)$ , we get

$$\mathbf{h}_1 * \mathbf{h}_2 = \mathbf{h}_1 \odot \mathbf{h}_2 - \frac{2}{7}\mathbf{1} \odot (\mathbf{h}_1 \mathbf{h}_2 + \mathbf{h}_2 \mathbf{h}_1) + \frac{2}{35} \text{tr}(\mathbf{h}_1 \mathbf{h}_2) \mathbf{1} \odot \mathbf{1}.$$

**3.2. Harmonic factorization theorem.** We now introduce the following  $\text{SO}(3, \mathbb{R})$ -equivariant mapping, from the direct sum  $\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3$  of  $n$  copies of  $\mathbb{R}^3$  to  $\mathbb{H}^n(\mathbb{R}^3)$ :

$$(3.1) \quad \Phi : (\mathbf{w}_1, \dots, \mathbf{w}_n) \mapsto \mathbf{h}(\mathbf{x}) := (\mathbf{x} \cdot \mathbf{w}_1) * \dots * (\mathbf{x} \cdot \mathbf{w}_n).$$

This mapping is obviously not *one-to-one*. It is however *surjective*. This result is not trivial and seems to have been demonstrated for the first time by Sylvester [48] (see also [3, Appendix 1]). We provide our own proof of Sylvester’s theorem in Appendix B, which relies on *binary forms*. Note however, that the use of binary forms for the proofs limits the validity of these theorems to dimension  $D = 3$  (see [3] for a counter-example in dimension  $D \geq 4$ ).

**Theorem 3.4** (Sylvester’s theorem). *Let  $\mathbf{h} \in \mathbb{H}^n(\mathbb{R}^3)$ , then there exist  $n$  vectors  $\mathbf{w}_1, \dots, \mathbf{w}_n$  in  $\mathbb{R}^3$  such that*

$$\mathbf{h}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{w}_1) * \dots * (\mathbf{x} \cdot \mathbf{w}_n).$$

*Example 3.5.* Let  $\mathbf{h} \in \mathbb{H}^2(\mathbb{R}^3)$  and  $\mathbf{f} \in \mathbb{S}_{\mathbb{R}}^4(\mathbb{C}^2)$  be the corresponding binary form (see example A.6). The four roots of  $\mathbf{f}$  are

$$\lambda_1, -\frac{1}{\lambda_1}, \lambda_2, -\frac{1}{\lambda_2}$$

with the possible couple  $(0, \infty)$  among them. Using the stereographic projection  $\tau$  (see remark A.4), we set

$$\mathbf{w}_i := \tau^{-1}(\lambda_i).$$

Note that

$$\tau^{-1}\left(-\frac{1}{\lambda_i}\right) = -\tau^{-1}(\lambda_i).$$

Then, we have

$$\mathbf{h}(\mathbf{x}) = k(\mathbf{x} \cdot \mathbf{w}_1) * (\mathbf{x} \cdot \mathbf{w}_2),$$

for some constant  $k \in \mathbb{R}$ .

The lack of injectivity for the mapping  $\Phi$  is detailed in the following proposition, which proof is provided in Appendix B.

**Proposition 3.6.** *We have*

$$\Phi(\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_n) = \Phi(\mathbf{w}_1, \dots, \mathbf{w}_n).$$

*if and only if*

$$(\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_n) = (\lambda_1 \mathbf{w}_{\sigma(1)}, \dots, \lambda_n \mathbf{w}_{\sigma(n)})$$

where  $(\mathbf{w}_{\sigma(1)}, \dots, \mathbf{w}_{\sigma(n)})$  is a permutation of the vectors  $(\mathbf{w}_1, \dots, \mathbf{w}_n)$  for some  $\sigma \in \mathfrak{S}_n$  and  $\lambda_1, \dots, \lambda_n$  are real numbers such that  $\lambda_1 \cdots \lambda_n = 1$ .

*Remark 3.7.* It is tempting to restrict the mapping  $\Phi$  defined in (3.1) to unit vectors, in order to try to reduce the indeterminacy in the choice of an  $n$ -uple of vectors  $(\mathbf{w}_1, \dots, \mathbf{w}_n)$ . However, this does not really solve the problem of indeterminacy. Of course, the indeterminacy due to the scaling by the  $\lambda_i$  is replaced by signs  $\epsilon_i$ , but the indeterminacy due to the permutation persists anyway.

Sylvester's theorem is equivalent to the following (in appearance, more general) theorem which leads to other important factorization results for harmonic polynomials (and thus harmonic tensors).

**Theorem 3.8** (Harmonic factorization theorem). *Let  $\mathbf{h} \in \mathbb{H}^{kn}(\mathbb{R}^3)$ . Then there exist harmonic polynomials  $h_1, \dots, h_k \in \mathbb{H}^n(\mathbb{R}^3)$  such that*

$$\mathbf{h} = h_1 * \dots * h_k.$$

*Remark 3.9.* The tensorial counterpart of Theorem 3.8 is that for every harmonic tensor  $\mathbf{H} \in \mathbb{H}^{kn}(\mathbb{R}^3)$  there exist harmonic tensors  $\mathbf{H}_1, \dots, \mathbf{H}_k \in \mathbb{H}^n(\mathbb{R}^3)$  such that

$$\mathbf{H} = \mathbf{H}_1 * \dots * \mathbf{H}_k.$$

A corollary of Theorem 3.8 is the following result, which applies to every even order harmonic tensor.

**Corollary 3.10.** *Let  $\mathbf{H} \in \mathbb{H}^{2n}(\mathbb{R}^3)$ . Then there exist  $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{H}^n(\mathbb{R}^3)$  such that*

$$(3.2) \quad \mathbf{H} = \mathbf{H}_1 * \mathbf{H}_1 - \mathbf{H}_2 * \mathbf{H}_2.$$

*Proof.* Let  $\mathbf{h} \in \mathbb{H}^{2n}(\mathbb{R}^3)$  corresponding to the harmonic tensor  $\mathbf{H}$ . Using Theorem 3.8, we can find  $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{H}^n(\mathbb{R}^3)$  such that  $\mathbf{h} = (\mathbf{p}_1 \mathbf{p}_2)_0$ . But

$$\mathbf{p}_1 \mathbf{p}_2 = \left( \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \right)^2 - \left( \frac{\mathbf{p}_1 - \mathbf{p}_2}{2} \right)^2.$$

Hence, if we set  $\mathbf{h}_1 = (\mathbf{p}_1 + \mathbf{p}_2)/2$  and  $\mathbf{h}_2 = (\mathbf{p}_1 - \mathbf{p}_2)/2$ , we get

$$\mathbf{h} = \mathbf{h}_1 * \mathbf{h}_1 - \mathbf{h}_2 * \mathbf{h}_2,$$

which tensorial form is precisely (3.2).  $\square$

*Remark 3.11.* This result has already been obtained in [15] for monoclinic harmonic fourth-order tensors, using direct elimination techniques.

*Remark 3.12.* In 2D, the factorization of the harmonic part  $\mathbf{H}^{2D} \in \mathbb{H}^4(\mathbb{R}^2)$  of the elasticity tensor has already been computed in [14], using previous results obtained in [53, 50]. In that case, we get

$$(3.3) \quad \mathbf{H}^{2D} = \mathbf{h} \otimes \mathbf{h} - \frac{1}{2}(\mathbf{h} \cdot \mathbf{h}) \mathbf{J}^{2D}$$

with second order tensor  $\mathbf{h}$  harmonic (*i.e.* symmetric deviatoric), and where  $\mathbf{J}^{2D} = \mathbf{I} - \frac{1}{2} \mathbf{1} \otimes \mathbf{1}$  and  $\mathbf{h} \cdot \mathbf{h} = \|\mathbf{h}\|^2$ . As in 3D, we can define a harmonic product between 2D harmonic tensors  $\mathbf{h}_k$

$$\mathbf{h}_1 * \mathbf{h}_2 := \mathbf{h}_1 \odot \mathbf{h}_2 - \frac{1}{4} \text{tr}(\mathbf{h}_1 \mathbf{h}_2) \mathbf{1} \odot \mathbf{1}$$

and equation (3.3) can be rewritten as

$$(3.4) \quad \mathbf{H}^{2D} = \mathbf{h} * \mathbf{h}.$$

Hence, any 2D harmonic fourth-order tensor can be expressed as the harmonic square  $\mathbf{h} * \mathbf{h}$ , *i.e.* the projection of diadic symmetric square  $\mathbf{h} \otimes \mathbf{h}$  onto the vector space of fourth-order harmonic tensors  $\mathbb{H}^4(\mathbb{R}^2)$ . In 3D, this result is generally false. Indeed, if a fourth-order harmonic tensor  $\mathbf{H}$  is a harmonic square, *i.e.*  $\mathbf{H} = \mathbf{h} * \mathbf{h}$ , then  $\mathbf{H}$  is either orthotropic or transversely isotropic (see section 4).

Although these various decompositions are interesting by themselves, there is however something frustrating. These decompositions are not unique in general and the solution is not enough constructive (they require root's extraction of possible high degree polynomials). In the next sections, we provide explicit geometric reconstructions.

#### 4. EQUIVARIANT RECONSTRUCTION OF A FOURTH-ORDER HARMONIC TENSOR

In this section, we consider the general problem of reconstructing a fourth-order harmonic tensor  $\mathbf{H}$  using lower order *covariants* (*i.e.* lower order tensors which depend on  $\mathbf{H}$  in an equivariant manner). This reconstruction is useful when applied to the fourth-order harmonic part  $\mathbf{H}$  of the elasticity tensor (2.6), since its other components  $\alpha, \beta, \mathbf{a}', \mathbf{b}'$  are already decomposed by means of second order covariants tensors and invariants.

From a mechanical point of view the definition of covariants of a given – effective elasticity – tensor is strongly related to the tensorial nature of the thermodynamics variables allowing to properly model a specific state

coupling with elasticity [29, 40, 30, 9, 31]. In the case of effective elasticity of cracked/damaged media for instance, either second order [25, 12, 28] or fourth order [11, 29] damage variables are considered, depending on the authors choice [30, 27, 31]. Considering the reconstruction by means of lower order covariants of the harmonic part of an effective elasticity tensor in this case would answer the question: do we have to use a fourth order damage variable – a fourth order covariant in next derivations – in order to reconstruct a given harmonic – effective elasticity – tensor? In 2D the answer is negative, as shown in [14, 15]: only one second order tensor  $\mathbf{h}$  is needed (see Eq. (3.4)).

**4.1. The reconstruction problem.** To illustrate the problem, let us first come back to Theorem 3.8. Applied to a fourth-order harmonic tensor  $\mathbf{H}$ , this theorem states that  $\mathbf{H}$  can be factorized as the harmonic product of two second order harmonic tensors

$$\mathbf{H} = \mathbf{h}_1 * \mathbf{h}_2.$$

In other words, the equivariant mapping

$$\mathcal{F} : \mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3) \rightarrow \mathbb{H}^4(\mathbb{R}^3), \quad (\mathbf{h}_1, \mathbf{h}_2) \mapsto \mathbf{h}_1 * \mathbf{h}_2$$

is surjective. What would be more interesting is to have an explicit mapping (i.e. an explicit construction of the solution)

$$\mathcal{S} : \mathbb{H}^4(\mathbb{R}^3) \rightarrow \mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3), \quad \mathbf{H} \mapsto (\mathbf{h}_1(\mathbf{H}), \mathbf{h}_2(\mathbf{H}))$$

such that:

$$(\mathcal{F} \circ \mathcal{S})(\mathbf{H}) = \mathbf{h}_1(\mathbf{H}) * \mathbf{h}_2(\mathbf{H}) = \mathbf{H}, \quad \forall \mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3),$$

and with geometric properties. A mapping like  $\mathcal{S}$  is called a *section* of  $\mathcal{F}$ . A geometric property would be for instance for  $\mathcal{S}$  to be equivariant, in which case each second order tensors  $\mathbf{h}_i(\mathbf{H})$  is a covariant tensor. The main problem is that there does not exist any *equivariant section*  $\mathcal{S}$  for  $\mathcal{F}$  defined on the whole space  $\mathbb{H}^4(\mathbb{R}^3)$ : for instance, for a tensor  $\mathbf{H}$  with cubic symmetry, all second order harmonic covariants vanish, which forbids the existence of such a general equivariant section. We will provide now the following definitions.

**Definition 4.1** (Decomposition). Let  $\mathbf{T}$  be some  $n$ th order tensor. A *decomposition* of a  $\mathbf{T}$  into other *tensors* (possibly scalars)  $\mathbf{T}_k$  ( $k = 1, \dots, N$ ) is a mapping

$$\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N).$$

The decomposition is *equivariant* if  $\mathcal{F}$  is an equivariant mapping.

**Definition 4.2** (Reconstruction). Given a decomposition

$$\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$$

of a tensor  $\mathbf{T}$  into other *tensors* (possibly scalars), a *reconstruction* of  $\mathbf{T}$  is a section  $\mathcal{S}$  of  $\mathcal{F}$ . The reconstruction is *equivariant* if both  $\mathcal{F}$  and  $\mathcal{S}$  are equivariant mappings.

*Remark 4.3.* In other words, a reconstruction is given by a mapping

$$\mathcal{S} : \mathbf{T} \mapsto (\kappa_1(\mathbf{T}), \dots, \kappa_N(\mathbf{T}))$$

such that

$$\mathbf{T} = \mathcal{F}(\kappa_1(\mathbf{T}), \dots, \kappa_N(\mathbf{T})).$$

If the reconstruction is equivariant the  $\mathbf{T}_k = \kappa_k(\mathbf{T})$  are *covariant tensors* (or invariants if there are scalars), which means that

$$g \star \mathbf{T}_k = \kappa_k(g \star \mathbf{T}), \quad \forall g \in \text{SO}(3, \mathbb{R}).$$

**4.2. Obstruction to an equivariant reconstruction.** The existence of an equivariant reconstruction may not be possible for certain symmetry classes. More precisely, we have the following result.

**Lemma 4.4.** *Consider an equivariant decomposition  $\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$  and suppose that there exists an equivariant reconstruction  $\mathcal{S}$ , so that  $\mathbf{T}_k$  are covariant tensors. Then*

$$(4.1) \quad G_{\mathbf{T}} = \bigcap_k G_{\mathbf{T}_k},$$

where

$$G_{\mathbf{T}} := \{g \in G; g \star \mathbf{T} = \mathbf{T}\}$$

is the symmetry group of the tensor  $\mathbf{T}$ .

*Proof.* Consider an equivariant decomposition  $\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$ . Then, we have

$$\bigcap_k G_{\mathbf{T}_k} \subset G_{\mathbf{T}}.$$

If moreover  $\mathbf{T}_k = \kappa_k(\mathbf{T})$  are covariant tensors, we get thus

$$g \star \mathbf{T}_k = \kappa_k(g \star \mathbf{T}) = \kappa_k(\mathbf{T}) = \mathbf{T}_k,$$

for all  $g \in G_{\mathbf{T}}$ , and hence  $G_{\mathbf{T}} \subset G_{\mathbf{T}_k}$ , which achieves the proof.  $\square$

According to lemma 4.4, the existence of an equivariant reconstruction associated to an equivariant decomposition

$$\mathbf{T} = \mathcal{F}(\kappa_1(\mathbf{T}), \dots, \kappa_N(\mathbf{T}))$$

requires some conditions on the symmetry groups of the involved covariants  $\mathbf{T}_k = \kappa_k(\mathbf{T})$ . For a decomposition involving second-order covariant tensors, we have the following result (for details on closed subgroups of  $\text{SO}(3, \mathbb{R})$ , see Appendix C).

**Corollary 4.5.** *If  $\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$  is an equivariant decomposition of  $\mathbf{T}$  into second-order symmetric covariant tensors  $\mathbf{T}_k$ , then*

$$[G_{\mathbf{T}}] \in \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\text{O}(2)]\},$$

where  $[H]$  means the conjugacy class of a subgroup  $H \subset \text{SO}(3, \mathbb{R})$ .

*Proof.* The proof is easily achieved using the *clips operator* introduced in [38]. This binary operator, noted  $\odot$  is defined on conjugacy classes (or finite families of conjugacy classes) of closed subgroups of  $\text{SO}(3, \mathbb{R})$ :

$$[H_1] \odot [H_2] = \{[H_1 \cap (gH_2g^{-1})], \quad g \in \text{SO}(3, \mathbb{R})\}.$$

It is an associative and commutative operation and

$$[H] \odot [\mathbf{1}] = \{[\mathbf{1}]\}, \quad [H] \odot [\mathrm{SO}(3, \mathbb{R})] = \{[H]\},$$

for all closed subgroup  $H$ . Moreover, we have in particular (see [38]):

$\odot$	$[\mathbb{Z}_2]$	$[\mathbb{D}_2]$	$[\mathrm{O}(2)]$
$[\mathbb{Z}_2]$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$
$[\mathbb{D}_2]$		$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$
$[\mathrm{O}(2)]$			$\{[\mathbb{Z}_2], [\mathbb{D}_2], [\mathrm{O}(2)]\}$

Thus, if each  $\mathbf{T}_k$  is a second-order symmetric covariant (or an invariant), then  $[G_{\mathbf{T}_k}]$  is either  $[\mathbb{D}_2]$ ,  $[\mathrm{O}(2)]$  or  $[\mathrm{SO}(3, \mathbb{R})]$  (see Appendix C) and we deduce that

$$[G_{\mathbf{T}}] \in [H_1] \odot [H_2] \dots \odot [H_N], \quad [H_i] \in \{[\mathbb{D}_2], [\mathrm{O}(2)], [\mathrm{SO}(3, \mathbb{R})]\},$$

which achieves the proof.  $\square$

*Remark 4.6.* Corollary 4.5 implies in particular that there is no equivariant reconstruction by means of second order symmetric covariant tensors of a harmonic fourth order tensor  $\mathbf{H}$  (and thus of the elasticity tensor  $\mathbf{E}$ ) for the cubic, the trigonal and the tetragonal symmetry classes. For a cubic tensor, the situation is even worse since all its second order symmetric covariants are spheric, whereas its first order covariants vanish.

*Remark 4.7.* The geometric framework introduced here is connected to the so-called *isotropic extension* of anisotropic constitutive functions via structural tensors developed by Boehler [6] and Liu [33] independently (see also [35]). In the case of a linear constitutive function, an isotropic extension is just an equivariant decomposition *limited to a given symmetry class*  $[G_H]$  and for which the arguments  $(\mathbf{T}_1, \dots, \mathbf{T}_N)$  (the structural tensors) of the decomposition satisfy (4.1). In that case, our approach is however more constructive since we are able to give an *explicit equivariant formula* in which the “structural tensors” are covariant tensors of the constitutive tensor. Moreover, condition (4.1) is a corollary of the theory and does not need to be imposed *a priori* as an hypothesis.

## 5. RECONSTRUCTIONS BY MEANS OF SECOND ORDER COVARIANTS

In this section, we provide explicit equivariant reconstructions for transverse-isotropic and orthotropic fourth order harmonic tensors. These reconstructions use the following second-order (polynomial) covariants introduced in [8]:

$$(5.1) \quad \begin{aligned} \mathbf{d}_2 &:= \mathrm{tr}_{13} \mathbf{H}^2, & \mathbf{d}_3 &:= \mathrm{tr}_{13} \mathbf{H}^3, & \mathbf{d}_4 &:= \mathbf{d}_2^2, \\ \mathbf{d}_5 &:= \mathbf{d}_2(\mathbf{H}\mathbf{d}_2), & \mathbf{d}_6 &:= \mathbf{d}_2^3, & \mathbf{d}_7 &:= \mathbf{d}_2^2(\mathbf{H}\mathbf{d}_2) \\ \mathbf{d}_8 &:= \mathbf{d}_2^2(\mathbf{H}^2\mathbf{d}_2), & \mathbf{d}_9 &:= \mathbf{d}_2^2(\mathbf{H}\mathbf{d}_2^2), & \mathbf{d}_{10} &:= \mathbf{d}_2^2(\mathbf{H}^2\mathbf{d}_2^2). \end{aligned}$$

where the following notations have been used.

- For two fourth-order tensors  $\mathbf{A}$  and  $\mathbf{B}$ ,  $\mathbf{AB}$  is the fourth-order tensor

$$(\mathbf{AB})_{ijkl} := A_{ijpq}B_{pqkl}.$$

- For a fourth-order tensor  $\mathbf{A}$  and a second-order tensor  $\mathbf{b}$ ,  $\mathbf{Ab}$  is the second-order tensor

$$(\mathbf{Ab})_{ij} := A_{ijkl}b_{kl}.$$

We also make use of the following invariants:

$$(5.2) \quad J_k := \operatorname{tr} \mathbf{d}_k, \quad k = 2, \dots, 10,$$

which constitute an *integrity basis* for  $\mathbb{H}^4(\mathbb{R}^3)$  (see [8]).

*Remark 5.1.* The second-order covariants  $\mathbf{d}_k$  ( $k = 2, 3, 4, 6$ ) are always *symmetric* covariants, while in general  $\mathbf{d}_k$  ( $k = 5, 7, 8, 9, 10$ ) are neither symmetric nor antisymmetric.

Finally, using the Kelvin representation, a fourth order harmonic tensor  $\mathbf{H} = (H_{ijkl})$  is represented by the symmetric matrix

$$\underline{\mathbf{H}} = \begin{pmatrix} H_{1111} & H_{1122} & H_{1133} & \sqrt{2}H_{1123} & \sqrt{2}H_{1113} & \sqrt{2}H_{1112} \\ H_{1122} & H_{2222} & H_{2233} & \sqrt{2}H_{2223} & \sqrt{2}H_{1223} & \sqrt{2}H_{1222} \\ H_{1133} & H_{2233} & H_{3333} & \sqrt{2}H_{2333} & \sqrt{2}H_{1333} & \sqrt{2}H_{1233} \\ \sqrt{2}H_{1123} & \sqrt{2}H_{2223} & \sqrt{2}H_{2333} & 2H_{2233} & 2H_{1233} & 2H_{1223} \\ \sqrt{2}H_{1113} & \sqrt{2}H_{1223} & \sqrt{2}H_{1333} & 2H_{1233} & 2H_{1133} & 2H_{1123} \\ \sqrt{2}H_{1112} & \sqrt{2}H_{1222} & \sqrt{2}H_{1233} & 2H_{1223} & 2H_{1123} & 2H_{1122} \end{pmatrix}$$

with only 9 independent components due to the traceless property, namely:

$$\begin{aligned} H_{1111} &= -H_{1122} - H_{1133}, & H_{2222} &= -H_{1122} - H_{2233}, \\ H_{3333} &= -H_{1133} - H_{2233}, & H_{2333} &= -H_{1123} - H_{2223}, \\ H_{1113} &= -H_{1223} - H_{2333}, & H_{1222} &= -H_{1112} - H_{1233}. \end{aligned}$$

**5.1. The transversely isotropic class.** A harmonic tensor  $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$  is transverse isotropic if and only if there exists  $g \in \operatorname{SO}(3, \mathbb{R})$  such that  $\mathbf{H} = g \star \mathbf{H}_0$  where  $\mathbf{H}_0$  has the following normal matrix form

$$\underline{\mathbf{H}}_0 = \begin{pmatrix} 3\delta & \delta & -4\delta & 0 & 0 & 0 \\ \delta & 3\delta & -4\delta & 0 & 0 & 0 \\ -4\delta & -4\delta & 8\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -8\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\delta \end{pmatrix},$$

where  $\delta \neq 0$ . It was moreover established in [2, Section 5.2] that

$$J_2 \neq 0, \quad \delta = \frac{7J_3}{18J_2}.$$

Thus, by a direct computation of the second order covariant  $\mathbf{d}_2$  (see Eq. (5.1)) and its deviatoric part  $\mathbf{d}'_2$ , we get the following result:

**Theorem 5.2.** *For any transversely isotropic harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$\mathbf{H} = \frac{63}{25} \frac{1}{J_3(\mathbf{H})} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H}),$$

*Remark 5.3.* If  $J_3(\mathbf{H}) > 0$ , then  $\mathbf{H}$  is a perfect harmonic square. The converse is also true and can be easily established, using binary forms, where the harmonic product corresponds to the ordinary product of polynomials.



**5.2. The orthotropic class.** A harmonic tensor  $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$  is orthotropic if and only if there exists  $g \in \text{SO}(3, \mathbb{R})$  such that  $\mathbf{H} = g \star \mathbf{H}_0$  where  $\mathbf{H}_0$  has the following normal matrix form

$$(5.3) \quad \underline{\mathbf{H}}_0 = \begin{pmatrix} \lambda_2 + \lambda_3 & -\lambda_3 & -\lambda_2 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 + \lambda_1 & -\lambda_1 & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & \lambda_2 + \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\lambda_3 \end{pmatrix}$$

where  $\lambda_1, \lambda_2, \lambda_3$  are three *distinct* real numbers. Note that this normal form is however not unique: any permutation of the  $\lambda_k$  provides an alternative normal form, but this is the only ambiguity. It is therefore useful to introduce the three elementary symmetric functions

$$(5.4) \quad \begin{aligned} \sigma_1 &:= \lambda_1 + \lambda_2 + \lambda_3, \\ \sigma_2 &:= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ \sigma_3 &:= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

which are independent of a particular normal form, as being invariant under any permutation of the  $\lambda_i$ . Moreover, it was shown in [2, Section 5.5] that the *discriminant*

$$\Delta_3 := (\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2,$$

which is *strictly positive*, is a polynomial invariant, and that the  $\sigma_k$  are themselves *rational invariants*. More precisely:

$$\Delta_3 = \frac{K_6}{432}, \quad \text{where} \quad K_6 := 6J_6 - 9J_2J_4 - 20J_3^2 + 3J_2^3,$$

and

$$\begin{aligned} \sigma_1 &= \frac{9(3J_7 - 3J_2J_5 + 3J_3J_4 - J_2^2J_3)}{2K_6}, \\ \sigma_2 &= \frac{4}{7}\sigma_1^2 - \frac{1}{14}J_2, \\ \sigma_3 &= -\frac{1}{24}J_3 + \frac{1}{7}\sigma_1^3 - \frac{1}{56}\sigma_1J_2. \end{aligned}$$

Consider now the matrix

$$\boldsymbol{\lambda}_0 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Since both  $\mathbf{H}_0$  and  $\boldsymbol{\lambda}_0$  have the same symmetry group, namely  $\mathbb{D}_2$  (as defined in Appendix C), the following definition

$$\boldsymbol{\lambda}(\mathbf{H}) := g \star \boldsymbol{\lambda}_0, \quad \text{if} \quad \mathbf{H} = g \star \mathbf{H}_0$$

does not depend on the rotation  $g$  and thus defines a covariant mapping from the orthotropic class in  $\mathbb{H}^4(\mathbb{R}^3)$  to the space of symmetric second order tensors. We have moreover the following result, which can be checked by a direct evaluation on  $\mathbf{H}_0$  and  $\boldsymbol{\lambda}'_0$ .

**Lemma 5.4.** *Let  $\mathbf{H}$  be an orthotropic harmonic fourth-order tensor. Then, the deviatoric part  $\boldsymbol{\lambda}'(\mathbf{H})$  of the second order covariant  $\boldsymbol{\lambda}(\mathbf{H})$  can be written as*

$$(5.5) \quad \boldsymbol{\lambda}'(\mathbf{H}) = \frac{1}{8\Delta_3} (\alpha_2 \mathbf{d}'_2(\mathbf{H}) + \alpha_3 \mathbf{d}'_3(\mathbf{H}) - 54\sigma_3 \mathbf{d}'_4(\mathbf{H}) + 11\sigma_2 \mathbf{d}'_5(\mathbf{H}))$$

where

$$\begin{aligned} \alpha_2 &:= 2(112\sigma_1^2\sigma_3 + 21\sigma_1\sigma_2^2 - 270\sigma_2\sigma_3), \\ \alpha_3 &:= 8(14\sigma_1\sigma_3 - 11\sigma_1^2\sigma_2 + 15\sigma_2^2). \end{aligned}$$

Note that

$$2(\sigma_1^2 - 3\sigma_2) = (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 > 0,$$

and we can thus introduce the positive invariant

$$\sigma_{eq} := \sqrt{\sigma_1^2 - 3\sigma_2}.$$

and the *Lode invariant* defined by

$$\mathcal{L} := \frac{1}{\sigma_{eq}^3} \left( \sigma_1^3 - \frac{9}{2}\sigma_1\sigma_2 + \frac{27}{2}\sigma_3 \right).$$

Since moreover

$$\begin{aligned} \Delta_3 &= \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2 \\ &= \frac{4}{27}\sigma_{eq}^6(1 - \mathcal{L}^2), \end{aligned}$$

and  $\Delta_3 > 0$ , we deduce that  $-1 < \mathcal{L} < 1$ .

With these notations, we obtain the following result which can be checked by a direct computation on the normal form.

**Theorem 5.5.** *For any orthotropic harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$(5.6) \quad \mathbf{H} = h_1 \boldsymbol{\lambda}'(\mathbf{H}) * \boldsymbol{\lambda}'(\mathbf{H}) + 2h_2 \boldsymbol{\lambda}'(\mathbf{H}) * (\boldsymbol{\lambda}'(\mathbf{H})^2)' + h_3 (\boldsymbol{\lambda}'(\mathbf{H})^2)' * (\boldsymbol{\lambda}'(\mathbf{H})^2)',$$

where  $\boldsymbol{\lambda}'(\mathbf{H})$  is defined by (5.5) and the three invariants  $h_k$  are given by

$$h_1 := \frac{5\sigma_1 + 7\mathcal{L}\sigma_{eq}}{2(1 - \mathcal{L}^2)\sigma_{eq}^2}, \quad h_2 := -\frac{3(5\mathcal{L}\sigma_1 + 7\sigma_{eq})}{2(1 - \mathcal{L}^2)\sigma_{eq}^3}, \quad h_3 := \frac{9(5\sigma_1 + 7\mathcal{L}\sigma_{eq})}{2(1 - \mathcal{L}^2)\sigma_{eq}^4}.$$

*Remark 5.6.* The three invariants  $h_k$  are in fact rational invariants of  $\mathbf{H}$ . Using invariants  $\sigma_k$  (Eq. (5.4)), we have indeed

$$\begin{cases} h_1 = \frac{(\sigma_1^2 - 3\sigma_2)(8\sigma_1^3 - 31\sigma_1\sigma_2 + 63\sigma_3)}{9\Delta_3}, \\ h_2 = -\frac{16\sigma_1^4 - 86\sigma_1^2\sigma_2 + 90\sigma_1\sigma_3 + 84\sigma_2^2}{3\Delta_3}, \\ h_3 = \frac{8\sigma_1^3 - 31\sigma_1\sigma_2 + 63\sigma_3}{\Delta_3}. \end{cases}$$

As in the transversely orthotropic case, the following theorem characterized orthotropic harmonic tensors which are a perfect harmonic squares.

**Theorem 5.7.** *An orthotropic harmonic fourth-order tensor  $\mathbf{H}$  is a perfect harmonic square  $\mathbf{H} = \mathbf{h} * \mathbf{h}$  if and only if*

$$(5.7) \quad \sigma_1 > 0 \quad \text{and} \quad 49\sigma_2 - 8\sigma_1^2 = 0.$$

*In that case the harmonic second order tensor  $\mathbf{h}$  is given by*

$$(5.8) \quad \mathbf{h} := \pm \sqrt{\frac{49}{10(1-\mathcal{L})\sigma_1}} \left( \boldsymbol{\lambda}'(\mathbf{H}) - \frac{21}{5\sigma_1} (\boldsymbol{\lambda}'(\mathbf{H})^2)' \right).$$

*Proof.* Let  $\mathbf{H}$  be an orthotropic fourth order harmonic tensor. Suppose first that conditions (5.7) are satisfied. Then, the coefficients  $h_1, h_2, h_3$  in (5.6) satisfy

$$h_2^2 - h_1 h_3 = 49\sigma_2 - 8\sigma_1^2 = 0$$

and we have thus a factorization  $\mathbf{H} = \mathbf{h} * \mathbf{h}$ , where  $\mathbf{h}$  is defined by (5.8).

Conversely, suppose that  $\mathbf{H}$  can be factorized as  $\mathbf{H} = \mathbf{h} * \mathbf{h}$ . Without loss of generality, we can assume that  $\mathbf{h}$  is diagonal, with diagonal elements  $a_1, a_2$  and  $-(a_1 + a_2)$ . Thus  $\mathbf{H}$  is fixed by  $\mathbb{D}_2$  and is represented by matrix normal form (5.3). Consider now the associated binary form  $\mathbf{g}$  of  $\mathbf{h}$  (resp.  $\mathbf{f}$  of  $\mathbf{H}$ ) – see Appendix A. We get then

$$\mathbf{g} = \alpha_0 u^4 + \alpha_2 u^2 v^2 + \alpha_0 v^2, \quad \alpha_0 := \frac{1}{4}(a_0 - a_1), \quad \alpha_2 = -\frac{3}{2}(a_0 + a_1),$$

and

$$\begin{aligned} \mathbf{f} = \frac{1}{16}(\lambda_1 + \lambda_2 + 8\lambda_3)u^8 + \frac{7}{4}u^6 v^2(\lambda_1 - \lambda_2) + \frac{35}{8}u^4 v^4(\lambda_1 + \lambda_2) \\ + \frac{7}{4}(\lambda_1 - \lambda_2)u^2 v^6 + \frac{1}{16}(\lambda_1 + \lambda_2 + 8\lambda_3)v^8. \end{aligned}$$

Now,  $\mathbf{f} = \mathbf{g}^2$  leads to

$$\begin{cases} \alpha_2^2 + 2\alpha_0^2 = \frac{35}{8}(\lambda_1 + \lambda_2) \geq 0, \\ \alpha_0 \alpha_2 = \frac{7}{8}(\lambda_1 - \lambda_2), \\ \alpha_0^2 = \frac{1}{16}(\lambda_1 + \lambda_2 + 8\lambda_3) \geq 0, \end{cases}.$$

and we get

$$\sigma_1 = \frac{\alpha_2^2 + 12\alpha_0^2}{5}, \quad \sigma_2 = \frac{8\alpha_2^4 + 192\alpha_0^2 \alpha_2^2 + 1152\alpha_0^4}{1225}$$

Therefore, we have

$$\sigma_1 \geq 0, \quad 8\sigma_1^2 - 49\sigma_2 = 0.$$

If  $\sigma_1 = 0$ , we get also  $\sigma_2 = 0$  and thus  $\Delta_3 = -243\sigma_3^2 \leq 0$ , which leads to a contradiction (since  $\mathbf{H}$  is supposed to be orthotropic). This concludes the proof.  $\square$

## 6. RECONSTRUCTIONS BY MEANS OF SECOND ORDER COVARIANTS WITH A CUBIC COVARIANT REMAINDER

In this section, we consider the problem of reconstruction of harmonic tensors for which an equivariant reconstruction into second-order tensors is not possible, namely the harmonic tensors within the trigonal and the tetragonal classes. For these two classes, we show that such a reconstruction

is possible up to a *fourth-order harmonic tensor with cubic symmetry*. All the closed  $\text{SO}(3, \mathbb{R})$  subgroups used in this section are defined in Appendix C.

**6.1. The tetragonal class.** A harmonic tensor  $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$  is tetragonal if and only if there exists  $g \in \text{SO}(3, \mathbb{R})$  such that  $\mathbf{H} = g \star \mathbf{H}_0$  where  $\mathbf{H}_0$  has the following normal matrix form

$$(6.1) \quad \underline{\mathbf{H}}_0 = \begin{pmatrix} 3\delta - \sigma & \delta + \sigma & -4\delta & 0 & 0 & 0 \\ \delta + \sigma & 3\delta - \sigma & -4\delta & 0 & 0 & 0 \\ -4\delta & -4\delta & 8\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -8\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\delta + 2\sigma \end{pmatrix}$$

where  $\sigma^2 - 25\delta^2 \neq 0$  and  $\sigma \neq 0$ .

*Remark 6.1.* Note that this normal form is however not unique: the change  $\sigma \mapsto -\sigma$ , provides an alternative normal form. Nevertheless, the choice  $\sigma > 0$  allows to fix this ambiguity. Note also that

$$\mathbf{H}_0(-\sigma, \delta) = r \star \mathbf{H}_0(\sigma, \delta),$$

where  $r = R(\mathbf{e}_3, \frac{\pi}{4})$  is the rotation by angle  $\frac{\pi}{4}$  around the  $(Oz)$  axis.

For tetragonal harmonic tensors, it was shown in [2, Section 5.4] that the following polynomial invariants do not vanish

$$(6.2) \quad K_4 := 3J_4 - J_2^2 > 0 \quad K_{10} := 2J_2K_4^2 - 35J_5^2 > 0$$

and that  $\delta$  and  $\sigma^2$  are rational invariants, given by

$$(6.3) \quad \delta = \frac{1}{4} \frac{J_5}{K_4}, \quad \sigma^2 = \frac{1}{8} (J_2 - 280\delta^2) = \frac{1}{16} \frac{K_{10}}{K_4^2}.$$

The choice  $\sigma > 0$  in the normal form (6.1) allows to write  $\sigma$  as follows:

$$(6.4) \quad \sigma = \frac{\sqrt{K_{10}}}{4K_4}.$$

*Remark 6.2.* Note that the condition  $\sigma^2 - 25\delta^2 = 0$  is equivalent to  $L_{10} = 0$  with

$$L_{10} := K_{10} - 25J_5^2,$$

and corresponds to the degeneracy when  $\mathbf{H}$  has at least a cubic symmetry. On the other hand, the condition  $\sigma = 0$  (with  $L_{10} \neq 0$ ) is equivalent to  $K_{10} = 0$  and corresponds to the degeneracy when  $\mathbf{H}$  has at least a transverse isotropic symmetry.

**6.1.1. Geometric picture for the tetragonal class.** The normal form (6.1) corresponds to the subspace  $\text{Fix}(\mathbb{D}_4)$ , where given a subgroup  $\Gamma$  of  $\text{SO}(3, \mathbb{R})$  the *fixed point set*  $\text{Fix}(\Gamma)$  is defined as

$$\text{Fix}(\Gamma) := \{ \mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3), \quad g \star \mathbf{H} = \mathbf{H}, \quad \forall g \in \Gamma \}.$$

Note now that there is only one subgroup in the conjugacy class  $[\text{O}(2)]$  which contains  $\mathbb{D}_4$ , it is  $\text{O}(2)$  itself. Its fix point set,  $\text{Fix}(\text{O}(2))$ , is the

one-dimensional subspace spanned by

$$(6.5) \quad \underline{\mathcal{T}}_0 := \begin{pmatrix} 3 & 1 & -4 & 0 & 0 & 0 \\ 1 & 3 & -4 & 0 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

However, there are two different subgroups  $\mathbb{O}_1$  and  $\mathbb{O}_2$  in the conjugacy class  $[\mathbb{O}]$  which contains  $\mathbb{D}_4$  and which are defined as follows. The first one,  $\mathbb{O}_1$ , is the symmetry group of the cube with edges  $(\pm 1; \pm 1; \pm 1)$ . The second one,  $\mathbb{O}_2$ , is obtained by rotating  $\mathbb{O}_1$  around the  $(Oz)$  axis by angle  $\frac{\pi}{4}$ . In other words,  $\mathbb{O}_2 := r\mathbb{O}_1r^{-1}$ , where  $r$  is the rotation by angle  $\frac{\pi}{4}$  around the  $(Oz)$  axis. The fixed point sets  $Fix(\mathbb{O}_1)$  and  $Fix(\mathbb{O}_2)$  are one-dimensional subspaces spanned respectively by

$$\underline{\mathcal{C}}_0^1 := \begin{pmatrix} 8 & -4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 \end{pmatrix}$$

and

$$\underline{\mathcal{C}}_0^2 := \begin{pmatrix} -2 & 6 & -4 & 0 & 0 & 0 \\ 6 & -2 & -4 & 0 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}.$$

Moreover, both  $Fix(\mathbb{O}_1)$  and  $Fix(\mathbb{O}_2)$  are supplementary subspaces of  $Fix(\mathbb{O}(2))$  in  $Fix(\mathbb{D}_4)$ , and thus we can write either

$$\mathbf{H}_0(\sigma, \delta) = \frac{5\delta + \sigma}{5}\mathcal{T}_0 - \frac{\sigma}{5}\mathcal{C}_0^1,$$

or

$$\mathbf{H}_0(\sigma, \delta) = \frac{5\delta - \sigma}{5}\mathcal{T}_0 + \frac{\sigma}{5}\mathcal{C}_0^2.$$

Let  $\mathbf{H}$  be a tetragonal harmonic fourth order tensor. Then there exists  $g \in \text{SO}(3, \mathbb{R})$  (defined up to a right composition by an element of  $\mathbb{D}_4$ ) and a unique couple of real numbers  $(\sigma, \delta)$  with  $\sigma > 0$  such that:

$$\mathbf{H} = g \star \mathbf{H}_0(\sigma, \delta).$$

Since  $\mathbb{D}_4 \subset \mathbb{O}(2)$ , the following definitions

$$\mathcal{T}^1(\mathbf{H}) := \frac{5\delta + \sigma}{5}g \star \mathcal{T}_0, \quad \mathcal{T}^2(\mathbf{H}) := \frac{5\delta - \sigma}{5}g \star \mathcal{T}_0,$$

do not depend on the rotation  $g$ , leading to well-defined covariant mappings  $\mathcal{T}^1, \mathcal{T}^2$  from the tetragonal class in  $\mathbb{H}^4(\mathbb{R}^3)$  to the transversely isotropic classes. Similarly, since  $\mathbb{O}_1$  and  $\mathbb{O}_2$  contains  $\mathbb{D}_4$ , the same is true for

$$\mathcal{C}^1(\mathbf{H}) := -\frac{\sigma}{5}g \star \mathcal{C}_0^1, \quad \mathcal{C}^2(\mathbf{H}) := \frac{\sigma}{5}g \star \mathcal{C}_0^2.$$

*Remark 6.3.* Let  $r = R(\mathbf{e}_3, \frac{\pi}{4})$  be the rotation by angle  $\frac{\pi}{4}$  around the  $(Oz)$  axis. We have then

$$r \star \mathcal{T}_0 = \mathcal{T}_0, \quad r \star \mathcal{C}_0^1 = \mathcal{C}_0^2.$$

Now if  $\mathbf{H} = g \star \mathbf{H}_0(\sigma, \delta)$ , we get

$$(grg^{-1}) \star \mathcal{T}^1(\mathbf{H}) = \mathcal{T}^2(\mathbf{H}), \quad (grg^{-1}) \star \mathcal{C}^1(\mathbf{H}) = \mathcal{C}^2(\mathbf{H}).$$

6.1.2. *Reconstruction theorem for the tetragonal class.*

**Theorem 6.4.** *For any tetragonal harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$\mathbf{H} = \mathcal{T}^k(\mathbf{H}) + \mathcal{C}^k(\mathbf{H}), \quad k = 1, 2$$

where

$$\mathcal{T}^k(\mathbf{H}) = \frac{7}{16} \frac{(5\delta + (-1)^{k+1}\sigma)}{(25\delta^2 - \sigma^2)^2} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H})$$

is a transversely isotropic covariant, and

$$\mathcal{C}^k(\mathbf{H}) = \left(1 - \frac{14\delta}{5\delta - (-1)^{k+1}\sigma}\right) \mathbf{H} + \frac{7}{2(5\delta - (-1)^{k+1}\sigma)} (\mathbf{H}^2)_0,$$

is a cubic covariant.

*Proof.* All the formulas can be checked on the normal form (6.1), for which  $\mathbf{d}'_2(\mathbf{H}) = 4(25\delta^2 - \sigma^2)(\mathbf{e}_3 \otimes \mathbf{e}_3)'$ . Since the formulas are covariant, this is enough to achieve the proof.  $\square$

*Remark 6.5.* In theorem 6.4, each formula, i.e.  $k = 1$  or  $k = 2$ , is self-sufficient. It is equivariant and applies in any basis to any tetragonal harmonic fourth order tensor  $\mathbf{H}$ . The two reconstruction formulas correspond to the choice of a frame and therefore of a cube, rotated by the angle  $\frac{\pi}{4}$  or not around  $(Oz)$ . They use the invariants  $\sigma > 0$  and  $\delta$  defined in (6.3) and (6.4).

The decomposition in Theorem 6.4 can be rewritten in terms of the invariants  $J_5, K_4, K_{10}$  and  $L_{10}$ , rather than  $\sigma, \delta$  by using their rational expressions (6.3) and (6.4).

**Corollary 6.6.** *For any tetragonal harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$\mathbf{H} = \frac{28K_4^3(5J_5 + \sqrt{K_{10}})}{L_{10}^2} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H}) + \mathcal{C}(\mathbf{H})$$

where

$$\mathcal{C}(\mathbf{H}) = \left(1 + \frac{14J_5(5J_5 + \sqrt{K_{10}})}{L_{10}}\right) \mathbf{H} - \frac{14K_4(5J_5 + \sqrt{K_{10}})}{L_{10}} (\mathbf{H}^2)_0$$

is a cubic fourth order covariant and

$$K_4 = 3J_4 - J_2^2 > 0, \quad K_{10} = 2J_2K_4^2 - 35J_5^2 > 0, \quad L_{10} = K_{10} - 25J_5^2 \neq 0.$$

**6.2. The trigonal class.** A harmonic tensor  $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$  is trigonal if and only if there exists  $g \in \text{SO}(3, \mathbb{R})$  such that  $\mathbf{H} = g \star \mathbf{H}_0$  where  $\mathbf{H}_0$  has the following normal matrix form

$$(6.6) \quad \underline{\mathbf{H}}_0 = \begin{pmatrix} 3\delta & \delta & -4\delta & -\sqrt{2}\sigma & 0 & 0 \\ \delta & 3\delta & -4\delta & \sqrt{2}\sigma & 0 & 0 \\ -4\delta & -4\delta & 8\delta & 0 & 0 & 0 \\ -\sqrt{2}\sigma & \sqrt{2}\sigma & 0 & -8\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\delta & -2\sigma \\ 0 & 0 & 0 & 0 & -2\sigma & 2\delta \end{pmatrix}$$

where  $\sigma^2 - 50\delta^2 \neq 0$  and  $\sigma \neq 0$ .

*Remark 6.7.* Note that this normal form is however not unique. Changing  $\sigma \mapsto -\sigma$ , provides an alternative normal form. Nevertheless, the choice  $\sigma > 0$  allows to fix this ambiguity. Note also that

$$\mathbf{H}_0(-\sigma, \delta) = r_t \star \mathbf{H}_0(\sigma, \delta),$$

where  $r_t = R(\mathbf{e}_3, \frac{\pi}{3})$  is the rotation by angle  $\frac{\pi}{3}$  around the  $(Oz)$  axis.

For trigonal harmonic tensors, it was shown in [2, Section 5.3] that the polynomial invariants  $K_4$  and  $K_{10}$ , defined by (6.2), are strictly positive and that  $\delta, \sigma^2$  are rational invariants, given by

$$(6.7) \quad \delta = \frac{1}{4} \frac{J_5}{K_4}, \quad \sigma^2 = \frac{1}{16} (J_2 - 280\delta^2) = \frac{1}{32} \frac{K_{10}}{K_4^2}.$$

The choice  $\sigma > 0$  in the normal form (6.6) allows to write  $\sigma$  as follows:

$$(6.8) \quad \sigma = \frac{1}{4\sqrt{2}} \frac{\sqrt{K_{10}}}{K_4}.$$

*Remark 6.8.* Note that the condition  $\sigma^2 - 50\delta^2 = 0$  is equivalent to  $M_{10} = 0$  with

$$M_{10} := K_{10} - 100J_5^2,$$

and corresponds to the degeneracy case when  $\mathbf{H}$  has at least a cubic symmetry. On the other hand, with  $M_{10} \neq 0$ , condition  $\sigma = 0$  is equivalent to  $K_{10} = 0$  and corresponds to the degeneracy case when  $\mathbf{H}$  has at least a transverse isotropic symmetry.

**6.2.1. Geometric picture for the trigonal class.** The geometric picture is similar to the tetragonal case:  $\mathbb{D}_3$  is contained in only one subgroup in the conjugacy class  $[\text{O}(2)]$  and two different subgroups  $\tilde{\mathbb{O}}_1, \tilde{\mathbb{O}}_2$  in the conjugacy class  $[\mathbb{O}]$ , which are defined as follows.

$$\tilde{\mathbb{O}}_1 := r_3 \mathbb{O}_1 r_3^{-1}, \quad \tilde{\mathbb{O}}_2 := r_t \tilde{\mathbb{O}}_1 r_t^{-1}$$

where  $\mathbb{O}_1$  is the symmetry group of the cube with edges  $(\pm 1; \pm 1; \pm 1)$  (already defined in section 6.1) and the rotations introduced are

$$r_3 := R\left(\mathbf{e}_3, \frac{\pi}{4}\right) \circ R\left(\mathbf{e}_1 - \mathbf{e}_2, \arccos\left(\frac{1}{\sqrt{3}}\right)\right), \quad r_t := R\left(\mathbf{e}_3, \frac{\pi}{3}\right).$$

The fixed point sets  $Fix(\tilde{\mathcal{O}}_1)$  and  $Fix(\tilde{\mathcal{O}}_2)$  are one-dimensional subspaces spanned respectively by

$$\underline{\tilde{\mathcal{C}}_0^1} := \begin{pmatrix} 3 & 1 & -4 & -10 & 0 & 0 \\ 1 & 3 & -4 & 10 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ -10 & 10 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & -10\sqrt{2} \\ 0 & 0 & 0 & 0 & -10\sqrt{2} & 2 \end{pmatrix}$$

and

$$\underline{\tilde{\mathcal{C}}_0^2} := \begin{pmatrix} 3 & 1 & -4 & 10 & 0 & 0 \\ 1 & 3 & -4 & -10 & 0 & 0 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ 10 & -10 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 10\sqrt{2} \\ 0 & 0 & 0 & 0 & 10\sqrt{2} & 2 \end{pmatrix}.$$

The normal form (6.6) can be decomposed as the sum of a transversely isotropic harmonic tensor and of a cubic harmonic tensor, either as

$$\mathbf{H}_0(\sigma, \delta) = \frac{10\delta - \sigma\sqrt{2}}{10} \mathcal{T}_0 + \frac{\sigma\sqrt{2}}{10} \underline{\tilde{\mathcal{C}}_0^1},$$

or as

$$\mathbf{H}_0(\sigma, \delta) = \frac{10\delta + \sigma\sqrt{2}}{10} \mathcal{T}_0 - \frac{\sigma\sqrt{2}}{10} \underline{\tilde{\mathcal{C}}_0^2},$$

where  $\mathcal{T}_0$  has been defined in (6.5).

As in the tetragonal case, we can give coherent definitions of transversely isotropic parts

$$\tilde{\mathcal{T}}^1(\mathbf{H}) := \frac{10\delta - \sigma\sqrt{2}}{10} g \star \mathcal{T}_0, \quad \tilde{\mathcal{T}}^2(\mathbf{H}) := \frac{10\delta + \sigma\sqrt{2}}{10} g \star \mathcal{T}_0,$$

and cubic parts

$$\tilde{\mathcal{C}}^1(\mathbf{H}) := \frac{\sigma\sqrt{2}}{10} g \star \underline{\tilde{\mathcal{C}}_0^1}, \quad \tilde{\mathcal{C}}^2(\mathbf{H}) := -\frac{\sigma\sqrt{2}}{10} g \star \underline{\tilde{\mathcal{C}}_0^2},$$

### 6.2.2. Reconstruction formulas for the trigonal class.

**Theorem 6.9.** *For any trigonal harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$\mathbf{H} = \tilde{\mathcal{T}}^k(\mathbf{H}) + \tilde{\mathcal{C}}^k(\mathbf{H}), \quad k = 1, 2,$$

where

$$\tilde{\mathcal{T}}^k(\mathbf{H}) = \frac{7(10\delta - (-1)^{k+1}\sigma\sqrt{2})}{8(50\delta^2 - \sigma^2)^2} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H}),$$

is a transversely isotropic covariant and

$$\tilde{\mathcal{C}}^k(\mathbf{H}) = \left(1 - \frac{7\delta}{10\delta + (-1)^{k+1}\sigma\sqrt{2}}\right) \mathbf{H} - \frac{7}{6(10\delta + (-1)^{k+1}\sigma\sqrt{2})} (\mathbf{H}^2)_0$$

is a cubic covariant.

*Proof.* All the formulas can be checked on the normal form (6.6), for which  $\mathbf{d}'_2(\mathbf{H}) = 2(50\delta^2 - \sigma^2)(\mathbf{e}_3 \otimes \mathbf{e}_3)'$ . Since the formulas are covariant, this is enough to achieve the proof.  $\square$



*Remark 6.10.* In theorem 6.9, each formula, i.e.  $k = 1$  or  $k = 2$ , is self-sufficient. It is equivariant and applies in any basis to any trigonal harmonic fourth order tensor  $\mathbf{H}$ . The two reconstruction formulas correspond to the choice of a frame and therefore of a cube, rotated by the angle  $\frac{\pi}{3}$  or not around  $(Oz)$ . They use the invariants  $\sigma > 0$  and  $\delta$  defined in Eq. (6.7) and (6.8).

The decomposition in Theorem 6.4 can be rewritten in terms of the invariants  $J_5, K_4, K_{10}$  and  $M_{10}$ , rather than  $\sigma, \delta$  by using their rational expressions (6.7) and (6.8).

**Corollary 6.11.** *For any trigonal harmonic fourth-order tensor  $\mathbf{H}$ , we have*

$$\mathbf{H} = \frac{224K_4^3(10J_5 + \sqrt{K_{10}})}{M_{10}^2} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H}) + \tilde{\mathcal{C}}(\mathbf{H})$$

where

$$\tilde{\mathcal{C}}(\mathbf{H}) = \left(1 + \frac{7J_5(10J_5 + \sqrt{K_{10}})}{M_{10}}\right) \mathbf{H} + \frac{14K_4(10J_5 + \sqrt{K_{10}})}{3M_{10}} (\mathbf{H}^2)_0$$

is a cubic fourth order covariant, and

$$K_4 = 3J_4 - J_2^2 > 0, \quad K_{10} = 2J_2K_4^2 - 35J_5^2 > 0, \quad M_{10} = K_{10} - 100J_5^2 \neq 0.$$

## 7. CONCLUSION

In this article, we have defined the *harmonic projection*  $(\mathbf{T})_0 \in \mathbb{H}^n(\mathbb{R}^3)$  of a totally symmetric  $n$ -th order tensor  $\mathbf{T}$ , and the *harmonic product*

$$\mathbf{H}_1 * \mathbf{H}_2 := (\mathbf{H}_1 \odot \mathbf{H}_2)_0 \in \mathbb{H}^{n_1+n_2}(\mathbb{R}^3)$$

of two harmonic tensors  $\mathbf{H}_1 \in \mathbb{H}^{n_1}(\mathbb{R}^3)$ ,  $\mathbf{H}_2 \in \mathbb{H}^{n_2}(\mathbb{R}^3)$ .

Using Sylvester's theorem 3.4, we have shown that the fourth order harmonic part  $\mathbf{H}$  of an elasticity tensor  $\mathbf{E}$  can be expressed as the harmonic product  $\mathbf{H} = \mathbf{h}_1 * \mathbf{h}_2$  of two second order harmonic tensors  $\mathbf{h}_1, \mathbf{h}_2$  (i.e. symmetric deviatoric). This decomposition is independent of any reference frame, even for triclinic materials. Nevertheless, such a factorization is non-unique and not really constructive. Moreover, a *globally defined* solution  $(\mathbf{h}_1(\mathbf{H}), \mathbf{h}_2(\mathbf{H}))$  can never be *covariant*.

We have therefore introduced an *equivariant reconstruction* problem of a tensor by means of *lower order covariants*, restricted to a given symmetry class. Finally, we have obtained explicit formulas for the reconstruction of the fourth order harmonic part  $\mathbf{H}$  of an elasticity tensor:

- for the *transversely isotropic* and *orthotropic* symmetry classes, by means of second order covariants (Theorems 5.2 and Theorem 5.5). Moreover, necessary and sufficient conditions for such tensors to be perfect harmonic squares, which means  $\mathbf{H} = \mathbf{h} * \mathbf{h}$ , have also been obtained (Remark 5.3 and Theorem 5.7);
- for the *tetragonal* and *trigonal* symmetry classes, by means of second order covariants and a fourth order cubic covariant (Theorem 6.4 and Theorem 6.9).

These reconstruction formulas are only valid inside each symmetry class, this is illustrated by the fact that they contain denominators which vanish when a degeneracy into a higher symmetry class occurs.

## APPENDIX A. BINARY FORMS

We have already described two models for the irreducible representations of the rotation group  $\mathrm{SO}(3, \mathbb{R})$ : the space of *harmonic tensors*  $\mathbb{H}^n(\mathbb{R}^3)$  and the space of harmonic polynomials  $\mathbb{H}^n(\mathbb{R}^3)$ . In this section, we shall derive a third model: the space of *binary forms*  $\mathbb{S}_{\mathbb{R}}^{2n}(\mathbb{C}^2)$ , whose construction is slightly more involved.

To start, let us recall that there is a well-known relation between the rotation group  $\mathrm{SO}(3, \mathbb{R})$  and the group of *special unitary complex matrices*

$$\mathrm{SU}(2) := \{ \gamma \in \mathrm{M}_2(\mathbb{C}); \bar{\gamma}^t \gamma = \mathbf{1}, \det \gamma = 1 \}.$$

Let

$$(A.1) \quad \mathbf{x} := (x, y, z) \mapsto M(\mathbf{x}) = \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix}$$

be a linear mapping from  $\mathbb{R}^3$  to the space of traceless, hermitian matrices of order 2 ( $\sigma_x = M(\mathbf{e}_1)$ ,  $\sigma_y = M(\mathbf{e}_2)$  and  $\sigma_z = M(\mathbf{e}_3)$  are the famous *Pauli matrices*). The group  $\mathrm{SU}(2)$  acts on this space by conjugacy

$$\mathrm{Ad}_\gamma : M \mapsto \gamma M \gamma^{-1}, \quad \gamma \in \mathrm{SU}(2),$$

and preserves the quadratic form

$$\det M = -(x^2 + y^2 + z^2).$$

It can be checked, moreover, that  $\det \mathrm{Ad}_\gamma = 1$  for all  $\gamma \in \mathrm{SU}(2)$ . Therefore, we deduce a group morphism

$$(A.2) \quad \pi : \gamma \mapsto \mathrm{Ad}_\gamma, \quad \mathrm{SU}(2) \rightarrow \mathrm{SO}(3, \mathbb{R}).$$

*Example A.1.* For

$$\gamma = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

we get

$$\pi(\gamma) = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis  $(\sigma_x, \sigma_y, \sigma_z)$ .

The group  $\mathrm{SU}(2)$  acts naturally on  $\mathbb{C}^2$  and more generally on complex polynomials in two variables  $(u, v)$  by the rule

$$(\gamma \star \mathbf{f})(u, v) := \mathbf{f}(\gamma^{-1} \star (u, v)), \quad \gamma \in \mathrm{SU}(2).$$

Let  $\mathbb{S}^n(\mathbb{C}^2)$  be the space of complex, homogeneous polynomials of degree  $n$  in two variables. An element of  $\mathbb{S}^n(\mathbb{C}^2)$

$$\mathbf{f}(\boldsymbol{\xi}) := \sum_{k=0}^n a_k u^k v^{n-k}, \quad \boldsymbol{\xi} := (u, v) \in \mathbb{C}^2$$

is called a *binary form* of degree  $n$ .

The main observation, due to Cartan (see [1] for the details), is that there is an *equivariant isomorphism* between the space  $\mathbb{S}^{2n}(\mathbb{C}^2)$  of binary forms of degree  $2n$  and the space  $\mathbb{H}^n(\mathbb{C}^3)$  of complex, harmonic polynomials of degree

$n$  in three variables  $x, y, z$ . This linear isomorphism  $\psi : \mathbb{H}^n(\mathbb{C}^3) \rightarrow \mathbb{S}^{2n}(\mathbb{C}^2)$  is given explicitly by

$$(A.3) \quad (\psi(\mathbf{h}))(u, v) := \mathbf{h} \left( \frac{u^2 - v^2}{2}, \frac{u^2 + v^2}{2i}, uv \right), \quad \mathbf{h} \in \mathbb{H}^n(\mathbb{C}^3).$$

It satisfies moreover the *equivariant relation*

$$\psi(\pi(\gamma) \star \mathbf{h}) = \gamma \star \psi(\mathbf{h}), \quad \gamma \in \mathrm{SU}(2), \quad \mathbf{h} \in \mathbb{H}^n(\mathbb{C}^3).$$

The proof that  $\psi$  is an equivariant isomorphism can be found in [1]. The inverse  $\psi^{-1}$  is obtained explicitly as followed. Let

$$\mathbf{f}(u, v) := \sum_{k=0}^{2n} a_k u^k v^{2n-k}.$$

For each  $k$  make the substitution

$$u^k v^{2n-k} \rightarrow \begin{cases} z^k (-x + iy)^{n-k}, & \text{if } 0 \leq k \leq n \\ z^{2n-k} (x + iy)^{k-n}, & \text{if } n \leq k \leq 2n. \end{cases}$$

We obtain this way a homogeneous polynomial in three variables  $\mathbf{p}$  of degree  $n$  such that

$$\mathbf{f}(u, v) = \mathbf{p} \left( \frac{u^2 - v^2}{2}, \frac{u^2 + v^2}{2i}, uv \right).$$

Now, let  $\mathbf{h} = (\mathbf{p})_0$  be the  $n$ -th order harmonic component of  $\mathbf{p}$  in the harmonic decomposition (2.2) of  $\mathbf{p}$ . Then  $\psi^{-1}(\mathbf{f}) := (\mathbf{p})_0 = \mathbf{h}$ .

*Remark A.2* (Real harmonic polynomials). Under this isomorphism, the space  $\mathbb{H}^n(\mathbb{R}^3)$  of *real* harmonic polynomials corresponds to the *real subspace*  $\mathbb{S}_{\mathbb{R}}^{2n}(\mathbb{C}^2) \subset \mathbb{S}^{2n}(\mathbb{C}^2)$  of binary forms  $\mathbf{f}$  which satisfy

$$(A.4) \quad \bar{\mathbf{f}}(-v, u) = (-1)^n \mathbf{f}(u, v).$$

These binary forms

$$\mathbf{f}(u, v) := \sum_{k=0}^{2n} a_k u^k v^{2n-k}$$

are also characterized by the following property:

$$(A.5) \quad a_{2n-k} = (-1)^{n-k} \overline{a_k}, \quad 0 \leq k \leq n.$$

The space  $\mathbb{S}_{\mathbb{R}}^{2n}(\mathbb{C}^2)$  is invariant under the action of  $\mathrm{SU}(2)$  and is moreover *irreducible* for this action.

*Remark A.3.* Note that if  $\mathbf{f}_1 \in \mathbb{S}_{\mathbb{R}}^{2n_1}(\mathbb{C}^2)$  and  $\mathbf{f}_2 \in \mathbb{S}_{\mathbb{R}}^{2n_2}(\mathbb{C}^2)$ , then  $\mathbf{f}_1 \mathbf{f}_2 \in \mathbb{S}_{\mathbb{R}}^{2(n_1+n_2)}(\mathbb{C}^2)$  and

$$\psi^{-1}(\mathbf{f}_1 \mathbf{f}_2) = (\psi^{-1}(\mathbf{f}_1) \psi^{-1}(\mathbf{f}_2))_0$$

*Example A.4* (Order-1 harmonic tensors). An order-1 harmonic tensor  $\mathbf{h}$  on  $\mathbb{R}^3$  is just a linear form  $\mathbf{h}$  on  $\mathbb{R}^3$ , which can be written as

$$\mathbf{h}(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} = w_i x_i,$$

with  $\mathbf{w} = (w_1, w_2, w_3)$ . The corresponding binary form  $\mathbf{f} = \psi(\mathbf{h}) \in \mathcal{B}_2$  is given by

$$\mathbf{f}(u, v) = \overline{a_0} u^2 + a_1 uv - a_0 v^2,$$

where

$$a_0 = \frac{1}{2}(w_1 + iw_2), \quad a_1 = w_3.$$

Conversely, given  $\mathbf{f} \in \mathcal{B}_2$ , the corresponding real 3-vector  $\mathbf{w}$  is given by

$$w_1 = a_0 + \overline{a_0}, \quad w_2 = -i(a_0 - \overline{a_0}), \quad w_3 = a_1.$$

*Remark A.5.* For order-1 tensors, there is a closed relation between the Cartan-map (A.3) and the stereographic projection

$$\tau : \mathbf{w}^0 = (w_1^0, w_2^0, w_3^0) \mapsto \lambda := \frac{w_1^0}{1 - w_3^0} + i \frac{w_2^0}{1 - w_3^0}, \quad S^2 \setminus \{\mathbf{n}\} \longrightarrow \mathbb{C}$$

where  $S^2$  is the unit sphere and  $\mathbf{n} = (0, 0, 1)$ . The stereographic projection can be extended to a bijection  $\tau : S^2 \rightarrow \mathbb{C} \cup \{\infty\}$  by setting  $\tau(\mathbf{n}) = \infty$ . Its inverse,  $\tau^{-1}$ , is given by

$$w_1^0 = \frac{2}{|\lambda|^2 + 1} \Re(\lambda), \quad w_2^0 = \frac{2}{|\lambda|^2 + 1} \Im(\lambda), \quad w_3^0 = \frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \quad \lambda \in \mathbb{C}.$$

Now let  $\mathbf{w} \neq (0, 0, 0)$ . Then we can write  $\mathbf{w} = t\mathbf{w}^0$ , where  $t = \|\mathbf{w}\|$  and  $\|\mathbf{w}^0\| = 1$ . If moreover  $\mathbf{w}^0 \neq (0, 0, 1)$ , we can write  $\mathbf{f} = \psi(\mathbf{h})$  as

$$(A.6) \quad \psi(\mathbf{h})(u, v) = t\alpha(u + \lambda v)(\overline{\lambda}u - v), \quad \alpha := \frac{1}{|\lambda|^2 + 1}, \quad \lambda = \tau(\mathbf{w}^0).$$

*Example A.6* (Order-2 harmonic tensors). Let  $\mathbf{h} = (h_{ij})$  be a second order real harmonic tensor and  $h$  be the corresponding real harmonic polynomial, then the corresponding binary form  $\mathbf{f} = \psi(\mathbf{h}) \in \mathcal{B}_4$  is given by

$$\mathbf{f}(u, v) = \overline{a_0}u^4 - \overline{a_1}u^3v + a_2u^2v^2 + a_1uv^3 + a_0v^4,$$

where

$$a_0 = \frac{1}{4}h_{11} - \frac{1}{4}h_{22} + \frac{1}{2}ih_{12}, \quad a_1 = -h_{13} - ih_{23}, \quad a_2 = \frac{3}{2}h_{33}$$

Conversely, given  $\mathbf{f} \in \mathcal{B}_4$ , the corresponding harmonic tensor  $\mathbf{h}$  is given by

$$(A.7) \quad \mathbf{h} := \begin{pmatrix} a_0 + \overline{a_0} - \frac{1}{3}a_2 & -i(a_0 - \overline{a_0}) & -\frac{1}{2}(a_1 + \overline{a_1}) \\ -i(a_0 - \overline{a_0}) & -a_0 - \overline{a_0} - \frac{1}{3}a_2 & \frac{1}{2}i(a_1 - \overline{a_1}) \\ -\frac{1}{2}(a_1 + \overline{a_1}) & \frac{1}{2}i(a_1 - \overline{a_1}) & \frac{2}{3}a_2 \end{pmatrix}$$

*Example A.7* (Order-4 harmonic tensors). Let  $\mathbf{H} = (H_{ijkl})$  be a fourth-order real harmonic tensor and  $h$  be the corresponding real harmonic polynomial, then the corresponding binary form  $\mathbf{f} = \psi(\mathbf{h}) \in \mathcal{B}_8$  is given by

$$\begin{aligned} \mathbf{f}(u, v) = & \overline{a_0}u^8 - \overline{a_1}u^7v + \overline{a_2}u^6v^2 - \overline{a_3}u^5v^3 + a_4u^4v^4 \\ & + a_3u^3v^5 + a_2u^2v^6 + a_1uv^7 + a_0v^8, \end{aligned}$$

where the independent components are

$$\begin{aligned} a_0 &= \frac{1}{16}(-8H_{1122} - H_{1133} - H_{2233}) + i\frac{1}{4}(2H_{1112} + H_{1233}), \\ a_1 &= \frac{1}{2}(4H_{1223} + H_{1333}) + i\frac{1}{2}(-3H_{1123} + H_{2223}), \\ a_2 &= \frac{7}{4}(H_{1133} - H_{2233}) + i\frac{7}{2}H_{1233}, \\ a_3 &= -\frac{7}{2}H_{1333} + i\frac{7}{2}(H_{1123} + H_{2223}), \\ a_4 &= -\frac{35}{8}(H_{1133} + H_{2233}) \end{aligned}$$

Conversely, given  $\mathbf{f} \in \mathcal{B}_8$ , the corresponding harmonic tensor in  $\mathbb{H}^4(\mathbb{R}^3)$  is given by:

$$\begin{aligned} H_{1111} &= -H_{1122} - H_{1133}, & H_{1112} &= \frac{i}{4} \left( 4\bar{a}_0 - 4a_0 - \frac{2}{7}\bar{a}_2 + \frac{2}{7}a_2 \right), \\ H_{1113} &= -H_{2223} - H_{2333}, & H_{1122} &= -\bar{a}_0 - a_0 + \frac{1}{35}a_4, \\ H_{1123} &= \frac{i}{4} \left( -\bar{a}_1 + a_1 + \frac{1}{7}\bar{a}_3 - \frac{1}{7}a_3 \right), & H_{1133} &= \frac{1}{7}\bar{a}_2 + \frac{1}{7}a_2 - \frac{4}{35}a_4, \\ H_{1222} &= -H_{1112} - H_{1233}, & H_{1223} &= \frac{1}{4} \left( \bar{a}_1 + a_1 + \frac{1}{7}\bar{a}_3 + \frac{1}{7}a_3 \right), \\ H_{1233} &= \frac{1}{7}i(\bar{a}_2 - a_2), & H_{1333} &= \frac{1}{7}(-\bar{a}_3 - a_3), \\ H_{2222} &= -H_{1122} - H_{2233}, & H_{2223} &= \frac{i}{4} \left( \bar{a}_1 - a_1 + \frac{3}{7}\bar{a}_3 - \frac{3}{7}a_3 \right), \\ H_{2233} &= -\frac{1}{7}\bar{a}_2 - \frac{1}{7}a_2 - \frac{4}{35}a_4, & H_{2333} &= -H_{1123} - H_{2223}, \\ H_{3333} &= -H_{1133} - H_{2233}. \end{aligned}$$

## APPENDIX B. PROOF OF SYLVESTER'S THEOREM

The key point to prove Sylvester's theorem 3.4 is a *root characterization* of binary forms in  $S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$ .

**Lemma B.1.** *Let  $\mathbf{f} \in S^{2n}(\mathbb{C}^2)$ . Then  $\mathbf{f} \in S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$  if and only if it can be written as*

$$\mathbf{f}(u, v) = \alpha u^r v^r \prod_{i=1}^{n-r} (u - \lambda_i v) (\bar{\lambda}_i u + v),$$

where  $\lambda_1, \dots, \lambda_{n-r} \in \mathbb{C}^*$  and  $\alpha \in \mathbb{R}$ .

*Proof.* Let  $\mathbf{f} \in S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$  which does not vanish identically. Due to (A.5), we deduce that for each  $0 \leq k \leq n$ ,  $a_{2n-k} = 0$  if and only if  $a_k = 0$ . Thus, there exists a non-negative integer  $r \leq n$  such that

$$\mathbf{f}(u, v) = u^r v^r \mathbf{g}(u, v),$$

where  $\mathbf{g} \in S_{\mathbb{R}}^{2(n-r)}(\mathbb{C}^2)$  with leading term  $b_{2(n-r)} = (-1)^{n-r} \bar{b}_0 \neq 0$ . Let

$$p(t) := \mathbf{g}(t, 1).$$

Then  $p$  is a polynomial of degree  $2(n-r)$  with no vanishing root. Due to (A.4), we deduce that if  $\lambda$  is a root of  $p$ , then  $-1/\bar{\lambda}$  is another root of  $p$ . Thus

$$p(t) = b_{2(n-r)} \prod_{i=1}^{n-r} (t - \lambda_i) \left( t + \frac{1}{\bar{\lambda}_i} \right),$$

and we get

$$\mathbf{g}(u, v) = \frac{b_{2(n-r)}}{\prod \bar{\lambda}_i} \prod_{i=1}^{n-r} (u - \lambda_i v) (\bar{\lambda}_i u + v)$$

Now, using (A.4), we have

$$\frac{\prod \lambda_i}{\prod \bar{\lambda}_i} = (-1)^{n-r} \frac{b_0}{b_{2(n-r)}} = \frac{\overline{b_{2(n-r)}}}{b_{2(n-r)}}$$

and thus

$$(B.1) \quad \alpha := \frac{b_{2(n-r)}}{\prod \bar{\lambda}_i} = \frac{\overline{b_{2(n-r)}}}{\prod \lambda_i}$$

is real. The converse can be immediately established using the functional characterization (A.4) of  $S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$ .  $\square$

*Remark B.2.* This factorization is not unique. Permuting a couple of paired roots  $\lambda_i, -1/\bar{\lambda}_i$ , leads to the transformation

$$(u - \lambda_i v) (\bar{\lambda}_i u + v) \rightarrow \frac{-1}{|\lambda_i|^2} (u - \lambda_i v) (\bar{\lambda}_i u + v),$$

and may result in a different value for  $\alpha$ .

We can now give the proof of Sylvester's theorem 3.4 and of proposition 3.6.

*Proof of theorem 3.4.* Using the isomorphism between  $H^n(\mathbb{R}^3)$  and  $S_{\mathbb{R}}^{2n}(\mathbb{C}^2)$  described in Appendix A, we set  $\mathbf{f} := \psi(p)$ . Using Lemma B.1, we can write

$$\mathbf{f}(u, v) = \alpha u^r v^r \prod_{i=1}^{n-r} (u - \lambda_i v) (\bar{\lambda}_i u + v).$$

Let

$$\mathbf{w}_i = \begin{cases} (u - \lambda_i v) (\bar{\lambda}_i u + v), & \text{if } i \leq n-r; \\ uv, & \text{if } i > n-r. \end{cases}$$

Each  $\mathbf{w}_i$  is in  $S_{\mathbb{R}}^2(\mathbb{C}^2)$  so  $\psi^{-1}(\mathbf{w}_i) \in H^1(\mathbb{R}^3)$  is a polynomial of degree 1 on  $\mathbb{R}^3$  and can be written as  $\mathbf{x} \cdot \mathbf{w}_i$  for some vector  $\mathbf{w}_i$  in  $\mathbb{R}^3$ . We get thus

$$\begin{aligned} p &= \psi^{-1}(\mathbf{f}) = \psi^{-1}(\mathbf{w}_1 \cdots \mathbf{w}_n) \\ &= \psi^{-1}(\mathbf{w}_1) * \cdots * \psi^{-1}(\mathbf{w}_n) = (\mathbf{x} \cdot \mathbf{w}_1) * \cdots * (\mathbf{x} \cdot \mathbf{w}_n), \end{aligned}$$

which achieves the proof.  $\square$

*Proof of Proposition 3.6.* Let  $\mathbf{w}_1, \dots, \mathbf{w}_n$  and  $\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_n$  be unit vectors. We can assume, rotating if necessary this configuration of  $2n$  vectors on the sphere, that none of the  $\mathbf{w}_k, \tilde{\mathbf{w}}_k$  are equal to  $(0, 0, 1)$  or  $(0, 0, -1)$ . Set

$$\lambda_k := \tau(\mathbf{w}_k), \quad \mu_k := \tau(\tilde{\mathbf{w}}_k),$$

and note that the  $\lambda_k, \mu_k$  belong to  $\mathbb{C} \setminus \{0\}$ . Using (A.6), the relation

$$(\mathbf{x} \cdot \mathbf{w}_1) * \cdots * (\mathbf{x} \cdot \mathbf{w}_n) = (\mathbf{x} \cdot \tilde{\mathbf{w}}_1) * \cdots * (\mathbf{x} \cdot \tilde{\mathbf{w}}_n), \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

can be recast as the binary form identity

$$\prod_{k=1}^n \alpha_k(u + \lambda_k v)(\overline{\lambda_k} u - v) = \prod_{k=1}^n \beta_k(u + \mu_k v)(\overline{\mu_k} u - v), \quad \forall (u, v) \in \mathbb{C}^2.$$

Therefore, there exists a permutation  $\sigma$  of  $\{1, \dots, n\}$ , such that either  $\mu_k = \lambda_{\sigma(k)}$  or  $\mu_k = -1/\overline{\lambda_{\sigma(k)}}$ . In the first case we deduce that

$$\tilde{\mathbf{w}}_k = \tau^{-1}(\lambda_{\sigma(k)}) = \mathbf{w}_{\sigma(k)}$$

while in the second case, we have

$$\tilde{\mathbf{w}}_k = \tau^{-1}\left(-\frac{1}{\overline{\lambda_{\sigma(k)}}}\right) = -\mathbf{w}_{\sigma(k)}.$$

We obtain therefore that  $\tilde{\mathbf{w}}_k = \epsilon_k \mathbf{w}_{\sigma(k)}$  where  $\epsilon_k = \pm 1$  for  $1 \leq k \leq n$  with  $\epsilon_1 \cdots \epsilon_n = 1$ . If the vectors  $\mathbf{w}_k, \tilde{\mathbf{w}}_k$  are not unit vectors, a normalisation process lead to the conclusion that  $\tilde{\mathbf{w}}_k = \lambda_k \mathbf{w}_{\sigma(k)}$  where  $\lambda_k \in \mathbb{R}$  and  $\lambda_1 \cdots \lambda_n = 1$ , which achieves the proof.  $\square$

### APPENDIX C. SYMMETRY CLASSES

The action of the rotation group  $\text{SO}(3, \mathbb{R})$  on the space of  $n$ -order harmonic tensors  $\mathbb{H}^n(\mathbb{R}^3)$  partitions this space into *symmetry classes*, where two tensors  $\mathbf{H}_1, \mathbf{H}_2$  belong to the same symmetry class if their respective symmetry group  $G_{\mathbf{H}_1}$  and  $G_{\mathbf{H}_2}$  are conjugates, that is

$$G_{\mathbf{H}_2} = gG_{\mathbf{H}_1}g^{-1}, \quad \text{for some } g \in \text{SO}(3, \mathbb{R}).$$

Symmetry classes are in correspondence with conjugacy classes

$$[K] := \{gKg^{-1}; g \in \text{SO}(3, \mathbb{R})\}$$

of closed subgroups  $K$  of  $\text{SO}(3, \mathbb{R})$  (several conjugacy classes may however correspond to empty symmetry classes). Moreover, symmetry classes are partially ordered by the following partial order defined on conjugacy classes

$$[K_1] \preceq [K_2] \quad \text{if} \quad \exists g \in \text{SO}(3, \mathbb{R}), \quad K_1 \subset gK_2g^{-1}.$$

Recall that a closed  $\text{SO}(3, \mathbb{R})$  subgroup is conjugate to one of the following list [19]:

$$\text{SO}(3), \text{O}(2), \text{SO}(2), \mathbb{D}_n(n \geq 2), \mathbb{Z}_n(n \geq 2), \mathbb{T}, \mathbb{O}, \mathbb{I}, \text{ and } \mathbb{1}$$

where:

- $\text{O}(2)$  is the subgroup generated by all the rotations around the  $z$ -axis and the order 2 rotation  $\sigma : (x, y, z) \mapsto (x, -y, -z)$  around the  $x$ -axis.
- $\text{SO}(2)$  is the subgroup of all the rotations around the  $z$ -axis.
- $\mathbb{Z}_n$  is the unique cyclic subgroup of order  $n$  of  $\text{SO}(2)$ , the subgroup of rotations around the  $z$ -axis.
- $\mathbb{D}_n$  is the *dihedral* group. It is generated by  $\mathbb{Z}_n$  and  $\sigma : (x, y, z) \mapsto (x, -y, -z)$ .

- $\mathbb{T}$  is the *tetrahedral* group, the (orientation-preserving) symmetry group of a tetrahedron. It has order 12.
- $\mathbb{O}$  is the *octahedral* group, the (orientation-preserving) symmetry group of a cube or octahedron. It has order 24.
- $\mathbb{I}$  is the *icosahedral* group, the (orientation-preserving) symmetry group of a icosahedra or dodecahedron. It has order 60.
- $\mathbb{1}$  is the trivial subgroup, containing only the unit element.

The symmetry classes of a second order symmetric tensor are described by the following partially ordered set:

$$\begin{array}{ccccc} \text{orthotropic} & \longrightarrow & \text{transversely isotropic} & \longrightarrow & \text{isotropic} \\ [\mathbb{D}_2] & & [\text{O}(2)] & & [\text{SO}(3, \mathbb{R})] \end{array}$$

For  $\mathbb{H}^4(\mathbb{R}^3)$ , it is known that there are only 8 symmetry classes (the same classes as for the Elasticity tensor [16]). The corresponding partial ordering is illustrated in figure 1, with the convention that a subgroup at the starting point of an arrow is conjugate to a subgroup of the group pointed by the arrow.

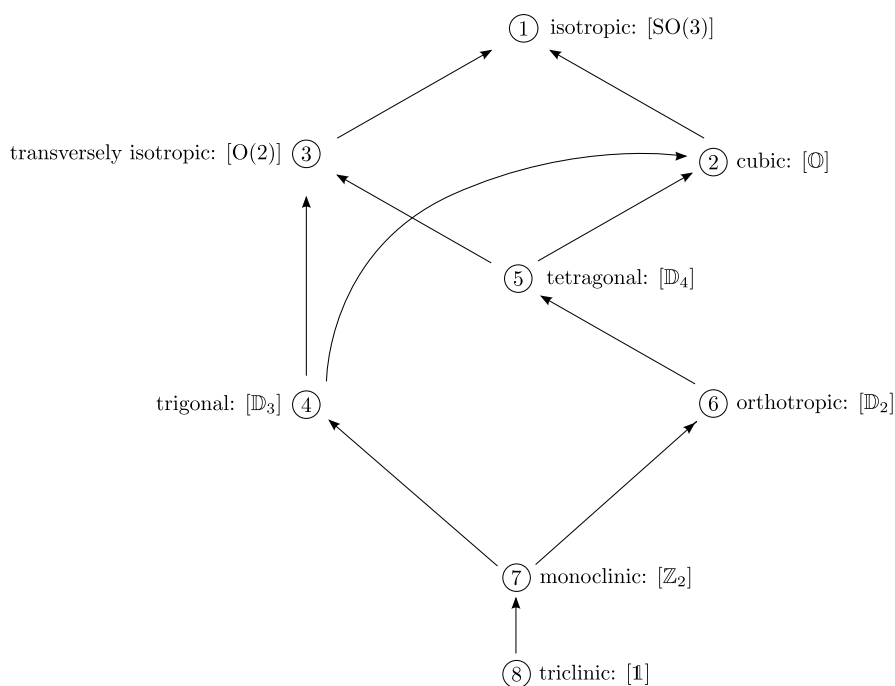


FIGURE 1. The partial ordering set for symmetry classes of  $\mathbb{H}^4(\mathbb{R}^3)$ .

REFERENCES

[1] N. Auffray, B. Kolev, and M. Olive. A minimal integrity basis for the elasticity tensor. Available at <https://hal.archives-ouvertes.fr/hal-01323543>, 2016.

[2] N. Auffray, B. Kolev, and M. Petitot. On Anisotropic Polynomial Relations for the Elasticity Tensor. *J. Elasticity*, 115(1):77–103, 2014.

[3] G. Backus. A geometrical picture of anisotropic elastic tensors. *Rev. Geophys.*, 8(3):633–671, 1970.



- [4] R. Baerheim. Classification of symmetry by means of maxwell multipoles. *Q. J. Mech. Appl. Math.*, 51:73–103, 1998.
- [5] J. Betten and W. Helisch. Integrity bases for a fourth-rank tensor. In *Anisotropy, inhomogeneity and nonlinearity in solid mechanics (Nottingham, 1994)*, volume 39 of *Solid Mech. Appl.*, pages 37–42. Kluwer Acad. Publ., Dordrecht.
- [6] J.-P. Boehler. A simple derivation of representations for nonpolynomial constitutive equations in some case of anisotropy. *Z. Angew. Math. Mech.*, 59(4):157–167, 1979.
- [7] J.-P. Boehler. Introduction to the invariant formulation of anisotropic constitutive equations. In *Applications of tensor functions in solid mechanics*, volume 292 of *CISM Courses and Lectures*, pages 13–30. Springer, Vienna, 1987.
- [8] J.-P. Boehler, A. A. Kirillov, Jr., and E. T. Onat. On the polynomial invariants of the elasticity tensor. *J. Elasticity*, 34(2):97–110, 1994.
- [9] A. Burr, F. Hild, and F. A. Leckie. Micro-mechanics and continuum damage mechanics. *Archive of Applied Mechanics*, 65:437–456, 1995.
- [10] A. Cayley. A seventh memoir on quantics. *Philosophical Transactions of the Royal Society of London*, 151:277–292, 1861.
- [11] J.-L. Chaboche. Le concept de contrainte effective appliqué à l'élasticité et à la viscoplasticité en présence d'un endommagement anisotrope. In J.-P. Boehler, editor, *Colloque Int. CNRS 295, Villard de Lans*, pages 737–760. Martinus Nijhoff Publishers and Editions du CNRS, 1982, 1979.
- [12] J. Cordebois and F. Sidoroff. Endommagement anisotrope en élasticité et plasticité. *J. Meca. Th. Appl., Special Volume*, pages 45–65, 1982.
- [13] F. Cormery and H. Weleman. A stress-based macroscopic approach for microcracks unilateral effect. *Comp. Mat. Sci.*, 47:727–738, 2010.
- [14] B. Desmorat and R. Desmorat. Tensorial polar decomposition of 2D fourth order tensors. *C. R. Mécanique*, 343:471–475, 2015.
- [15] B. Desmorat and R. Desmorat. Second order tensorial framework for 2D medium with open and closed cracks. *European Journal of Mechanics A/Solids*, 58:262–277, 2016.
- [16] S. Forte and M. Vianello. Symmetry classes for elasticity tensors. *J. Elasticity*, 43(2):81–108, 1996.
- [17] M. Francois, Y. Berthaud, and G. Geymonat. Une nouvelle analyse des symétries d'un matériau élastique anisotrope. exemple d'utilisation à partir de mesures ultrasonores. *Comptes rendus de l'Académie des Sciences. Série IIb*, 322:87–94, 1996.
- [18] G. Geymonat and T. Weller. Symmetry classes of piezoelectric solids. *Comptes rendus de l'Académie des Sciences. Série I*, 335:847–8524, 2002.
- [19] M. Golubitsky, I. Stewart, and D. G. Schaeffer. *Singularities and groups in bifurcation theory. Vol. II*, volume 69 of *Applied Mathematical Sciences*. Springer-Verlag, 1988.
- [20] P. Gordan. Beweis, dass jede Covariante und Invariante einer Bineren Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist. *Journal für die reine und angewandte Mathematik*, 69:323–354, 1868.
- [21] P. Gordan. *Über das Formensystem Binaerer Formen*. 1875.
- [22] P. Gordan. *Vorlesungen über Invariantentheorie*. Chelsea Publishing Co., New York, second edition, 1987. Erster Band: Determinanten. [Vol. I: Determinants], Zweiter Band: Binäre Formen. [Vol. II: Binary forms], Edited by Georg Kerschensteiner.
- [23] J. H. Grace and A. Young. *The algebra of invariants*. Cambridge Library Collection. Cambridge University Press, Cambridge, 2010. Reprint of the 1903 original.
- [24] G. A. Holzapfel. *Nonlinear solid mechanics*. John Wiley & Sons, Ltd., Chichester, 2000. A continuum approach for engineering.
- [25] M. Kachanov. On continuum theory of medium with cracks. *Mech. Solids*, 7:54–59, 1972.
- [26] K. Kanatani. Distribution of directional data and fabric tensors. *Int. J. Eng. Sci.*, 22:14–164, 1984.
- [27] D. Krajcinovic. *Damage Mechanics*. Applied Mathematics and Mechanics. North Holland, 1996.
- [28] P. Ladevèze. Sur une théorie de l'endommagement anisotrope. Technical report, Internal report 34 of LMT-Cachan, 1983.

- [29] F. A. Leckie and E. T. Onat. *Tensorial nature of damage measuring internal variables*, chapter Physical Non-Linearities in Structural Analysis, pages 140–155. J. Hult and J. Lemaitre eds, Springer Berlin, 1980.
- [30] J. Lemaitre and J.-L. Chaboche. *Mécanique des matériaux solides*. Dunod, english translation 1990 'Mechanics of Solid Materials' Cambridge University Press, 1985.
- [31] J. Lemaitre and R. Desmorat. *Engineering Damage Mechanics : Ductile, Creep, Fatigue and Brittle Failures*. Springer, 2005.
- [32] R. Lercier and C. Ritzenthaler. Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects. *J. Algebra*, 372:595–636, 2012.
- [33] I.-S. Liu. On representations of anisotropic invariants. *Int. J. Engrg. Sci.*, 20(10):1099–1109, 1982.
- [34] J.-G. Luque. Invariants des hypermatrices. Available at <http://tel.archives-ouvertes.fr/tel-00250312>, 2007.
- [35] C.-S. Man. Remarks on isotropic extension of anisotropic constitutive functions via structural tensors. In *XXIV ICTAM, 21-26 August 2016*, Aug. 2016.
- [36] R. D. Mindlin. Second gradient of strain and surface-tension in linear elasticity. *Int. J. Solids Structures*, 1:417–438, 1965.
- [37] M. Montemurro, A. Vincenti, and P. Vannucci. A two-level procedure for the global optimum design of composite modular structures—application to the design of an aircraft wing. *Journal of Optimization Theory and Applications*, 155(1):24–53, 2012.
- [38] M. Olive. *Géométrie des espaces de tenseurs, une approche effective appliquée à la mécanique des milieux continus*. PhD thesis, 2014.
- [39] M. Olive and N. Auffray. Isotropic invariants of a completely symmetric third-order tensor. *Journal of Mathematical Physics*, 55(9):092901, 2014.
- [40] E. T. Onat. Effective properties of elastic materials that contain penny shaped voids. *nt. J. Eng. Sci.*, 22:1013–1021, 1984.
- [41] N. I. Ostrosablin. On invariants of a fourth-rank tensor of elasticity moduli. *Sib. Zh. Ind. Mat.*, 1(1):155–163, 1998.
- [42] J. A. Schouten. *Tensor Analysis for Physicists*. Oxford, At the Clarendon Press, 1951.
- [43] G. F. Smith. On isotropic integrity bases. *Arch. Rational Mech. Anal.*, 18:282–292, 1965.
- [44] G. F. Smith and G. Bao. Isotropic invariants of traceless symmetric tensors of orders three and four. *Int. J. Eng. Sci.*, 35(15):1457–1462, 1997.
- [45] A. Spencer. A note on the decomposition of tensors into traceless symmetric tensors. *Int. J. Engng Sci.*, 8:475–481, 1970.
- [46] A. Spencer. *Constitutive theory for strongly anisotropic solids*, volume 1, pages 1–32. A. J. M. Spencer Ed., CISM Courses and Lectures No. 282, Springer Wien., 1984.
- [47] S. Sternberg. *Group theory and physics*. Cambridge University Press, 1994.
- [48] J. J. Sylvester. Note on spherical harmonics. In *Collected Mathematical Papers*, volume 3, pages 37–51. Cambridge University Press, 1909.
- [49] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics*. Handbuch fr Physik III/3. Springer-Verlag, 1965.
- [50] P. Vannucci. Plane anisotropy by the polar method. *Meccanica*, 40:437–454, 2005.
- [51] P. Vannucci and J. Pouget. Laminates with given piezoelectric expansion coefficients. *Mechanics of Advanced Materials and Structures*, 13(5):419–427, 2006.
- [52] P. Vannucci and G. Verchery. Stiffness design of laminates using the polar method. *International Journal of Solids and Structures*, 38:9281–9894, 2001.
- [53] G. Verchery. Les invariants des tenseurs d'ordre 4 du type de l'élasticité. In J.-P. Boehler, editor, *Colloque Int. CNRS 295, Villard de Lans*, pages 93–104. Martinus Nijhoff Publishers and Editions du CNRS, 1982, 1979.
- [54] M. Vianello. An integrity basis for plane elasticity tensors. *Arch. Mech.*, 49:197–208, 1997.
- [55] H. Xiao. On isotropic invariants of the elasticity tensor. *J. Elasticity*, 46(2):115–149, 1997.
- [56] J. Yang. *Special topics in the theory of piezoelectricity*. Springer, New York, 2009.

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