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**Equidistribution and counting under equilibrium states
in negatively curved spaces and graphs of groups.
Applications to non-Archimedean Diophantine
approximation**

Anne Broise-Alamichel, Jouni Parkkonen, Frédéric Paulin

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Equidistribution and counting under equilibrium states in
negatively curved spaces and graphs of groups. Applications
to non-Archimedean Diophantine approximation

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December 19, 2016

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Chapter 1

Introduction

In this book, we study equidistribution and counting problems concerning locally geodesic arcs in negatively curved spaces endowed with potentials, and we deduce, from the special case of tree quotients, various arithmetic applications to equidistribution and counting problems in non-Archimedean local fields.

For several decades, tools in ergodic theory and dynamical systems have been used to obtain geometric equidistribution and counting results on manifolds, both inspired by and with applications to arithmetic and number theoretic problems, in particular in Diophantine approximation. Especially pioneered by Margulis, this field has produced a huge corpus of works, by Bowen, Cosentino, Clozel, Dani, Einseidler, Eskin, Gorodnik, Ghosh, Guivarc'h, Kim, Kleinbock, Kontorovich, Lindenstraus, Margulis, McMullen, Michel, Mohammadi, Mozes, Nevo, Oh, Pollicott, Roblin, Shah, Sharp, Sullivan, Ullmo, Weiss and the last two authors, just to mention a few contributors. We refer for now to the surveys [Bab2, Oh, PaP16a, PaP16d] and we will explain in more details in this introduction the relation of our work with previous works.

In this text, we consider geometric equidistribution and counting problems weighted with a potential function in quotient spaces of $\text{CAT}(-1)$ spaces by discrete groups of isometries. The $\text{CAT}(-1)$ spaces form a huge class of metric spaces that contains (but is not restricted to) metric trees, hyperbolic buildings and simply connected Riemannian manifolds with sectional curvature bounded above by -1 . See [BridH] and Chapter 2 for a review of some basic properties of these spaces. Although some of the equidistribution and counting results with potentials on negatively curved manifolds are known (see for instance [PauPS]), as well as some of such results on $\text{CAT}(-1)$ spaces without potential (see for instance [Rob2]), bringing together these two aspects and producing new results and applications is one of the goals of this book.

We extend the theory of Patterson-Sullivan, Bowen-Margulis and skinning measures to $\text{CAT}(-1)$ spaces with potentials, with a special emphasis on trees endowed with a system of conductances. We prove that under natural nondegeneracy, mixing and finiteness assumptions, the pushforward under the geodesic flow of the skinning measure of properly immersed locally convex closed subsets of $\text{CAT}(-1)$ spaces equidistributes to the Gibbs measure, generalising the main result of [PaP14a].

We also prove that the (appropriate generalisations of) the initial and terminal tangent vectors of the common perpendiculars to any two properly immersed locally convex closed subsets jointly equidistribute to the skinning measures when the lengths of the common per-

perpendiculars tend to $+\infty$. This result is then used to prove asymptotic results on weighted counting functions of common perpendiculars whose lengths tend to $+\infty$. Common perpendiculars have been studied, in various particular cases, sometimes not explicitly, by Basmajian, Bridgeman, Bridgeman-Kahn, Eskin-McMullen, Herrmann, Huber, Kontorovich-Oh, Margulis, Martin-McKee-Wambach, Meyerhoff, Mirzakhani, Oh-Shah, Pollicott, Roblin, Shah, the last two authors and many others. See the comments after Theorem 1.5 below, and the survey [PaP16a] for references.

In the Part III of this book, we apply the geometric results obtained for trees to deduce arithmetic applications in non-Archimedean local fields. In particular, we prove equidistribution and counting results for rationals and quadratic irrationals in any completion of any function field over a finite field.

Let us now describe more precisely the content of this book, restricted to special cases for the sake of the exposition.

Geometric and dynamical tools

Let Y be a geodesically complete connected proper locally $\text{CAT}(-1)$ space (or good orbispace), such that the fundamental group of Y is not virtually nilpotent. In this introduction, we will mainly concentrate on the cases where Y is either a metric graph (or graph of finite groups in the sense of Bass and Serre, see [Ser3]) or a Riemannian manifold (or good orbifold) of dimension at least 2 with sectional curvature at most -1 . Let $\mathcal{G}Y$ be the space of locally geodesic lines of Y , on which the geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{R}}$ acts by real translations on the source. When Y is a simplicial¹ graph (of finite groups), we consider the discrete time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{Z}}$, see Section 2.7. If Y is a Riemannian manifold, then $\mathcal{G}Y$ is naturally identified with the unit tangent bundle T^1Y by the map that associates to a locally geodesic line its tangent vector at time 0. In general, we define T^1Y as the space of germs of locally geodesic lines in Y , and $\mathcal{G}Y$ maps onto T^1Y with possibly uncountable fibers.

Let $F : T^1Y \rightarrow \mathbb{R}$ be a continuous map, called a *potential*, which plays the same role in the construction of Gibbs measures/equilibrium states as the energy function in Bowen's treatment of the thermodynamic formalism of symbolic dynamical systems in [Bowe2, Sect. 1]. In this introduction, we assume that F is bounded in order to simplify the statements. We define in Section 3.3 the *critical exponent* δ_F associated with F , which describes the logarithmic growth of an orbit of the fundamental group on the universal cover of Y weighted by the (lifted) potential F , and which coincides with the classical critical exponent when $F = 0$. When Y is a metric graph, we associate in Section 3.5 a potential F_c to a *system of conductances* c (that is, a map from the set of edges of Y to \mathbb{R}), in such a way that the correspondence $c \mapsto F_c$ is bijective at the level of cohomology classes, and we denote δ_{F_c} by δ_c . We assume in the remainder of this introduction that δ_F is finite and positive.

We say that the pair (Y, F) satisfies the *HC-property* if the integral of F on compact locally geodesic segments of Y varies in a Hölder-continuous way on its extremities (see Definition 3.4). The pairs which have the HC-property include Riemannian manifolds with pinched sectional curvature at most -1 and Hölder-continuous potentials, and metric graphs with any potential. This HC-property is the new technical idea compared to [PauPS] which allows the extensions to our very general framework. See also [ConLT], under the strong assumption that Y is compact.

¹that is, if its edges all have lengths 1

In Chapter 4, building on the works of [Rob2] when $F = 0$ and of [PauPS]² when Y is a Riemannian manifold, we generalise, to locally $\text{CAT}(-1)$ spaces Y endowed with a potential F satisfying the HC-property, the construction and basic properties of the *Patterson densities* at infinity of the universal cover of Y associated with F and the *Gibbs measure* m_F on $\mathcal{G}Y$ associated with F .

Using the Patterson-Sullivan-Bowen-Margulis approach, the Patterson densities are limits of renormalised measures on the orbit points of the fundamental group, weighted by the potential, and the Gibbs measures on $\mathcal{G}Y$ are local products of Patterson densities on the endpoints of the geodesic line, with the Lebesgue measure on the time parameter, weighted by the Gibbs cocycle defined by the potential.

Generalizing a result of [CoP2], we prove in Section 6.2 that when Y is a regular simplicial graph and c is an anti-reversible system of conductances, then the Patterson measures, normalized to be probability measures, are harmonic measures (or hitting measures) on $\partial_\infty Y$ for a transient random walk on the vertices, whose transition probabilities are constructed using the total mass of the Patterson measures.

Gibbs measures were first introduced in statistical mechanics, and are naturally associated via the thermodynamic formalism³ with symbolic dynamics. We prove in Section 4.2 that our Gibbs measures satisfy a Gibbs property analogous to the one in symbolic dynamics. If $F = 0$, the Gibbs measure m_F is the *Bowen-Margulis measure* m_{BM} . If Y is a compact Riemannian manifold and F is the *strong unstable Jacobian* $v \mapsto -\frac{d}{dt}|_{t=0} \ln \text{Jac}(\mathbf{g}^t|_{W^-(v)})(v)$, then m_F is the Liouville measure and $\delta_F = 0$ (see [PauPS, Chap. 7] for more general assumptions on Y). Thus, one interesting aspect of Gibbs measure is that they form a natural family of measures invariant under the geodesic flow that interpolates between the Liouville measure and the Bowen-Margulis measure (which in variable curvature are in general not in the same measure class). Another interesting point is that such measures are plentiful: a recent result of Belarif [Bel] proves that when Y is a geometrically finite Riemannian manifold with pinched negative curvature and topologically mixing geodesic flow, the finite and mixing Gibbs measures associated with bounded Hölder-continuous potentials are, once normalised, dense (for the weak-star topology) in the whole space of probability measures invariant under the geodesic flow.

The Gibbs measures are remarkable measures for $\text{CAT}(-1)$ spaces endowed with potentials due to their unique ergodic-theoretic properties. Let $(Z, (\phi_t)_{t \in \mathbb{R}})$ be a topological space endowed with a continuous one-parameter group of homeomorphisms and let $\psi : Z \rightarrow \mathbb{R}$ be a bounded continuous map. Let \mathcal{M} be the set of Borel probability measures m on Z invariant under the flow $(\phi_t)_{t \in \mathbb{R}}$. Let $h_m(\phi^1)$ be the (metric) entropy of the geodesic flow with respect to $m \in \mathcal{M}$. The *metric pressure* for ψ of a measure $m \in \mathcal{M}$ and the *pressure* of ψ are respectively

$$P_\psi(m) = h_m(\phi_1) + \int_Z \psi \, dm \quad \text{and} \quad P_\psi = \sup_{m \in \mathcal{M}} P_\psi(m).$$

An element $m \in \mathcal{M}$ is an *equilibrium state* for ψ if the least upper bound defining P_ψ is attained on m .

Let $F^\sharp : \mathcal{G}Y \rightarrow \mathbb{R}$ be the composition of the canonical map $\mathcal{G}Y \rightarrow T^1Y$ with F , and note that $F^\sharp = F$ if Y is a Riemannian manifold. When $F = 0$ and Y is a Riemannian manifold, whose sectional curvatures and their first derivatives are bounded, by [OP, Thm. 2], the

²itself building on the works of Ledrappier [Led], Hamenstädt, Coudène, Mohsen

³see for instance [Rue3, Kel, Zin]

pressure P_F coincides with the entropy of the geodesic flow, it is equal to the critical exponent of the fundamental group of Y , and the Bowen-Margulis measure $m_F = m_{\text{BM}}$, normalised to be a probability measure, is the measure of maximal entropy. When Y is a Riemannian manifold whose sectional curvatures and their first derivatives are bounded and F is Hölder-continuous, by [PauPS, Thm. 6.1], we have $P_F = \delta_F$. If furthermore the Gibbs measure m_F is finite and normalised to be a probability measure, then m_F is an equilibrium state for F .

We prove an analog of these results for the potential F^\sharp when Y is a metric graph of groups. The case when Y is a finite simplicial graph⁴ is classical by the work of Bowen [Bowe2], as it reduces to arguments of subshifts of finite type (see for instance [CoP1]). When Y is a compact⁵ locally CAT(-1)-space,⁶ a complete statement about existence, uniqueness and Gibbs property of equilibrium states for any Hölder-continuous potential is given in [ConLT].

Theorem 1.1 (The variational principle for metric graphs of groups). *Assume that Y is a metric graph of finite groups, with a positive lower bound and finite upper bound on the lengths of edges. If the critical exponent δ_F is finite, if the Gibbs measure m_F is finite, then $P_{F^\sharp} = \delta_F$ and the Gibbs measure normalised to be a probability measure is the unique equilibrium state for F^\sharp .*

The main tool is a natural coding of the discrete time geodesic flow by a topological Markov shift (see Section 5.1). This coding is delicate when the vertex stabilisers are nontrivial, as in particular it does not satisfy in general the Markovian property of dependence only on the immediate past (see Section 5.2). We then apply results of Buzzi and Sarig in symbolic dynamics over a countable alphabet (see Appendix A written by J. Buzzi), and suspension techniques introduced in Section 5.3.

In Chapter 7, we generalise for nonconstant potentials on any geodesically complete connected proper locally CAT(-1) space Y the construction of the *skinning measures* σ_D^+ and σ_D^- on the outer and inner unit normal bundles of a connected proper nonempty properly immersed closed locally convex subset D of Y . By definition, D is the image, by the universal covering map, of a proper nonempty closed convex subset of the universal cover of Y , whose family of images under the universal covering group is locally finite. We refer to Section 2.5 for the appropriate definition of the outer and inner unit normal bundles of D when the boundary of D is not smooth. We construct these measures σ_D^+ and σ_D^- as the induced measures on Y of appropriate pushforwards of the Patterson densities associated with the potential F to the outer and inner unit normal bundles of the lift of D in the universal cover of Y . This construction generalises the one in [PaP14a] when $F = 0$, which itself generalises the one in [OhS1, OhS2] when M has constant curvature and D is a ball, a horoball or a totally geodesic submanifold.

In Section 10.1, we prove the following result on the equidistribution of equidistant hypersurfaces in CAT(-1) spaces. This result is a generalisation of [PaP14a, Theo. 1] (valid in Riemannian manifolds with zero potential) which itself generalised the ones in [Mar2, EM, PaP12] when Y has constant curvature, $F = 0$ and D is a ball, a horoball or a totally geodesic submanifold. See also [Rob2] when Y is a CAT(-1) space, $F = 0$ and D is a ball or a horoball.

⁴that is, a finite graph of trivial groups with edge lengths 1

⁵a very strong assumption that we do not want to make in this text

⁶not in the orbifold sense, hence this excludes for instance the case of graphs of groups with some nontrivial vertex stabiliser

Theorem 1.2. *Let (Y, F) be a locally CAT(-1) space endowed with a potential satisfying the HC-property. Assume that the Gibbs measure m_F on $\mathcal{G}Y$ is finite and mixing for the geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{R}}$, and that the skinning measure σ_D^+ is finite and nonzero. Then, as t tends to $+\infty$, the pushforwards $(\mathbf{g}^t)_* \sigma_D^+$ of the skinning measure of D by the geodesic flow weak-star converges towards the Gibbs measure m_F (after normalisation as probability measures).*

We prove in Theorem 10.4 an analog of Theorem 1.2 for the discrete time geodesic flow on simplicial graphs and, more generally, simplicial graphs of groups. As a special case, we recover known results on nonbacktracking simple random walks on regular graphs. The equidistribution of the pushforward of the skinning measure of a subgraph is a weighted version of the following classical result, see for instance [ABLS], which under further assumptions on the spectral properties on the graph gives precise rates of convergence.

Corollary 1.3. *Let \mathbb{Y} be a finite regular graph which is not bipartite. Let \mathbb{Y}' be a nonempty connected subgraph. Then the n -th vertex of the non-backtracking simple random walk on \mathbb{Y} starting transversally to \mathbb{Y}' converges in distribution to the uniform distribution as $n \rightarrow +\infty$.*

See Chapter 10 for more details and for the extensions to nonzero potential and to graphs of groups, as well as Section 10.4 for error terms.

The distribution of common perpendiculars

Let D^- and D^+ be connected proper nonempty properly immersed locally convex closed subsets of Y . A *common perpendicular* from D^- to D^+ is a locally geodesic path in Y starting perpendicularly from D^- and arriving perpendicularly to D^+ .⁷ We denote the length of a common perpendicular α from D^- to D^+ by $\lambda(\alpha)$, and its initial and terminal unit tangent vectors by v_α^- and v_α^+ . In the general CAT(-1) case, v_α^\pm are two different parametrisations (by $\mp[0, \lambda(\alpha)]$) of α , considered as elements of the space $\mathcal{G}Y$ of generalised locally geodesic lines in Y , see [BartL] or Section 2.3. For all $t > 0$, we denote by $\text{Perp}(D^-, D^+, t)$ the set of common perpendiculars from D^- to D^+ with length at most t (considered with multiplicities), and we define the counting function with weights by

$$\mathcal{N}_{D^-, D^+, F}(t) = \sum_{\alpha \in \text{Perp}(D^-, D^+, t)} e^{\int_\alpha F},$$

where $\int_\alpha F = \int_0^{\lambda(\alpha)} F(\mathbf{g}^t v_\alpha^-) dt$. We refer to Section 12.1 for the definition of the multiplicities in the manifold case, which are equal to 1 if D^- and D^+ are embedded and disjoint. Higher multiplicities for common perpendiculars α can occur for instance when D^- is a non-simple closed geodesic and the initial point of α is a multiple point of D^- .

Let $\text{Perp}(D^-, D^+)$ be the set of all common perpendiculars from D^- to D^+ (considered with multiplicities). The family $(\lambda(\alpha))_{\alpha \in \text{Perp}(D^-, D^+)}$ is called the *marked ortholength spectrum* from D^- to D^+ . The set of lengths (with multiplicities) of elements of $\text{Perp}(D^-, D^+)$ is called the *ortholength spectrum* of D^-, D^+ . This second set has been introduced by Basmajian [Basm] (under the name “full orthogonal spectrum”) when M has constant curvature, and D^- and D^+ are disjoint or equal embedded totally geodesic hypersurfaces or embedded horospherical cusp neighbourhoods or embedded balls. We refer to the paper [BridK] which

⁷See Section 2.6 for explanations when the boundary of D^- or D^+ is not smooth.

proves that the ortholength spectrum with $D^\pm = \partial M$ determines the volume of a compact hyperbolic manifold M with totally geodesic boundary (see also [Cal] and [MasM]).

We prove in Chapter 12 that the critical exponent δ_F of F is the exponential growth rate of $\mathcal{N}_{D^-, D^+, F}(t)$, and we give an asymptotic formula of the form $\mathcal{N}_{D^-, D^+, F}(t) \sim c e^{\delta_F t}$ as $t \rightarrow +\infty$, with error term estimates in appropriate situations. The constants c that will appear in such asymptotic formulas will be explicit, in terms of the measures naturally associated with the (normalised) potential F : the Gibbs measure m_F and the skinning measures of D^- and D^+ .

When $F = 0$ and Y is a Riemannian manifold with pinched sectional curvature and finite and mixing Bowen-Margulis measure, the asymptotics of the counting function $\mathcal{N}_{D^-, D^+, 0}(t)$ are described in [PaP16c, Thm. 1]. The only restriction on the two convex sets D^\pm is that their skinning measures are finite. Here, we generalise that result by allowing for nonzero potential and more general CAT(-1) spaces than just manifolds.

The counting function $\mathcal{N}_{D^-, D^+, 0}(t)$ has been studied in negatively curved manifolds since the 1950's and in a number of more recent works, sometimes in a different guise. A number of special cases (all with $F = 0$ and covered by the results of [PaP16c]) were known:

- D^- and D^+ are reduced to points, by for instance [Hub2], [Mar1] and [Rob2],
- D^- and D^+ are horoballs, by [BeHP], [HeP3], [Cos] and [Rob2] without an explicit form of the constant in the asymptotic expression,
- D^- is a point and D^+ is a totally geodesic submanifold, by [Her], [EM] and [OhS3] in constant curvature,
- D^- is a point and D^+ is a horoball, by [Kon] and [KonO] in constant curvature, and [Kim] in rank one symmetric spaces,
- D^- is a horoball and D^+ is a totally geodesic submanifold, by [OhS1] and [PaP12] in constant curvature, and
- D^- and D^+ are (properly immersed) locally geodesic lines in constant curvature and dimension 3, by [Pol2].

We refer to the survey [PaP16a] for more details on the manifold case.

When X is a compact metric or simplicial graph and D^\pm are points, the asymptotics of $\mathcal{N}_{D^-, D^+, 0}(t)$ as $t \rightarrow +\infty$ is treated in [Gui], as well as [Rob2]. Under the same setting, see also the work of Kiro-Smilansky-Smilansky announced in [KiSS] for a counting result of paths (not assumed to be locally geodesic) in finite metric graphs with rationally independent edge lengths and vanishing potential.

The proofs of the asymptotic results on the counting function $\mathcal{N}_{D^-, D^+, F}$ are based on the following simultaneous equidistribution result that shows that the initial and terminal tangent vectors of the common perpendiculars equidistribute to the skinning measures of D^- and D^+ . We denote the unit Dirac mass at a point z by Δ_z and the total mass of any measure m by $\|m\|$.

Theorem 1.4. *Assume that Y is a nonelementary Riemannian manifold with pinched sectional curvature at most -1 or a metric graph. Let $F : T^1 Y \rightarrow \mathbb{R}$ be a potential, with finite and positive critical exponent δ_F , which is bounded and Hölder-continuous when Y is a manifold. Let D^\pm be as above. Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow. For the weak-star convergence of measures on $\check{\mathcal{G}}Y \times \check{\mathcal{G}}Y$, we have*

$$\lim_{t \rightarrow +\infty} \delta_F \|m_F\| e^{-\delta_F t} \sum_{\alpha \in \text{Perp}(D^-, D^+, t)} e^{\int_\alpha F} \Delta_{v_\alpha^-} \otimes \Delta_{v_\alpha^+} = \sigma_{D^-}^+ \otimes \sigma_{D^+}^-.$$

There is a similar statement for nonbipartite simplicial graphs and for more general graphs of groups on which the discrete time geodesic flow is mixing for the Gibbs measure, see the end of Chapter 11 and Section 12.4. Again, the results can then be interpreted in terms of nonbacktracking random walk.

In Section 12.2, we deduce our counting results for common perpendiculars of the subsets D^- and D^+ from the above simultaneous equidistribution theorem.

Theorem 1.5. (1) *Let Y, F, D^\pm be as in Theorem 1.4. Assume that the Gibbs measure m_F is finite and mixing for the continuous time geodesic flow and that the skinning measures $\sigma_{D^-}^+$ and $\sigma_{D^+}^-$ are finite and nonzero. Then, as $s \rightarrow +\infty$,*

$$\mathcal{N}_{D^-, D^+, F}(s) \sim \frac{\|\sigma_{D^-}^+\| \|\sigma_{D^+}^-\|}{\|m_F\|} \frac{e^{\delta_F s}}{\delta_F}.$$

(2) *If Y is a finite nonbipartite simplicial graph, then*

$$\mathcal{N}_{D^-, D^+, F}(n) \sim \frac{e^{\delta_F n} \|\sigma_{D^-}^+\| \|\sigma_{D^+}^-\|}{(e^{\delta_F} - 1) \|m_F\|} e^{\delta_F n}.$$

The above Assertion (1) is valid when Y is a good orbifold instead of a manifold or a metric graph of finite groups instead of a metric graph (for the appropriate notion of multiplicities), and when D^- and D^+ are replaced by locally finite families. See Section 12.4 for generalisations of Assertion (2) to (possibly infinite) simplicial graphs of finite groups and Sections 12.3 and 12.6 for error terms.

We avoid any compactness assumption on Y , we only assume that the Gibbs measure m_F of F is finite and that it is mixing for the geodesic flow. By Babillot's theorem [Bab1], if the length spectrum of Y is not contained in a discrete subgroup of \mathbb{R} , then m_F is mixing if finite. If Y is a Riemannian manifold, this condition is satisfied for instance if the limit set of a fundamental group of Y is not totally disconnected, see for instance [Dal1, Dal2]. When Y is a metric graph, Babillot's mixing condition is in particular satisfied if the lengths of the edges of Y are rationally independent.

As in [PaP16c], we have very weak finiteness and curvature assumptions on the space and the convex subsets we consider. Furthermore, the space Y is no longer required to be a manifold and we extend the theory to non-constant weights using equilibrium states. Such weighted counting has only been used in the orbit-counting problem in manifolds with pinched negative curvature in [PauPS]. The approach is based on ideas from Margulis's thesis to use the mixing of the geodesic flow. Our measures are much more general. As in [PaP16c], we push simultaneously the unit normal vectors to the two convex sets D^- and D^+ in opposite directions.

Classically, an important characterization of the Bowen-Margulis measure on closed negatively curved Riemannian manifolds ($F = 0$) is that it coincides with the weak-star limit of properly normalised sums of Lebesgue measures supported on periodic orbits. The result was extended to CAT(-1) spaces with zero potential in [Rob2] and to Gibbs measures in the manifold case in [PauPS, Thm. 9.11]. As a corollary of the simultaneous equidistribution result Theorem 1.4, we obtain a weighted version for simplicial and metric graphs of groups. The following is a simplified version of such a result for Gibbs measures of metric graphs.

Let $\mathfrak{Per}'(t)$ be the set of prime periodic orbits of the geodesic flow on Y . Let $\lambda(g)$ denote the length of a closed orbit g . Let \mathcal{L}_g be the Lebesgue measure along g and let $\mathcal{L}_g(F^\sharp)$ be the period of g for the potential F .

Theorem 1.6. *Assume that Y is a finite metric graph, that the critical exponent δ_F is positive and that the Gibbs measure m_F is mixing for the (continuous time) geodesic flow. As $t \rightarrow +\infty$, the measures*

$$\delta_F e^{\delta_F t} \sum_{g \in \mathfrak{Pct}'(t)} e^{\mathcal{L}_g(F)} \mathcal{L}_g$$

and

$$\delta_F t e^{\delta_F t} \sum_{g \in \mathfrak{Pct}'(t)} e^{\mathcal{L}_g(F)} \frac{\mathcal{L}_g}{\lambda(g)}$$

converge to $\frac{m_F}{\|m_F\|}$ for the weak-star convergence of measures.

See Section 13.2 for the proof of the full result and for a similar statement for (possibly infinite) simplicial graphs of finite groups. As a corollary, we obtain counting results of simple loops in metric and simplicial graphs, generalising results of [ParP], [Gui].

Corollary 1.7. *Assume that Y is a finite metric graph with all vertices of degree at least 3 such that the critical exponent δ_F is positive.*

(1) *If the Gibbs measure is mixing for the geodesic flow, then*

$$\sum_{g \in \mathfrak{Pct}'(t)} e^{\mathcal{L}_g(F)} \sim \frac{e^{\delta_F t}}{\delta_F t}$$

as $t \rightarrow +\infty$.

(2) *If Y is simplicial and if the Gibbs measure is mixing for the discrete time geodesic flow, then*

$$\sum_{g \in \mathfrak{Pct}'(t)} e^{\mathcal{L}_g(F)} \sim \frac{e^{\delta_F}}{e^{\delta_F} - 1} \frac{e^{\delta_F t}}{t}$$

as $t \rightarrow +\infty$.

In the cases when error bounds are known for the mixing property of the continuous time or discrete time geodesic flow on $\mathcal{G}Y$, we obtain corresponding error terms in the equidistribution result of Theorem 1.2 generalising [PaP14a, Theo. 20] (where $F = 0$) and in the approximation of the counting function $\mathcal{N}_{D^-, D^+, 0}$ by the expression introduced in Theorem 1.5. In the manifold case, see [KM1], [Clo], [Dol1], [Sto], [Live], [GLP], and Section 12.3 for definitions and precise references. Here is an example of such a result in the manifold case.

Theorem 1.8. *Assume that Y is a compact Riemannian manifold and m_F is exponentially mixing under the geodesic flow for the Hölder regularity, or that Y is a locally symmetric space, the boundary of D^\pm is smooth, m_F is finite, smooth, and exponentially mixing under the geodesic flow for the Sobolev regularity. Assume that the strong stable/unstable ball masses by the conditionals of m_F are Hölder-continuous in their radius.*

(1) *As t tends to $+\infty$, the pushforwards $(\mathbf{g}^t)_* \sigma_{D^-}^+$ of the skinning measure of D^- by the geodesic flow equidistribute towards the Gibbs measure m_F with exponential speed.*

(2) *There exists $\kappa > 0$ such that, as $t \rightarrow +\infty$,*

$$\mathcal{N}_{D^-, D^+, F}(t) = \frac{\|\sigma_{D^-}^+\| \|\sigma_{D^+}^-\|}{\delta_F \|m_F\|} e^{\delta_F t} (1 + O(e^{-\kappa t})).$$

See Section 12.3 for a discussion of the assumptions and the dependence of $O(\cdot)$ on the data. Similar (sometimes more precise) error estimates were known earlier for the counting function in special cases of D^\pm in constant curvature geometrically finite manifolds (often in small dimension) through the work of Huber, Selberg, Patterson, Lax and Phillips [LaxP], Cosentino [Cos], Kontorovich and Oh [KonO], Lee and Oh [LeO].

When Y is a finite volume hyperbolic manifold and the potential F is constant 0, the Gibbs measure is proportional to the Liouville measure and the skinning measures of totally geodesic submanifolds, balls and horoballs are proportional to the induced Riemannian measures of the unit normal bundles of their boundaries. In this situation, there are very explicit forms of the counting results in finite-volume hyperbolic manifolds, see [PaP16c, Cor.21], [PaP16a]. These results are extended to complex hyperbolic space in [PaP16b].

As an example of this result, if D^- and D^+ are closed geodesics of Y of lengths ℓ_- and ℓ_+ , respectively, then the number $\mathcal{N}(s)$ of common perpendiculars (counted with multiplicity) from D^- to D^+ of length at most s satisfies, as $s \rightarrow +\infty$,

$$\mathcal{N}_{D^-, D^+, 0}(s) \sim \frac{\pi^{\frac{n}{2}-1} \Gamma(\frac{n-1}{2})^2}{2^{n-2} (n-1) \Gamma(\frac{n}{2})} \frac{\ell_- \ell_+}{\text{Vol}(Y)} e^{(n-1)s}. \quad (1.1)$$

Counting in weighted graphs of groups

From now on in this introduction, we only consider metric or simplicial graphs or graphs of groups.

Let \mathbb{Y} be a connected finite graph with set of vertices $V\mathbb{Y}$ and set of edges $E\mathbb{Y}$ (see [Ser3] for the conventions). We assume that the degree of the graph at each vertex is at least 3. Let $\lambda : E\mathbb{Y} \rightarrow]0, +\infty[$ with $\lambda(\bar{e}) = \lambda(e)$ for every $e \in E\mathbb{Y}$ be an *edge length map*, let $Y = |\mathbb{Y}|_\lambda$ be the geometric realisation of \mathbb{Y} where the geometric realisation of every edge $e \in E\mathbb{Y}$ has length $\lambda(e)$, and let $c : E\mathbb{Y} \rightarrow \mathbb{R}$ be a map, called a (logarithmic) system of conductances in the analogy between graphs and electrical networks, see for instance [Zem].

Let \mathbb{Y}^\pm be proper nonempty subgraphs of \mathbb{Y} . For every $t \geq 0$, we denote by $\text{Perp}(\mathbb{Y}^-, \mathbb{Y}^+, t)$ the set of edge paths $\alpha = (e_1, \dots, e_k)$ in \mathbb{Y} without backtracking, of length $\lambda(\alpha) = \sum_{i=1}^k \lambda(e_i)$ at most t , of conductance $c(\alpha) = \sum_{i=1}^k c(e_i)$, starting from a vertex of \mathbb{Y}^- but not by an edge of \mathbb{Y}^- , ending at a vertex of \mathbb{Y}^+ but not by an edge of \mathbb{Y}^+ . Let

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(t) = \sum_{\alpha \in \text{Perp}(\mathbb{Y}^-, \mathbb{Y}^+, t)} e^{c(\alpha)}$$

be the number of paths without backtracking from \mathbb{Y}^- to \mathbb{Y}^+ of length at most t , counted with weights defined by the system of conductances.

Recall that a real number x is *Diophantine* if it is badly approximable by rational numbers, that is, if there exist $\alpha, \beta > 0$ such that $|x - \frac{p}{q}| \geq \alpha q^{-\beta}$ for all $p, q \in \mathbb{Z}$ with $q > 0$. We obtain the following result, which is a very simplified version of our results for the sake of this introduction.

Theorem 1.9. (1) *If Y has two cycles whose ratio of lengths is Diophantine, then there exists $C > 0$ such that for every $k \in \mathbb{N} - \{0\}$, as $t \rightarrow +\infty$,*

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(t) = C e^{\delta_c t} (1 + O(t^{-k})).$$

(2) If $\lambda \equiv 1$, then there exists $C', \kappa > 0$ such that, as $n \in \mathbb{N}$ tends to $+\infty$,

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n) = C' e^{\delta_c n} (1 + O(e^{-\kappa n})) .$$

Note that the Diophantine assumption on Y in Theorem 1.9 (1) is standard in the theory of quantum graphs (see for instance [BerK]).

The constants $C = C_{\mathbb{Y}^\pm, c, \lambda} > 0$ and $C' = C'_{\mathbb{Y}^\pm, c} > 0$ in the above asymptotic formulas are explicit. When $c \equiv 0$ and $\lambda \equiv 1$, the constants can often be determined concretely, as indicated in the two examples below.⁸ Among the ingredients in these computations are the explicit expressions of the total mass of many Bowen-Margulis measures and skinning measures obtained in Chapter 8.

See Sections 12.4, 12.5 and 12.6 for generalisations of Theorem 1.9 when the graphs \mathbb{Y}^\pm are not embedded in \mathbb{Y} , and for versions in (possibly infinite) metric graphs of finite groups. In particular, Assertion (2) remains valid if Y is the quotient of a uniform simplicial tree by a geometrically finite lattice in the sense of [Pau2], such as an arithmetic lattice in PGL_2 over a non-Archimedean local field, see [Lub1]. Recall that a locally finite metric tree X is uniform if it admits a discrete and cocompact group of isometries, and that a lattice Γ of X is a lattice in the locally compact group of isometries of X preserving without inversions the simplicial structure. We refer for instance to [BasK, BasL] for uncountably many examples of tree lattices.

Example 1.10. (1) When \mathbb{Y} is a $(q+1)$ -regular finite graph with constant edge length map $\lambda \equiv 1$ and vanishing system of conductances $c \equiv 0$, then $\delta_c = \ln q$, and if furthermore \mathbb{Y}^+ and \mathbb{Y}^- are vertices, then (see Equation (12.10))

$$C' = \frac{q+1}{(q-1) \mathrm{Card}(V\mathbb{Y})} .$$

(2) When \mathbb{Y} is biregular of degrees $p+1$ and $q+1$ with $p, q \geq 2$, when $\lambda \equiv 1$ and $c \equiv 0$, then $\delta_c = \ln \sqrt{pq}$, and if furthermore the subgraphs \mathbb{Y}^\pm are simple cycles of lengths L^\pm , then (see Equation (12.11))

$$C' = \frac{(\sqrt{q} + \sqrt{p})^2 L^- L^+}{2(pq-1) \mathrm{Card}(E\mathbb{Y})} .$$

The main tool in order to obtain the error terms in Theorem 1.9 and its more general versions is to study the error terms in the mixing property of the geodesic flow. Using the already mentioned coding (given in Section 5.2) of the discrete time geodesic flow by a two-sided topological Markov shift, classical reduction to one-sided topological Markov shift, and results of Young [You1] on the decay of correlations for Young towers with exponentially small tails, we in particular obtain the following simple criteria for the exponential decay of correlation of the discrete time geodesic flow. See Theorem 9.1 for the complete result. In particular, we do not assume \mathbb{Y} to be finite.

Theorem 1.11. *Assume that the Gibbs measure m_F is finite and mixing for the discrete time geodesic flow on \mathbb{Y} . Assume moreover that there exist a finite subset E of $V\mathbb{Y}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have*

$$m_F(\{\ell \in \mathcal{G}\mathbb{Y} : \ell(0) \in E \text{ and } \forall k \in \{1, \dots, n\}, \ell(k) \notin E\}) \leq C' e^{-\kappa' n} .$$

Then the discrete time geodesic flow has exponential decay of Hölder correlations for m_F .

⁸See Section 12.4 for more examples.

The assumption of having exponentially small mass of geodesic lines which have a big return time to a given finite subset of $V\mathbb{Y}$ is in particular satisfied (see Section 9.2) if \mathbb{Y} is the quotient of a uniform simplicial tree by a geometrically finite lattice in the sense of [Pau2], such as an arithmetic lattice in PGL_2 over a non-Archimedean local field, see [Lub1], but also by many other examples of \mathbb{Y} .

These results allow to prove in Section 9.3, under Diophantine assumptions, the rapid mixing property for the continuous time geodesic flow, that leads to Assertion (1) of Theorem 1.9, see Section 12.6. The proof uses suspension techniques due to Dolgopyat [Dol2] when Y is a compact metric tree, and to Melbourne [Mel1] otherwise.

As a corollary of the general version of the counting result Theorem 1.5, we have the following asymptotic for the orbital counting function in conjugacy classes for groups acting on trees. Given $x_0 \in X$ and a nontrivial conjugacy class \mathfrak{K} in a discrete group Γ of isometries of X , we consider the counting function

$$N_{\mathfrak{K}, x_0}(t) = \mathrm{Card}\{\gamma \in \mathfrak{K} : d(x_0, \gamma x_0) \leq t\},$$

introduced by Huber [Hub1] when X is replaced by the real hyperbolic plane and Γ is a lattice. We refer to [PaP15] for many results on the asymptotic growth of such orbital counting functions in conjugacy classes, when X is replaced by a finitely generated group with a word metric, or a complete simply connected pinched negatively curved Riemannian manifold. See also [ChaP, ArCT].

Theorem 1.12. *Let X be a uniform metric tree with vertices of degree ≥ 3 , let δ be the Hausdorff dimension of $\partial_\infty X$, let Γ be a discrete group of isometries of X , let x_0 be a vertex of X with trivial stabiliser in Γ , and let \mathfrak{K} be a loxodromic conjugacy class in Γ .*

(1) *If the metric graph $\Gamma \backslash X$ is compact and has two cycles whose ratio of lengths is Diophantine, then there exists $C > 0$ such that for every $k \in \mathbb{N} - \{0\}$, as $t \rightarrow +\infty$,*

$$\mathcal{N}_{\mathfrak{K}, x_0}(t) = C e^{\frac{\delta}{2}t} (1 + O(t^{-k})).$$

(2) *If X is simplicial and Γ is a geometrically finite lattice of X , then there exist $C', \kappa > 0$ such that, as $n \in \mathbb{N}$ tends to $+\infty$,*

$$\mathcal{N}_{\mathfrak{K}, x_0}(n) = C' e^{\delta \lfloor (n - \lambda(\gamma))/2 \rfloor} (1 + O(e^{-\kappa n})).$$

We refer to Theorem 13.1 for a more general version, including a version with a system of conductances in the counting function, and when \mathfrak{K} is elliptic. When $\Gamma \backslash X$ is compact and Γ is torsion free,⁹ Assertion (1) of this result is due to Kenison and Sharp [KeS], who proved it using transfer operator techniques for suspensions of subshifts of finite type. Up to strengthening the Diophantine assumption, using work of Melbourne [Mel1] on the decay of correlations of suspensions of Young towers, we are able to extend Assertion (1) to all geometrically finite lattices Γ of X in Chapter 13.1.

The constants $C = C_{\mathfrak{K}, x_0}$ and $C' = C'_{\mathfrak{K}, x_0}$ are explicit. For instance in Assertion (2), if X is the geometric realisation of a regular simplicial tree \mathbb{X} of degree $q + 1$, if x_0 is a vertex of \mathbb{X} , if \mathfrak{K} is the conjugacy class of γ_0 with translation length $\lambda(\gamma_0)$ on X , if

$$\mathrm{Vol}(\Gamma \backslash \mathbb{X}) = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|}$$

⁹in particular Γ then has the very restricted structure of a free group

is the volume¹⁰ of the quotient graph of groups $\Gamma \backslash \mathbb{X}$, then

$$C' = \frac{\lambda(\gamma_0)}{[Z_\Gamma(\gamma_0) : \gamma_0^{\mathbb{Z}}] \text{Vol}(\Gamma \backslash \mathbb{X})},$$

where $Z_\Gamma(\gamma_0)$ is the centraliser of γ_0 in Γ . When furthermore Γ is torsion free and $\Gamma \backslash \mathbb{X}$ is finite, as $\delta = \ln q$, we get

$$\mathcal{N}_{\mathbb{R}, x_0}(n) = \frac{\lambda(\gamma_0)}{\text{Card}(\Gamma \backslash \mathbb{X})} q^{\lfloor (n - \lambda(\gamma_0))/2 \rfloor} + O(q^{(1 - \kappa') n/2})$$

as $n \in \mathbb{N}$ tends to $+\infty$, thus recovering the result of [Dou] who used the spectral theory of the discrete Laplacian.

Selected arithmetic applications

We end this introduction by giving a sample of our arithmetic applications (see Part III of this book) of the ergodic and dynamical results on the discrete time geodesic flow on simplicial trees described in Part II of this book, as summarized above. Our equidistribution and counting results of common perpendiculars between subtrees indeed produce equidistribution and counting results of rationals and quadratic irrationals in non-Archimedean local fields. We refer to [BrPP] for an announcement of the results of Part III, with a presentation different from the one in this introduction.

To motivate what follows, consider $R = \mathbb{Z}$ the ring of integers, $K = \mathbb{Q}$ its field of fractions, $\widehat{K} = \mathbb{R}$ the completion of \mathbb{Q} for the usual Archimedean absolute value $|\cdot|$, and $\text{Haar}_{\widehat{K}}$ the Lebesgue measure of \mathbb{R} (which is the Haar measure of the additive group \mathbb{R} normalised so that $\text{Haar}_{\widehat{K}}([0, 1]) = 1$).

The following equidistribution result of rationals, due to Neville [Nev], is a quantitative statement on the density of K in \widehat{K} : For the weak-star convergence of measures on \widehat{K} , as $s \rightarrow +\infty$, we have

$$\lim_{s \rightarrow +\infty} \frac{\pi}{6} s^{-2} \sum_{p, q \in R : pR + qR = R, |q| \leq s} \Delta_{\frac{p}{q}} = \text{Haar}_{\widehat{K}}.$$

Furthermore, there exists $\ell \in \mathbb{N}$ such that for every smooth function $\psi : \widehat{K} \rightarrow \mathbb{C}$ with compact support, there is an error term in the above equidistribution claim evaluated on ψ , of the form $O(s(\ln s)\|\psi\|_\ell)$ where $\|\psi\|_\ell$ is the Sobolev norm of ψ . The following counting result due to Mertens on the asymptotic behaviour of the average of Euler's totient function $\varphi : k \mapsto \text{Card}(R/kR)^\times$, follows from the above equidistribution one:

$$\sum_{k=1}^n \varphi(k) = \frac{3}{\pi} n^2 + O(n \ln n).$$

See [PaP14b] for an approach using methods similar to the ones in this text, and for instance [HaW, Th. 330] for a more traditional proof, as well as [Wal] for a better error term.

Let us now switch to a non-Archimedean setting, restricting to positive characteristic in this introduction. See Part III for analogous applications in characteristic zero.

¹⁰See for instance [BasK, BasL].

Let \mathbb{F}_q be a finite field of order q . Let $R = \mathbb{F}_q[Y]$ be the ring of polynomials in one variable Y with coefficients in \mathbb{F}_q . Let $K = \mathbb{F}_q(Y)$ be the field of rational fractions in Y with coefficients in \mathbb{F}_q , which is the field of fractions of R . Let $\hat{K} = \mathbb{F}_q((Y^{-1}))$ be the field of formal Laurent series in the variable Y^{-1} with coefficients in \mathbb{F}_q , which is the completion of K for the (ultrametric) absolute value $|\frac{P}{Q}| = q^{\deg P - \deg Q}$. Let $\mathcal{O} = \mathbb{F}_q[[Y^{-1}]]$ be the ring of formal power series in Y^{-1} with coefficients in \mathbb{F}_q , which is the ball of centre 0 and radius 1 in \hat{K} for this absolute value.

Note that \hat{K} is locally compact, and we endow the additive group \hat{K} with the Haar measure $\text{Haar}_{\hat{K}}$ normalised so that $\text{Haar}_{\hat{K}}(\mathcal{O}) = 1$. The following results extend (with appropriate constants) when K is replaced by any function field of a nonsingular projective curve over \mathbb{F}_q and \hat{K} any completion of K , see Part III.

The following equidistribution result¹¹ of elements of K in \hat{K} gives an analog of Neville's equidistribution results for function fields. Note that when $G = \text{GL}_2(R)$, we have $(P, Q) \in G(1, 0)$ if and only if $\langle P, Q \rangle = R$. We denote by H_x the stabiliser of any element x of any set endowed with any action of any group H .

Theorem 1.13. *Let G be any finite index subgroup of $\text{GL}_2(R)$. For the weak-star convergence of measures on \hat{K} , we have*

$$\lim_{t \rightarrow +\infty} \frac{(q+1) [\text{GL}_2(R) : G]}{(q-1) q^2 [\text{GL}_2(R)_{(1,0)} : G_{(1,0)}]} q^{-2t} \sum_{(P,Q) \in G(1,0), \deg Q \leq t} \Delta_{\frac{P}{Q}} = \text{Haar}_{\hat{K}} .$$

We emphasise the fact that we are not assuming G to be a congruence subgroup of $\text{GL}_2(R)$. This is made possible by our geometric and ergodic methods.

The following variation of this result is more interesting when the class number of the function field K is larger than 1 (see Corollary 16.7 in Chapter 16).

Theorem 1.14. *Let \mathfrak{m} be a nonzero fractional ideal of R with norm $\mathfrak{N}(\mathfrak{m})$. For the weak-star convergence of measures on \hat{K} , we have*

$$\lim_{t \rightarrow +\infty} \frac{q+1}{(q-1) q^2} s^{-2} \sum_{\substack{(x,y) \in \mathfrak{m} \times \mathfrak{m} \\ \mathfrak{N}(\mathfrak{m})^{-1} \mathfrak{N}(y) \leq s, Rx + Ry = \mathfrak{m}}} \Delta_{\frac{x}{y}} = \text{Haar}_{\hat{K}} .$$

In the next two statements, we assume that the characteristic of K is different from 2. If $\alpha \in \hat{K}$ is quadratic irrational over K ,¹² let α^σ be the Galois conjugate of α ,¹³ let $\text{tr}(\alpha) = \alpha + \alpha^\sigma$ and $\mathfrak{n}(\alpha) = \alpha \alpha^\sigma$, and let

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|} .$$

This is an appropriate complexity for quadratic irrationals in a given orbit by homographies under $\text{PGL}_2(R)$. See Section 17.2 and for instance [HeP4, §6] for motivations and results. Note that although there are only finitely many orbits by homographies of $\text{PGL}_2(R)$ on K (and exactly one in the particular case of this introduction), there are infinitely many orbits of $\text{PGL}_2(R)$ in the set of quadratic irrationals in \hat{K} over K . The following result gives in particular that any orbit of quadratic irrationals under $\text{PGL}_2(R)$ equidistributes in \hat{K} , when the complexity tends to infinity. See Theorem 17.5 in Section 17.2 for a more general version. We denote by \cdot the action by homographies of $\text{GL}_2(\hat{K})$ on $\mathbb{P}_1(\hat{K}) = \hat{K} \cup \{\infty = [1 : 0]\}$.

¹¹See Theorem 16.4 in Chapter 16 for a more general version.

¹²that is, α does not belong to K and satisfies a quadratic equation with coefficients in K

¹³that is, the other root of the polynomial defining α

Theorem 1.15. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$. Let $\alpha_0 \in \widehat{K}$ be a quadratic irrational over K . For the weak-star convergence of measures on \widehat{K} , we have*

$$\lim_{s \rightarrow +\infty} \frac{(\ln q) (q+1) m_0 [\mathrm{GL}_2(R) : G]}{2 q^2 (q-1)^3 |\ln |\mathrm{tr} g_0||} s^{-1} \sum_{\alpha \in G \cdot \alpha_0, h(\alpha) \leq s} \Delta_\alpha = \mathrm{Haar}_{\widehat{K}} .$$

where $g_0 \in G$ fixes α_0 with $|\mathrm{tr} g_0| > 1$, and m_0 is the index of $g_0^{\mathbb{Z}}$ in G_{α_0} .

Another equidistribution result of an orbit of quadratic irrationals under $\mathrm{PGL}_2(R)$ is obtained by taking another complexity, constructed using crossratios with a fixed quadratic irrational. We denote by $[a, b, c, d] = \frac{(c-a)(d-b)}{(c-b)(d-a)}$ the crossratio of four pairwise distinct elements in \widehat{K} . If $\alpha, \beta \in \widehat{K}$ are two quadratic irrationals over K such that $\alpha \notin \{\beta, \beta^\sigma\}$,¹⁴ let

$$h_\beta(\alpha) = \max\{ |[\alpha, \beta, \beta^\sigma, \alpha^\sigma] |, |[\alpha^\sigma, \beta, \beta^\sigma, \alpha] | \} ,$$

which is also an appropriate complexity when α varies in a given orbit of quadratic irrationals by homographies under $\mathrm{PGL}_2(R)$. See Section 18.1 and for instance [PaP14b, §4] for motivations and results in the Archimedean case.

Theorem 1.16. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$. Let $\alpha_0, \beta \in \widehat{K}$ be two quadratic irrationals over K . For the weak-star convergence of measures on $\widehat{K} - \{\beta, \beta^\sigma\}$, we have, with g_0 and m_0 as in the statement of Theorem 1.15,*

$$\lim_{s \rightarrow +\infty} \frac{(\ln q) (q+1) m_0 [\mathrm{GL}_2(R) : G]}{2 q^2 (q-1)^3 |\beta - \beta^\sigma| |\ln |\mathrm{tr} g_0||} s^{-1} \sum_{\alpha \in G \cdot \alpha_0, h_\beta(\alpha) \leq s} \Delta_\alpha = \frac{d \mathrm{Haar}_{\widehat{K}}(z)}{|z - \beta| |z - \beta^\sigma|} .$$

The fact that the measure towards which we have an equidistribution is only absolutely continuous with respect to the Haar measure is explained by the invariance of $\alpha \mapsto h_\beta(\alpha)$ under the (infinite) stabiliser of β in $\mathrm{PGL}_2(R)$. See Theorem 18.4 in Section 18.1 for a more general version.

The last statement of this introduction is an equidistribution result for the integral representations of quadratic norm forms

$$(x, y) \mapsto \mathfrak{n}(x - y\alpha)$$

on $K \times K$, where $\alpha \in \widehat{K}$ is a quadratic irrational over K . See Theorem 19.1 in Section 19 for a more general version, and for instance [PaP14b, §5.3] for motivations and results in the Archimedean case.

There is an extensive bibliography on the integral representation of norm forms and more generally decomposable forms over function fields, we only refer to [Sch1, Maso1, Gyo, Maso2]. These references mostly consider higher degrees, with an algebraically closed ground field of characteristic 0, instead of \mathbb{F}_q .

Theorem 1.17. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$ and let $\beta \in K_v$ be a quadratic irrational over K . For the weak-star convergence of measures on $\widehat{K} - \{\beta, \beta^\sigma\}$, we have*

$$\lim_{s \rightarrow +\infty} \frac{(q+1) [\mathrm{GL}_2(R_v) : G]}{q^2 (q-1)^3 [\mathrm{GL}_2(R_v)_{(1,0)} : G_{(1,0)}]} s^{-1} \sum_{\substack{(x,y) \in G(1,0), \\ |x^2 - xy \mathrm{tr}(\beta) + y^2 \mathfrak{n}(\beta)| \leq s}} \Delta_{\frac{x}{y}} = \frac{d \mathrm{Haar}_{\widehat{K}}(z)}{|z - \beta| |z - \beta^\sigma|} .$$

¹⁴See Section 18.1 when this condition is not satisfied.

Furthermore, we have error estimates in the arithmetic applications: There exists $\kappa > 0$ such that for every locally constant function with compact support $\psi : \widehat{K} \rightarrow \mathbb{C}$ in Theorems 1.13, 1.14 and 1.15, or $\psi : \widehat{K} - \{\beta, \beta^\sigma\} \rightarrow \mathbb{C}$ in Theorems 1.16 and 1.17, there are error terms in the above equidistribution claims evaluated on ψ , of the form $O(s^{-\kappa})$ where $s = q^t$ in Theorem 1.13, with for each result an explicit control on the test function ψ involving only some norm of ψ , see in particular Section 15.4.

The link between the geometry described in the first part of this introduction and the above arithmetic statements is provided by the Bruhat-Tits tree of $(\mathrm{PGL}_2, \widehat{K})$, see [Ser3] and Section 15.1 for background. We refer to Part III for more general arithmetic applications.

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Part I

Geometry and dynamics in negative curvature

Chapter 2

Negatively curved geometry

2.1 General notation

Here is some general notation that will be used in this text. We recommend the use of the List of symbols (mostly in alphabetical order by the first letter) and of the Index for easy references to the definitions in the text.

Let A be a subset of a set E . We denote by $\mathbb{1}_A : E \rightarrow \{0, 1\}$ the *characteristic* (or indicator) *function* of A : $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise. We denote by ${}^cA = E - A$ the complementary subset of A in E .

We denote by \ln the natural logarithm (with $\ln(e) = 1$).

We denote by $\|\mu\|$ the total mass of a finite positive measure μ .

If (X, \mathcal{A}) and (Y, \mathcal{B}) are measurable spaces, $f : X \rightarrow Y$ a measurable map, and μ a measure on X , we denote by $f_*\mu$ the image measure of μ by f , with $f_*\mu(B) = \mu(f^{-1}(B))$ for every $B \in \mathcal{B}$.

If (X, d) is a metric space, then $B(x, r)$ is the closed ball with centre $x \in X$ and radius $r > 0$.

For every subset A of a metric space and for every $\epsilon > 0$, we denote by $\mathcal{N}_\epsilon A$ the closed ϵ -neighbourhood of A , and by convention $\mathcal{N}_0 A = \bar{A}$.

If X is a uniquely geodesic space and $x, y \in X$, then $[x, y]$ is the unique geodesic segment from x to y .

Given a topological space Z , we denote by $\mathcal{C}_c(Z)$ the vector space of continuous maps from Z to \mathbb{R} with compact support.

The *negative part* of a real-valued map f is $f^- = \max\{0, -f\}$.

We denote by Δ_x the unit Dirac mass at a point x in any measurable space.

Finally, the symbol \square right at the end of a statement indicates that this statement won't be given a proof, either since a reference is given or since it is an immediate consequence of previous statements.

2.2 Background on CAT(-1) spaces

Let X be a geodesically complete proper CAT(-1) space, let $x_0 \in X$ be an arbitrary basepoint, and let Γ be a nonelementary discrete group of isometries of X .

We refer for example to [BridH] for the relevant definitions and complements on these notions. Recall that a geodesic metric space X is *geodesically complete* (or has *extendible*

geodesics) if any isometric map from an interval in \mathbb{R} to X extends to at least one isometric map from \mathbb{R} to X . We will put a special emphasis on the case when X is a (proper, geodesically complete) \mathbb{R} -tree, that is, a uniquely arcwise connected geodesic metric space. In the Introduction, we have denoted by Y the geodesically complete proper locally CAT(-1) good orbispace $\Gamma \backslash X$, see for instance [GH, Ch. 11] for the terminology.

We denote by $\partial_\infty X$ the space at infinity of X , which consists of the asymptotic classes of geodesic rays in X . It coincides with the space of (Freudenthal's) ends of X when X is an \mathbb{R} -tree. We denote by $\Lambda\Gamma$ the limit set of Γ and by $\mathcal{C}\Lambda\Gamma$ the convex hull in X of $\Lambda\Gamma$.

When X is an \mathbb{R} -tree, then a subset D of X is convex if and only if it is connected, and we will call it a *subtree*. In particular, if X is an \mathbb{R} -tree, then $\mathcal{C}\Lambda\Gamma$ is equal to the union of the geodesic lines between pairs of distinct points in $\Lambda\Gamma$, since this union is connected and contained in $\mathcal{C}\Lambda\Gamma$.

A point $\xi \in \partial_\infty X$ is called a *conical limit point* if there exists a sequence of orbit points of x_0 under Γ converging to ξ while staying at bounded distance from a geodesic ray ending at ξ . The set of conical limit points is the *conical limit set* $\Lambda_c\Gamma$ of Γ .

A point $p \in \Lambda\Gamma$ is a *bounded parabolic limit point* of Γ if its stabiliser Γ_p in Γ acts properly discontinuously with compact quotient on $\Lambda\Gamma - \{p\}$. The discrete nonelementary group of isometries Γ of X is said to be *geometrically finite* if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic limit point of Γ (see for instance [Bowd], as well as [Pau2] when X is an \mathbb{R} -tree, and [DaSU] for a very interesting study of equivalent conditions in an even greater generality).

For all $x \in X \cup \partial_\infty X$ and $A \subset X$, the *shadow of A seen from x* is the subset $\mathcal{O}_x A$ of $\partial_\infty X$ consisting of the positive endpoints of the geodesic rays starting at x and meeting A if $x \in X$, and of the geodesic lines starting at x and meeting A if $x \in \partial_\infty X$.

We denote by $\text{Isom}(X)$ the isometry group of X . The *translation length* of an isometry $\gamma \in \text{Isom}(X)$ is

$$\lambda(\gamma) = \inf_{x \in X} d(x, \gamma x).$$

Recall that $\gamma \in \text{Isom}(X)$ is *loxodromic* if $\lambda(\gamma) > 0$, and that then

$$\text{Ax}_\gamma = \{x \in X : d(x, \gamma x) = \lambda(\gamma)\}$$

is (the image of) a geodesic line in X , called the *translation axis* of γ .

We will need the following well-known lemma later on. An element of Γ is *primitive* if there is no $\gamma_0 \in \Gamma$ and $k \in \mathbb{N} - \{0, 1\}$ such that $\gamma = \gamma_0^k$.

Lemma 2.1. (1) *For every loxodromic element $\gamma \in \Gamma$, there exists a primitive loxodromic element $\gamma_0 \in \Gamma$ and $k \in \mathbb{N} - \{0\}$ such that $\gamma = \gamma_0^k$.*

(2) *If $\alpha, \gamma \in \Gamma$ are loxodromic with $\text{Ax}_\gamma = \text{Ax}_\alpha$, then there exists $p, q \in \mathbb{Z}$ and a primitive loxodromic element $\gamma_0 \in \Gamma$ such that $\alpha = \gamma_0^p$ and $\gamma = \gamma_0^q$.*

(3) *For all $\alpha, \gamma \in \Gamma$ such that α is loxodromic, if γ centralises¹ α , then either γ pointwise fixes the translation axis of α , or γ is loxodromic, with $\text{Ax}_\gamma = \text{Ax}_\alpha$.*

(4) *For every loxodromic element $\gamma \in \Gamma$, for all $A > 0$ and $r > 0$, there exists $L > 0$ such that for every loxodromic element $\alpha \in \Gamma$, if $\lambda(\alpha) = \lambda(\gamma) \leq A$ and if Ax_γ and Ax_α have segments of length at least L at Hausdorff distance at most r , then $\text{Ax}_\gamma = \text{Ax}_\alpha$.*

¹that is, commutes with α

Proof. We only give a proof of Claim (4) and refer to for instance [BridH] for proofs of the first three classical assertions.

Since the action of Γ on X is properly discontinuous, and by the compactness of $\gamma^{\mathbb{Z}} \setminus \text{Ax}_\gamma$, there exists $N \geq 1$ such that for all $x \in \text{Ax}_\gamma$, the cardinality of $\{\beta \in \Gamma : d(x, \beta x) \leq 2r\}$ is at most N . Let $L = AN$. For every loxodromic element $\alpha \in \Gamma$ with $\lambda(\alpha) = \lambda(\gamma) \leq A$, assume that $[x, y]$ and $[x', y']$ are segments in Ax_γ and Ax_α respectively, with length exactly L such that $d(x, x'), d(y, y') \leq r$. We may assume, up to replacing them by their inverses, that γ translates from x towards y and α translates from x' towards y' . In particular for $k = 0, \dots, N$, we have $d(\alpha^{-k}\gamma^k x, x) \leq d(\gamma^k x, \alpha^k x') + d(x', x) \leq 2r$ since $\gamma^k x$ and $\alpha^k x'$ are respectively the points at distance $k\lambda(\gamma) \leq kA \leq L$ from x and x' on the segments $[x, y]$ and $[x', y']$. Hence by the definition of N , there exists $k \neq k'$ such that $\alpha^{-k}\gamma^k = \alpha^{-k'}\gamma^{k'}$. Therefore $\gamma^{k-k'} = \alpha^{k-k'}$, which implies by Assertion (3) that $\text{Ax}_\gamma = \text{Ax}_\alpha$. \square

For every $x \in X$, recall that the Gromov-Bourdon *visual distance* d_x on $\partial_\infty X$ seen from x (see [Bou]) is defined by

$$d_x(\xi, \eta) = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}(d(\xi_t, \eta_t) - d(x, \xi_t) - d(x, \eta_t))}, \quad (2.1)$$

where $\xi, \eta \in \partial_\infty X$ and $t \mapsto \xi_t, \eta_t$ are any geodesic rays ending at ξ, η respectively. By the triangle inequality, for all $x, y \in \tilde{M}$ and $\xi, \eta \in \partial_\infty \tilde{M}$, we have

$$e^{-d(x, y)} \leq \frac{d_x(\xi, \eta)}{d_y(\xi, \eta)} \leq e^{d(x, y)}. \quad (2.2)$$

In particular, the identity map from $(\partial_\infty X, d_x)$ to $(\partial_\infty X, d_y)$ is a bilipschitz homeomorphism. Under our assumptions, $(\partial_\infty X, d_{x_0})$ is hence a compact metric space, on which $\text{Isom}(X)$ acts by bilipschitz homeomorphisms. The following well-known result compares shadows of balls to balls for the visual distance.

Lemma 2.2. *For every geodesic ray ρ in X , starting from $x \in X$ and ending at $\xi \in \partial_\infty X$, for all $R \geq 0$ and $t \in]R, +\infty[$, we have*

$$B_{d_x}(\xi, R e^{-t}) \subset \mathcal{O}_x B(\rho(t), R) \subset B_{d_x}(\xi, e^R e^{-t}).$$

Proof. The lower bound is for instance the lower bound in [HeP2, Lem. 3.1] (which only uses the CAT(-1) property). In order to prove the upper bound, let $\xi' \in \mathcal{O}_x B(\rho(t), R)$ and let ρ' be the geodesic ray from x to ξ' . The closest point p to $\rho(t)$ on the image of ρ' satisfies $d(p, \rho(t)) \leq R$, hence $d(x, p) \geq t - R$. Then

$$\begin{aligned} d_x(\xi', \xi) &\leq \limsup_{t' \rightarrow +\infty} e^{\frac{1}{2}(d(\rho(t'), \rho(t)) + d(\rho(t), p) + d(p, \rho'(t))) - t'} \\ &\leq \lim_{t' \rightarrow +\infty} e^{\frac{1}{2}((t' - t) + R + (t' - t + R)) - t'} = e^R e^{-t}. \end{aligned}$$

Therefore $\xi' \in B_{d_x}(\xi, e^R e^{-t})$. \square

The *Busemann cocycle* of X is the map $\beta : \partial_\infty X \times X \times X \rightarrow \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto \beta_\xi(x, y) = \lim_{t \rightarrow +\infty} d(\xi_t, x) - d(\xi_t, y),$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ . The above limit exists and is independent of x_0 , and we have

$$|\beta_\xi(x, y)| \leq d(x, y). \quad (2.3)$$

The *horosphere* with centre $\xi \in \partial_\infty X$ through $x \in X$ is $\{y \in X : \beta_\xi(x, y) = 0\}$, and $\{y \in X : \beta_\xi(x, y) \leq 0\}$ is the (closed) *horoball* centred at ξ bounded by this horosphere. Horoballs are nonempty proper closed (strictly) convex subsets of X . Given a horoball \mathcal{H} and $t \geq 0$, we denote by $\mathcal{H}[t] = \{x \in \mathcal{H} : d(x, \partial\mathcal{H}) \geq t\}$ the horoball contained in \mathcal{H} (hence centred at the same point at infinity as \mathcal{H}) whose boundary is at distance t from the boundary of \mathcal{H} .

2.3 Generalised geodesic lines

Let $\check{\mathcal{G}}X$ be the space of 1-Lipschitz maps $w : \mathbb{R} \rightarrow X$ which are isometric on a closed interval and locally constant outside it.² This space has been introduced by Bartels and Lück in [BartL], to which we refer for the following basic properties. The elements of $\check{\mathcal{G}}X$ are called the *generalised geodesic lines* of X . We endow $\check{\mathcal{G}}X$ with the distance $d = d_{\check{\mathcal{G}}X}$ defined by

$$\forall w, w' \in \check{\mathcal{G}}X, \quad d(w, w') = \int_{-\infty}^{+\infty} d(w(t), w'(t)) e^{-2|t|} dt. \quad (2.4)$$

The group $\text{Isom}(X)$ acts isometrically on $\check{\mathcal{G}}X$ by postcomposition. The distance d induces the topology of uniform convergence on compact subsets on $\check{\mathcal{G}}X$, and $\check{\mathcal{G}}X$ is a proper metric space.

The *geodesic flow* $(\mathbf{g}^t)_{t \in \mathbb{R}}$ on $\check{\mathcal{G}}X$ is the one-parameter group of homeomorphisms of the space $\check{\mathcal{G}}X$ defined by $\mathbf{g}^t w : s \mapsto w(s + t)$ for all $w \in \check{\mathcal{G}}X$ and $t \in \mathbb{R}$. It commutes with the action of $\text{Isom}(X)$. If w is isometric exactly on the interval I , then $\mathbf{g}^{-t} w$ is isometric exactly on the interval $t + I$. Note that for all $w \in \check{\mathcal{G}}X$ and $s \in \mathbb{R}$, we have

$$d(w, \mathbf{g}^s w) \leq |s|, \quad (2.5)$$

with equality if $w \in \mathcal{G}X$.

The *footpoint projection* is the $\text{Isom}(X)$ -equivariant $\frac{1}{2}$ -Hölder-continuous map $\pi : \check{\mathcal{G}}X \rightarrow X$ defined by $\pi(w) = w(0)$ for all $w \in \check{\mathcal{G}}X$. The *antipodal map* of $\check{\mathcal{G}}X$ is the $\text{Isom}(X)$ -equivariant isometric map $\iota : \check{\mathcal{G}}X \rightarrow \check{\mathcal{G}}X$ defined by $\iota w : s \mapsto w(-s)$ for all $w \in \check{\mathcal{G}}X$, which satisfies $\iota \circ \mathbf{g}^t = \mathbf{g}^{-t} \circ \iota$ for every $t \in \mathbb{R}$ and $\pi \circ \iota = \pi$.

The *positive* and *negative* endpoint maps are the continuous maps from $\check{\mathcal{G}}X$ to $X \cup \partial_\infty X$ defined by

$$w \mapsto w_\pm = \lim_{t \rightarrow \pm\infty} w(t).$$

The space $\mathcal{G}X$ of *geodesic lines* in X is the $\text{Isom}(X)$ -invariant closed subspace of $\check{\mathcal{G}}X$ consisting of the elements $\ell \in \check{\mathcal{G}}X$ with $\ell_\pm \in \partial_\infty X$. Note that the distances on $\mathcal{G}X$ considered in [BartL] and [PauPS] are topologically equivalent, although slightly different from the restriction to $\mathcal{G}X$ of the distance defined in Equation (2.4). The factor $e^{-2|t|}$ in this equation,

²that is, constant on each complementary component

sufficient in order to deal with Hölder-continuity issues, is replaced by $e^{-t^2}/\sqrt{\pi}$ in [PauPS] and by $e^{-|t|}/2$ in [BartL] (so that the above $\frac{1}{2}$ -Hölder-continuity claim of π does follow from the one in [BartL]).

We will also consider the $\text{Isom}(X)$ -invariant closed subspaces

$$\mathcal{G}_{\pm}X = \{w \in \check{\mathcal{G}}X : w_{\pm} \in \partial_{\infty}X\},$$

and their $\text{Isom}(X)$ -invariant closed subspaces $\mathcal{G}_{\pm,0}X$ consisting of the elements $\rho \in \mathcal{G}_{\pm}X$ which are isometric exactly on $\pm[0, +\infty[$.

The subspaces $\mathcal{G}X$ and $\mathcal{G}_{\pm}X$ satisfy $\mathcal{G}_-X \cap \mathcal{G}_+X = \mathcal{G}X$ and they are invariant under the geodesic flow. The antipodal map ι preserves $\mathcal{G}X$, and maps $\mathcal{G}_{\pm}X$ to $\mathcal{G}_{\mp}X$ as well as $\mathcal{G}_{\pm,0}X$ to $\mathcal{G}_{\mp,0}X$. We denote again by $\iota : \Gamma \backslash \check{\mathcal{G}}X \rightarrow \Gamma \backslash \check{\mathcal{G}}X$ and by $\mathbf{g}^t : \Gamma \backslash \check{\mathcal{G}}X \rightarrow \Gamma \backslash \check{\mathcal{G}}X$ the quotient maps of ι and \mathbf{g}^t , for every $t \in \mathbb{R}$.

Let $w \in \check{\mathcal{G}}X$ be isometric exactly on an interval I of \mathbb{R} . If I is compact then w is a (*generalised*) *geodesic segment*, and if $I =]-\infty, a]$ or $I = [a, +\infty[$ for some $a \in \mathbb{R}$, then w is a (*generalised*) (*negative or positive*) *geodesic ray* in X . Any geodesic line $\hat{w} \in \mathcal{G}X$ such that $\hat{w}|_I = w|_I$ is an *extension* of w . Note that \hat{w} is an extension of w if and only if $\gamma\hat{w}$ is an extension of γw for any $\gamma \in \text{Isom}(X)$, if and only if $\iota\hat{w}$ is an extension of ιw , and if and only if $\mathbf{g}^s\hat{w}$ is an extension of $\mathbf{g}^s w$ for any $s \in \mathbb{R}$. For any subset Ω of $\check{\mathcal{G}}X$ and any subset A of \mathbb{R} , let

$$\Omega|_A = \{w|_A : w \in \Omega\}.$$

Remark 2.3. Let $(\ell_i)_{i \in \mathbb{N}}$ be a sequence of generalised geodesic lines such that $[t_i^-, t_i^+]$ is the maximal segment on which ℓ_i is isometric. Let $(s_i)_{i \in \mathbb{N}}$ be a sequence in \mathbb{R} such that $t_i^{\pm} - s_i \rightarrow \pm\infty$ as $i \rightarrow +\infty$ and $\ell_i(s_i)$ stays in a compact subset of X , then $d(\ell_i, \mathcal{G}X) \rightarrow 0$ as $i \rightarrow +\infty$. Furthermore if $(s_i)_{i \in \mathbb{N}}$ is bounded, then up to extracting a subsequence, $(\ell_i)_{i \in \mathbb{N}}$ converges to an element in $\mathcal{G}X$.

This conceptually important observation explains how it is conceivable that long common perpendicular segments may equidistribute towards measures supported on geodesic lines. See Chapter 11 for further developments of these ideas.

2.4 The unit tangent bundle

In this work, we define the *unit tangent bundle* T^1X of X as the space of germs at 0 of the geodesic lines in X , that is the quotient space

$$T^1X = \mathcal{G}X / \sim$$

where $\ell \sim \ell'$ if and only if there exists $\epsilon > 0$ such that $\ell|_{[-\epsilon, \epsilon]} = \ell'|_{[-\epsilon, \epsilon]}$. The canonical projection from $\mathcal{G}X$ to T^1X will be denoted by $\ell \mapsto v_{\ell}$. When X is a Riemannian manifold, the spaces $\mathcal{G}X$ and T^1X canonically identify with the usual unit tangent bundle of X , but in general, the map $\ell \mapsto v_{\ell}$ has infinite fibers.

We endow T^1X with the quotient distance $d = d_{T^1X}$ of the distance of $\mathcal{G}X$, defined by:

$$\forall v, v' \in T^1X, \quad d_{T^1X}(v, v') = \inf_{\ell, \ell' \in \mathcal{G}X : v=v_{\ell}, v'=v_{\ell'}} d(\ell, \ell'). \quad (2.6)$$

It is easy to check that this distance is indeed Hausdorff, hence that T^1X is locally compact, and that it induces on T^1X the quotient topology of the compact-open topology of $\mathcal{G}X$. The map $\ell \mapsto v_\ell$ is 1-Lipschitz.

The action of $\text{Isom}(X)$ on $\mathcal{G}X$ induces an isometric action of $\text{Isom}(X)$ on T^1X . The antipodal map and the footpoint projection restricted to $\mathcal{G}X$ respectively induce an $\text{Isom}(X)$ -equivariant isometric map $\iota : T^1X \rightarrow T^1X$ and an $\text{Isom}(X)$ -equivariant $\frac{1}{2}$ -Hölder-continuous map $\pi : T^1X \rightarrow X$ called the *antipodal map* and *footpoint projection* of T^1X . The canonical projection from $\mathcal{G}X$ to T^1X is $\text{Isom}(X)$ -equivariant and commutes with the antipodal map: For all $\gamma \in \text{Isom}(X)$ and $\ell \in \mathcal{G}X$, we have $\gamma v_\ell = v_{\gamma\ell}$, $\iota v_\ell = v_{\iota\ell}$ and $\pi(v_\ell) = \pi(\ell)$. We denote again by $\iota : \Gamma \backslash T^1X \rightarrow \Gamma \backslash T^1X$ the quotient map of ι .

Let $\partial_\infty^2 X$ be the subset of $\partial_\infty X \times \partial_\infty X$ which consists of pairs of distinct points at infinity of X . *Hopf's parametrisation* of $\mathcal{G}X$ is the homeomorphism which identifies $\mathcal{G}X$ with $\partial_\infty^2 X \times \mathbb{R}$, by the map $\ell \mapsto (\ell_-, \ell_+, t)$, where t is the signed distance from the closest point to the basepoint x_0 on the geodesic line ℓ to $\ell(0)$.³ We have $\mathbf{g}^s(\ell_-, \ell_+, t) = (\ell_-, \ell_+, t + s)$ for all $s \in \mathbb{R}$, and for all $\gamma \in \Gamma$, we have $\gamma(\ell_-, \ell_+, t) = (\gamma\ell_-, \gamma\ell_+, t + t_{\gamma, \ell_-, \ell_+})$ where $t_{\gamma, \ell_-, \ell_+} \in \mathbb{R}$ depends only on γ , ℓ_- and ℓ_+ . In Hopf's parametrisation, the restriction of the antipodal map to $\mathcal{G}X$ is the map $(\ell_-, \ell_+, t) \mapsto (\ell_+, \ell_-, -t)$.

The *strong stable leaf* of $w \in \mathcal{G}_+ X$ is

$$W^+(w) = \{ \ell \in \mathcal{G}X : \lim_{t \rightarrow +\infty} d(\ell(t), w(t)) = 0 \},$$

and the *strong unstable leaf* of $w \in \mathcal{G}_- X$ is

$$W^-(w) = \iota W^+(\iota w) = \{ \ell \in \mathcal{G}X : \lim_{t \rightarrow -\infty} d(\ell(t), w(t)) = 0 \}.$$

For every $w \in \mathcal{G}_\pm X$, let $d_{W^\pm(w)}$ be *Hamenstädt's distance* on $W^\pm(w)$ defined as follows:⁴ for all $\ell, \ell' \in W^\pm(w)$, let

$$d_{W^\pm(w)}(\ell, \ell') = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(\ell(\mp t), \ell'(\mp t)) - t}.$$

The above limits exist, and Hamenstädt's distances are distances inducing the original topology on $W^\pm(w)$. For all $\ell, \ell' \in W^\pm(w)$ and $\gamma \in \text{Isom}(X)$, we have

$$d_{W^\pm(\gamma w)}(\gamma\ell, \gamma\ell') = d_{W^\pm(w)}(\ell, \ell') = d_{W^\mp(\iota w)}(\iota\ell, \iota\ell').$$

Furthermore, for every $s \in \mathbb{R}$, we have for all $\ell, \ell' \in W^\pm(w)$

$$d_{W^\pm(\mathbf{g}^s w)}(\mathbf{g}^s \ell, \mathbf{g}^s \ell') = e^{\mp s} d_{W^\pm(w)}(\ell, \ell'). \quad (2.7)$$

If X is an \mathbb{R} -tree, for all $w \in \mathcal{G}_+ X$ and $\ell, \ell' \in W^+(w)$, if $[s, +\infty[$ is the maximal interval on which ℓ and ℓ' agree, then $d_{W^+(w)}(\ell, \ell') = e^s$.

The following lemma compares the distance in $\mathcal{G}X$ with Hamenstädt's distance for two leaves in the same strong (un)stable leaf.

Lemma 2.4. *There exists a universal constant $c > 0$ such that for all $w \in \mathcal{G}_\pm X$ and $\ell, \ell' \in W^\pm(w)$, we have*

$$d(\ell, \ell') \leq c d_{W^\pm(w)}(\ell, \ell').$$

³More precisely, $\ell(t)$ is the closest point to x_0 on ℓ .

⁴See [HeP1, Appendix] and compare with [Ham1].

Proof. We refer to [PaP14a, Lem. 3] for a proof of this result. Note that the distance on $\mathcal{G}X$ considered in loc. cit. is slightly different from the one in this text, but the proof adapts easily. \square

Let \mathcal{H} be a horoball in X , centred at $\xi \in \partial_\infty X$. The strong stable leaves $W^+(w)$ are equal for all geodesic rays w starting at time $t = 0$ from a point of ∂H and converging to ξ . Using the homeomorphism $\ell \mapsto \ell_-$ from $W^+(w)$ to $\partial_\infty X - \{\xi\}$, Hamenstädt's distance on $W^+(w)$ defines a distance $d_{\mathcal{H}}$ on $\partial_\infty X - \{\xi\}$ that we also call *Hamenstädt's distance*. For all $\ell, \ell' \in W^+(w)$, we have

$$d_{\mathcal{H}}(\ell_-, \ell_+) = d_{W^+(w)}(\ell, \ell'),$$

and for all $\eta, \eta' \in \partial_\infty X - \{\xi\}$, we have

$$d_{\mathcal{H}}(\eta, \eta') = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(\ell_\eta(-t), \ell_{\eta'}(-t)) - t}, \quad (2.8)$$

where $\ell_\eta, \ell_{\eta'}$ are the geodesic lines starting from η, η' respectively, ending at ξ , and passing through the boundary of \mathcal{H} at time $t = 0$. Note that for every $t \geq 0$, if $\mathcal{H}[t]$ is the horoball contained in \mathcal{H} whose boundary is at distance t from the boundary of \mathcal{H} , then we have

$$d_{\mathcal{H}[t]} = e^{-t} d_{\mathcal{H}}. \quad (2.9)$$

Let $w \in \mathcal{G}_\pm X$ and $\eta' > 0$. We define $B^\pm(w, \eta')$ as the set of $\ell \in W^\pm(w)$ such that there exists an extension $\hat{w} \in \mathcal{G}X$ of w with $d_{W^\pm(w)}(\ell, \hat{w}) < \eta'$. In particular, $B^\pm(w, \eta')$ contains all the extensions of w , and is the union of the open balls centred at the extensions of w , of radius η' , for Hamenstädt's distance on $W^\pm(w)$.

The union over $t \in \mathbb{R}$ of the images under g^t of the strong stable leaf of $w \in \mathcal{G}_+ X$ is the *stable leaf*

$$W^{0+}(w) = \bigcup_{t \in \mathbb{R}} g^t W^+(w)$$

of w , which consists of the elements $\ell \in \mathcal{G}X$ with $\ell_+ = w_+$. Similarly, the *unstable leaf* of $w \in \mathcal{G}_- X$

$$W^{0-}(w) = \bigcup_{t \in \mathbb{R}} g^t W^-(w),$$

consists of the elements $\ell \in \mathcal{G}X$ with $\ell_- = w_-$. Note that the (strong) (un)stable leaves are subsets of the space of geodesic lines $\mathcal{G}X$.

The *unstable horosphere* $H_-(w)$ of $w \in \mathcal{G}_- X$ is the horosphere in X centred at w_- and passing through $\hat{w}(0)$ for any extension $\hat{w} \in \mathcal{G}X$ of w . The *stable horosphere* $H_+(w)$ of $w \in \mathcal{G}_+ X$ is the horosphere in X centred at w_+ and passing through $\hat{w}(0)$ for any extension $\hat{w} \in \mathcal{G}X$ of w . These horospheres $H_\pm(w)$ do not depend on the chosen extensions \hat{w} of $w \in \mathcal{G}_\pm X$. The *unstable horoball* $HB_-(w)$ of $w \in \mathcal{G}_- X$ and *stable horoball* $HB_+(w)$ of $w \in \mathcal{G}_+ X$ are the horoballs bounded by these horospheres. Note that $\pi(W^\pm(w)) = H_\pm(w)$ for every $w \in \mathcal{G}_\pm X$, and that $w(0)$ belongs to $H_\pm(w)$ if and only if w is isometric at least on $\pm[0, +\infty[$.

2.5 Normal bundles and dynamical neighbourhoods

In this Section, adapting [PaP16c, §2.2] to the present context, we define spaces of geodesic rays that generalise the unit normal bundles of submanifolds of negatively curved Riemannian manifolds. When X is a manifold, these normal bundles are submanifolds of the unit tangent bundle of X , which identifies with $\mathcal{G}X$. In general and in particular in trees, it is essential to use geodesic rays to define normal bundles, and not geodesic lines.

Let D be a nonempty proper (that is, different from X) closed convex subset in X . We denote by ∂D its boundary in X and by $\partial_\infty D$ its set of points at infinity. Let

$$P_D : X \cup (\partial_\infty X - \partial_\infty D) \rightarrow D$$

be the (continuous) *closest point map to D* , defined on $\xi \in \partial_\infty X - \partial_\infty D$ by setting $P_D(\xi)$ to be the unique point in D that minimises the function $y \mapsto \beta_\xi(y, x_0)$ from D to \mathbb{R} . The *outer unit normal bundle* $\partial_+^1 D$ of (the boundary of) D is

$$\partial_+^1 D = \{\rho \in \mathcal{G}_{+,0}X : P_D(\rho_+) = \rho(0)\}.$$

The *inner unit normal bundle* $\partial_-^1 D$ of (the boundary of) D is

$$\partial_-^1 D = \iota \partial_+^1 D = \{\rho \in \mathcal{G}_{-,0}X : P_D(\rho_-) = \rho(0)\}.$$

Note that $\partial_+^1 D$ and $\partial_-^1 D$ are spaces of geodesic rays. If X is a smooth manifold, then these spaces have a natural identification with subsets of $\mathcal{G}X$ because every geodesic ray is the restriction of a unique geodesic line. But this does not hold in general.

Remark 2.5. As X is assumed to be proper with extendible geodesics, we have $\pi(\partial_\pm^1 D) = \partial D$. To see this, let $x \in \partial D$ and let $(x_k)_{k \in \mathbb{N}}$ be a sequence of points in the complement of D converging to x . For all $k \in \mathbb{N}$, let $\rho_k \in \partial_+^1 D$ be a geodesic ray with $\rho_k(0) = P_D(x_k)$ and such that the image of ρ_k contains x_k . As the closest point map does not increase distances, the sequence $(P_D(x_k))_{k \in \mathbb{N}}$ converges to x . Since X is proper, the space $\partial_\infty X$ is compact and the sequence $((\rho_k)_+)_{k \in \mathbb{N}}$ has a subsequence that converges to a point $\xi \in \partial_\infty X$. The claim follows from the continuity of the closest point map.

The failure of the equality when X is not proper is easy to see for example, when X is the \mathbb{R} -tree constructed by starting with the Euclidean line $D = \mathbb{R}$ and attaching a copy of the halfline $[0, +\infty[$ to each $x \in D$ such that $x > 0$. Then $0 \in \partial D - \pi(\partial_\pm^1 D)$.

The restriction of the positive (respectively negative) endpoint map to $\partial_+^1 D$ (respectively $\partial_-^1 D$) is a homeomorphism to its image $\partial_\infty X - \partial_\infty D$. We denote its inverse map by P_D^\pm . Note that $P_D = \pi \circ P_D^\pm$. For every isometry γ of X , we have $\partial_\pm^1(\gamma D) = \gamma \partial_\pm^1 D$ and $P_{\gamma D}^\pm \circ \gamma = \gamma \circ P_D^\pm$. In particular, $\partial_\pm^1 D$ is invariant under the isometries of X that preserve D .

For every $w \in \mathcal{G}_\pm X$, we have a canonical homeomorphism $N_w^\pm : W^\pm(w) \rightarrow \partial_\mp^1 HB_\pm(w)$, that associates to each geodesic line $\ell \in W^\pm(w)$ the unique geodesic ray $\rho \in \partial_\mp^1 HB_\pm(w)$ such that $\ell_\mp = \rho_\mp$, or, equivalently, such that $\ell(t) = \rho(t)$ for every $t \in \mathbb{R}$ with $\mp t > 0$. It is easy to check that $N_{\gamma w}^\pm \circ \gamma = \gamma \circ N_w^\pm$ for every $\gamma \in \text{Isom}(X)$.

We define

$$\mathcal{U}_D^\pm = \{\ell \in \mathcal{G}X : \ell_\pm \notin \partial_\infty D\}. \quad (2.10)$$

Note that \mathcal{U}_D^\pm is an open subset of $\mathcal{G}X$, invariant under the geodesic flow. We have $\mathcal{U}_{\gamma D}^\pm = \gamma \mathcal{U}_D^\pm$ for every isometry γ of X and, in particular, \mathcal{U}_D^\pm is invariant under the isometries of X

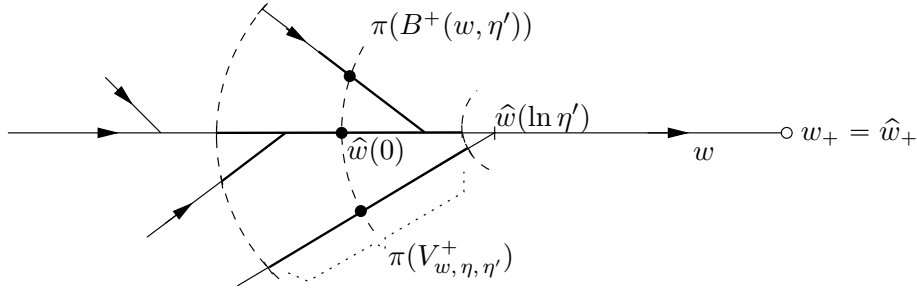
preserving D . Define $f_D^\pm : \mathcal{U}_D^\pm \rightarrow \partial_\pm^1 D$ as the composition of the continuous endpoint map $\ell \mapsto \ell_\pm$ from \mathcal{U}_D^\pm onto $\partial_\infty X - \partial_\infty D$ and the homeomorphism P_D^\pm from $\partial_\infty X - \partial_\infty D$ to $\partial_\pm^1 D$. The continuous map f_D^\pm takes $\ell \in \mathcal{U}_D^\pm$ to the unique element $\rho \in \partial_\pm^1 D$ such that $\rho_\pm = \ell_\pm$. The fiber of $\rho \in \partial_+^1 D$ for f_D^+ is exactly the stable leaf $W^{0+}(\rho)$, and the fiber of $\rho \in \partial_-^1 D$ for f_D^- is the unstable leaf $W^{0-}(\rho)$. For all $\gamma \in \text{Isom}(X)$ and $t \in \mathbb{R}$, we have

$$f_{\gamma D}^\pm \circ \gamma = \gamma \circ f_D^\pm \quad \text{and} \quad f_D^\pm \circ \mathbf{g}^t = f_D^\pm. \quad (2.11)$$

Let $w \in \mathcal{G}_\pm X$ and $\eta, \eta' > 0$. We define the *dynamical (η, η') -neighbourhood* of w by

$$V_{w, \eta, \eta'}^\pm = \bigcup_{s \in]-\eta, \eta[} \mathbf{g}^s B^\pm(w, \eta'). \quad (2.12)$$

Example 2.6. If X is an \mathbb{R} -tree, $w \in \mathcal{G}_+ X$ and $\eta < \ln \eta'$, then $V_{w, \eta, \eta'}^+$ is as in the following picture.



Clearly, $B^\pm(w, \eta') = \iota B^\mp(\iota w, \eta')$, and hence we have $V_{w, \eta, \eta'}^\pm = \iota V_{\iota w, \eta, \eta'}^\mp$. Furthermore, for every $s \in \mathbb{R}$,

$$\mathbf{g}^s B^\pm(w, \eta') = B^\pm(\mathbf{g}^s w, e^{\mp s} \eta') \quad \text{hence} \quad \mathbf{g}^s V_{w, \eta, \eta'}^\pm = V_{\mathbf{g}^s w, \eta, e^{\mp s} \eta'}^\pm. \quad (2.13)$$

For every $\gamma \in \text{Isom}(X)$, we have $\gamma B^\pm(w, \eta') = B^\pm(\gamma w, \eta')$ and $\gamma V_{w, \eta, \eta'}^\pm = V_{\gamma w, \eta, \eta'}^\pm$. The map from $]-\eta, \eta[\times B^\pm(w, \eta')$ to $V_{w, \eta, \eta'}^\pm$ defined by $(s, \ell') \mapsto \mathbf{g}^s \ell'$ is a homeomorphism.

For all subsets Ω^- of $\mathcal{G}_+ X$ and Ω^+ of $\mathcal{G}_- X$, let

$$\mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp) = \bigcup_{w \in \Omega^\mp} V_{w, \eta, \eta'}^\pm, \quad (2.14)$$

that we call the *dynamical neighbourhoods* of Ω^\mp . Note that they are subsets of $\mathcal{G}X$, not of $\mathcal{G}_\pm X$. The families $(\mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp))_{\eta, \eta' > 0}$ are nondecreasing in η and in η' . For every $\gamma \in \text{Isom}(X)$, we have $\gamma \mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp) = \mathcal{V}_{\eta, \eta'}^\pm(\gamma \Omega^\mp)$ and for every $t \geq 0$, we have

$$\mathbf{g}^{\pm t} \mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp) = \mathcal{V}_{\eta, e^{-t} \eta'}^\pm(\mathbf{g}^{\pm t} \Omega^\mp). \quad (2.15)$$

Note that

$$\bigcup_{\eta, \eta' > 0} \mathcal{V}_{\eta, \eta'}^\pm(\partial_\pm^1 D) = \mathcal{U}_D^\pm,$$

and that $\bigcap_{\eta, \eta' > 0} \mathcal{V}_{\eta, \eta'}^\pm(\partial_\pm^1 D)$ is the set of all extensions in $\mathcal{G}X$ of the elements of $\partial_\pm^1 D$. Assume that Ω^\mp is a subset of $\partial_\pm^1 D$. The restriction of f_D^\pm to $\mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp)$ is a continuous map onto Ω^\mp , with fiber over $w \in \Omega^\mp$ the open subset $V_{w, \eta, \eta'}^\pm$ of $W^{0\pm}(w)$.

We will need the following elementary lemma in Section 10.4.

Lemma 2.7. *There exists a universal constant $c' > 0$ such that for every $w \in \mathcal{G}_+ X$ which is isometric on $[s_w, +\infty[$ and every $\ell \in V_{w, \eta, \eta'}^+$, we have*

$$d(\ell, w) \leq c'(\eta + \eta' + e^{s_w}).$$

Proof. By Equation (2.12) and by the definition of $B^+(w, \eta')$ in Section 2.3, there exist $s \in]-\eta, +\eta[$ and an extension $\hat{w} \in \mathcal{G}X$ of w such that $d_{W^+(w)}(\mathbf{g}^s \ell, \hat{w}) \leq \eta'$. By Equation (2.5), we have $d(\ell, \mathbf{g}^s \ell) \leq |s| \leq \eta$. By Lemma 2.4, we have $d(\mathbf{g}^s \ell, \hat{w}) \leq c d_{W^+(w)}(\mathbf{g}^s \ell, \hat{w}) \leq c \eta'$. By the definition of the distance on $\check{\mathcal{G}}X$ (see Equation (2.4)), we have

$$d(\hat{w}, w) \leq \int_{-\infty}^{s_w} |s_w - t| e^{-2|t|} dt = O(e^{s_w}).$$

Therefore the result follows from the triangle inequality

$$d(\ell, w) \leq d(\ell, \mathbf{g}^s \ell) + d(\mathbf{g}^s \ell, \hat{w}) + d(\hat{w}, w). \quad \square$$

2.6 Creating common perpendiculars

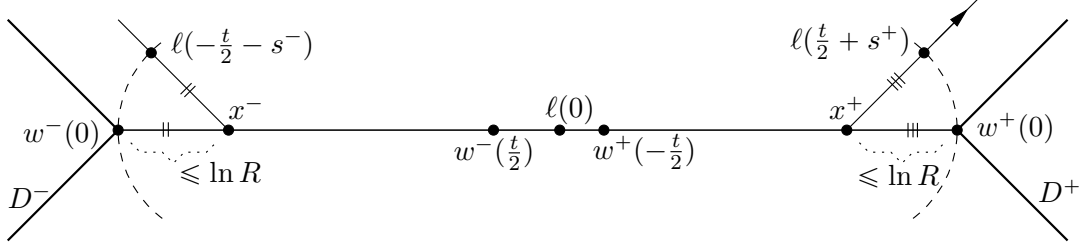
Let D^- and D^+ be two nonempty proper closed convex subsets of X , where X is as in the beginning of Section 2.2. A geodesic arc $\alpha : [0, T] \rightarrow X$, where $T > 0$, is a *common perpendicular of length T from D^- to D^+* if there exist $w^\mp \in \partial_\pm^1 D^\mp$ such that $w^-|_{[0, T]} = \mathbf{g}^{-T} w^+|_{[0, T]} = \alpha$. Since X is CAT(-1), this geodesic arc α is the unique shortest geodesic segment from a point of D^- to a point of D^+ . There is a common perpendicular from D^- to D^+ if and only if the closures of D^- and D^+ in $X \cup \partial_\infty X$ are disjoint. When X is an \mathbb{R} -tree, then two closed subtrees of X have a common perpendicular if and only if they are nonempty and disjoint.

One of the aims of this text is to count orbits of common perpendiculars between two equivariant families of closed convex subsets of X . The crucial remark is that two nonempty proper closed convex subsets D^- and D^+ of X have a common perpendicular α of length a given $T > 0$ if and only if the subsets $\mathbf{g}^{T/2} \partial_+^1 D^-|_{[-\frac{T}{2}, \frac{T}{2}]}$ and $\mathbf{g}^{-T/2} \partial_-^1 D^+|_{[-\frac{T}{2}, \frac{T}{2}]}$ of $\check{\mathcal{G}}X$ intersect. This intersection then consists of the common perpendicular from D^- to D^+ reparametrised by $[-\frac{T}{2}, \frac{T}{2}]$. As a controlled perturbation of this remark, we now give an effective creation result of common perpendiculars in \mathbb{R} -trees. It has a version satisfied for X in the generality of Section 2.2, see the end of this Section.

Lemma 2.8. *Assume that X is an \mathbb{R} -tree. For all $R > 1$, $\eta \in]0, 1[$ and $t \geq 2 \ln R + 4$, for all nonempty closed connected subsets D^-, D^+ in X , and for every geodesic line $\ell \in \mathbf{g}^{t/2} \mathcal{V}_{\eta, R}^+(\partial_+^1 D^-) \cap \mathbf{g}^{-t/2} \mathcal{V}_{\eta, R}^-(\partial_-^1 D^+)$, there exist $s \in]-2\eta, 2\eta[$ and a common perpendicular \tilde{c} from D^- to D^+ such that*

- the length of \tilde{c} is $t + s$,
- the endpoint of \tilde{c} in D^\mp is $w^\mp(0)$ where $w^\mp = f_{D^\mp}^\pm(\ell)$,
- the footpoint $\ell(0)$ of ℓ lies on \tilde{c} , and

$$\max \left\{ d(w^-(\frac{t}{2}), \ell(0)), d(w^+(-\frac{t}{2}), \ell(0)) \right\} \leq \eta.$$



Proof. Let R, η, t, D^\pm, ℓ be as in the statement. By the definition of the sets $\mathcal{V}_{\eta, R}^\mp(\partial_\mp^1 D^\pm)$, there exist geodesic rays $w^\pm \in \partial_\mp^1 D^\pm$, geodesic lines $\hat{w}^\pm \in \mathcal{G}X$ extending w^\pm , and $s^\pm \in]-\eta, +\eta[$, such that $\ell_\pm = (w^\mp)_\pm$ and

$$d_{W^\pm(w^\mp)}(\mathbf{g}^{\mp \frac{t}{2} \mp s^\mp} \ell, \hat{w}^\mp) \leq R.$$

Let x_\pm be the closest point to $w^\pm(0)$ on ℓ . By the definition of Hamenstädt's distances, we have

$$d(w^\pm(0), x^\pm) = d(\ell(\pm \frac{t}{2} \pm s^\pm), x^\pm) \leq \ln R,$$

and in particular $x_\pm = w^\pm(0)$ if and only if $\ell(\pm \frac{t}{2} \pm s^\pm) = w^\pm(0)$. As $t \geq 2 \ln R + 4$ and $|s^\pm| \leq 2\eta \leq 2$, the points $\ell(-\frac{t}{2} - s^-)$, x^- , $\ell(0)$, x^+ , $\ell(\frac{t}{2} + s^+)$ are in this order on ℓ . In particular, the segment $[w^-(0), x^-] \cup [x^-, x^+] \cup [x^+, w^+(0)]$ is a nontrivial geodesic segment from a point of D^- to a point of D^+ that meets D^\mp only at an endpoint. Hence, D^- and D^+ are disjoint, and $[w^-(0), w^+(0)]$ is the image of the common perpendicular from D^- to D^+ .

Let $s = s_- + s_+$. The length of \tilde{c} is $(\frac{t}{2} + s^+) - (-\frac{t}{2} - s^-) = t + s$. The point $\ell(0)$ lies on \tilde{c} , we have $w^\mp = f_{D^\mp}^\pm(\ell)$ and the endpoints of \tilde{c} are $w^\pm(0)$. Furthermore,

$$d(w^\mp(\pm \frac{t}{2}), \ell(0)) = |d(w^\mp(\pm \frac{t}{2}), w^\mp(0)) - d(\ell(0), \ell(\mp \frac{t}{2} \mp s^\mp))| = |s^\mp| \leq \eta. \quad \square$$

When X is as in the beginning of Section 2.2, the statement and the proof of the following analog of Lemma 2.8 is slightly more technical. We refer to [PaP16c, Lem. 7] for a proof in the Riemannian case, and we leave the extension to the reader, since we will not need it in this book.

Lemma 2.9. *Let X be as in the beginning of Section 2.2. For every $R > 0$, there exist $t_0, c_0 > 0$ such that for all $\eta \in]0, 1[$ and all $t \in [t_0, +\infty[$, for all nonempty closed convex subsets D^-, D^+ in X , and for all $w \in \mathbf{g}^{t/2} \mathcal{V}_{\eta, R}^+(\partial_+^1 D^-) \cap \mathbf{g}^{-t/2} \mathcal{V}_{\eta, R}^-(\partial_-^1 D^+)$, there exist $s \in]-2\eta, 2\eta[$ and a common perpendicular \tilde{c} from D^- to D^+ such that*

- the length of \tilde{c} is contained in $[t + s - c_0 e^{-\frac{t}{2}}, t + s + c_0 e^{-\frac{t}{2}}]$,
- if $w^\mp = f_{D^\mp}^\pm(w)$ and if p^\pm is the endpoint of \tilde{c} in D^\pm , then $d(\pi(w^\pm), p^\pm) \leq c_0 e^{-\frac{t}{2}}$,
- the basepoint $\pi(w)$ of w is at distance at most $c_0 e^{-\frac{t}{2}}$ from a point of \tilde{c} , and

$$\max\{d(\pi(\mathbf{g}^{\frac{t}{2}} w^-), \pi(w)), d(\pi(\mathbf{g}^{-\frac{t}{2}} w^+), \pi(w))\} \leq \eta + c_0 e^{-\frac{t}{2}}. \quad \square$$

2.7 Metric and simplicial trees, and graphs of groups

Metric and simplicial trees and graphs of groups are important examples throughout the rest of this work. In this Section, we recall the definitions and basic properties of these objects.

We use Serre's definitions in [Ser3, §2.1] concerning a graph \mathbb{X} , with $V\mathbb{X}$ and $E\mathbb{X}$ its set of vertices and edges, and $o(e)$, $t(e)$ and \bar{e} the initial vertex, terminal vertex and opposite edge of an edge $e \in E\mathbb{X}$. Recall that a connected graph is *bipartite* if it is endowed with a partition of its set of vertices into two nonempty subsets such that any two elements of either subset are not related by an edge.

The *degree* of a vertex $x \in V\mathbb{X}$ is the cardinality of the set $\{e \in E\mathbb{X} : o(e) = x\}$. For all $j, k \in \mathbb{N}$, a graph \mathbb{X} is *k-regular* if the degree of each vertex $x \in V\mathbb{X}$ is k , and it is *(j, k)-biregular* if it is bipartite with the partition of its vertices into the two subsets consisting of vertices with degree j and with degree k respectively.

A *metric graph* (\mathbb{X}, λ) is a pair consisting of a graph \mathbb{X} and a map $\lambda : E\mathbb{X} \rightarrow]0, +\infty[$ with a positive lower bound⁵ such that $\lambda(\bar{e}) = \lambda(e)$, called its *edge length map*. A *simplicial graph* \mathbb{X} is a metric graph whose edge length map is constant equal to 1.

The *topological realisation* of a graph \mathbb{X} is the topological space obtained from the collection $(I_e)_{e \in E\mathbb{X}}$ of closed unit intervals I_e by the finest equivalence relation that identifies intervals corresponding to an edge and its opposite edge by the map $t \mapsto 1 - t$ and identifies the origins of the intervals I_{e_1} and I_{e_2} if and only if $o(e_1) = o(e_2)$, see [Ser3, Sect. 2.1].

The *geometric realisation* of a metric tree (\mathbb{X}, λ) is the topological realisation of \mathbb{X} endowed with the maximal geodesic metric that gives length $\lambda(e)$ to the topological realisation of each edge $e \in E\mathbb{X}$, and we denote it by $X = |\mathbb{X}|_\lambda$. We identify $V\mathbb{X}$ with its image in X . The metric space X determines (\mathbb{X}, λ) up to subdivision of edges, hence we will often not make a strict distinction between X and (\mathbb{X}, λ) . In particular, we will refer to convex subsets of (\mathbb{X}, λ) as convex subsets of X , etc.

If \mathbb{X} is a tree, the metric space X is an \mathbb{R} -tree, hence it is a CAT(-1) space. Since λ is bounded from below by a positive constant, the \mathbb{R} -tree X is geodesically complete if and only if \mathbb{X} has no *terminal vertex* (that is, no vertex of degree 1).

We will denote by $\text{Aut}(\mathbb{X}, \lambda)$, and $\text{Aut } \mathbb{X}$ in the simplicial case, the group of edge-preserving isometries of X that have no inversions.⁶ Since the edge length map has a positive lower bound, the metric space X is proper if and only if \mathbb{X} is locally finite. In this case, the nonelementary discrete subgroups Γ of isometries of X we will consider will always be edge-preserving and without inversion.

A locally finite metric tree (\mathbb{X}', λ) is *uniform* if there exists some discrete subgroup Γ' of $\text{Aut}(\mathbb{X}', \lambda)$ such that $\Gamma' \backslash \mathbb{X}'$ is a finite graph. See [BasK, BasL] for characterisations of this property in the case of simplicial trees.

The *space of generalised discrete geodesic lines* of a locally finite simplicial tree \mathbb{X} is the locally compact space $\check{\mathcal{G}}\mathbb{X}$ of 1-Lipschitz mappings w from \mathbb{R} to its geometric realisation $X = |\mathbb{X}|_1$ which are isometric on a closed interval with endpoints in $\mathbb{Z} \cup \{-\infty, +\infty\}$ and locally constant outside it, such that $w(0) \in V\mathbb{X}$ (or equivalently $w(\mathbb{Z}) \subset V\mathbb{X}$). Note that $\check{\mathcal{G}}\mathbb{X}$ is hence a proper subset of $\check{\mathcal{G}}X$.

By restriction to $\check{\mathcal{G}}\mathbb{X}$, or intersection with $\check{\mathcal{G}}\mathbb{X}$, of the objects defined in Sections 2.3 and 2.5 for $\check{\mathcal{G}}X$, we define the distance d on $\check{\mathcal{G}}\mathbb{X}$, the subspaces $\mathcal{G}_\pm\mathbb{X}$, $\mathcal{G}\mathbb{X}$, $\mathcal{G}_{\pm,0}\mathbb{X}$, the strong stable/unstable leaves $W^\pm(w)$ of $w \in \mathcal{G}_\pm\mathbb{X}$ and their Hamenstädt distances $d_{W^\pm(w)}$, the stable/unstable leaves $W^{0\pm}(w)$ of $w \in \mathcal{G}_\pm\mathbb{X}$, the outer and inner unit normal bundles $\partial_\pm^1 \mathbb{D}$ of a nonempty proper simplicial subtree \mathbb{D} of \mathbb{X} , the dynamical neighbourhoods $\mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp)$ of

⁵This assumption, though not necessary at this stage, will be used repeatedly in this text, hence we prefer to add it to the definition.

⁶An automorphism g of a graph has an *inversion* if there exists an edge e of the graph such that $ge = \bar{e}$.

subsets Ω^\mp of $\partial_\pm^1 \mathbb{D}$ as well as the fibrations

$$f_{\mathbb{D}}^\pm : \mathcal{U}_{\mathbb{D}}^\pm = \{\ell \in \mathcal{GX} : \ell_\pm \notin \partial_\infty \mathbb{D}\} \rightarrow \partial_\pm^1 \mathbb{D},$$

whose fiber over $\rho \in \partial_\pm^1 \mathbb{D}$ is $W^{0\pm}(\rho)$. Note that some definitions actually simplify when considering generalised discrete geodesic lines. For instance, for all $w \in \mathcal{G}_\pm \mathbb{X}$, $\eta' > 0$ and $\eta \in]0, 1[$, the dynamical neighbourhood $V_{w, \eta, \eta'}^\pm$ is equal to $B^\pm(w, \eta')$, and is hence independent of $\eta \in]0, 1[$.

Besides the map $\pi : \mathcal{GX} \rightarrow V\mathbb{X}$ defined as in the continuous case by $\ell \mapsto \ell(0)$, we have another natural map $T\pi : \mathcal{GX} \rightarrow E\mathbb{X}$, which associates to ℓ the edge e with $o(e) = \ell(0)$ and $t(e) = \ell(1)$. This map is equivariant under the group of automorphisms (without inversions) $\text{Aut}(\mathbb{X})$ of \mathbb{X} , and we also denote by $T\pi : \Gamma \backslash \mathcal{GX} \rightarrow \Gamma \backslash E\mathbb{X}$ its quotient map, for every subgroup Γ of $\text{Aut}(\mathbb{X})$.

If \mathbb{X} has no terminal vertex, for every $e \in E\mathbb{X}$, let

$$\partial_e \mathbb{X} = \{\ell_+ : \ell \in \mathcal{GX}, T\pi(\ell) = e\}$$

be the set of points at infinity of the geodesic rays whose initial (oriented) edge is e .

Given $x_0 \in V\mathbb{X}$, the *discrete Hopf parametrisation* now identifies \mathcal{GX} with $\partial_\infty^2 \mathbb{X} \times \mathbb{Z}$ by the map $\ell \mapsto (\ell_-, \ell_+, t)$ where $t \in \mathbb{Z}$ is the signed distance from the closest vertex to the basepoint x_0 on the geodesic line ℓ to the vertex $\ell(0)$.

The *discrete time geodesic flow* $(\mathbf{g}^t)_{t \in \mathbb{Z}}$ on $\check{\mathcal{G}}\mathbb{X}$ is the one-(discrete-)parameter group of homeomorphisms of $\check{\mathcal{G}}\mathbb{X}$ consisting of (the restriction to $\check{\mathcal{G}}\mathbb{X}$ of) the integral time maps of the continuous time geodesic flow of the geometric realisation of \mathbb{X} : we have $\mathbf{g}^t w : s \mapsto w(s + t)$ for all $w \in \check{\mathcal{G}}\mathbb{X}$ and $t \in \mathbb{Z}$.

Recall (see for instance [Ser3, BasL]) that a *graph of groups* (\mathbb{Y}, G_*) consists of

- a graph \mathbb{Y} , which is connected unless otherwise stated,
- a group G_v for every vertex $v \in V\mathbb{Y}$,
- a group G_e for every edge $e \in E\mathbb{Y}$ such that $G_e = G_{\bar{e}}$,
- an injective group morphism $\rho_e : G_e \rightarrow G_{t(e)}$ for every edge $e \in V\mathbb{Y}$.

We will identify G_e with its image in $G_{t(e)}$ by ρ_e , unless the meaning is not clear (which might be the case for instance if $o(e) = t(e)$).

A *subgraph of subgroups* of (\mathbb{Y}, G_*) is a graph of groups (\mathbb{Y}', G'_*) where

- \mathbb{Y}' is a subgraph of \mathbb{Y} ,
- for every $v \in V\mathbb{Y}'$, the group G'_v is a subgroup of G_v ,
- for every $e \in E\mathbb{Y}'$, the group G'_e is a subgroup of G_e , and the monomorphism $G'_e \rightarrow G'_{t(e)}$ is the restriction of the monomorphism $G_e \rightarrow G_{t(e)}$, and

$$G'_{t(e)} \cap \rho_e(G_e) = \rho_e(G'_e).$$

This condition, first introduced in [Bass, Coro. 1.14], is equivalent to the injectivity of the natural map $G'_{t(e)}/\rho_e(G'_e) \rightarrow G_{t(e)}/\rho_e(G_e)$. It implies by [Bass, 2.15] that when the underlying basepoint is chosen in \mathbb{Y}' , the fundamental group of (\mathbb{Y}', G'_*) injects in the fundamental group

of (\mathbb{Y}, G_*) , and the Bass-Serre tree \mathbb{X}' of (\mathbb{Y}', G'_*) injects in an equivariant way in the Bass-Serre tree \mathbb{X} of (\mathbb{Y}, G_*) .

Note that the fundamental group of (\mathbb{Y}, G_*) does not always act faithfully on its Bass-Serre tree \mathbb{X} , that is, the kernel of its action might be nontrivial.

The *edge-indexed graph* (\mathbb{Y}, i) of the graph of groups (\mathbb{Y}, G_*) is the graph \mathbb{Y} endowed with the map $i : E\mathbb{Y} \rightarrow \mathbb{N} - \{0\}$ defined by $i(e) = [G_{o(e)} : G_e]$ (see for instance [BasK, BasL]).

In Section 12.4, we will consider *metric graphs of groups* $(\mathbb{Y}, G_*, \lambda)$ which are graphs of groups endowed with an edge length function $\lambda : E\mathbb{Y} \rightarrow]0, +\infty[$ (with $\lambda(\bar{e}) = \lambda(e)$ for every $e \in E\mathbb{Y}$).

Example 2.10. The main examples of graphs of groups that we will consider in this text are the following ones. Let \mathbb{X} be a simplicial tree and let Γ be a subgroup of $\text{Aut}(\mathbb{X})$. The *quotient graph of groups* $\Gamma \backslash \mathbb{X}$ is the following graph of groups (\mathbb{Y}, G_*) (having finite vertex groups if \mathbb{X} is locally finite and Γ is discrete). Its underlying graph \mathbb{Y} is the quotient graph $\Gamma \backslash \mathbb{X}$. Fix a lift $\tilde{z} \in V\mathbb{X} \cup E\mathbb{X}$ for every $z \in V\mathbb{Y} \cup E\mathbb{Y}$. For every $e \in E\mathbb{Y}$, assume that $\tilde{e} = \tilde{\bar{e}}$, and fix an element $g_e \in \Gamma$ such that $g_e t(\tilde{e}) = t(\tilde{\bar{e}})$. For every $y \in V\mathbb{Y} \cup E\mathbb{Y}$, take as G_y the stabiliser $\Gamma_{\tilde{y}}$ in Γ of the fixed lift \tilde{y} . Take as monomorphism $\rho_e : G_e \rightarrow G_{t(e)}$ the restriction to $\Gamma_{\tilde{e}}$ of the conjugation $\gamma \mapsto g_e^{-1} \gamma g_e$ by g_e^{-1} .

The *volume form* of a graph of finite groups (\mathbb{Y}, G_*) is the measure $\text{vol}_{(\mathbb{Y}, G_*)}$ on the discrete set $V\mathbb{Y}$, such that for every $y \in V\mathbb{Y}$,

$$\text{vol}_{(\mathbb{Y}, G_*)}(\{y\}) = \frac{1}{|G_y|},$$

where $|G_y|$ is the order of the finite group G_y . Its total mass, called the *volume* of (\mathbb{Y}, G_*) , is

$$\text{Vol}(\mathbb{Y}, G_*) = \|\text{vol}_{(\mathbb{Y}, G_*)}\| = \sum_{y \in V\mathbb{Y}} \frac{1}{|G_y|}.$$

We denote by $\mathbb{L}^2(\mathbb{Y}, G_*) = \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$ the Hilbert space of square integrable maps $V\mathbb{Y} \rightarrow \mathbb{C}$ for this measure $\text{vol}_{(\mathbb{Y}, G_*)}$, and by $f \mapsto \|f\|_2$ and $(f, g) \mapsto \langle f, g \rangle_2$ its norm and scalar product. Let

$$\mathbb{L}_0^2(\mathbb{Y}, G_*) = \{f \in \mathbb{L}^2(\mathbb{Y}, G_*) : \int f d\text{vol}_{(\mathbb{Y}, G_*)} = 0\}.$$

When $\text{Vol}(\mathbb{Y}, G_*)$ is finite, $\mathbb{L}_0^2(\mathbb{Y}, G_*)$ is the orthogonal subspace to the constant functions.

We also consider a *(edge-)volume form* $\text{Tvol}_{(\mathbb{Y}, G_*)}$ on the discrete set $E\mathbb{Y}$ such that for every $e \in E\mathbb{Y}$,

$$\text{Tvol}_{(\mathbb{Y}, G_*)}(\{e\}) = \frac{1}{|G_e|},$$

with total mass

$$\text{TVol}(\mathbb{Y}, G_*) = \|\text{Tvol}_{(\mathbb{Y}, G_*)}\| = \sum_{e \in E\mathbb{Y}} \frac{1}{|G_e|}.$$

The *(edge-)volume form* of a metric graph of groups $(\mathbb{Y}, G_*, \lambda)$ is given by

$$\text{Tvol}(\mathbb{Y}, G_*, \lambda) = \frac{ds}{|G_e|}$$

on each edge e of \mathbb{Y} parameterised by its arclength s , so that its total mass is

$$\mathbf{TVol}(\mathbb{Y}, G_*, \lambda) = \|\mathbf{Tvol}_{(\mathbb{Y}, G_*, \lambda)}\| = \sum_{e \in E\mathbb{Y}} \frac{\lambda(e)}{|G_e|}.$$

For $\lambda \equiv 1$, this total mass agrees with that of the discrete definition above.

Remark 2.11. Note that $\mathbf{TVol}(\mathbb{Y}, G_*) = \text{Card}(E\mathbb{Y})$ when the edge groups are trivial. We have

$$\mathbf{TVol}(\mathbb{Y}, G_*) = \sum_{e \in E\mathbb{Y}} \frac{1}{|G_e|} = \sum_{y \in V\mathbb{Y}} \frac{1}{|G_y|} \sum_{e \in E\mathbb{Y}, o(e)=y} \frac{|G_y|}{|G_e|} = \sum_{y \in V\mathbb{Y}} \frac{\deg(\tilde{y})}{|G_y|},$$

where \tilde{y} is any lift of y in the Bass-Serre tree of (\mathbb{Y}, G_*) . In particular, if \mathbb{X} is a uniform simplicial tree and Γ is discrete subgroup of $\text{Aut } \mathbb{X}$, then the finiteness of $\text{Vol}(\Gamma \backslash \mathbb{X})$ and of $\mathbf{TVol}(\Gamma \backslash \mathbb{X})$ are equivalent. Defining the volume form on $V\mathbb{Y}$ by $\{y\} \mapsto \frac{\deg(\tilde{y})}{|G_y|}$ sometimes makes formulas simpler, but we will follow the convention which occurs in the classical references (see for instance [BasL]).

If the Bass-Serre tree of (\mathbb{Y}, G_*) is $(q+1)$ -regular, then

$$\pi_* \mathbf{Tvol}_{\mathbb{Y}, G_*} = (q+1) \text{vol}_{\mathbb{Y}, G_*} \quad \text{and} \quad \mathbf{TVol}(\mathbb{Y}, G_*) = (q+1) \text{Vol}(\mathbb{Y}, G_*). \quad (2.16)$$

We say that a discrete group of isometries Γ of a locally finite metric or simplicial tree (\mathbb{X}, λ) is a *(tree) lattice* of (\mathbb{X}, λ) if the quotient graph of groups $\Gamma \backslash \mathbb{X}$ has finite volume. If \mathbb{X} is simplicial, then this implies that Γ is a lattice in the locally compact group $\text{Aut}(\mathbb{X})$ (hence that $\text{Aut}(\mathbb{X})$ is unimodular), the converse being true for instance if \mathbb{X} is regular or biregular (see [BasK]). For instance, if Γ is a *uniform lattice* of \mathbb{X} (or (\mathbb{X}, λ)), that is, if Γ is a discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$ and if the quotient graph $\Gamma \backslash \mathbb{X}$ is finite, then Γ is a lattice of (\mathbb{X}, λ) .

A graph of finite groups (\mathbb{Y}, G_*) is a *cuspidal ray* if \mathbb{Y} is a simplicial ray such that the homomorphisms $G_{e_n} \rightarrow G_{o(e_n)}$ are surjective for its sequence of consecutive edges $(e_i)_{i \in \mathbb{N}}$ oriented towards the unique end of \mathbb{Y} . By [Pau1], a discrete group Γ' of $\text{Aut}(\mathbb{X})$ (hence of $\text{Isom}(|\mathbb{X}|_1)$) is geometrically finite if and only if it is nonelementary and if the quotient graph of groups by Γ' of its minimal nonempty invariant subtree is the union of a finite graph of groups and a finite number of cuspidal rays attached to the finite graph at their finite endpoints.

Remark 2.12. If \mathbb{X} is a locally finite simplicial tree and if Γ' is a geometrically finite discrete group of $\text{Aut}(\mathbb{X})$ such that the convex hull of its limit set $\mathcal{C}\Lambda\Gamma'$ is a uniform tree, then Γ' is a lattice of $\mathcal{C}\Lambda\Gamma'$.

Proof. Since $\mathcal{C}\Lambda\Gamma'$ is uniform, there is a uniform upper bound on the length of an edge path in $\mathcal{C}\Lambda\Gamma'$ which injects in $\Gamma' \backslash \mathcal{C}\Lambda\Gamma'$ such that the stabiliser of each edge of this edge path is equal to the stabilisers of both endpoints of this edge. It is hence easy to see that the volume of each of the (finitely many) cuspidal rays in $\Gamma' \backslash \mathcal{C}\Lambda\Gamma'$ is finite, by a geometric series argument. Hence the volume of $\Gamma' \backslash \mathcal{C}\Lambda\Gamma'$ is finite. \square

Note that contrarily to the case of Riemannian manifolds, there are many more (tree) lattices than there are geometrically finite (tree) lattices, even in regular trees, see for instance [BasL].

In Part [III](#) of this text, we will consider simplicial graphs of groups that arise from arithmetics of non-Archimedean local fields. We say that Γ is *algebraic* if there exists a non-Archimedean local field \widehat{K} (a finite extension of \mathbb{Q}_p for some prime p or the field of formal Laurent series over a finite field) and a connected semi-simple algebraic group \underline{G} with finite centre defined over \widehat{K} , of \widehat{K} -rank one, such that \mathbb{X} identifies with the Bruhat-Tits tree of \underline{G} in such a way that Γ identifies with a lattice of $\underline{G}(\widehat{K})$. If Γ is algebraic, then Γ is geometrically finite by [\[Lub1\]](#). Note that \mathbb{X} is then bipartite, see Section 2 of op. cit. for a discussion and references. See Sections [14](#) and [15.1](#) for more details, and the subsequent Sections for arithmetic applications arising from algebraic lattices.

Chapter 3

Potentials, critical exponents and Gibbs cocycles

Let X be a geodesically complete proper $\text{CAT}(-1)$ space, let $x_0 \in X$ be an arbitrary basepoint, and let Γ be a nonelementary discrete group of isometries of X .

In this Chapter, we define potentials on T^1X , which are new data on X in addition to its geometry. We introduce the fundamental tools associated with potentials, and we give some of their basic properties. The development follows [PauPS] with modifications to fit the present more general context.

In Section 3.5, we introduce a natural method to associate a (Γ -invariant) potential $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ to a Γ -invariant function $c : E\mathbb{X} \rightarrow \mathbb{R}$ defined on the set of edges of a simplicial or metric tree \mathbb{X} , with geometric realisation X , that we call a system of conductances on \mathbb{X} . This construction gives a nonsymmetric generalisation of electric networks.

3.1 Background on (uniformly local) Hölder-continuity

In this preliminary Section, we recall the notion of Hölder-continuity we will use in this text, which needs to be defined appropriately in order to deal with noncompactness issues. The Hölder-continuity will be used on one hand for potentials when X is a Riemannian manifold in Section 3.2, and on the other hand for error term estimates in Chapters 9, 10 and 11.

As in [PauPS], we will use the following uniformly local definition of Hölder-continuous maps. Let E and E' be two metric spaces, and let $\alpha \in]0, 1]$. A map $f : E \rightarrow E'$ is

- *α -Hölder-continuous* if there exist $c, \epsilon > 0$ such that for all $x, y \in E$ with $d(x, y) \leq \epsilon$, we have

$$d(f(x), f(y)) \leq c d(x, y)^\alpha .$$

- *locally α -Hölder-continuous* if for every $x \in E$, there exists a neighbourhood U of x such that the restriction of f to U is α -Hölder-continuous;
- *Hölder-continuous* (respectively *locally Hölder-continuous*) if there exists $\alpha \in]0, 1]$ such that f is α -Hölder-continuous (respectively locally α -Hölder-continuous);
- *Lipschitz* if it is 1-Hölder-continuous and *locally Lipschitz* if it is locally 1-Hölder-continuous.

Let E and E' be two metric spaces. We say that a map $f : E \rightarrow E'$ has

- *at most linear growth* if there exists $a, b \geq 0$ such that $d(f(x), f(y)) \leq a d(x, y) + b$ for all $x, y \in E$,

- *subexponential growth* if for every $a > 0$, there exists $b \geq 0$ such that $d(f(x), f(y)) \leq b e^{a d(x,y)}$ for all $x, y \in E$.

Remark 3.1. When E is a geodesic space, a consequence of the (uniformly local) Hölder-continuous property of $f : E \rightarrow E'$ is that f then has at most linear growth: the definition implies that $d(f(x), f(y)) \leq c \epsilon^{\alpha-1} d(x, y) + c \epsilon^\alpha$ for all x, y in X , by subdividing the geodesic segment in E from x to y in $\lceil \frac{d(x,y)}{\epsilon} \rceil$ segments of equal lengths at most ϵ and using the triangle inequality in E' .

For any metric space Z and $\alpha \in]0, 1]$, the *Hölder norm* of a bounded α -Hölder-continuous function $f : Z \rightarrow \mathbb{R}$ is

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{x, y \in Z \\ 0 < d(x,y) \leq 1}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

When the diameter of Z is bounded by 1,¹ this coincides with the usual definition. Note that even if the constant ϵ in the above definition of a α -Hölder-continuous map is less than 1, this norm is finite, since

$$\sup_{\substack{x, y \in Z \\ \epsilon \leq d(x,y) \leq 1}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq 2 \epsilon^{-\alpha} \|f\|_\infty.$$

Note that for all bounded α -Hölder-continuous maps $f, g : Z \rightarrow \mathbb{R}$, we have

$$\|fg\|_\alpha \leq \|f\|_\alpha \|g\|_\infty + \|f\|_\infty \|g\|_\alpha. \quad (3.1)$$

We denote by $\mathcal{C}_c^\alpha(Z)$ (respectively $\mathcal{C}_b^\alpha(Z)$) the space of α -Hölder-continuous real-valued functions with compact support (respectively which are bounded) on Z , endowed with this norm. Note that $\mathcal{C}_b^\alpha(Z)$ is a Banach space.²

A stronger assumption than the Hölder regularity is the locally constant regularity, that we now define. Although it is only useful for totally disconnected metric spaces, several error terms estimates in the literature use this stronger regularity (see for instance [AtGP, KemaPS] and Part III of this text).

Let $\epsilon > 0$. For every metric space E and every set E' , we say that a map $f : E \rightarrow E'$ is ϵ -locally constant if f is constant on every closed ball of radius ϵ (or equivalently of radius at most ϵ) in E . We say that $f : E \rightarrow E'$ is *locally constant* if there exists $\epsilon > 0$ such that f is ϵ -locally constant.

Note that if E is a geodesic metric space and $f : E \rightarrow E'$ is locally constant, then f is constant. But when E is for instance an ultrametric space, since two distinct closed balls of the same radius are disjoint, the above definition turns out to be very interesting (and much used in representation theory in positive characteristic, for instance). For example, the characteristic function $\mathbb{1}_A$ of a subset A of E is ϵ -locally constant if and only if for every $x \in A$, the closed ball $B(x, \epsilon)$ is contained in A . In particular, the characteristic function of a closed ball of radius ϵ in an ultrametric space is ϵ -locally constant.

The next result says that the Hölder regularity is indeed stronger than the locally constant one.

¹This is in particular the case for the sequence spaces of symbolic dynamical systems, see Sections 5.2 and 9.2.

²The standard proof using Arzela-Ascoli's theorem applies with our slightly different definition of the Hölder norms.

Remark 3.2. Let E and E' be two metric spaces. If a map $f : E \rightarrow E'$ is ϵ -locally constant, then it is α -Hölder-continuous for every $\alpha \in]0, 1]$. Indeed, for all $x, y \in E$, if $d(x, y) \leq \epsilon$ then $d(f(x), f(y)) = 0 \leq c d(x, y)^\alpha$ for all $c > 0$. If furthermore $E' = \mathbb{R}$ and f is bounded, then

$$\sup_{x, y \in E, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = \sup_{x, y \in E, d(x, y) > \epsilon} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq \frac{2}{\epsilon^\alpha} \|f\|_\infty.$$

For all $\epsilon \in]0, 1]$ and $\beta > 0$, we denote by $\mathcal{C}_b^{\epsilon \text{lc}, \beta}(E)$ the vector space³ of ϵ -locally constant functions $f : E \rightarrow \mathbb{R}$ endowed with the ϵlc -norm of exponent β defined by

$$\|f\|_{\epsilon \text{lc}, \beta} = \epsilon^{-\beta} \|f\|_\infty.$$

The above remark proves that if $\beta \in]0, 1]$, the inclusion map from $\mathcal{C}_b^{\epsilon \text{lc}, \beta}(E)$ into $\mathcal{C}_b^\beta(E)$ is continuous. We will only use the ϵlc -norms in Section 15.4.

3.2 Potentials

In this text, a *potential* for Γ is a continuous Γ -invariant function $\tilde{F} : T^1X \rightarrow \mathbb{R}$. The quotient function $F : \Gamma \backslash T^1X \rightarrow \mathbb{R}$ of \tilde{F} is called a *potential* on $\Gamma \backslash T^1X$. The function \tilde{F} defines a continuous Γ -invariant function from $\mathcal{G}X$ to \mathbb{R} , also denoted by \tilde{F} , by $\tilde{F}(\ell) = \tilde{F}(v_\ell)$.

For all $x, y \in X$, and any geodesic line $\ell \in \mathcal{G}X$ such that $\ell(0) = x$ and $\ell(d(x, y)) = y$, let

$$\int_x^y \tilde{F} = \int_0^{d(x, y)} \tilde{F}(v_{g^t \ell}) dt.$$

Note that for all $t \in]0, d(x, y)[$, the germ $v_{g^t \ell}$ is independent on the choice of such a line ℓ , hence $\int_x^y \tilde{F}$ does not depend on the extension ℓ of the geodesic segment $[x, y]$. The following properties are easy to check using the Γ -invariance of \tilde{F} and the basic properties of integrals: For all $\gamma \in \Gamma$

$$\int_{\gamma x}^{\gamma y} \tilde{F} = \int_x^y \tilde{F},$$

for the antipodal map ι

$$\int_y^x \tilde{F} = \int_x^y \tilde{F} \circ \iota, \quad (3.2)$$

and, for any $z \in [x, y]$,

$$\int_x^y \tilde{F} = \int_x^z \tilde{F} + \int_z^y \tilde{F}. \quad (3.3)$$

The *period* of a loxodromic isometry γ of X for the potential \tilde{F} is

$$\text{Per}_F(\gamma) = \int_x^{\gamma x} \tilde{F}$$

for any x in the translation axis of γ . Note that, for all $\alpha \in \Gamma$ and $n \in \mathbb{N} - \{0\}$, we have

$$\text{Per}_F(\alpha \gamma \alpha^{-1}) = \text{Per}_F(\gamma), \quad \text{Per}_F(\gamma^n) = n \text{Per}_F(\gamma) \quad \text{and} \quad \text{Per}_F(\gamma^{-1}) = \text{Per}_{F \circ \iota}(\gamma). \quad (3.4)$$

In trees, we have the following Lipschitz-type control on the integrals of the potentials along segments.

³Note that a linear combination of ϵ -locally constant functions is again a ϵ -locally constant function.

Lemma 3.3. *When \tilde{F} is constant or when X is an \mathbb{R} -tree, for all $x, x', y, y' \in X$, we have*

$$\left| \int_x^y \tilde{F} - \int_{x'}^{y'} \tilde{F} \right| \leq d(x, x') \sup_{\pi^{-1}([x, x'])} |\tilde{F}| + d(y, y') \sup_{\pi^{-1}([y, y'])} |\tilde{F}|.$$

Proof. When \tilde{F} is constant, the result follows from the triangle inequality.

Assume that X is an \mathbb{R} -tree. Consider the case $x = x'$. Let $z \in X$ be such that $[x, z] = [x, y] \cap [x, y']$. Using Equation (3.3) and the fact that $d(y, z) + d(z, y') = d(y, y')$, the claim follows. The general case follows by combining this case $x = x'$ and a similar estimate for the case $y = y'$. \square

Some form of uniform Hölder-type control of the potential, analogous to the Lipschitz-type one in the previous lemma, will be crucial throughout the present work. The following Definition 3.4 formalises this (weaker) assumption.

Definition 3.4. *The triple (X, Γ, \tilde{F}) satisfies the HC-property (Hölder-type control) if \tilde{F} has subexponential growth when X is not an \mathbb{R} -tree and if there exists $\kappa_1 \geq 0$ and $\kappa_2 \in]0, 1]$ such that for all $x, y, x', y' \in X$ with $d(x, x'), d(y, y') \leq 1$, we have*

$$\begin{aligned} & \left| \int_x^y \tilde{F} - \int_{x'}^{y'} \tilde{F} \right| && \text{(HC)} \\ & \leq \left(\kappa_1 + \max_{\pi^{-1}(B(x, d(x, x')))} |\tilde{F}| \right) d(x, x')^{\kappa_2} + \left(\kappa_1 + \max_{\pi^{-1}(B(y, d(y, y')))} |\tilde{F}| \right) d(y, y')^{\kappa_2}. \end{aligned}$$

By Equation (3.2), $(X, \Gamma, \tilde{F} \circ \iota)$ satisfies the HC-property if and only if (X, Γ, \tilde{F}) does. By Equation (2.3), for every $\kappa \in \mathbb{R}$, the triple $(X, \Gamma, \tilde{F} + \kappa)$ satisfies the HC-property (up to changing the constant κ_1) if and only if (X, Γ, \tilde{F}) does.

When X is assumed to be a Riemannian manifold with pinched sectional curvature, requiring the potentials to be Hölder-continuous as in [PauPS] is sufficient to have the HC-property, as we will see below.

Proposition 3.5. *The triple (X, Γ, \tilde{F}) satisfies the HC-property if one of the following conditions is satisfied:*

- \tilde{F} is constant,
- X is an \mathbb{R} -tree,
- X is a Riemannian manifold with pinched sectional curvature and \tilde{F} is Hölder-continuous.

Proof. The first two cases are treated in Lemma 3.3, and we may take for them $\kappa_1 = 0$ and $\kappa_2 = 1$ in the definition of the HC-property.

The claim for Riemannian manifolds follows from the property of at most linear growth of the Hölder-continuous maps (see Remark 3.1) and from [PauPS, Lem. 3.2], with constants $\kappa_1 > 0$ and $\kappa_2 \in]0, 1]$ depending only on the Hölder-continuity constants of \tilde{F} and on the bounds on the sectional curvature of X . \square

Remark 3.6. (1) If $X = \tilde{M}$ is a Riemannian manifold, then $T^1 X$ is naturally identified with the usual Riemannian unit tangent bundle of X . If the potential $\tilde{F} : T^1 \tilde{M} \rightarrow \mathbb{R}$ is Hölder-continuous for Sasaki's Riemannian metric on $T^1 \tilde{M}$, it is a potential as defined in [Rue1] and [PauPS]. Furthermore, the definition of $\int_x^y \tilde{F}$ coincides with the one in these references.

(2) The quotient function F is Hölder-continuous when \tilde{F} is Hölder-continuous.

Let $\tilde{F}, \tilde{F}^* : T^1X \rightarrow \mathbb{R}$ be potentials for Γ . We say that \tilde{F}^* is *cohomologous* to \tilde{F} (see for instance [Livš]) if there exists a continuous Γ -invariant function $\tilde{G} : T^1X \rightarrow \mathbb{R}$, such that, for every $\ell \in \mathcal{G}X$, the map $t \mapsto \tilde{G}(v_{\mathbf{g}^t\ell})$ is differentiable and

$$\tilde{F}^*(v_\ell) - \tilde{F}(v_\ell) = \frac{d}{dt}\Big|_{t=0} \tilde{G}(v_{\mathbf{g}^t\ell}). \quad (3.5)$$

A potential \tilde{F} is said to be *reversible* if \tilde{F}^+ and \tilde{F}^- are cohomologous.

When working with Hölder-continuous potentials, the regularity requirement is for \tilde{G} to also be Hölder-continuous. Note that the right-hand side of Equation (3.5) does not depend on the choice of the representative ℓ of its germ v_ℓ . In particular, $\text{Per}_F(\gamma) = \text{Per}_{F^*}(\gamma)$ for any loxodromic isometry γ if \tilde{F} and \tilde{F}^* are cohomologous potentials.

3.3 Poincaré series and critical exponents

Let us fix a potential $\tilde{F} : T^1X \rightarrow \mathbb{R}$ for Γ , and $x, y \in X$.

The *critical exponent* of (Γ, F) is the element $\delta = \delta_{\Gamma, F}$ of the extended real line $[-\infty, +\infty]$ defined by

$$\delta = \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \sum_{\gamma \in \Gamma, n-1 < d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \tilde{F}}.$$

The *Poincaré series* of (Γ, F) is the map $Q = Q_{\Gamma, F, x, y} : \mathbb{R} \rightarrow [0, +\infty]$ defined by

$$Q : s \mapsto \sum_{\gamma \in \Gamma} e^{\int_x^{\gamma y} (\tilde{F} - s)}.$$

If $\delta < +\infty$, we say that (Γ, F) is of *divergence type* if the series $Q_{\Gamma, F, x, y}(\delta)$ diverges, and of *convergence type* otherwise.

When $F = 0$, the critical exponent $\delta_{\Gamma, 0}$ is the usual critical exponent $\delta_\Gamma \in]0, +\infty]$ of Γ , the Poincaré series $Q_{\Gamma, 0, x, y}$ is the usual Poincaré series of Γ , and we recover the usual notion of divergence or convergence type of Γ .

The Poincaré series of (Γ, F) and its critical exponent make sense even if Γ is elementary (see for instance Lemma 3.7 (10)). The following result collects some of the basic properties of the critical exponent. The proofs from [PauPS, Lem. 3.3] generalise to the current setting, replacing the use of [PauPS, Lem. 3.2] by the HC-property.

Lemma 3.7. *Assume that (X, Γ, \tilde{F}) satisfies the HC-property. Then,*

- (1) *the critical exponent $\delta_{\Gamma, F}$ and the divergence or convergence of $Q_{\Gamma, F, x, y}(s)$ are independent of the points $x, y \in X$; they depend only on the cohomology class of \tilde{F} ;*
- (2) *$Q_{\Gamma, F \circ \iota, x, y} = Q_{\Gamma, F, y, x}$ and $\delta_{\Gamma, F \circ \iota} = \delta_{\Gamma, F}$; in particular, (Γ, F) is of divergence type if and only if $(\Gamma, F \circ \iota)$ is of divergence type;*
- (3) *the Poincaré series $Q(s)$ diverges if $s < \delta_{\Gamma, F}$ and converges if $s > \delta_{\Gamma, F}$;*
- (4) *$\delta_{\Gamma, F + \kappa} = \delta_{\Gamma, F} + \kappa$ for any $\kappa \in \mathbb{R}$, and (Γ, F) is of divergence type if and only if $(\Gamma, F + \kappa)$ is of divergence type;*
- (5) *if Γ' is a nonelementary subgroup of Γ , denoting by $F' : \Gamma' \backslash T^1X \rightarrow \mathbb{R}$ the map induced by \tilde{F} , then $\delta_{\Gamma', F'} \leq \delta_{\Gamma, F}$;*

(6) if $\delta_\Gamma < +\infty$, then $\delta_\Gamma + \inf_{\pi^{-1}(\mathcal{C}\Lambda\Gamma)} \tilde{F} \leq \delta_{\Gamma, F} \leq \delta_\Gamma + \sup_{\pi^{-1}(\mathcal{C}\Lambda\Gamma)} \tilde{F}$;

(7) $\delta_{\Gamma, F} > -\infty$;

(8) the map $\tilde{F} \mapsto \delta_{\Gamma, F}$ is convex, sub-additive, and 1-Lipschitz for the uniform norm on the vector space of real continuous maps on $\pi^{-1}(\mathcal{C}\Lambda\Gamma)$;⁴

(9) if Γ'' is a discrete cocompact group of isometries of X such that \tilde{F} is Γ'' -invariant, denoting by $F'' : \Gamma'' \backslash T^1 X \rightarrow \mathbb{R}$ the map induced by \tilde{F} , then

$$\delta_{\Gamma, F} \leq \delta_{\Gamma'', F''} ;$$

(10) if Γ is infinite cyclic, generated by a loxodromic isometry γ of X , then (Γ, F) is of divergence type and

$$\delta_{\Gamma, F} = \max \left\{ \frac{\text{Per}_F(\gamma)}{\lambda(\gamma)}, \frac{\text{Per}_{F \circ \iota}(\gamma)}{\lambda(\gamma)} \right\}. \quad \square$$

Examples 3.8. (1) If δ_Γ is finite and \tilde{F} is bounded, then the critical exponent δ is finite by Lemma 3.7 (6).

(2) If X is a Riemannian manifold with pinched negative curvature or when X has a compact quotient, then δ_Γ is finite. See for instance [Bou].

(3) There are examples of (X, Γ) with $\delta_\Gamma = +\infty$ (and hence $\delta = +\infty$ if \tilde{F} is constant), for instance when X is the complete ideal hyperbolic triangle complex with 3 ideal triangles along each edge, see [GaP], and Γ its isometry group. Hence the finiteness assumption of the critical exponent is nonempty in general. For the type of results treated in this book, it is however natural and essential.

We may replace upper limits by limits in the definition of the critical exponents, as follows.

Theorem 3.9. Assume that (X, Γ, \tilde{F}) satisfies the HC-property. If $c > 0$ is large enough, then

$$\delta = \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \sum_{\gamma \in \Gamma, n-c < d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \tilde{F}} .$$

If $\delta > 0$, then

$$\delta = \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \tilde{F}} .$$

Proof. The proofs of [PauPS, Theo. 4.2], either using the original arguments of [Rob1] valid when \tilde{F} is constant, or the super-multiplicativity arguments of [DaPS], extend, using the HC-property (see Definition 3.4) instead of [PauPS, Lem. 3.2]. \square

In what follows, we fix a potential \tilde{F} for Γ such that (X, Γ, \tilde{F}) satisfies the HC-property. We define $\tilde{F}^+ = \tilde{F}$ and $\tilde{F}^- = \tilde{F} \circ \iota$, we denote by $F^\pm : \Gamma \backslash T^1 X \rightarrow \mathbb{R}$ their induced maps, and we assume that $\delta = \delta_{\Gamma, F^+} = \delta_{\Gamma, F^-}$ is finite.

⁴That is, if $\tilde{F}, \tilde{F}^* : T^1 X \rightarrow \mathbb{R}$ are potentials for Γ satisfying the HC-property, inducing $F, F^* : \Gamma \backslash T^1 X \rightarrow \mathbb{R}$, and if $\delta_{\Gamma, F}, \delta_{\Gamma, F^*} < +\infty$, then $\delta_{\Gamma, tF+(1-t)F^*} \leq t\delta_{\Gamma, F} + (1-t)\delta_{\Gamma, F^*}$ for every $t \in [0, 1]$,

$$\begin{aligned} \delta_{\Gamma, F+F^*} &\leq \delta_{\Gamma, F} + \delta_{\Gamma, F^*} , \\ |\delta_{\Gamma, F^*} - \delta_{\Gamma, F}| &\leq \sup_{v \in \pi^{-1}(\mathcal{C}\Lambda\Gamma)} |\tilde{F}^*(v) - \tilde{F}(v)| . \end{aligned}$$

3.4 Gibbs cocycles

The (normalised) Gibbs cocycle associated with the group Γ and the potential \tilde{F}^\pm is the map $C^\pm = C_{\Gamma, F^\pm}^\pm : \partial_\infty X \times X \times X \rightarrow \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C_\xi^\pm(x, y) = \lim_{t \rightarrow +\infty} \int_y^{\xi_t} (\tilde{F}^\pm - \delta) - \int_x^{\xi_t} (\tilde{F}^\pm - \delta),$$

where $t \mapsto \xi_t$ is any geodesic ray with endpoint $\xi \in \partial_\infty X$.

We will prove in Proposition 3.10 below that this map is well defined, that is, the above limits exist for all $(\xi, x, y) \in \partial_\infty X \times X \times X$ and they are independent of the choice of the geodesic rays $t \mapsto \xi_t$. If $\tilde{F}^\pm = 0$, then $C^- = C^+ = \delta_\Gamma \beta$, where β is the Busemann cocycle. If X is an \mathbb{R} -tree, then

$$C_\xi^\pm(x, y) = \int_y^p (\tilde{F}^\pm - \delta) - \int_x^p (\tilde{F}^\pm - \delta), \quad (3.6)$$

where $p \in X$ is the point for which $[p, \xi[= [x, \xi[\cap [y, \xi[$; in particular, the map $\xi \mapsto C_\xi^\pm(x, y)$ is locally constant on the totally discontinuous space $\partial_\infty X$.

The Gibbs cocycles satisfy the following equivariance and cocycle properties: For all $\xi \in \partial_\infty X$ and $x, y, z \in X$, and for every isometry γ of X , we have

$$C_{\gamma\xi}^\pm(\gamma x, \gamma y) = C_\xi^\pm(x, y) \quad \text{and} \quad C_\xi^\pm(x, z) + C_\xi^\pm(z, y) = C_\xi^\pm(x, y). \quad (3.7)$$

For every $\ell \in \mathcal{G}X$, for all x and y on the image of the geodesic line ℓ , with ℓ_-, x, y, ℓ_+ in this order on ℓ , we have

$$C_{\ell_-}^-(x, y) = C_{\ell_+}^+(y, x) = -C_{\ell_+}^+(x, y) = \int_x^y (\tilde{F}^+ - \delta). \quad (3.8)$$

Proposition 3.10. *Assume that (X, Γ, \tilde{F}) satisfies the **HC**-property and that $\delta < +\infty$.*

- (1) *The maps $C^\pm : \partial_\infty X \times X \times X \rightarrow \mathbb{R}$ are well-defined.*
- (2) *For all $x, y \in X$ and $\xi \in \partial_\infty X$, if $d(x, y) \leq 1$, then*

$$|C_\xi^\pm(x, y)| \leq (\kappa_1 + \delta + \max_{\pi^{-1}(B(x, d(x, y)))} |\tilde{F}|) d(x, y)^{\kappa_2},$$

*with the constants κ_1, κ_2 of the **HC**-property. If X is an \mathbb{R} -tree, then for all $x, y \in X$ and $\xi \in \partial_\infty X$, we have*

$$|C_\xi^\pm(x, y)| \leq d(x, y) \max_{\pi^{-1}([x, y])} |\tilde{F}^\pm - \delta|.$$

- (3) *The maps $C^\pm : \partial_\infty X \times X \times X \rightarrow \mathbb{R}$ are locally Hölder-continuous (and locally Lipschitz when X is an \mathbb{R} -tree). In particular, they are continuous.*
- (4) *For all $r > 0$, $x, y \in X$ and $\xi \in \partial_\infty X$, if ξ belongs to the shadow $\mathcal{O}_x B(y, r)$ of the ball $B(y, r)$ seen from x , then with the constants κ_1, κ_2 of the **HC**-property,*

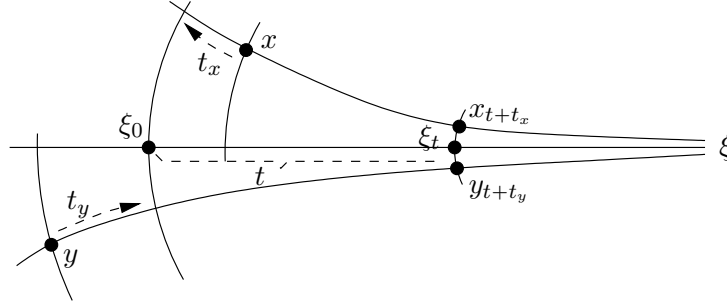
$$\left| C_\xi^\pm(x, y) + \int_x^y (\tilde{F}^\pm - \delta) \right| \leq 2(\kappa_1 + \delta + \max_{\pi^{-1}(B(y, r))} |\tilde{F}|) r^{\kappa_2}.$$

If X is an \mathbb{R} -tree, then

$$\left| C_\xi^\pm(x, y) + \int_x^y (\tilde{F}^\pm - \delta) \right| \leq 2r \max_{\pi^{-1}(B(y, r))} |\tilde{F}^\pm - \delta|.$$

Proof. (1) The fact that $C_{\xi}^{\pm}(x, y)$ is well defined when X is an \mathbb{R} -tree follows from Equation (3.6).

When X is not an \mathbb{R} -tree, let $\rho : t \mapsto \xi_t$ be any geodesic ray with endpoint $\xi \in \partial_{\infty} X$, let $t \mapsto x_t$ (respectively $t \mapsto y_t$) be the geodesic ray from x (respectively y) to ξ . Let $t_x = \beta_{\xi}(x, \xi_0)$ and $t_y = \beta_{\xi}(y, \xi_0)$, so that the quantity $\beta = t_y - t_x$ is equal to $\beta_{\xi}(y, x)$ (which is independent of ρ), and for every t big enough, we have $\beta_{\xi}(\xi_t, x_{t+t_x}) = \beta_{\xi}(\xi_t, y_{t+t_y}) = 0$.



Since X is CAT(-1), if t is big enough, then the distances $d(\xi_t, x_{t+t_x})$ and $d(\xi_t, y_{t+t_y})$ are at most one, and converge, in a nonincreasing way, exponentially fast to 0 as $t \rightarrow +\infty$. For $s \geq 0$, let $a_s = \int_y^{y_s} (\tilde{F}^{\pm} - \delta) - \int_x^{x_{s-\beta}} (\tilde{F}^{\pm} - \delta)$ (which is independent of ρ). We have, using Equation (HC),

$$\begin{aligned} & \left| \left(\int_y^{\xi_t} (\tilde{F}^{\pm} - \delta) - \int_x^{\xi_t} (\tilde{F}^{\pm} - \delta) \right) - a_{t+t_y} \right| \\ &= \left| \left(\int_y^{\xi_t} (\tilde{F}^{\pm} - \delta) - \int_y^{y_{t+t_y}} (\tilde{F}^{\pm} - \delta) \right) + \left(\int_x^{x_{t+t_x}} (\tilde{F}^{\pm} - \delta) - \int_x^{\xi_t} (\tilde{F}^{\pm} - \delta) \right) \right| \\ &\leq 2(\kappa_1 + \max_{\pi^{-1}(B(\xi_t, 1))} |\tilde{F} - \delta|) \max\{d(\xi_t, x_{t+t_x}), d(\xi_t, y_{t+t_y})\}^{\kappa_2}, \end{aligned}$$

which converges to 0 since \tilde{F} has subexponential growth by the assumptions of the HC-property. Hence in order to prove Assertion (1), we only have to prove that $\lim_{s \rightarrow +\infty} a_s$ exists.

For all $s \geq t \geq |\beta|$, we have, by the additivity of the integral along geodesics (see Equation (3.3)) and by using again Equation (HC),

$$\begin{aligned} |a_t - a_s| &= \left| \int_{y_t}^{y_s} (\tilde{F}^{\pm} - \delta) - \int_{x_{t-\beta}}^{x_{s-\beta}} (\tilde{F}^{\pm} - \delta) \right| \\ &\leq (\kappa_1 + \max_{\pi^{-1}(B(x_{t-\beta}, 1))} |\tilde{F} - \delta|) d(y_t, x_{t-\beta})^{\kappa_2} + (\kappa_1 + \max_{\pi^{-1}(B(y_s, 1))} |\tilde{F} - \delta|) d(y_s, x_{s+\beta})^{\kappa_2}. \end{aligned}$$

Again by the subexponential growth of \tilde{F} , the above expression converges to 0 as $t \rightarrow +\infty$ uniformly in s , hence $\lim_{s \rightarrow +\infty} a_s$ exists by a Cauchy type of argument.

(2) Let $(\xi, x, y) \in \partial_{\infty} X \times X \times X$. Assertion (2) of Proposition 3.10 follows from Equation (3.6) when X is an \mathbb{R} -tree, since $[x, y] = [x, p] \cup [p, y]$ where p be the closest point to y on $[x, \xi[$. When X is not an \mathbb{R} -tree, Assertion (2) follows immediately from the HC-property of $(X, \Gamma, \tilde{F}^{\pm} - \delta)$.

(3) Let $(\xi, x, y), (\xi', x', y') \in \partial_{\infty} X \times X \times X$. By the cocycle property (3.7), we have

$$|C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x', y')| \leq |C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x, y)| + |C_{\xi'}^{\pm}(x, x')| + |C_{\xi'}^{\pm}(y, y')|. \quad (3.9)$$

First assume that X is an \mathbb{R} -tree. Let K be a compact subset of X , and let

$$\epsilon_K = \inf_{x,y \in K} e^{-d(x,x_0)-d(x,y)} > 0.$$



Let p, q be the points in X such that $[x, \xi[\cap [y, \xi[= [p, \xi[$ and $[x, \xi[\cap [x, \xi'[= [x, q]$. If $d_{x_0}(\xi, \xi') \leq \epsilon_K$, then by the definition of the visual distance and by Equation (2.2), we have

$$e^{-d(x,q)} = d_x(\xi, \xi') \leq e^{-d(x,y)} \leq e^{-d(x,p)}.$$

In particular $q \in [p, \xi[$, so that $[x, \xi'[\cap [y, \xi'[= [p, \xi'[$. Thus by Equation (3.6), we have

$$|C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x, y)| = 0.$$

Therefore, by Equation (3.9) and by the \mathbb{R} -tree case of Assertion (2), if $d_{x_0}(\xi, \xi') \leq \epsilon_K$, if $x, y \in K$ and $d(x, x'), d(y, y') \leq 1$, then

$$|C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x', y')| \leq d(x, x') \max_{\pi^{-1}([x, x'])} |\tilde{F}^{\pm} - \delta| + d(y, y') \max_{\pi^{-1}([y, y'])} |\tilde{F}^{\pm} - \delta|.$$

Since \tilde{F} is bounded on compact subsets of $T^1 X$, this proves that C^{\pm} is locally Lipschitz.

Let us now consider the case when X is general. For all distinct $\xi, \xi' \in \partial_{\infty} X$, let $t \mapsto \xi_t$ and $t \mapsto \xi'_t$ be the geodesic rays from x_0 to ξ and ξ' respectively. By the end of the proof of Assertion (1), for every compact subset K of X , there exists $a_1, a_2 > 0$ such that for every $x, y \in K$, we have for $\eta \in \{\xi, \xi'\}$,

$$\left| C_{\eta}^{\pm}(x, y) - \left(\int_y^{\eta_t} (\tilde{F}^{\pm} - \delta) - \int_x^{\eta_t} (\tilde{F}^{\pm} - \delta) \right) \right| \leq a_1 e^{-a_2 t}.$$

Let $T = -\frac{1}{2} \ln d_{x_0}(\xi, \xi')$. If $T \geq 0$, by the properties of CAT(-1)-spaces, there exist constants $a_3, a_4 > 0$ such that $d(\xi_{2T}, \xi'_{2T}) \leq a_3$ and $d(\xi_T, \xi'_T) \leq a_4 e^{-T}$. Hence by Assertion (2), if $d_{x_0}(\xi, \xi') \leq \min\{\frac{1}{a_4}, 1\}$ (so that $T \geq 0$ and $d(\xi_T, \xi'_T) \leq 1$), we have

$$\begin{aligned} & |C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x, y)| \\ & \leq \left| \int_y^{\xi_T} (\tilde{F}^{\pm} - \delta) - \int_y^{\xi'_T} (\tilde{F}^{\pm} - \delta) \right| + \left| \int_x^{\xi_T} (\tilde{F}^{\pm} - \delta) - \int_x^{\xi'_T} (\tilde{F}^{\pm} - \delta) \right| + 2a_1 e^{-a_2 T} \\ & \leq 2(\kappa_1 + \max_{\pi^{-1}(B(\xi_T, a_4))} |\tilde{F}^{\pm} - \delta|) d(\xi_T, \xi'_T)^{\kappa_2} + 2a_1 e^{-a_2 T}. \end{aligned}$$

By the subexponential growth of \tilde{F} , there exists $a_5 > 0$ such that

$$|C_{\xi}^{\pm}(x, y) - C_{\xi'}^{\pm}(x, y)| \leq a_5 e^{-\frac{\kappa_2}{2} T} + 2a_1 e^{-a_2 T} \leq (a_5 + 2a_1) d_{x_0}(\xi, \xi')^{\min\{\frac{\kappa_2}{4}, \frac{a_2}{2}\}}.$$

We now conclude from Equation (3.9) and Assertion (2) as in the end of the above tree case that C^{\pm} is locally Hölder-continuous.

(4) Let $r > 0$, $x, y \in X$ and $\xi \in \partial_\infty X$ be such that $\xi \in \mathcal{O}_x B(y, r)$. Let p be the closest point to y on $[x, \xi[$, so that $d(p, y) \leq r$. By Equations (3.8) and (3.7), we have

$$\begin{aligned} \left| C_\xi^\pm(x, y) + \int_x^y (\tilde{F}^\pm - \delta) \right| &= \left| C_\xi^\pm(x, y) - C_\xi^\pm(x, p) - \int_x^p (\tilde{F}^\pm - \delta) + \int_x^y (\tilde{F}^\pm - \delta) \right| \\ &\leq |C_\xi^\pm(p, y)| + \left| \int_x^p (\tilde{F}^\pm - \delta) - \int_x^y (\tilde{F}^\pm - \delta) \right|. \end{aligned} \quad (3.10)$$

First assume that X is an \mathbb{R} -tree. Then by Assertion (2) and by Lemma 3.3, we deduce from Equation (3.10) that

$$\left| C_\xi^\pm(x, y) + \int_x^y (\tilde{F}^\pm - \delta) \right| \leq 2d(y, p) \max_{\pi^{-1}(\{y, p\})} |\tilde{F}^\pm - \delta| \leq 2r \max_{\pi^{-1}(B(y, r))} |\tilde{F}^\pm - \delta|.$$

In the general case, the result then follows similarly from Equation (3.10) by using Assertion (2) and the HC-property. \square

3.5 Systems of conductances on trees and generalised electrical networks

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices, let $X = |\mathbb{X}|_\lambda$ be its geometric realisation, and let Γ be a nonelementary discrete subgroup of $\text{Isom}(\mathbb{X}, \lambda)$.

Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant function, called a *system of (logarithmic) conductances* for Γ . We denote by $c : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ the function induced by $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$, which we also call a *system of conductances* on $\Gamma \backslash \mathbb{X}$.

Classically, an *electric network*⁵ (without sources or reactive elements) is a pair (G, e^e) , where G is a graph and $c : EG \rightarrow \mathbb{R}$ a function, such that c is *reversible*: $c(e) = c(\bar{e})$ for all $e \in EG$, see for example [NaW], [Zem]. In this text, we do not assume our system of conductances \tilde{c} to be reversible. In Chapter 6, we will even sometimes assume that the system of conductances is *anti-reversible*, that is, satisfying $c(\bar{e}) = -c(e)$ for every $e \in E\mathbb{X}$.

Two systems of conductances $\tilde{c}, \tilde{c}' : E\mathbb{X} \rightarrow \mathbb{R}$ are said to be *cohomologous*, if there exists a Γ -invariant map $f : V\mathbb{X} \rightarrow \mathbb{R}$ such that

$$\tilde{c}' - \tilde{c} = df,$$

where for all $e \in E\mathbb{X}$, we have

$$df(e) = \frac{f(t(e)) - f(o(e))}{\lambda(e)}.$$

Proposition 3.11. *Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances for Γ . There exists a potential \tilde{F} on $T^1 X$ for Γ such that for all $x, y \in V\mathbb{X}$, if (e_1, \dots, e_n) is the edge path in \mathbb{X} without backtracking such that $x = o(e_1)$ and $y = t(e_n)$, then*

$$\int_x^y \tilde{F} = \sum_{i=1}^n \tilde{c}(e_i) \lambda(e_i).$$

⁵A potential in this work is not the analog of a potential in an electric network, we follow the dynamical systems terminology as in for example [PauPS].

Proof. Any germ $v \in T^1X$ determines a unique edge e_v of the tree \mathbb{X} , the first one into which it enters: if ℓ is any geodesic line whose class in T^1X is v , the edge e_v is the unique edge of \mathbb{X} containing $\pi(v)$ whose terminal vertex is the first vertex of \mathbb{X} encountered at a positive time by ℓ . The function $\tilde{F} : T^1X \rightarrow \mathbb{R}$ defined by

$$\tilde{F}(v) = \frac{4\tilde{c}(e_v)}{\lambda(e_v)} \min \{d(\pi(v), o(e_v)), d(\pi(v), t(e_v))\} \quad (3.11)$$

is a potential on the \mathbb{R} -tree X , with $\tilde{F}(v) = 0$ if $\pi(v) \in V\mathbb{X}$.

Let us now compute $\int_x^y \tilde{F}$, for all $x, y \in X$. For every $\lambda > 0$, let $\psi_\lambda : [0, \lambda] \rightarrow \mathbb{R}$ be the continuous map defined by $\psi_\lambda(t) = \frac{t^2}{2}$ if $t \in [0, \frac{\lambda}{2}]$ and $\psi_\lambda(t) = \frac{\lambda^2}{4} - \frac{(\lambda-t)^2}{2}$ if $t \in [\frac{\lambda}{2}, \lambda]$. Let (e_0, e_1, \dots, e_n) be the edge path in \mathbb{X} without backtracking such that $x \in e_0 - \{t(e_0)\}$ and $y \in e_n - \{o(e_n)\}$. An easy computation shows that

$$\int_x^y \tilde{F} = \sum_{i=0}^{n-1} \tilde{c}(e_i) \lambda(e_i) + \frac{4\tilde{c}(e_n)}{\lambda(e_n)} \psi_{\lambda(e_n)}(d(y, o(e_n))) - \frac{4\tilde{c}(e_0)}{\lambda(e_0)} \psi_{\lambda(e_0)}(d(x, o(e_0))).$$

If x and y are vertices, the expression simplifies to the sum of the lengths of the edges weighted by the conductances. \square

We denote by \tilde{F}_c the potential defined by Equation (3.11) in the above proof, and by $F_c : \Gamma \backslash T^1X \rightarrow \mathbb{R}$ the induced potential. Note that F_c is bounded if c is bounded. We call \tilde{F}_c and F_c the potentials *associated with* the system of conductances \tilde{c} and c . This is by no means the unique potential with the property required in Proposition 3.11. The following result proves that the choice is unimportant.

Given a potential $\tilde{F} : T^1X \rightarrow \mathbb{R}$ for Γ , let us define a map $\tilde{c}_F : E\mathbb{X} \rightarrow \mathbb{R}$ by

$$\tilde{c}_F : e \mapsto \tilde{c}_F(e) = \frac{1}{\lambda(e)} \int_{o(e)}^{t(e)} \tilde{F}. \quad (3.12)$$

Note that \tilde{c}_F is Γ -invariant, hence it is a system of conductances for Γ . We denote by $c_F : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ the function induced by $\tilde{c}_F : E\mathbb{X} \rightarrow \mathbb{R}$. Note that $\tilde{c}_{F+\kappa} = \tilde{c}_F + \kappa$ for every constant $\kappa \in \mathbb{R}$, that \tilde{c}_F is bounded if \tilde{F} is bounded, and that $c_{F_c} = c$ by the above proposition.

Proposition 3.12. (1) *Every potential for Γ is cohomologous to a potential associated with a system of conductances for Γ .*

(2) *If two systems of conductances \tilde{c}' and \tilde{c} are cohomologous, then their associated potentials $\tilde{F}_{c'}$ and \tilde{F}_c are cohomologous.*

(3) *If \mathbb{X} has no vertex of degree 2, if two potentials \tilde{F}^* and \tilde{F} for Γ are cohomologous, then the systems of conductances \tilde{c}_{F^*} and \tilde{c}_F for Γ are cohomologous.*

Hence if \mathbb{X} has no vertex of degree 2, the map $[F] \mapsto [c_F]$ from the set of cohomology classes of potentials for Γ to the set of cohomology classes of systems of conductances for Γ is bijective, with inverse $[c] \mapsto [F_c]$.

Proof. (1) Let \tilde{F} be a potential for Γ , and let $\tilde{F}^* = \tilde{F}_{c_F}$ be the potential associated with the system of conductances \tilde{c}_F . For all $e \in E\mathbb{X}$ and $t \in]0, \lambda(e)[$, let $v_{e,t} \in T^1X$ be the germ of any geodesic line passing at time 0 through the point of e at distance t from $o(e)$. Let

$\tilde{G} : T^1X \rightarrow \mathbb{R}$ be the map defined by $G(v) = 0$ if $\pi(v) \in V\mathbb{X}$ and such that for all $e \in E\mathbb{X}$ and $t \in]0, \lambda(e)[$,

$$\tilde{G}(v_{e,t}) = \int_0^t (\tilde{F}^*(v_{e,s}) - \tilde{F}(v_{e,s})) ds .$$

Since $\int_{o(e)}^{t(e)} \tilde{F} = \lambda(e) \tilde{c}_F(e)$ by construction and $\lambda(e) \tilde{c}_F(e) = \int_{o(e)}^{t(e)} \tilde{F}^*$ by Proposition 3.11, the map $\tilde{G} : T^1X \rightarrow \mathbb{R}$ is continuous. Let ℓ be a geodesic line. The map $t \mapsto \tilde{G}(v_{gt\ell})$ is obviously differentiable at time $t = 0$ if $\pi(\ell) \notin V\mathbb{X}$, with derivative $\tilde{F}^*(v_\ell) - \tilde{F}(v_\ell)$. Since $\tilde{F}^*(v)$ vanishes if $\pi(v) \in V\mathbb{X}$ by Equation (3.11), and by continuity of \tilde{F} at such a point, this is still true if $\pi(\ell) \in V\mathbb{X}$. Hence \tilde{F}^* and \tilde{F} are cohomologous, and this proves the first claim.

(2) Assume that \tilde{c}' and \tilde{c} are cohomologous systems of conductances for Γ , and let $f : V\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant function such that $\tilde{c}' - \tilde{c} = df$. For all $e \in E\mathbb{X}$ and $t \in]0, \lambda(e)[$, define

$$\tilde{G}(v_{e,t}) = \lambda(e) \int_0^t (\tilde{F}_{c'}(v_{e,s}) - \tilde{F}_c(v_{e,s})) ds + f(o(e)) ,$$

which is Γ -invariant, whose limit as $t \rightarrow 0$ is $f(o(e))$ (independent of the edge e with given origin), and whose limit as $t \rightarrow \lambda(e)$ is

$$\lambda(e)(\tilde{c}'(e) - \tilde{c}(e)) + f(o(e)) = \lambda(e) df(e) + f(o(e)) = f(t(e))$$

(independent of the edge e with given extremity). As above, this proves that \tilde{G} is continuous, and that $F_{c'}$ and F_c are cohomologous.

(3) In order to prove the third claim, assume that \tilde{F}^* and \tilde{F} are two cohomologous potentials for Γ , and let $\tilde{G} : T^1X \rightarrow \mathbb{R}$ be as in the definition of cohomologous potentials, see Equation (3.5). By the continuity of \tilde{G} , for all elements v and v' in T^1X such that $\pi(v) = \pi(v') \in V\mathbb{X}$, we have $\tilde{G}(v) = \tilde{G}(v')$, since (by the assumption on the degrees of vertices) the two edges (possibly equal) into which v and v' enter can be extended to geodesic lines with a common negative subray. Hence for every $x \in V\mathbb{X}$, the value $f(x) = \tilde{G}(v_x)$ for every $v_x \in T^1X$ such that $\pi(v_x) = x$ does not depend on the choice of v_x . The map $f : V\mathbb{X} \rightarrow \mathbb{R}$ thus defined is Γ -invariant. With the above notation and by Equation (3.12), we hence have, for every $e \in E\mathbb{X}$,

$$\begin{aligned} \tilde{c}_{F^*}(e) - \tilde{c}_F(e) &= \frac{1}{\lambda(e)} \int_0^{\lambda(e)} (\tilde{F}^*(v_{e,t}) - \tilde{F}(v_{e,t})) dt = \frac{1}{\lambda(e)} \int_0^{\lambda(e)} \frac{d}{dt} \tilde{G}(v_{e,t}) dt \\ &= \frac{1}{\lambda(e)} (\tilde{G}(v_{t(e)}) - \tilde{G}(v_{o(e)})) = \frac{f(t(e)) - f(o(e))}{\lambda(e)} = df(e) . \end{aligned}$$

Hence \tilde{c}_{F^*} and \tilde{c}_F are cohomologous. □

Given a metric tree (X, λ) , we define the *critical exponent* of a Γ -invariant system of conductances $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ (or of the induced system of conductances $c : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$) as the critical exponent of (Γ, \tilde{F}_c) where \tilde{F}_c is the potential for Γ associated with \tilde{c} :

$$\delta_c = \delta_{\Gamma, F_c} .$$

By Proposition 3.12 (2) and Lemma 3.7 (1), this does not depend on the choice of a potential \tilde{F}_c satisfying Proposition 3.11.

Chapter 4

Patterson-Sullivan and Bowen-Margulis measures with potential on $\text{CAT}(-1)$ spaces

Let X, x_0, Γ be as in the beginning of Section 2.2,¹ and let \tilde{F} be a potential for Γ . From now on, we assume that the triple (X, Γ, \tilde{F}) satisfies the **HC**-property of Definition 3.4 and that the critical exponent $\delta = \delta_{\Gamma, F^\pm}$ is finite.

In this chapter, we discuss geometrically and dynamically relevant measures on the boundary at infinity of X and on the space of geodesic lines $\mathcal{G}X$. We extend the theory of Gibbs measures from the case of manifolds with pinched negative sectional curvature treated in [PauPS]² to $\text{CAT}(-1)$ spaces with the **HC**-property.

4.1 Patterson densities

A family $(\mu_x^\pm)_{x \in X}$ of finite nonzero (positive Borel) measures on $\partial_\infty X$, whose support is $\Lambda\Gamma$, is a (*normalised*) *Patterson density* for the pair (Γ, \tilde{F}^\pm) if

$$\gamma_* \mu_x^\pm = \mu_{\gamma x}^\pm \quad (4.1)$$

for all $\gamma \in \Gamma$ and $x \in X$, and if the following Radon-Nikodym derivatives exist for all $x, y \in X$ and satisfy for (almost) all $\xi \in \partial_\infty X$

$$\frac{d\mu_x^\pm}{d\mu_y^\pm}(\xi) = e^{-C_\xi^\pm(x, y)}. \quad (4.2)$$

In particular, the measures μ_x are in the same measure class for all $x \in X$, and, by Proposition 3.10, they depend continuously on x for the weak-star convergence of measures. Note that a Patterson density for (Γ, F^\pm) is also a Patterson density for $(\Gamma, F^\pm + s)$ for every $s \in \mathbb{R}$, since the definition involves only the normalised potential $\tilde{F}^\pm - \delta$. If $F = 0$, we get the usual notion of a Patterson-Sullivan density (of dimension δ_Γ) for the group Γ , see for instance [Pat2, Sul2, Nic, Coo, Bou, Rob2].

¹That is, X is a geodesically complete proper $\text{CAT}(-1)$ space, $x_0 \in X$ is a basepoint, and Γ is a nonelementary discrete group of isometries of X .

²See also the previous works [Led, Ham2, Cou, Moh].

Proposition 4.1. *There exists at least one Patterson density for the pair (Γ, \tilde{F}^\pm) .*

Proof. The Patterson construction (see [Pat1], [Coo]) modified as in [PauPS, Section 3.6] gives the result. \square

We refer to Theorem 4.5 for the uniqueness up to scalar multiple of the Patterson density when (Γ, F^\pm) is of divergence type and to [DaSU, Coro. 17.1.8] for a characterisation of the uniqueness.

The Patterson densities satisfy the following extension of the classical Sullivan shadow lemma (which gives the claim when \tilde{F} is constant, see [Rob2]), and its corollaries.

If μ is a Borel positive measure on a metric space (X, d) , the triple (X, d, μ) is called a *metric measure space*. A metric measure space (X, d, μ) is *doubling* if there exists $c \geq 1$ such that for all $x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq c \mu(B(x, r)).$$

Note that, up to changing c , the number 2 may be replaced by any constant larger than 1. See for instance [Hei]) for more details on doubling metric measure spaces. We refer for instance to [DaSU, Ex. 17.4.12] for examples of non-doubling Patterson(-Sullivan) measures, and to [DaSU, Prop. 17.4.4] for a characterisation of the doubling property of the Patterson measures when Γ is geometrically finite and $F = 0$.

A family $(X, \mu_i, d_i)_{i \in I}$ of Borel positive measures μ_i and distances d_i on a common set X is called *uniformly doubling* if there exists $c \geq 1$ such that for all $i \in I$, $x \in X$ and $r > 0$

$$\mu_i(B_{d_i}(x, 2r)) \leq c \mu_i(B_{d_i}(x, r)).$$

Lemma 4.2. *Let $(\mu_x^\pm)_{x \in X}$ be a Patterson density for the pair (Γ, F^\pm) , and let K be a compact subset of X .*

- (1) [Mohsen's shadow lemma] *If R is large enough, there exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$,*

$$\frac{1}{C} e^{\int_x^{\gamma y} (\tilde{F}^\pm - \delta)} \leq \mu_x^\pm(\mathcal{O}_x B(\gamma y, R)) \leq C e^{\int_x^{\gamma y} (\tilde{F}^\pm - \delta)}.$$

- (2) *For all $x, y \in X$, there exists $c > 0$ such that for every $n \in \mathbb{N}$*

$$\sum_{\gamma \in \Gamma : n-1 < d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} (\tilde{F}^\pm - \delta)} \leq c.$$

- (3) *For every $R > 0$ large enough, there exists $C = C(R) > 0$ such that for all $\gamma \in \Gamma$ and all $x, y \in K$*

$$\mu_x^\pm(\mathcal{O}_x B(\gamma y, 5R)) \leq C \mu_x^\pm(\mathcal{O}_x B(\gamma y, R)).$$

- (4) *If Γ is convex-cocompact, then the metric measure space $(\Lambda\Gamma, d_x, \mu_x^\pm)$ is doubling for every x in X , and the family of metric measure spaces $(\Lambda\Gamma, d_x, \mu_x^\pm)_{x \in C\Lambda\Gamma}$ is uniformly doubling.*

Proof. For the first assertion, the proof of [PauPS, Lem. 3.10] (see also [Cou, Lem. 4] with the multiplicative rather than additive convention, as well as [Moh]) extends, using Proposition 3.10 (2), (4) instead of [PauPS, Lem. 3.4 (i), (ii)]. The second assertion is similar to the one of [PauPS, Lem. 3.11 (i)], and the proof of the last two assertions is similar to the one of [PauPS, Prop. 3.12], using Lemma 2.2 instead of [HeP4, Lem. 2.1]. The uniformity in the last assertion follows from the compactness of $\Gamma \backslash \mathcal{C}\Lambda\Gamma$ and the invariance and continuity properties of the Patterson densities. \square

4.2 Gibbs measures

We fix from now on two Patterson densities $(\mu_x^\pm)_{x \in X}$ for the pairs (Γ, F^\pm) .

The *Gibbs measure* \tilde{m}_F on $\mathcal{G}X$ (associated with this ordered pair of Patterson densities) is the measure \tilde{m}_F on $\mathcal{G}X$ given by the density

$$d\tilde{m}_F(\ell) = e^{C_{\ell_-}^-(x_0, \ell(0)) + C_{\ell_+}^+(x_0, \ell(0))} d\mu_{x_0}^-(\ell_-) d\mu_{x_0}^+(\ell_+) dt \quad (4.3)$$

in Hopf's parametrisation with respect to the basepoint x_0 . The Gibbs measure \tilde{m}_F is independent of x_0 , and it is invariant under the actions of the group Γ and of the geodesic flow. Thus, it defines a measure m_F on $\Gamma \backslash \mathcal{G}X$ which is invariant under the quotient geodesic flow, called the *Gibbs measure* on $\Gamma \backslash \mathcal{G}X$. The proofs of these claims are analogous to the ones of [PauPS, §3.7]. If $F = 0$ and the Patterson densities $(\mu_x^+)_{x \in X}$ and $(\mu_x^-)_{x \in X}$ coincide, the Gibbs measure m_F coincides with the *Bowen-Margulis measure* m_{BM} (associated with this Patterson density), see for instance [Rob2].

Remark 4.3. The (Borel positive) measure given by the density

$$d\lambda(\xi, \eta) = e^{C_\xi^-(x_0, p) + C_\eta^+(x_0, p)} d\mu_{x_0}^-(\xi) d\mu_{x_0}^+(\eta) \quad (4.4)$$

on $\partial_\infty^2 X$ is independent of $p \in]\xi, \eta[$, locally finite and invariant under the diagonal action of Γ on $\partial_\infty^2 X$. It is a *geodesic current* for the action of Γ on the Gromov-hyperbolic proper metric space X in the sense of Ruelle-Sullivan-Bonahon, see for instance [Bon] and references therein.

Let us now indicate why the terminology of Gibbs measures is indeed appropriate. This explanation will be the point of Proposition 4.4, but we need to give some definitions first.

For all $\ell \in \mathcal{G}X$ and $r > 0$, $T, T' \geq 0$, the *dynamical (or Bowen) ball* around ℓ is

$$B(\ell; T, T', r) = \{ \ell' \in \mathcal{G}X : \sup_{t \in [-T', T]} d(\ell(t), \ell'(t)) < r \} .$$

Bowen balls have the following invariance properties: for all $s \in [-T', T]$ and $\gamma \in \Gamma$,

$$\mathbf{g}^s B(\ell; T, T', r) = B(\mathbf{g}^s \ell; T - s, T' + s, r) \quad \text{and} \quad \gamma B(\ell; T, T', r) = B(\gamma \ell; T, T', r) .$$

The following inclusion properties of the dynamical balls are immediate: If $r \leq s$, $T \geq S$, $T' \geq S'$, then $B(\ell; T, T', r)$ is contained in $B(\ell; S, S', s)$. The dynamical balls are almost independent on r (see [PauPS, Lem. 3.14]): For all $r' \geq r > 0$, there exists $T_{r, r'} \geq 0$ such that

for all $\ell \in \mathcal{G}X$ and $T, T' \geq 0$, the dynamical ball $B(\ell; T + T_{r,r'}, T' + T_{r,r'}, r')$ is contained in $B(\ell; T, T', r)$.

For every $\ell \in \Gamma \backslash \mathcal{G}X$, let us define $B(\ell; T, T', r')$ as the image by the canonical projection $\mathcal{G}X \rightarrow \Gamma \backslash \mathcal{G}X$ of $B(\tilde{\ell}; T, T', r')$, for any preimage $\tilde{\ell}$ of ℓ in $\mathcal{G}X$.

A $(\mathbf{g}^t)_{t \in \mathbb{R}}$ -invariant measure m' on $\Gamma \backslash \mathcal{G}X$ satisfies the Gibbs property for the potential F with constant $c(F) \in \mathbb{R}$ if for every compact subset K of $\Gamma \backslash \mathcal{G}X$, there exist $r > 0$ and $c_{K,r} \geq 1$ such that for all large enough $T, T' \geq 0$, for every $\ell \in \Gamma \backslash \mathcal{G}X$ with $\mathbf{g}^{-T'}\ell, \mathbf{g}^T\ell \in K$, we have

$$\frac{1}{c_{K,r}} \leq \frac{m'(B(\ell; T, T', r))}{e^{\int_{-T'}^T (F(v_{\mathbf{g}^t\ell}) - c(F)) dt}} \leq c_{K,r}.$$

We refer to [PauPS, Sect. 3.8] for equivalent variations on the definition of the Gibbs property. The following result shows that the Gibbs measures indeed satisfy the Gibbs property on the dynamical balls of the geodesic flow, thereby justifying their names. We refer for instance to [PauPS, Sect. 3.8] for the explanations of the connection with symbolic dynamics mentioned in the introduction. See also Proposition 4.12 for a discussion of the case when X is a simplicial tree – here the correspondence with symbolic dynamics is particularly clear.

Proposition 4.4. *Let m_F be the Gibbs measure on $\Gamma \backslash \mathcal{G}X$ associated with a pair of Patterson densities $(\mu_x^\pm)_{x \in \tilde{M}}$ for (Γ, \tilde{F}^\pm) . Then m_F satisfies the Gibbs property for the potential F , with constant $c(F) = \delta$.*

Proof. The proof is similar to the one of [PauPS, Prop. 3.16] (in which the key Lemma 3.17 uses only CAT(−1) arguments), up to replacing the use of [PauPS, Lem. 3.4 (1)] by finitely many applications of Proposition 3.10 (2). \square

The basic ergodic properties of the Gibbs measures are summarised in the following result. The case when \tilde{F} is constant is due to [Rob2], see also [BuM, §6].

Theorem 4.5 (Hopf-Tsuji-Sullivan-Roblin). *The following conditions are equivalent*

- (i) *The pair (Γ, F) is of divergence type.*
- (ii)[−] *The conical limit set of Γ has positive measure with respect to μ_x^- for some (equivalently every) $x \in X$.*
- (ii)⁺ *The conical limit set of Γ has positive measure with respect to μ_x^+ for some (equivalently every) $x \in X$.*
- (iii) *The dynamical system $(\partial_\infty^2 X, \Gamma, (\mu_x^- \otimes \mu_x^+)_{|\partial_\infty^2 X})$ is ergodic and conservative for some (equivalently every) $x \in X$.*
- (iv) *The dynamical system $(\Gamma \backslash \mathcal{G}X, (\mathbf{g}^t)_{t \in \mathbb{R}}, m_F)$ is ergodic and conservative.*

If one of the above conditions is satisfied, then

- (1) *the measures μ_x^\pm have no atoms for any $x \in X$,*
- (2) *the diagonal of $\partial_\infty X \times \partial_\infty X$ has measure 0 for $\mu_x^- \otimes \mu_x^+$,*
- (3) *the Patterson densities $(\mu_x^\pm)_{x \in X}$ are unique up to a scalar multiple, and*

(4) for all $x, y \in X$, as $n \rightarrow +\infty$,

$$\max_{\gamma \in \Gamma, n-1 < d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \tilde{F}^\pm} = o(e^{\delta n}).$$

Proof. The proof of the equivalence claim is similar to the one of [PauPS, Theo. 5.4], using Proposition 3.10 (2), (4) instead of [PauPS, Lem. 3.4], and Lemma 4.2 instead of [PauPS, Lem. 3.10]. The claims (1) to (4) are proved as in [PauPS, Sect. 5.3]. \square

The following corollary follows immediately from Poincaré's recurrence theorem and the Hopf-Tsuji-Sullivan-Roblin theorem, see [PauPS, Thm. 5.15] for the argument written for the manifold case.

Corollary 4.6. *If m_F is finite, then*

- (1) *the pair (Γ, F^\pm) is of divergence type,*
- (2) *the Patterson densities $(\mu_x^\pm)_{x \in X}$ are unique up to a multiplicative constant and the Gibbs measure m_F is uniquely defined up to a multiplicative constant.*
- (3) *the support of m_F is the image $\Omega\Gamma$ in $\Gamma \backslash \mathcal{G}X$ of $\tilde{\Omega}\Gamma = \{\ell \in \mathcal{G}X : \ell_\pm \in \Lambda_c\Gamma\}$, and*
- (4) *the geodesic flow is ergodic for m_F .* \square

As the finiteness of the Gibbs measures will be a standing hypothesis in many of the following results, we now give criteria for Gibbs measures to be finite. Recall (see Section 2.2) that the discrete nonelementary group of isometries Γ of X is geometrically finite if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic limit point of Γ .

Theorem 4.7. *Assume that Γ is a geometrically finite discrete group of isometries of X .*

- (1) *If (Γ, F^\pm) is of divergence type, then the Gibbs measure m_F is finite if and only if for every bounded parabolic limit point p of Γ , the series*

$$\sum_{\alpha \in \Gamma_p} d(x, \alpha y) e^{\int_x^{\alpha y} (\tilde{F}^\pm - \delta)}$$

converges, where Γ_p is the stabiliser of p in Γ .

- (2) *If we have $\delta_{\Gamma_p, F_p^\pm} < \delta$, for every bounded parabolic limit point p of Γ with stabiliser Γ_p in Γ and with $F_p^\pm : \Gamma_p \backslash X \rightarrow \mathbb{R}$ the map induced by \tilde{F}^\pm , then (Γ, F) is of divergence type. In particular, m_F is finite.*

When X is a manifold, this result is due to [DaOP, Theo. B] for the case $F = 0$, and to [Cou] and [PauPS, Theo. 8.3, 8.4] for the general case of Hölder-continuous potentials. When $F = 0$ but on much more general assumptions on X with optimal generality, this result is due to [DaSU, Theo. 17.1.2].

Proof. The proof is similar to the manifold case in [PauPS], which follows closely the proof of [DaOP]. Note that the convergence or divergence of the above series does not depend on the choice of the sign \pm .

Let Par_Γ be the set of bounded parabolic limit points of Γ . By [Rob2, Lem. 1.9]³, there exists a Γ -equivariant family $(\mathcal{H}_p)_{p \in \text{Par}_\Gamma}$ of pairwise disjoint closed horoballs, with \mathcal{H}_p centred at p , such that the quotient

$$M_0 = \Gamma \backslash (\mathcal{C}\Lambda\Gamma - \bigcup_{p \in \text{Par}_\Gamma} \mathcal{H}_p)$$

is compact. Using Theorem 4.5, the HC-property and Proposition 3.10 instead of [PauPS, Lem. 3.4], the proofs of [PauPS, Theo. 8.3, 8.4] then extend to our situation. \square

Recall that the *length spectrum* of Γ on X is the subgroup of \mathbb{R} generated by the translation lengths in X of the elements of Γ .

Recall that a continuous-time 1-parameter group $(h^t)_{t \in \mathbb{R}}$ of homeomorphisms of a topological space Z is *topologically mixing* if for all nonempty open subsets U, V of Z , there exists $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$, we have $U \cap h^t V \neq \emptyset$.

We have the following result, due to [Bab1, Thm. 1] in the manifold case, with developments by [Rob2] when $\tilde{F} = 0$, and by [PauPS, Sect. 8] for manifolds with pinched negative curvature.

Theorem 4.8. *If the Gibbs measure is finite, then the following assertions are equivalent :*

- (1) *the geodesic flow of $\Gamma \backslash X$ is mixing for the Gibbs measure,*
- (2) *the geodesic flow of $\Gamma \backslash X$ is topologically mixing on its nonwandering set, which is the quotient under Γ of the space of geodesic lines in X both of whose endpoints belong to $\Lambda\Gamma$.*
- (3) *the length spectrum of Γ on X is not contained in a discrete subgroup of \mathbb{R} .* \square

In the manifold case, the third assertion of Theorem 4.8 is satisfied, for example, if Γ has a parabolic element, if $\Lambda\Gamma$ is not totally disconnected (hence if $\Gamma \backslash X$ is compact), or if X is a surface or a (rank-one) symmetric space, see for instance [Dal1, Dal2].

Error terms for the mixing property will be described in Chapter 9. The above result holds for the continuous time geodesic flow when X is a metric tree. See Proposition 4.15 for a version of this theorem for the discrete time geodesic flow on simplicial trees. At least when X is an \mathbb{R} -tree and Γ is a uniform lattice (so that $\Gamma \backslash X$ is a finite metric graph), we have a stronger result under additional regularity assumptions, see Section 9.2.

We end this Section by an elementary remark on the independence of Gibbs measures upon replacement of the potential F by a cohomologous one.

Remark 4.9. Let $\tilde{F}^* : T^1X \rightarrow \mathbb{R}$ be a potential for Γ cohomologous to \tilde{F} and satisfying the (HC)-property. As usual, let $\tilde{F}^{*+} = \tilde{F}^*$ and $\tilde{F}^{*-} = \tilde{F}^* \circ \iota$, and let $F^* : \Gamma \backslash T^1X \rightarrow \mathbb{R}$ be the induced map. Let $\tilde{G} : T^1X \rightarrow \mathbb{R}$ be a continuous Γ -invariant function such that, for every $\ell \in \mathcal{G}X$, the map $t \mapsto \tilde{G}(v_{g^t\ell})$ is differentiable and $\tilde{F}^*(v_\ell) - \tilde{F}(v_\ell) = \frac{d}{dt}|_{t=0} \tilde{G}(v_{g^t\ell})$.

For all $x \in X$ and $\xi \in \partial_\infty X$, let $\ell_{x,\xi}$ be any geodesic line with footpoint $\ell_{x,\xi}(0) = x$ and endpoint $(\ell_{x,\xi})_+ = \xi$, and let $\ell_{\xi,x}$ be any geodesic line with $\ell_{\xi,x}(0) = x$ and origin $(\ell_{\xi,x})_- = \xi$. Note that the value $\tilde{G}(v_{\ell_{x,\xi}})$ is independent of the choice of $\ell_{x,\xi}$, by the continuity of \tilde{G} , and similarly for $\tilde{G}(v_{\ell_{\xi,x}})$. In particular, for all $\gamma \in \Gamma$, by the Γ -invariance of \tilde{G} , we have

$$\tilde{G}(v_{\ell_{x,\gamma^{-1}\xi}}) = \tilde{G}(v_{\ell_{\gamma x,\xi}}) \quad \text{and} \quad \tilde{G} \circ \iota(v_{\ell_{x,\xi}}) = \tilde{G}(v_{\ell_{\xi,x}}). \quad (4.5)$$

³See also [Pau2] for the case of simplicial trees and [DaSU, Thm. 12.4.5] for a greater generality on X .

Note that $\tilde{F}^{*-} = \tilde{F}^* \circ \iota$ and $\tilde{F}^- = \tilde{F} \circ \iota$ are cohomologous, since if $\tilde{G}^* = -\tilde{G} \circ \iota$, for every $\ell \in \mathcal{G}X$, we have

$$\begin{aligned} \tilde{F}^* \circ \iota(v_\ell) - \tilde{F} \circ \iota(v_\ell) &= \tilde{F}^*(v_{\iota\ell}) - \tilde{F}(v_{\iota\ell}) = \frac{d}{dt}|_{t=0} \tilde{G}(v_{\mathbf{g}^t \iota\ell}) \\ &= \frac{d}{dt}|_{t=0} \tilde{G}(\iota v_{\mathbf{g}^{-t}\ell}) = \frac{d}{dt}|_{t=0} \tilde{G}^*(v_{\mathbf{g}^t \ell}). \end{aligned}$$

As already seen in Lemma 3.7, the critical exponent $\delta_{\Gamma, F^{*\pm}}$ is equal to the critical exponent δ_{Γ, F^\pm} , and independent of the choice of \pm , and we denote by δ the common value in the definition of the Gibbs cocycle.

Note that if $C^{*\pm} = C_{\Gamma, F^{*\pm}}^\pm$ is the Gibbs cocycle associated with $(\Gamma, F^{*\pm})$, then $C^{*\pm}$ and C^\pm are cohomologous:

$$C_\xi^{*+}(x, y) - C_\xi^+(x, y) = \tilde{G}(v_{\ell_y, \xi}) - \tilde{G}(v_{\ell_x, \xi}), \quad (4.6)$$

and similarly

$$C_\xi^{*-}(x, y) - C_\xi^-(x, y) = \tilde{G}^*(v_{\ell_y, \xi}) - \tilde{G}^*(v_{\ell_x, \xi}). \quad (4.7)$$

Let $(\mu_x^\pm)_{x \in \tilde{M}}$ be a Patterson density for (Γ, F^\pm) . In order to simplify the notation, let $\tilde{G}^+ = \tilde{G}$ and $\tilde{G}^- = \tilde{G}^*$. The family of measures $(\mu_x^{*\pm})_{x \in X}$ defined by setting, for all $x \in X$ and $\xi \in \partial_\infty X$,

$$d\mu_x^{*\pm}(\xi) = e^{\tilde{G}^\pm(v_{\ell_x, \xi})} d\mu_x^\pm(\xi), \quad (4.8)$$

is also a Patterson density for $(\Gamma, F^{*\pm})$. Indeed, the equivariance property (4.1) for $(\mu_x^{*\pm})_{x \in X}$ follows from the one for $(\mu_x^\pm)_{x \in X}$ and from Equation (4.5). The absolutely continuous property (4.2) for $(\mu_x^{*\pm})_{x \in X}$ follows from the one for $(\mu_x^\pm)_{x \in X}$ and Equations (4.6) and (4.7).

Assume that the Patterson density for $(\Gamma, F^{*\pm})$ defined by Equation (4.8) is chosen in order to construct the Gibbs measure \tilde{m}_{F^*} for (Γ, F^*) on $\mathcal{G}X$. Then using

- Hopf's parametrisation with respect to the base point x_0 and Equation (4.3) with F replaced by F^* for the first equality,
- Equations (4.6), (4.7), (4.8) and cancellations for the second equality,
- the definition of $\tilde{G}^* = -\tilde{G} \circ \iota$ and again Equation (4.3) for the third equality,
- Equation (4.5) so that $\iota v_{\ell_{\ell(0), \ell_-}} = v_{\ell_{-\ell(0)}}$, and the fact that we may choose $\ell_{-\ell(0)} = \ell$ and $\ell_{\ell(0), \ell_+} = \ell$ for the last equality,

we have

$$\begin{aligned} d\tilde{m}_{F^*}(\ell) &= e^{C_{\ell_-}^{*-}(x_0, \ell(0)) + C_{\ell_+}^{*+}(x_0, \ell(0))} d\mu_{x_0}^{*-}(\ell_-) d\mu_{x_0}^{*+}(\ell_+) dt \\ &= e^{C_{\ell_-}^-(x_0, \ell(0)) + \tilde{G}^*(v_{\ell_{\ell(0), \ell_-})} + C_{\ell_+}^+(x_0, \ell(0)) + \tilde{G}(v_{\ell_{\ell(0), \ell_+})} d\mu_{x_0}^-(\ell_-) d\mu_{x_0}^+(\ell_+) dt \\ &= e^{-\tilde{G} \circ \iota(v_{\ell_{\ell(0), \ell_-})} + \tilde{G}(v_{\ell_{\ell(0), \ell_-})} d\tilde{m}_F(\ell) \\ &= e^{-\tilde{G}(v_\ell) + \tilde{G}(v_\ell)} d\tilde{m}_F(\ell), \end{aligned}$$

hence $\tilde{m}_{F^*} = \tilde{m}_F$.

In particular, since the Gibbs measure, when finite, is independent up to a multiplicative constant on the choice of the Patterson densities by Corollary 4.6, we have that m_F is finite if and only if m_{F^*} is finite, and then

$$\frac{m_{F^*}}{\|m_{F^*}\|} = \frac{m_F}{\|m_F\|}. \quad (4.9)$$

4.3 Patterson densities for simplicial trees

In this Section and the following one, we specialise and modify the general framework of the previous sections to treat simplicial trees. Recall⁴ that a simplicial tree \mathbb{X} is a metric tree whose edge length map is constant equal to 1. The time 1 map of the geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{R}}$ on the space $\check{\mathcal{G}}X$ of all generalised geodesic lines of the geometric realisation $X = |\mathbb{X}|_1$ of \mathbb{X} preserves for instance its subset of generalised geodesic lines whose footpoints are at distance at most $1/4$ from vertices. Since both this subset and its complement have nonempty interior in $\check{\mathcal{G}}X$, the geodesic flow on $\check{\mathcal{G}}X$ has no mixing or ergodic measure with full support. This is why we considered the discrete time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{Z}}$ on $\check{\mathcal{G}}\mathbb{X}$ in Section 2.7.

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$. Let $\tilde{F} : T^1 X \rightarrow \mathbb{R}$ be a potential for Γ , and let $\tilde{F}^+ = \tilde{F}$, $\tilde{F}^- = \tilde{F} \circ \iota$. Let $C^\pm : \partial_\infty X \times X \times X \rightarrow \mathbb{R}$ be the associated Gibbs cocycles. Let $(\mu_x^\pm)_{x \in X}$ be two Patterson densities on $\partial_\infty X$ for the pairs (Γ, F^\pm)

Note that only the restrictions of the cocycles C^\pm to $\partial_\infty X \times V\mathbb{X} \times V\mathbb{X}$ are useful and that it is often convenient and always sufficient to replace the cocycles by finite sums involving a system of conductances (as defined in Section 3.5), see below. Furthermore, only the restriction $(\mu_x^\pm)_{x \in V\mathbb{X}}$ of the family of Patterson densities to the set of vertices of \mathbb{X} is useful.

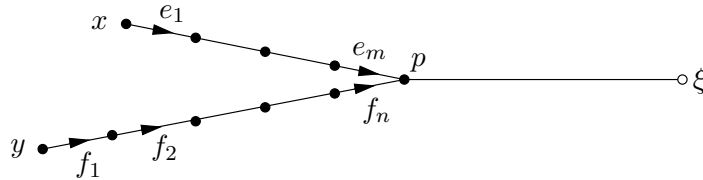
Example 4.10. Let \mathbb{X} be a simplicial tree and let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances on \mathbb{X} . For all x, y in $V\mathbb{X}$ and $\xi \in \partial_\infty \mathbb{X}$, with the usual convention on the empty sums, let

$$c_\xi^+(x, y) = \sum_{i=1}^m \tilde{c}(e_i) - \sum_{j=1}^n \tilde{c}(f_j)$$

and

$$c_\xi^-(x, y) = \sum_{i=1}^m \tilde{c}(\bar{e}_i) - \sum_{j=1}^n \tilde{c}(\bar{f}_j),$$

where, if $p \in V\mathbb{X}$ is such that $[p, \xi[= [x, \xi[\cap [y, \xi[$, then (e_1, e_2, \dots, e_m) is the geodesic edge path in \mathbb{X} from $x = o(e_1)$ to $p = t(e_m)$ and (f_1, f_2, \dots, f_n) is the geodesic edge path in \mathbb{X} from $y = o(f_1)$ to $p = t(f_n)$.



With δ_c defined in the end of Section 3.5 and with C^\pm the Gibbs cocycles for (Γ, \tilde{F}_c) , by Proposition 3.11, we have, for all $\xi \in \partial_\infty \mathbb{X}$ and $x, y \in V\mathbb{X}$,

$$C_\xi^\pm(x, y) = -c_\xi^\pm(x, y) + \delta_c d(x, y),$$

and Equation (4.2) gives

$$d\mu_x^\pm(\xi) = e^{c_\xi^\pm(x, y) - \delta_c d(x, y)} d\mu_y^\pm(\xi).$$

⁴See Section 2.7.

Using the particular structure of trees, we can prove a version of the Shadow Lemma 4.2 where one can take the radius R to be 0. When $F = 0$, this result is due to Coornaert [Coo].

Lemma 4.11 (Mohsen's shadow lemma for trees). *Let K be a finite subset of $V\mathbb{X}$. There exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$ with $y \in \mathcal{C}\Lambda\Gamma$, we have*

$$\frac{1}{C} e^{\int_x^{\gamma y} (\tilde{F}^\pm - \delta)} \leq \mu_x^\pm(\mathcal{O}_x\{\gamma y\}) \leq C e^{\int_x^{\gamma y} (\tilde{F}^\pm - \delta)}.$$

Proof. The structure of the proof is the same as in Lemma 4.2 with some differences in the details towards the end of the argument. Note that $C_\xi^\pm(x, \gamma y) + \int_x^{\gamma y} (\tilde{F}^\pm - \delta) = 0$ if $\xi \in \mathcal{O}_x\{\gamma y\}$ (that is, if $\gamma y \in [x, \xi[$), by Equation (3.8).

First, one argues as in the proof of [PauPS, Lem. 3.10], that it suffices to prove that there exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$ with $\gamma y \in \mathcal{C}\Lambda\Gamma$, we have

$$\frac{1}{C} \leq \mu_{\gamma y}^\pm(\mathcal{O}_x\{\gamma y\}) \leq C.$$

Now, the argument for proving the upper bound is the same as in loc. cit., using Proposition 3.10 (2) instead of [PauPS, Lem. 3.4 (i)].

In order to prove the lower bound, we assume by contradiction that there exist sequences $(x_i)_{i \in \mathbb{N}}$, $(y_i)_{i \in \mathbb{N}}$ in K with $y_i \in \mathcal{C}\Lambda\Gamma$ and $(\gamma_i)_{i \in \mathbb{N}}$ in Γ such that $\mu_{\gamma_i y_i}^\pm(\mathcal{O}_{x_i}\{\gamma_i y_i\})$ converges to 0 as $i \rightarrow +\infty$. Up to extracting a subsequence, since K is finite, we may assume that the sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ are constant, say with value x and y respectively. Since $y \in \mathcal{C}\Lambda\Gamma$, as every point in $\mathcal{C}\Lambda\Gamma$ belongs to at least one geodesic line between two limit points of Γ , the geodesic segment from x to $\gamma_i y$ may be extended to a geodesic ray from x to a limit point. Since the support of μ_z^\pm is equal to $\Lambda\Gamma$ for any $z \in X$, we have $\mu_{\gamma_i y}^\pm(\mathcal{O}_x\{\gamma_i y\}) > 0$ for all $i \in \mathbb{N}$. Thus, up to taking a subsequence, we can assume that $\gamma_i^{-1}x$ converges to $\xi \in \Lambda\Gamma$ (otherwise by discreteness, we may extract a subsequence so that $(\gamma_i)_{i \in \mathbb{N}}$ is constant, and $\mu_{\gamma_i y}^\pm(\mathcal{O}_x\{\gamma_i y\})$ cannot converge to 0).

Since \mathbb{X} is a tree, there exists a positive integer N such that $\mathcal{O}_{\gamma_i^{-1}x}\{y\} = \mathcal{O}_{\gamma_N^{-1}x}\{y\} = \mathcal{O}_\xi\{y\}$ for all $i \geq N$. As above, $\mathcal{O}_\xi\{y\}$ meets $\Lambda\Gamma$ since $y \in \mathcal{C}\Lambda\Gamma$, thus $\mu_y^\pm(\mathcal{O}_\xi\{y\}) > 0$. But for every $i \geq N$,

$$\mu_y^\pm(\mathcal{O}_\xi\{y\}) = (\gamma_i^{-1})_* \mu_{\gamma_i y}^\pm(\mathcal{O}_{\gamma_i^{-1}x}\{y\}) = \mu_{\gamma_i y}^\pm(\mathcal{O}_x\{\gamma_i y\})$$

tends to 0 as $i \rightarrow +\infty$, a contradiction. \square

Let $\tilde{\phi}_\mu^\pm : V\mathbb{X} \rightarrow [0, +\infty[$ be the *total mass functions of the Patterson densities*:

$$\tilde{\phi}_\mu^\pm(x) = \|\mu_x^\pm\|$$

for every $x \in V\mathbb{X}$. These maps are Γ -invariant by Equation (4.1), hence they induce maps $\phi_\mu^\pm : \Gamma \backslash V\mathbb{X} \rightarrow [0, +\infty[$. In the case of real hyperbolic manifolds and vanishing potentials, the total mass functions have important links to the spectrum of the hyperbolic Laplacian (see [Sul1]). See also [CoP3, CoP5] for the case of simplicial trees and the discrete Laplacian, Section 6.1 for a generalisation of the result of Coornaert and Papadopoulos, and for instance [BerK] for developments in the field of quantum graphs.

4.4 Gibbs measures for metric and simplicial trees

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices, and let $X = |\mathbb{X}|_\lambda$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$. Let $\tilde{F} : T^1 X \rightarrow \mathbb{R}$ be a potential for Γ . Let $(\mu_x^\pm)_{x \in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty X$ for the pairs (Γ, F^\pm) .

The *Gibbs measure* \tilde{m}_F on the space of discrete geodesic lines $\mathcal{G}\mathbb{X}$ of \mathbb{X} , invariant under Γ and under the discrete time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{Z}}$ of $\tilde{\mathcal{G}}\mathbb{X}$, is defined analogously with the continuous time case, using the discrete Hopf parametrisation for any basepoint $x_0 \in V\mathbb{X}$, by

$$d\tilde{m}_F(\ell) = e^{C_{\ell_-}^-(x_0, \ell(0)) + C_{\ell_+}^+(x_0, \ell(0))} d\mu_{x_0}^-(\ell_-) d\mu_{x_0}^+(\ell_+) dt, \quad (4.10)$$

where now dt is the counting measure on \mathbb{Z} . We again denote by m_F the measure it induces on $\Gamma \backslash \mathcal{G}\mathbb{X}$.

In this Section, we prove that the Gibbs measures in the case of trees satisfy a Gibbs property even closer to the one in symbolic dynamics, we give an analytic finiteness criterion of the Gibbs measure for metric trees, and recall the ergodic properties of tree lattices.

As recalled in the introduction, Gibbs measures were first introduced in statistical mechanics and consequently in symbolic dynamics, see for example [Bowe2], [ParP], [PauPS]. In order to motivate the terminology used here, we recall the definition of a Gibbs measure for the full two-sided shift on a finite alphabet.⁵ Let $\Sigma_n = \{1, 2, \dots, n\}^{\mathbb{Z}}$ be the product space of sequences $x = (x_n)_{n \in \mathbb{Z}}$ indexed by \mathbb{Z} in the finite set $\{1, 2, \dots, n\}$, and let $\sigma : \Sigma_n \rightarrow \Sigma_n$ be the *shift map* defined by $\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$. A shift-invariant probability measure μ on Σ_n *satisfies the Gibbs property* for an energy function $\phi : \Sigma_n \rightarrow \mathbb{R}$ if

$$\frac{1}{C} \leq \frac{\mu([a_{-m_-}, a_{-m_-+1}, \dots, a_{m_+-1}, a_{m_+}])}{e^{-P(m_-+m_++1) + \sum_{k=-m_-}^{m_+} \phi(\sigma^k x)}} \leq C$$

for some constants $C \geq 1$ and $P \in \mathbb{R}$ (called the *pressure*) and for all m_\pm in \mathbb{N} and x in the cylinder $[a_{-m_-}, a_{-m_-+1}, \dots, a_{m_+-1}, a_{m_+}]$ that consists of those $x \in \Sigma_n$ for which $x_k = a_k$ for all $k \in [-m_-, m_+]$.

Let $x_-, x_+ \in V\mathbb{X}$ and let $x_0 \in V\mathbb{X} \cap [x_-, x_+]$. Let us define the *tree cylinder* of the triple (x_-, x_0, x_+) by

$$[x_-, x_0, x_+] = \{\ell \in \mathcal{G}\mathbb{X} : \ell_\pm \in \mathcal{O}_{x_0}\{x_\pm\}, \ell(0) = x_0\}.$$

These cylinders are close to dynamical balls that have been introduced in Section 4.2, and the parallel with the symbolic case is obvious, as this cylinder is the set of geodesic lines which coincides on $[-m_-, m_+]$, where $m_\pm = d(x_0, x_\pm)$, with a given geodesic line passing through x_\pm and through x_0 at time $t = 0$. The Gibbs measure \tilde{m}_F on the space of discrete geodesic lines $\mathcal{G}\mathbb{X}$ satisfies a variant of the Gibbs property which is even closer to the one in symbolic dynamics than the general case described in Proposition 4.4.

Proposition 4.12 (Gibbs property). *Let K be a finite subset of $V\mathbb{X} \cap \mathcal{C}\Lambda\Gamma$. There exists $C' > 1$ such that for all $\gamma \in \Gamma$ and $x_\pm \in \Gamma K$ and all $x_0 \in V\mathbb{X} \cap [x_-, x_+]$,*

$$\frac{1}{C'} \leq \frac{\tilde{m}_F([x_-, x_0, x_+])}{e^{-\delta d(x_-, x_+) + \int_{x_-}^{x_+} \tilde{F}}} \leq C'$$

⁵See Section 5.1 for the appropriate definition when the alphabet is countable.

Proof. The result is immediate if $d(x_-, x_+)$ is bounded, since the above denominator and numerator take only finitely many values, and the numerator is nonzero since $x_{\pm} \in \mathcal{C}\Lambda\Gamma$, hence $[x_-, x_0, x_+]$ meets the support of \tilde{m}_F . We may hence assume that $d(x_-, x_+) \geq 2$. Using the invariance of \tilde{m}_F under the discrete time geodesic flow, we may thus assume that $x_0 \neq x_-, x_+$.

Using the discrete Hopf parametrisation with respect to the vertex x_0 , we have, by Lemma 4.11,

$$\begin{aligned} \tilde{m}_F([x_-, x_0, x_+]) &= \mu_{x_0}^-(\mathcal{O}_{x_0}\{x_-\}) \mu_{x_0}^+(\mathcal{O}_{x_0}\{x_+\}) \\ &\leq C^2 e^{\int_{x_0}^{x_-} (\tilde{F} \circ \iota - \delta)} e^{\int_{x_0}^{x_+} (\tilde{F} - \delta)} = C^2 e^{\int_{x_-}^{x_+} (\tilde{F} - \delta)}. \end{aligned}$$

This proves the upper bound in Proposition 4.12 with $C' = C^2$ and the lower bound follows similarly. \square

Next, we give a finiteness criterion of the Gibbs measure for metric trees in terms of the total mass functions of the Patterson densities, extending the case when Γ is torsion free and $\tilde{F} = 0$, due to [CoP4, Theo. 1.1].

Proposition 4.13. *Let $(\mathbb{X}, \lambda, \Gamma, \tilde{F})$ be as in the beginning of this Section.*

(1) *If (\mathbb{X}, λ) is simplicial and $\|\cdot\|_2$ is the Hilbert norm of $\mathbb{L}^2(\Gamma \backslash V\mathbb{X}, \text{vol}_{\Gamma \backslash \mathbb{X}})$, we have*

$$\|m_F\| \leq \|\phi_{\mu}^+\|_2 \|\phi_{\mu}^-\|_2.$$

(2) *In general, with $\|\cdot\|_2$ the Hilbert norm of $\mathbb{L}^2(\Gamma \backslash E\mathbb{X}, \text{Tvol}_{\Gamma \backslash \mathbb{X}, \lambda})$, we have⁶*

$$\|m_F\| \leq \|\phi_{\mu}^+ \circ o\|_2 \|\phi_{\mu}^- \circ o\|_2.$$

Proof. (1) The simplicial assumption on (\mathbb{X}, λ) means that all edges have length 1. The space $\Gamma \backslash \mathcal{G}\mathbb{X}$ is the disjoint union of the subsets $\{\ell \in \Gamma \backslash \mathcal{G}\mathbb{X} : \pi(\ell) = \ell(0) = [x]\}$ as the class $[x] = \Gamma x$ of $x \in V\mathbb{X}$ ranges over $\Gamma \backslash V\mathbb{X}$. By Equation (4.10), using Hopf's decomposition with respect to the basepoint x , we have

$$d(\tilde{m}_F)_{\{\ell \in \mathcal{G}\mathbb{X} : \ell(0)=x\}}(\ell) = d\mu_x^-(\ell_-) d\mu_x^+(\ell_+).$$

Let $\Delta_{[x]}$ be the unit Dirac mass at $[x]$. By ramified covering arguments, we hence have the following equality of measures on the discrete set $\Gamma \backslash V\mathbb{X}$:

$$\pi_* m_F = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \mu_x^- \times \mu_x^+ (\{(\ell_-, \ell_+) \in \partial_{\infty}^2 \mathbb{X} : x \in]\ell_-, \ell_+[\}) \Delta_{[x]}. \quad (4.11)$$

Thus, using the Cauchy-Schwarz inequality,

$$\|m_F\| = \|\pi_* m_F\| \leq \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \|\mu_x^- \times \mu_x^+\| = \langle \phi_{\mu}^-, \phi_{\mu}^+ \rangle_2 \leq \|\phi_{\mu}^+\|_2 \|\phi_{\mu}^-\|_2.$$

This proves Assertion (1) of Proposition 4.13.

⁶Recall that $o : \Gamma \backslash E\mathbb{X} \rightarrow \Gamma \backslash V\mathbb{X}$ is the initial vertex map, see Section 2.7.

(2) The argument is similar to the proof of the simplicial case. Since the singletons in \mathbb{R} have zero Lebesgue measure, the space $\Gamma \backslash \mathcal{G}X$ is, up to a measure zero subset for m_F , the disjoint union for $[e] \in \Gamma \backslash E\mathbb{X}$ of the sets $A_{[e]}$ consisting of the elements $\ell \in \Gamma \backslash \mathcal{G}X$ such that $\ell(0)$ belongs to the interior of the edge $[e]$. We fix a representative e of each $[e] \in \Gamma \backslash E\mathbb{X}$. For every $t \in [0, \lambda(e)]$, let e_t be the point of e at distance t from $o(e)$. By Equation (4.3), using Hopf's decomposition with respect to the basepoint $o(e)$ in $A_{[e]}$, we have as above

$$\|m_F\| = \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{1}{|\Gamma_e|} \int_{\ell_- \in \partial_{\bar{e}}\mathbb{X}} \int_{\ell_+ \in \partial_e\mathbb{X}} \int_0^{\lambda(e)} e^{C_{\ell_-}^-(o(e), e_t) + C_{\ell_+}^+(o(e), e_t)} d\mu_{o(e)}^-(\ell_-) d\mu_{o(e)}^+(\ell_+) dt .$$

Since $\ell_-, o(e), e_t, \ell_+$ are in this order on the geodesic line ℓ with $\ell_- \in \partial_{\bar{e}}\mathbb{X}$ and $\ell_+ \in \partial_e\mathbb{X}$, we have $C_{\ell_-}^-(o(e), e_t) + C_{\ell_+}^+(o(e), e_t) = 0$ by Equation (3.8). Hence

$$\begin{aligned} \|m_F\| &= \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{\lambda(e)}{|\Gamma_e|} \mu_{o(e)}^-(\partial_{\bar{e}}\mathbb{X}) \mu_{o(e)}^+(\partial_e\mathbb{X}) \\ &\leq \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{\lambda(e)}{|\Gamma_e|} \|\mu_{o(e)}^-\| \|\mu_{o(e)}^+\| = \langle \phi_\mu^- \circ o, \phi_\mu^+ \circ o \rangle_2 \leq \|\phi_\mu^- \circ o\|_2 \|\phi_\mu^+ \circ o\|_2 , \end{aligned}$$

which finishes the proof. \square

Let us give some corollaries of this proposition in the case of simplicial trees. It follows from Assertion (1) of Proposition 4.13 that if the \mathbb{L}_2 -norms of the total mass of the Patterson densities are finite, then the Gibbs measure m_F is finite. Taking $\tilde{F} = 0$ and $(\mu_x^+)_{x \in V\mathbb{X}} = (\mu_x^-)_{x \in V\mathbb{X}}$, so that the Gibbs measure m_F is the Bowen-Margulis measure m_{BM} , it follows from this proposition that

$$\|m_{\text{BM}}\| \leq \|\phi_\mu^\pm\|_2^2 \leq \text{Vol}(\Gamma \backslash \mathbb{X}) \sup_{x \in V\mathbb{X}} \|\mu_x^\pm\|^2 . \quad (4.12)$$

In particular, if Γ is a lattice in \mathbb{X} and if the total mass of the Patterson density is bounded, then the Bowen-Margulis measure m_{BM} is finite.

The following statement summarises the basic ergodic properties of the lattices of \mathbb{X} when $F = 0$.

Proposition 4.14. *Let (\mathbb{X}, λ) be a metric or simplicial tree, with geometric realisation X . Assume that (\mathbb{X}, λ) is uniform and that Γ is a lattice in $\text{Aut}(\mathbb{X}, \lambda)$. Then*

- (1) Γ is of divergence type, and its critical exponent δ_Γ is the Hausdorff dimension of any visual distance d_x on $\partial_\infty X = \Lambda\Gamma$;
- (2) the Patterson density $(\mu_x)_{x \in X}$ coincides, up to a scalar multiple, with the family of Hausdorff measures $(\mu_x^{\text{Haus}})_{x \in X}$ of dimension δ_Γ of the visual distances $(\partial_\infty X, d_x)$; in particular, it is $\text{Aut}(\mathbb{X}, \lambda)$ -equivariant: for all $\gamma \in \text{Aut}(\mathbb{X}, \lambda)$, we have $\gamma_*\mu_x = \mu_{\gamma x}$;
- (3) the Bowen-Margulis measure \tilde{m}_{BM} of Γ on $\mathcal{G}X$ is $\text{Aut}(\mathbb{X}, \lambda)$ -invariant, and the Bowen-Margulis measure m_{BM} of Γ on $\Gamma \backslash \mathcal{G}X$ is finite.

Proof. Let Γ' be any uniform lattice of (\mathbb{X}, λ) , which exists since the metric tree (\mathbb{X}, λ) is uniform. It is well-known (see for instance [Bou]) that the critical exponent $\delta_{\Gamma'}$ of Γ' is finite and equal to the Hausdorff dimension of any visual distance $(\partial_\infty X, d_x)$, and that the family $(\mu_x^{\text{Haus}})_{x \in V\mathbb{X}}$ of Hausdorff measures of the visual distances $(\partial_\infty X, d_x)$ is a Patterson density for any discrete nonelementary subgroup of $\text{Aut}(\mathbb{X}, \lambda)$ with critical exponent equal to $\delta_{\Gamma'}$.

By [BuM, Coro. 6.5(2)], the lattice Γ in $\text{Aut}(\mathbb{X}, \lambda)$ is of divergence type and $\delta_\Gamma = \delta_{\Gamma'}$. By the uniqueness property of the Patterson densities when Γ is of divergence type (see Theorem 4.5), the family $(\mu_x)_{x \in V\mathbb{X}}$ coincides, up to a scalar multiple, with $(\mu_x^{\text{Haus}})_{x \in V\mathbb{X}}$.

As the graph $\Gamma \backslash \mathbb{X}$ is compact, the total mass function of the Hausdorff measures of the visual distances is bounded, hence so is $(\|\mu_x\|)_{x \in V\mathbb{X}}$. By Proposition 4.13, since Γ is a lattice of \mathbb{X} , this implies that the Bowen-Margulis measure m_{BM} of Γ is finite. \square

Note that as in [DaOP], when (\mathbb{X}, λ) (or its minimal nonempty Γ -invariant subtree) is not assumed to be uniform, there are examples of Γ that are lattices (or are geometrically finite) whose Bowen-Margulis measure m_{BM} is infinite, see Section 15.5 for more details.

Let us now discuss the mixing properties of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ for the Gibbs measure m_F .

Let L_Γ be the *length spectrum* of Γ , which is, in this simplicial case, the subgroup of \mathbb{Z} generated by the translation lengths of the elements of Γ .

Since $d(x, \gamma x) = 2d(x, \text{Ax}_\gamma) + \ell(\gamma)$ if an isometry γ of X is loxodromic and $d(x, \gamma x) = 2d(x, \text{Fix}_\gamma)$ if γ is elliptic with fixed point set Fix_γ , the following assertions are equivalent :

$$\begin{aligned} (1) \quad & L_\Gamma \subset 2\mathbb{Z} \\ (2) \quad & \forall x \in X, \quad \forall \gamma \in \Gamma, \quad d(x, \gamma x) \in 2\mathbb{Z}. \end{aligned} \tag{4.13}$$

Let $\mathcal{G}_{\text{even}}\mathbb{X}$ (respectively $\check{\mathcal{G}}_{\text{even}}\mathbb{X}$) be the subset of $\mathcal{G}\mathbb{X}$ (respectively $\check{\mathcal{G}}\mathbb{X}$) that consists of the geodesic lines (respectively generalised geodesic lines) in \mathbb{X} whose origin is at an even distance from the basepoint x_0 .

Recall that a discrete-time 1-parameter group $(h^n)_{n \in \mathbb{Z}}$ of homeomorphisms of a topological space Z is *topologically mixing* if for all nonempty open subsets U, V of Z , there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $U \cap h^n V \neq \emptyset$.

The following result is well-known. When $\tilde{F} = 0$, see for instance [Rob2] for the equivalence of the first, second and fourth claims, and the arguments of [BrP2, Prop. 3.3] for what remains to be proved.

Proposition 4.15. *Assume that the smallest nonempty Γ -invariant simplicial subtree of \mathbb{X} is uniform, without vertices of degree 2, and that m_F is finite. Then the following assertions are equivalent:*

- the length spectrum of Γ is nonarithmetic, that is $L_\Gamma = \mathbb{Z}$;
- the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ is topologically mixing on its nonwandering set;
- the quotient graph $\Gamma \backslash \mathbb{X}$ is not bipartite;
- the Gibbs measure m_F is mixing under the discrete time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{Z}}$ on $\Gamma \backslash \mathcal{G}\mathbb{X}$.

Otherwise $L_\Gamma = 2\mathbb{Z}$, and the square of the discrete time geodesic flow $(\mathbf{g}^{2t})_{t \in \mathbb{Z}}$ is topologically mixing on the nonwandering subset of $\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}$ and mixing for the restriction of the Gibbs measure m_F to $\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}$. \square

By Proposition 4.14, the general assumptions of Proposition 4.15 are satisfied if \mathbb{X} is uniform, without vertices of degree 2, Γ is a lattice of \mathbb{X} and $\tilde{F} = 0$. Thus, if we assume furthermore that $\Gamma \backslash \mathbb{X}$ is not bipartite, then the Bowen-Margulis measure m_{BM} of Γ is mixing under the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$.

Chapter 5

Symbolic dynamics of geodesic flows on trees

5.1 Two-sided topological Markov shifts

In this short and independent Section, that will be used in Sections 5.2, 5.3, 5.4, 9.2 and 9.3, we recall some definitions concerning symbolic dynamics on countable alphabets.¹

A (two-sided, topological) *Markov shift*² is a topological dynamical system (Σ, σ) constructed from a countable discrete *alphabet* \mathcal{A} and a *transition matrix* $A = (A_{i,j})_{i,j \in \mathcal{A}} \in \{0,1\}^{\mathcal{A} \times \mathcal{A}}$, where Σ is the closed subset of the topological product space $\mathcal{A}^{\mathbb{Z}}$ defined by

$$\Sigma = \{x = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} : \forall n \in \mathbb{Z}, A_{x_n, x_{n+1}} = 1\},$$

and $\sigma : \Sigma \rightarrow \Sigma$ is the (two-sided) *shift* defined by

$$(\sigma(x))_n = x_{n+1}$$

for all $x \in \Sigma$ and $n \in \mathbb{Z}$. Note that to be given (\mathcal{A}, A) is equivalent to be given an oriented graph with countable set of vertices \mathcal{A} (and set of oriented edges a subset of $\mathcal{A} \times \mathcal{A}$) and with incidence matrix A such that $A_{i,j} = 1$ if there is an oriented edge from the vertex i to the vertex j and $A_{i,j} = 0$ otherwise.

For all $p \leq q \in \mathbb{Z}$, a finite sequence $(a_n)_{p \leq n \leq q} \in \mathcal{A}^{\{p, \dots, q\}}$ is *admissible* (or *A-admissible* when we need to make A precise) if $A_{a_n, a_{n+1}} = 1$ for all $n \in \{p, \dots, q-1\}$. A topological Markov shift is *transitive* if for all $x, y \in \mathcal{A}$, there exists an admissible finite sequence $(a_n)_{p \leq n \leq q}$ with $a_p = x$ and $a_q = y$. This is equivalent to require the dynamical system (Σ, σ) to be *topologically transitive*: for all nonempty open subsets U, V in Σ , there exists $n \in \mathbb{Z}$ such that $A \cap \sigma^n(B) \neq \emptyset$.

Note that the product space $\mathcal{A}^{\mathbb{Z}}$ is not locally compact when \mathcal{A} is infinite. When the matrix A has only finitely many nonzero entries on each line and each column, then (Σ, σ) is also called a *subshift of finite type (on a countable alphabet)*. The topological space Σ is then

¹See for instance [Kit, Sar2].

²Note that the terminology could be misleading, a topological Markov shift comes a priori without a measure, and many probability measures invariant under the shift do not satisfy the Markov chain property that the probability to pass from one state to another depends only on the previous state, not of all past states.

locally compact: By diagonal extraction, for all $p \leq q$ in \mathbb{Z} and $a_p, a_{p+1}, \dots, a_{q-1}, a_q$ in \mathcal{A} , every cylinder

$$[a_p, a_{p+1}, \dots, a_{q-1}, a_q] = \{(x_n)_{n \in \mathbb{Z}} \in \Sigma : \forall n \in \{p, \dots, q\}, x_n = a_n\}$$

is a compact open subset of Σ .

Given a continuous map $F_{\text{symp}} : \Sigma \rightarrow \mathbb{R}$ and a constant $c_{F_{\text{symp}}} \in \mathbb{R}$, we say that a measure \mathbb{P} on Σ , invariant under the shift σ , *satisfies the Gibbs property*³ with Gibbs constant $c_{F_{\text{symp}}}$ for the potential F_{symp} if for every finite subset E of the alphabet \mathcal{A} , there exists $C_E \geq 1$ such that for all $p \leq q$ in \mathbb{Z} and for every $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma$ such that $x_p, x_q \in E$, we have

$$\frac{1}{C_E} \leq \frac{\mathbb{P}([x_p, x_{p+1}, \dots, x_{q-1}, x_q])}{e^{-c_{F_{\text{symp}}}(q-p+1) + \sum_{n=p}^q F_{\text{symp}}(\sigma^n x)}} \leq C_E. \quad (5.1)$$

5.2 Coding discrete time geodesic flows on simplicial trees

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, with $X = |\mathbb{X}|_1$ its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut } \mathbb{X}$, and let $\tilde{F} : T^1 X \rightarrow \mathbb{R}$ be a potential for Γ .

In this Section, we give a coding of the discrete time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{Z}}$ on the nonwandering subset of $\Gamma \backslash \mathcal{G}\mathbb{X}$ by a locally compact transitive (two-sided) topological Markov shift. This explicit construction will be useful later on to study the variational principle (see Section 5.4) and rates of mixing (see Section 9.2).

The main technical aspect of this construction, building on [BrP2, §6], is to allow the case when Γ has torsion. When Γ is torsion free and Γ/\mathbb{X} is finite, the construction is well-known, we refer for instance to [CoP6] for a more general setting when the potential is 0. In order to consider for instance non-uniform tree lattices, it is important to allow torsion in Γ . Our direct approach also avoids the assumption that the discrete subgroup Γ is *full*, that is, equal to the subgroup consisting of the elements $g \in \text{Aut}(\mathbb{X})$ such that $p \circ g = p$ where $p : \mathbb{X} \rightarrow \Gamma \backslash \mathbb{X}$ is the canonical projection, as in [Kwo] (building on [BuM, 7.3]).

Let \mathbb{X}' be the minimal nonempty Γ -invariant simplicial subtree of \mathbb{X} , whose geometric realisation is $\mathcal{C}\Lambda\Gamma$. Since we are only interested in the support of the Gibbs measures, we will only code the geodesic flow on the non-wandering subset $\Gamma \backslash \mathbb{X}'$ of $\Gamma \backslash \mathcal{G}\mathbb{X}$. The same construction works with the full space $\Gamma \backslash \mathcal{G}\mathbb{X}$, but the resulting Markov shift is then not necessarily transitive.

Let $(\mathbb{Y}, G_*) = \Gamma \backslash \mathbb{X}'$ be the quotient graph of groups of \mathbb{X}' by Γ (see for instance Example 2.10), and let $p : \mathbb{X}' \rightarrow \mathbb{Y} = \Gamma \backslash \mathbb{X}'$ be the canonical projection. We denote by $[1] = H$ the trivial double coset in any double coset set $H \backslash G / H$ of a group G by a subgroup H .

We consider the alphabet \mathcal{A} consisting of the triples (e^-, h, e^+) where

- $e^\pm \in E\mathbb{Y}$ satisfy $t(e^-) = o(e^+)$ and
- $h \in \rho_{e^-}(G_{e^-}) \backslash G_{o(e^+)}/\rho_{e^+}(G_{e^+})$ satisfy $h \neq [1]$ if $\overline{e^+} = e^-$.

This set is countable (and finite if and only if the quotient graph $\Gamma \backslash \mathbb{X}'$ is finite), we endow it with the discrete topology. We consider the (two-sided) topological Markov shift with alphabet \mathcal{A} and transition matrix $A_{(e^-, h, e^+), (e'^-, h', e'^+)} = 1$ if $e^+ = e'^-$ and 0 otherwise.

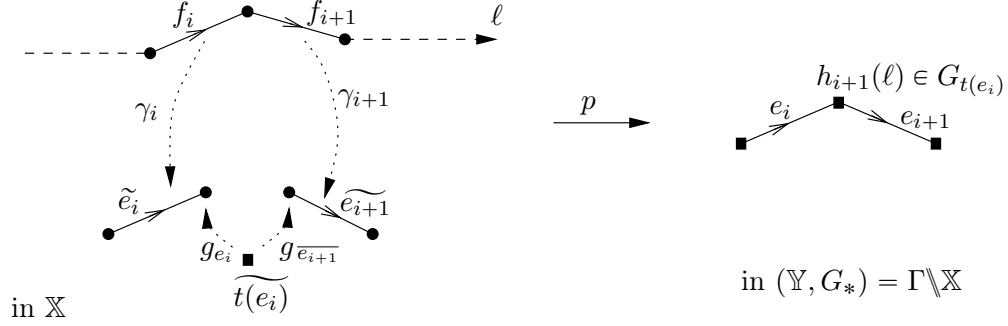
³Note that some references have a stronger notion of Gibbs measure (see for instance [Sar1]), with the constant C independent of E .

Note that this matrix $A = (A_{i,j})_{i,j \in \mathcal{A}}$ has only finitely many nonzero entries on each line and each column, since \mathbb{X}' is locally finite and Γ has finite vertex stabilisers in \mathbb{X}' . We consider the subspace

$$\Sigma = \{(e_i^-, h_i, e_i^+)_{i \in \mathbb{Z}} : \forall i \in \mathbb{Z}, e_{i-1}^+ = e_i^-\}$$

of the product space $\mathcal{A}^{\mathbb{Z}}$, and the *shift* $\sigma : \Sigma \rightarrow \Sigma$ defined by $(\sigma(x))_i = x_{i+1}$ for all $(x_i)_{i \in \mathbb{Z}}$ in Σ and $i \in \mathbb{Z}$. As seen above, Σ is locally compact.

Let us now construct a natural coding map Θ from $\Gamma \backslash \mathcal{G}\mathbb{X}'$ to Σ , by slightly modifying the construction of [BrP2, §6].



For every discrete geodesic line $\ell \in \mathcal{G}\mathbb{X}'$, for every $i \in \mathbb{Z}$, let $f_i = f_i(\ell)$ be the edge of \mathbb{X}' whose geometric realisation is $\ell([i, i+1])$ with origin $f(i)$ and endpoint $f(i+1)$, and let $e_i = p(f_i)$, which is an edge in \mathbb{Y} . Let us use the notation of Example 2.10: we fix lifts \tilde{e} and \tilde{v} of every edge e and vertex v of \mathbb{Y} in \mathbb{X}' such that $\tilde{e} = \tilde{e}$, and elements $g_e \in \Gamma$ such that $g_e \tilde{t}(e) = t(\tilde{e})$. Since $p(\tilde{e}_i) = e_i = p(f_i)$, there exists $\gamma_i = \gamma_i(\ell) \in \Gamma$, well defined up to multiplication on the left by an element of $G_{e_i} = \Gamma_{\tilde{e}_i}$, such that $\gamma_i f_i = \tilde{e}_i$ for all $i \in \mathbb{Z}$.

We define $e_i^-(\ell) = e_{i-1}$, $e_i^+(\ell) = e_i$, and

$$h_i(\ell) = g_{e_i^-}^{-1} \gamma_{i-1}(\ell) \gamma_i(\ell)^{-1} g_{e_i^+}^{-1}. \quad (5.2)$$

Since for every edge e of \mathbb{Y} the structural monomorphism

$$\rho_e : G_e = \Gamma_{\tilde{e}} \longrightarrow G_{t(e)} = \Gamma_{\tilde{t}(e)}$$

is the map $g \mapsto g_e^{-1} g g_e$, the double coset of $h_i(\ell)$ in $\rho_{e_i^-}(G_{e_i^-}) \backslash G_{o(e_i^+)} / \rho_{e_i^+}(G_{e_i^+})$ does not depend on the choice of the γ_i 's, and we again denote it by $h_i(\ell)$.

The next result shows that, under the only assumptions on Γ that it is discrete and nonelementary, the time-one discrete geodesic flow \mathbf{g}^1 on its nonwandering subset of $\Gamma \backslash \mathcal{G}\mathbb{X}$ is topologically conjugate to a locally compact transitive (two-sided) topological Markov shift.

Theorem 5.1. *If $\mathbb{X}' = \mathcal{C}\Lambda\Gamma$, the map $\Theta : \Gamma \backslash \mathcal{G}\mathbb{X}' \rightarrow \Sigma$ defined by*

$$\Gamma \ell \mapsto (e_i^-(\ell), h_i(\ell), e_i^+(\ell))_{i \in \mathbb{Z}}$$

is a homeomorphism which conjugates the time-one discrete geodesic flow \mathbf{g}^1 and the shift σ , and the topological Markov shift (Σ, σ) is a locally compact and transitive.

Furthermore, if we endow $\Gamma \backslash \mathcal{GX}'$ with the quotient distance of⁴

$$d(\ell, \ell') = e^{-\max \{n \in \mathbb{N} : \ell|_{[-n, n]} = \ell'|_{[-n, n]}\}}$$

on \mathcal{GX}' and Σ with the distance⁵

$$d(x, x') = e^{-\max \{n \in \mathbb{N} : \forall i \in \{-n, \dots, n\}, x_i = x'_i\}},$$

then Θ is a bilipschitz homeomorphism.

Finally, if \mathbb{X}' is a uniform tree without vertices of degree at most 2, if the Gibbs measure m_F of Γ is finite, and if the length spectrum L_Γ of Γ is equal to \mathbb{Z} , then the topological Markov shift (Σ, σ) is topologically mixing.

The following diagram hence commutes

$$\begin{array}{ccc} \Gamma \backslash \mathcal{GX}' & \xrightarrow{\mathfrak{g}^1} & \Gamma \backslash \mathcal{GX}' \\ \Theta \downarrow & & \downarrow \Theta \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array} .$$

Note that when \mathbb{Y} is finite (or equivalently when Γ is cocompact), the alphabet \mathcal{A} is finite (hence (Σ, σ) is a standard subshift of finite type). When furthermore the vertex groups of (\mathbb{Y}, G_*) are trivial (or equivalently when Γ acts freely, and in particular is a finitely generated free group), this result is well-known, but it is new if the vertex groups are not trivial. Compare with the construction of [CoP6], whose techniques might be applied since Γ is word-hyperbolic if \mathbb{Y} is finite, up to replacing Gromov's (continuous time) geodesic flow of Γ by the (discrete time) geodesic flow on \mathcal{GX}' , thus avoiding the suspension part (see also the end of loc. cit. when Γ is a free group).

Proof. For all $\ell \in \mathcal{GX}'$ and $\gamma \in \Gamma$, we can take $\gamma_i(\gamma\ell) = \gamma_i(\ell)\gamma^{-1}$, and since $p(\gamma f_i) = p(f_i)$, we have $e_i^\pm(\gamma\ell) = e_i^\pm(\ell)$ and $h_i(\gamma\ell) = h_i(\ell)$, hence the map Θ is well defined.

By construction, the map Θ is equivariant for the actions of \mathfrak{g}^1 on $\Gamma \backslash \mathcal{GX}'$ and σ .

For every $N \in \mathbb{N}$, if $\ell, \ell' \in \mathcal{GX}'$ are close enough, then for $-N - 1 \leq i \leq N + 1$, we have $f_i(\ell) = f_i(\ell')$ and we may take $\gamma_i(\ell) = \gamma_i(\ell')$, so that $e_i^\pm(\ell) = e_i^\pm(\ell')$ and $h_i(\ell) = h_i(\ell')$ for $-N \leq i \leq N$. Therefore Θ is continuous.

Furthermore, with the distances indicated in the statement of Theorem 5.1, if $\ell, \ell' \in \mathcal{GX}'$ satisfy $\ell|_{[-n, n]} = \ell'|_{[-n, n]}$ for some $n \in \mathbb{N}$, then we have $e_i^\pm(\ell) = e_i^\pm(\ell')$ for $-n \leq i \leq n - 1$, and we may take $\gamma_i(\ell) = \gamma_i(\ell')$ for $-n \leq i \leq n - 1$, so that $h_i(\ell) = h_i(\ell')$ for $-n \leq i \leq n - 1$. Therefore, we have

$$d(\Theta(\Gamma\ell), \Theta(\Gamma\ell')) \leq e d(\Gamma\ell, \Gamma\ell'),$$

and Θ is Lipschitz.

Let us construct an inverse $\Psi : \Sigma \rightarrow \Gamma \backslash \mathcal{GX}'$ of θ , by a more general construction that will be useful later on. Let I be a nonempty interval of consecutive integers in \mathbb{Z} , either finite or equal to \mathbb{Z} (the definition of the inverse of θ only requires the second case $I = \mathbb{Z}$). For all $e^-, e^+ \in E\mathbb{Y}$ such that $t(e^-) = o(e^+)$, we fix once and for all a representative of every double coset in $\rho_{e^-}(G_{e^-}) \backslash G_{o(e^+)} / \rho_{e^+}(G_{e^+})$, and we will denote this double coset by its representative.

⁴ with the convention that $\max \emptyset = 0$ when we consider the empty set \emptyset as a subset of \mathbb{N} .

⁵See previous footnote.

Let $w = (e_i^-, h_i, e_i^+)_{i \in I}$ be a sequence indexed by I in the alphabet \mathcal{A} such that for all $i \in I$ such that $i - 1 \in I$, we have $e_{i-1}^+ = e_i^-$ (when I is finite, this simply means that w is an A -admissible sequence in \mathcal{A} , and when $I = \mathbb{Z}$, this simply means that $w \in \Sigma$). In particular, the element $h_i \in G_{o(e_i^+)} = \Gamma_{\widetilde{o(e_i^+)}}$ is the chosen representative of its double coset $\rho_{e_i^-}(G_{e_i^-}) h_i \rho_{e_i^+}(G_{e_i^+})$.

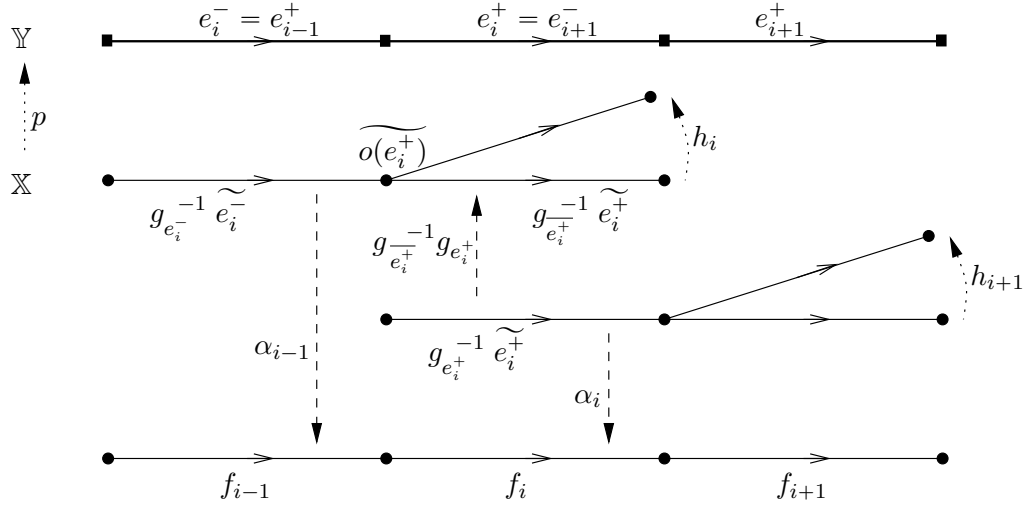
For every $i \in I$, note that

$$o(h_i g_{e_i^+}^{-1} \widetilde{e_i^+}) = o(g_{e_i^+}^{-1} \widetilde{e_i^+}) = \widetilde{o(e_i^+)} = \widetilde{t(e_i^-)} = t(g_{e_i^-}^{-1} \widetilde{e_i^-}).$$

But $h_i g_{e_i^+}^{-1} \widetilde{e_i^+}$ is not the opposite edge of the edge $g_{e_i^-}^{-1} \widetilde{e_i^-}$, since the double coset of h_i is not the trivial one [1] when $e_i^+ = \overline{e_i^-}$, hence h_i does not fix $g_{e_i^-}^{-1} \widetilde{e_i^-}$. Therefore the length 2 edge path (see the picture below)

$$(g_{e_i^-}^{-1} \widetilde{e_i^-}, h_i g_{e_i^+}^{-1} \widetilde{e_i^+})$$

is geodesic.



Let us construct by induction a geodesic segment \widetilde{w} in \mathbb{X}' (which will be a discrete geodesic line if $I = \mathbb{Z}$), well defined up to the action of Γ , as follows.

We fix $i_0 \in I$ (for instance $i_0 = 0$ if $I = \mathbb{Z}$ or $i_0 = \min I$ if I is finite), and $\alpha_{i_0} \in \Gamma$. Let us define

$$f_{i_0} = f_{i_0}(w) = \alpha_{i_0} g_{e_{i_0}^+}^{-1} \widetilde{e_{i_0}^+}.$$

Let us then define

$$\alpha_{i_0-1} = \alpha_{i_0-1}(w) = \alpha_{i_0} g_{e_{i_0}^+}^{-1} g_{e_{i_0}^+}^{-1} h_{i_0}^{-1} \quad \text{and} \quad f_{i_0-1} = f_{i_0-1}(w) \alpha_{i_0-1} g_{e_{i_0}^-}^{-1} \widetilde{e_{i_0}^-}.$$

We have $\alpha_{i_0-1} h_{i_0} g_{e_{i_0}^+}^{-1} \widetilde{e_{i_0}^+} = f_{i_0}$ and (f_{i_0-1}, f_{i_0}) is a geodesic edge path of length 2 (as the image by α_{i_0-1} of such a path).

Let $i - 1, i' \in I$ be such that $i' \leq i_0 \leq i - 1$. Assume by increasing induction on i and decreasing induction on i' that a geodesic edge path $(f_{i'-1} = f_{i'-1}(w), \dots, f_{i-1} = f_{i-1}(w))$ in \mathbb{X}' and a sequence $(\alpha_{i'} = \alpha_{i'-1}(w), \dots, \alpha_{i-1} = \alpha_{i-1}(w))$ in Γ have been constructed such that

$$f_j = \alpha_j g_{e_j^+}^{-1} \widetilde{e}_j^+ \quad \text{and} \quad \alpha_j = \alpha_{j-1} h_j g_{e_j^+}^{-1} g_{e_j^+}$$

for every $j \in \mathbb{N}$ such that $i' - 1 \leq j \leq i - 1$, with besides $j \geq i'$ for the equality on the right.

If i does not belong to I , we stop the construction on the right hand side at $i - 1$. If on the contrary $i \in I$, let us define (see the above picture)

$$\alpha_i = \alpha_{i-1} h_i g_{e_i^+}^{-1} g_{e_i^+} \quad \text{and} \quad f_i = f_i(w) = \alpha_i g_{e_i^+}^{-1} \widetilde{e}_i^+ .$$

Then

$$(f_{i-1}, f_i) = (\alpha_{i-1} g_{e_i^-}^{-1} \widetilde{e}_i^-, \alpha_{i-1} h_i g_{e_i^+}^{-1} \widetilde{e}_i^+) ,$$

is a geodesic edge path of length 2 (as the image by α_{i-1} of such a path). As an edge path is geodesic if and only if it has no back-and-forth, $(f_{i'}, \dots, f_i)$ is a geodesic edge path in \mathbb{X}' . Thus the construction holds at rank i on the right.

If $i' - 1$ does not belong to I , we stop the construction on the left side at i' . Otherwise we proceed as for the construction of α_{i_0-1} and f_{i_0-1} in order to construct α_{i-1} and f_{i-1} with the required properties.

If $I = [p, q] \cap \mathbb{Z}$ with $p \leq q \in \mathbb{Z}$, let $I' = [p - 1, q] \cap \mathbb{Z}$. If $I = \mathbb{Z}$, let $I' = \mathbb{Z}$. We have thus constructed a geodesic edge path

$$(f_i)_{i \in I'} = (f_i(w))_{i \in I'} \tag{5.3}$$

in \mathbb{X}' . We denote by \widetilde{w} its parametrisation by \mathbb{R} if $I = \mathbb{Z}$ and by $[p - 1, q + 1]$ if $I = [p, q] \cap \mathbb{Z}$, in such a way that $\widetilde{w}(i) = o(f_i)$ for all $i \in I$. In particular, $f_i = \widetilde{w}([i, i + 1])$ for all $i \in I'$. When $I = [p, q] \cap \mathbb{Z}$, we consider \widetilde{w} as a generalised discrete geodesic line, by extending it to a constant on $] - \infty, p - 1]$ and on $[q, +\infty[$.

The orbit $\Gamma \widetilde{w}$ of \widetilde{w} does not depend on the choice of α_{i_0} , since replacing α_{i_0} by α'_{i_0} replaces f_i by $\alpha'_{i_0} \alpha_{i_0}^{-1} f_i$ for all $i \in I'$, hence replaces \widetilde{w} by $\alpha'_{i_0} \alpha_{i_0}^{-1} \widetilde{w}$. This also implies that $\Gamma \widetilde{w}$ does not depend on the choice of $i_0 \in I$.

Assume from now on that $I = \mathbb{Z}$, and define $\Psi : \Sigma \rightarrow \Gamma \backslash \mathcal{G}\mathbb{X}'$ by

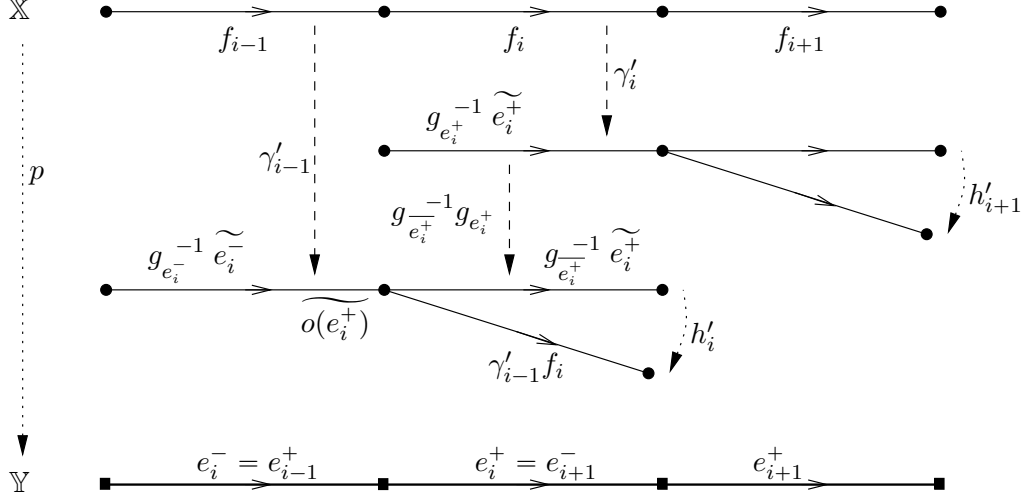
$$\Psi(w) = \Gamma \widetilde{w} .$$

With the distances indicated in the statement of Theorem 5.1, let $w = (e_i^-, h_i, e_i^+)_{i \in \mathbb{Z}}$ and $w' = (e'_i, h'_i, e'^+_i)_{i \in \mathbb{Z}}$ in Σ satisfy $e_i^\pm = e'^\pm_i$ and $h_i = h'_i$ for all $i \in \{-n, \dots, n\}$ for some $n \in \mathbb{N}$. Then we may take the same $i_0 = 0$ and α_{i_0} in the construction of \widetilde{w} and \widetilde{w}' . We thus have $\alpha_i(w) = \alpha_i(w')$ and $f_i(w) = f_i(w')$ for $-n \leq i \leq n$. Therefore, we have

$$d(\Psi(w), \Psi(w')) \leq d(w, w') ,$$

and Ψ is Lipschitz.

Let us prove that Ψ is indeed the inverse of Θ . As in the construction of Θ , for all $\ell \in \mathcal{G}\mathbb{X}'$ and $i \in \mathbb{Z}$, we define $f_i = \ell([i, i + 1])$, $e_i^+ = p(f_i)$ and $e_i^- = e_{i-1}^+$. We denote by $\gamma_i \in \Gamma$ an element sending f_i to $g_{e_i^-}^{-1} \widetilde{e}_i^-$ for all $i \in \mathbb{Z}$ (see the picture below): with the notation above the statement of Theorem 5.1, we have $\gamma_i = g_{e_i^-}^{-1} \gamma_i(\ell)$.



Then γ'_i is well defined up to multiplication on the left by an element of $\Gamma_{g_{e_i^-}^{-1} \widetilde{e}_i^-} = \rho_{e_i^-}(G_{e_i^-})$.

Let h'_i be an element in $G_{o(e_i^+)}$ sending $g_{e_i^+}^{-1} \widetilde{e}_i^+$ to $\gamma'_{i-1} f_i$. It exists since these two edges have the same origin $\widetilde{o(e_i^+)}$, and same image by p :

$$p(\gamma'_{i-1} f_i) = p(f_i) = e_i^+ = p(\widetilde{e}_i^+) = p(g_{e_i^+}^{-1} \widetilde{e}_i^+).$$

Furthermore, it is well defined up to multiplication on the right by an element of $\Gamma_{g_{e_i^+}^{-1} \widetilde{e}_i^+} = \rho_{e_i^+}(G_{e_i^+})$, and we have (see the above picture)

$$\gamma'_{i-1} \gamma_i'^{-1} g_{e_i^-}^{-1} g_{e_i^+} \in h'_i \rho_{e_i^+}(G_{e_i^+})$$

By the construction of Θ (see Equation (5.2) with $\gamma'_i = g_{e_i^-}^{-1} \gamma_i(\ell)$ for all $i \in \mathbb{Z}$), we have

$$\Theta(\ell) = (e_i^-, \rho_{e_i^-}(G_{e_i^-}) h'_i \rho_{e_i^+}(G_{e_i^+}), e_i^+(\ell))_{i \in \mathbb{Z}}.$$

Let h_i be the chosen representative of the double coset $\rho_{e_i^-}(G_{e_i^-}) h'_i \rho_{e_i^+}(G_{e_i^+})$: there exists $\alpha \in \rho_{e_i^-}(G_{e_i^-})$ and $\beta \in \rho_{e_i^+}(G_{e_i^+})$ such that $h_i = \alpha h'_i \beta$. Up to replacing γ'_i by $\alpha^{-1} \gamma'_i$ and h'_i by $h'_i \beta$, we then may have $h'_i = h_i$. By taking $\alpha_{i_0} = \gamma'_{i_0}{}^{-1}$, we have $\alpha_i = \gamma_i'^{-1}$ for all $i \in \mathbb{Z}$, and an inspection of the above two constructions gives that $\Theta \circ \Psi = \text{id}$ and $\Psi \circ \Theta = \text{id}$.

Since the discrete time geodesic flow is topologically transitive on its nonwandering subset and by conjugation, the topological Markov shift (Σ, σ) is topologically transitive.

If \mathbb{X}' is a uniform tree without vertices of degree at most 2, and if the length spectrum of Γ is nonarithmetic and if the Gibbs measure m_F for Γ is finite, then by Proposition 4.15, the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}'$ is topologically mixing, hence by conjugation by Θ , the topological Markov shift (Σ, σ) is topologically mixing. This concludes the proof of Theorem 5.1. \square

When the length spectrum L_Γ of Γ is different from \mathbb{Z} , the topological Markov shift (Σ, σ) constructed above is not always topologically mixing. We now modify the above construction in order to take care of this problem.

Recall that $\mathbb{X}' = \mathcal{C}\Lambda\Gamma$ and that $\mathcal{G}_{\text{even}}\mathbb{X}'$ is the space of geodesic lines $\ell \in \mathcal{G}\mathbb{X}'$ whose origin $\ell(0)$ is at even distance from the basepoint x_0 (we assume that $x_0 \in \mathbb{X}'$), which is invariant under the time-two discrete geodesic flow g^2 and, when $L_\Gamma = 2\mathbb{Z}$, under Γ .

Consider $\mathcal{A}_{\text{even}}$ the alphabet consisting of the quintuples $(f^-, h^-, f^0, h^+, f^+)$ where the triples (f^-, h^-, f^0) and (f^0, h^+, f^+) belong to \mathcal{A} and $o(f^0)$ is at even distance from the image in $\mathbb{Y} = \Gamma\backslash\mathbb{X}'$ of the basepoint x_0 . Let $A_{\text{even}} = (A_{\text{even},i,j})_{i,j \in \mathcal{A}_{\text{even}}}$ be the transition matrix with line and column indices in $\mathcal{A}_{\text{even}}$ such that for all $i = (f^-, h^-, f^0, h^+, f^+)$ and $j = (f_*^-, h_*^-, f_*^0, h_*^+, f_*^+)$, we have $A_{\text{even},i,j} = 1$ if and only if $f^+ = f_*^-$. We denote by $(\Sigma_{\text{even}}, \sigma_{\text{even}})$ the associated topological Markov shift. We endow Σ_{even} with the slightly modified distance

$$d_{\text{even}}(x, x') = e^{-\frac{1}{2} \max \{n \in \mathbb{N} : \forall k \in \{-n, \dots, n\}, x_k = x'_k\}},$$

where $x = (x_k)_{k \in \mathbb{Z}}$ and $x' = (x'_k)_{k \in \mathbb{Z}}$ are in Σ_{even} .

We have a canonical injection $\text{inj} : \Sigma_{\text{even}} \rightarrow \Sigma$ sending the sequence $(f_n^-, h_n^-, f_n^0, h_n^+, f_n^+)_{n \in \mathbb{Z}}$ to $(e_n^-, h_n, e_n^+)_{n \in \mathbb{Z}}$ with, for every $n \in \mathbb{Z}$,

$$e_{2n}^- = f_n^-, h_{2n} = h_n^-, e_{2n}^+ = f_n^0, e_{2n+1}^- = f_n^0, h_{2n+1} = h_n^+, e_{2n+1}^+ = f_n^+,$$

By construction, inj is clearly a homeomorphism onto its image, and

$$\Theta(\Gamma\backslash\mathcal{G}_{\text{even}}\mathbb{X}') = \text{inj}(\Sigma_{\text{even}}).$$

If two sequences in Σ_{even} coincide between $-n$ and n , then their images by inj coincide between $-2n$ and $2n$. Conversely, if the images by inj of two sequences in Σ_{even} coincide between $-2n-1$ and $2n+1$, then these sequences coincide between $-n$ and n . Hence inj is bilipschitz, for the above distances.

Let us define $\Theta_{\text{even}} = \text{inj}^{-1} \circ \Theta|_{\Gamma\backslash\mathcal{G}_{\text{even}}\mathbb{X}'} : \Gamma\backslash\mathcal{G}_{\text{even}}\mathbb{X}' \rightarrow \Sigma_{\text{even}}$. The following diagram hence commutes

$$\begin{array}{ccc} \Gamma\backslash\mathcal{G}_{\text{even}}\mathbb{X}' & \xrightarrow{\Theta_{\text{even}}} & \Sigma_{\text{even}} \\ \downarrow & & \downarrow \text{inj} \\ \Gamma\backslash\mathcal{G}\mathbb{X}' & \xrightarrow{\Theta} & \Sigma, \end{array}$$

where the vertical map on the left hand side is the inclusion map.

Theorem 5.2. *Assume that $\mathbb{X}' = \mathcal{C}\Lambda\Gamma$ is a uniform tree without vertices of degree at most 2, that the Gibbs measure m_F of Γ is finite, and that the length spectrum L_Γ of Γ is equal to $2\mathbb{Z}$. Then the map $\Theta_{\text{even}} : \Gamma\backslash\mathcal{G}_{\text{even}}\mathbb{X}' \rightarrow \Sigma_{\text{even}}$ is a bilipschitz homeomorphism which conjugates the time-two discrete geodesic flow g^2 and the shift σ_{even} , and the topological Markov shift $(\Sigma_{\text{even}}, \sigma_{\text{even}})$ is locally compact and topologically mixing.*

Proof. The only claims that remains to be proven is the last one, which follows from Proposition 4.15, by conjugation. \square

Let us now study the properties of the image by the coding map Θ of finite Gibbs measures on $\Gamma\backslash\mathcal{G}\mathbb{X}$.

Let $(\mu_x^\pm)_{x \in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty\mathbb{X}$ for the pairs (Γ, F^\pm) , where as previously $\tilde{F}^+ = \tilde{F}$, $\tilde{F}^- = \tilde{F} \circ \iota$. Assume that the associated Gibbs measure m_F on $\Gamma\backslash\mathcal{G}\mathbb{X}$ (using the convention for discrete time of Section 4.3) is finite.

Let us define

$$\mathbb{P} = \frac{1}{\|m_F\|} \Theta_* m_F \quad (5.4)$$

as the image of the Gibbs measure m_F (whose support is $\Gamma \setminus \mathcal{G}\mathbb{X}'$) by the homeomorphism Θ , normalised to be a probability measure. It is a probability measure on Σ , invariant under the shift σ .

Let $(Z_n)_{n \in \mathbb{Z}}$ be the random process classically associated with the full shift σ on Σ : it is the random process on the Borel space Σ indexed by \mathbb{Z} with values in the discrete alphabet \mathcal{A} , where $Z_n : \Sigma \rightarrow \mathcal{A}$ is the (continuous hence measurable) n -th projection $(x_k)_{k \in \mathbb{N}} \mapsto x_n$ for all $n \in \mathbb{Z}$.

The following result summarises the properties of the probability measure \mathbb{P} . We start by recalling and giving some notation used in this proposition.

For every admissible finite sequence $w = (a_p, \dots, a_q)$ in \mathcal{A} , where $p \leq q \in \mathbb{Z}$, we denote

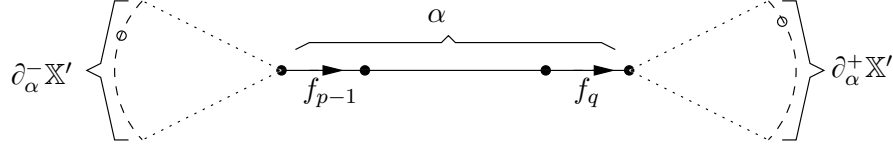
- by $[w] = [a_p, \dots, a_q] = \{(x_n)_{n \in \mathbb{Z}} \in \Sigma : \forall n \in \{p, \dots, q\}, x_n = a_n\}$ the associated cylinder in Σ ,
- by \hat{w} the associated geodesic edge path in \mathbb{X} with length $q - p + 2$ constructed in the proof of Theorem 5.1 (see Equation (5.3)), with origin \hat{w}_- and endpoint \hat{w}_+ .

For every geodesic edge path $\alpha = (f_{p-1}, \dots, f_q)$ in \mathbb{X}' , we define (See Section 2.7 for the notation, and the picture below)

$$\partial_\alpha^+ \mathbb{X}' = \partial_{f_q} \mathbb{X}' \quad \text{and} \quad \partial_\alpha^- \mathbb{X}' = \partial_{f_{p-1}} \mathbb{X}' ,$$

and

$$\mathcal{G}_\alpha \mathbb{X} = \{\ell \in \mathcal{G}\mathbb{X} : \ell(p-1) = o(f_{p-1}) \quad \text{and} \quad \ell(q+1) = t(f_q)\} .$$



We define a map $F_{\text{symb}} : \Sigma \rightarrow \mathbb{R}$ by

$$F_{\text{symb}}(x) = \int_{o(e_0^+)}^{t(e_0^+)} F \quad (5.5)$$

if $x = (x_i)_{i \in \mathbb{Z}}$ with $x_0 = (e_0^-, h_0, e_0^+)$. Note that F_{symb} is locally constant (constant on each cylinder of length 1 at time 0), hence continuous: for all $(x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in \Sigma$, if $x_0 = y_0$, then $F_{\text{symb}}(x) = F_{\text{symb}}(y)$.

For instance, if $F = F_c$ is the potential associated with a system of conductances $c : \Gamma \setminus E\mathbb{X}' \rightarrow \mathbb{R}$ (see Section 3.5), then

$$F_{\text{symb}}(x) = c(e_0^+) .$$

Note that if $c, c' : \Gamma \setminus E\mathbb{X}' \rightarrow \mathbb{R}$ are cohomologous systems of conductances on $\Gamma \setminus E\mathbb{X}'$, then the corresponding maps $F_{\text{symb}}, F'_{\text{symb}} : \Sigma \rightarrow \mathbb{R}$ are cohomologous. Indeed if $f : \Gamma \setminus V\mathbb{X} \rightarrow \mathbb{R}$ is a map such that $c'(e) - c(e) = f(t(e)) - f(o(e))$ for all $e \in \Gamma \setminus E\mathbb{X}$, with $G : \Sigma \rightarrow \mathbb{R}$ the map defined by $G(x) = f(o(e_0^+))$ if $x = (x_i)_{i \in \mathbb{Z}}$ with $x_0 = (e_0^-, h_0, e_0^+)$, then G is locally constant, hence continuous, and since $t(e_0^+) = o(e_1^+)$, we have, for every $x \in \Sigma$,

$$F'_{\text{symb}}(x) - F_{\text{symb}}(x) = G(\sigma x) - G(x) .$$

Definition 5.3. Let \mathbb{X} be a locally finite simplicial tree. A nonelementary discrete subgroup Γ' of $\text{Aut}(\mathbb{X})$ is *Markov-good* if for every $n \in \mathbb{N} - \{0\}$ and every geodesic edge path (e_0, \dots, e_{n+1}) in $\mathcal{C}\Lambda\Gamma'$, we have

$$|\Gamma'_{e_0} \cap \dots \cap \Gamma'_{e_n}| |\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} \cap \Gamma'_{e_{n+1}}| = |\Gamma'_{e_0} \cap \dots \cap \Gamma'_{e_{n+1}}| |\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n}|. \quad (5.6)$$

Remark 5.4. (1) Note that Equation (5.6) is automatically satisfied if $n = 1$ and that Γ' is Markov-good if Γ' acts freely on \mathbb{X} .

(2) A group action on a tree is *2-acylindrical*⁶ if the stabiliser of any geodesic edge path of length 2 is trivial. If Γ' is 2-acylindrical on \mathbb{X} , then Γ' is Markov-good, since all groups appearing in Equation 5.6 are trivial.

(3) If \mathbb{X} has degrees at least 3 and if Γ' is a noncocompact geometrically finite lattice of \mathbb{X} , then Γ' is not Markov-good.

Proof. (3) Since the quotient graph $\Gamma' \backslash \mathbb{X}$ is infinite, the graph of groups $\Gamma' \backslash \mathbb{X}$ contains at least one cuspidal ray. Consider a geodesic ray in \mathbb{X} with consecutive edges $(f_n)_{n \in \mathbb{N}}$ mapping injectively onto this cuspidal ray, pointing towards its end. Their stabilisers in Γ' are hence nondecreasing: we have $\Gamma'_{f_n} \subset \Gamma'_{f_{n+1}}$ for all $n \in \mathbb{N}$. By the finiteness of the volume, there exists $n \geq 3$ such that $\Gamma'_{f_{n-2}}$ is strictly contained in $\Gamma'_{f_{n-1}}$. Since \mathbb{X} has degrees at least 3, there exists $\gamma \in \Gamma'$ fixing $t(f_{n-1})$ but not fixing f_{n-1} . Let $e_0 = f_0, \dots, e_{n-1} = f_{n-1}, e_n = \overline{\gamma f_{n-1}}$ and $e_{n+1} = \overline{\gamma f_{n-2}}$. Then (e_0, \dots, e_{n+1}) is a geodesic edge path in \mathbb{X} (equal to $\mathcal{C}\Lambda\Gamma'$ since Γ' is a lattice). Since $\Gamma'_{e_0} \cap \dots \cap \Gamma'_{e_n} = \Gamma'_{f_0}$, $\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} \cap \Gamma'_{e_{n+1}} = \Gamma'_{f_{n-2}}$, $\Gamma'_{e_0} \cap \dots \cap \Gamma'_{e_{n+1}} = \Gamma'_{f_0}$, $\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} = \Gamma'_{f_{n-1}}$ and $|\Gamma'_{f_{n-2}}| \neq |\Gamma'_{f_{n-1}}|$, the subgroup Γ' is not Markov-good. \square

Recall that a random process $(Z_n)_{n \in \mathbb{Z}}$ on (Σ, \mathbb{P}) is a *Markov chain* if and only if for all $p \leq q$ in \mathbb{Z} and a_p, \dots, a_q, a_{q+1} in \mathcal{A} , we have

$$\mathbb{P}(Z_{q+1} = a_{q+1} \mid Z_q = a_q, \dots, Z_p = a_p) = \mathbb{P}(Z_{q+1} = a_{q+1} \mid Z_q = a_q). \quad (5.7)$$

Proposition 5.5. (1) For every admissible finite sequence w in \mathcal{A} , we have

$$\mathbb{P}([w]) = \frac{\mu_{\hat{w}_-}^- (\partial_{\hat{w}}^- \mathbb{X}') \mu_{\hat{w}_+}^+ (\partial_{\hat{w}}^+ \mathbb{X}') e^{\int_{\hat{w}_-}^{\hat{w}_+} (\tilde{F} - \delta)}}{|\Gamma_{\hat{w}}| \|m_F\|}.$$

(2) The random process $(Z_n)_{n \in \mathbb{Z}}$ on (Σ, \mathbb{P}) is a Markov chain if and only if Γ is Markov-good.

(3) The measure \mathbb{P} on the topological Markov shift Σ satisfies the Gibbs property with Gibbs constant δ for the potential F_{symb} .

It follows from the above Assertion (2) and from Remark 5.4 that when \mathbb{X} has degrees at least 3 and Γ is a noncocompact geometrically finite lattice of \mathbb{X} , then $(Z_n)_{n \in \mathbb{Z}}$ is not a Markov chain. The fact that codings of discrete time geodesic flows on trees might not satisfy the Markov chain property had been noticed by Burger and Mozes around the time the paper [BuM] was published.⁷ When proving the variational principle in Section 5.4 and the exponential decay of correlations in Section 9.2, we will hence have to use tools that are not using the Markov chain property.

⁶See for instance [Sel, GuL], which require other minor hypothesis that are not relevant here.

⁷Personal communication.

Proof. (1) Let $w = (a_p, \dots, a_q)$ where $p \leq q \in \mathbb{Z}$, be an admissible finite sequence in \mathcal{A} . By the construction of Θ , the preimage $\Theta^{-1}([w])$ is equal to the image $\Gamma \mathcal{G}_{\hat{w}} \mathbb{X}'$ of $\mathcal{G}_{\hat{w}} \mathbb{X}'$ in $\Gamma \backslash \mathcal{G} \mathbb{X}'$. Hence, since $\Gamma_{\hat{w}}$ is the stabiliser of $\mathcal{G}_{\hat{w}} \mathbb{X}'$ in Γ ,

$$\mathbb{P}([w]) = \frac{1}{\|m_F\|} m_F(\Gamma \mathcal{G}_{\hat{w}} \mathbb{X}') = \frac{1}{|\Gamma_{\hat{w}}| \|m_F\|} \tilde{m}_F(\mathcal{G}_{\hat{w}} \mathbb{X}').$$

In the expression of \tilde{m}_F given by Equation (4.10), let us use as basepoint x_0 the origin \hat{w}_- of the edge path \hat{w} , and note that all elements of $\mathcal{G}_{\hat{w}} \mathbb{X}'$ pass through \hat{w}_- at time $t = p - 1$, so that by invariance of \tilde{m}_F under the geodesic flow, we have

$$\begin{aligned} \tilde{m}_F(\mathcal{G}_{\hat{w}} \mathbb{X}') &= \int_{\ell \in \mathcal{G}_{\hat{w}} \mathbb{X}'} d\tilde{m}_F(g^{1-p}\ell) = \int_{\ell_- \in \partial_{\hat{w}_+}^- \mathbb{X}'} \int_{\ell_+ \in \partial_{\hat{w}_+}^+ \mathbb{X}'} d\mu_{\hat{w}_-}^-(\ell_-) d\mu_{\hat{w}_-}^+(\ell_+) \\ &= \mu_{\hat{w}_-}^-(\partial_{\hat{w}_+}^- \mathbb{X}') \mu_{\hat{w}_-}^+(\partial_{\hat{w}_+}^+ \mathbb{X}') = \mu_{\hat{w}_-}^-(\partial_{\hat{w}_+}^- \mathbb{X}') \mu_{\hat{w}_+}^+(\partial_{\hat{w}_+}^+ \mathbb{X}') e^{\int_{\hat{w}_-}^{\hat{w}_+} (F-\delta)}, \end{aligned}$$

where this last equality follows by Equations (4.2) and (3.8) with $x = \hat{w}_-$ and $y = \hat{w}_+$, since for every $\ell_+ \in \partial_{\hat{w}_+}^+ \mathbb{X}'$, we have $\hat{w}_+ \in [\hat{w}_+, \ell_+]$.

(2) Let us fix $p \leq q$ in \mathbb{Z} and a_p, \dots, a_q, a_{q+1} in \mathcal{A} , and let us verify Equation (5.7). We may assume that $\alpha = (a_p, \dots, a_q, a_{q+1})$ is an admissible sequence. Let $\alpha_* = (a_p, \dots, a_q)$, which is also an admissible sequence. Let us consider

$$Q_\alpha = \frac{\mathbb{P}(Z_{q+1} = a_{q+1} \mid Z_q = a_q, \dots, Z_p = a_p)}{\mathbb{P}(Z_{q+1} = a_{q+1} \mid Z_q = a_q)} = \frac{\mathbb{P}([a_p, \dots, a_{q+1}]) \mathbb{P}([a_q])}{\mathbb{P}([a_p, \dots, a_q]) \mathbb{P}([a_q, a_{q+1}])}.$$

Let us replace each one of the four terms in this ratio by its value given by Assertion (1). Since $\partial_{\hat{\alpha}}^- \mathbb{X}' = \partial_{\hat{\alpha}_*}^- \mathbb{X}'$, $\partial_{\hat{\alpha}}^+ \mathbb{X}' = \partial_{\widehat{a_q, a_{q+1}}}^+ \mathbb{X}'$, $\partial_{\hat{\alpha}_*}^+ \mathbb{X}' = \partial_{\hat{a}_q}^+ \mathbb{X}'$ and $\partial_{\hat{a}_q}^- \mathbb{X}' = \partial_{\widehat{a_q, a_{q+1}}}^- \mathbb{X}'$, all Patterson measure terms cancel. Denoting by y_1 the common origin of $\hat{\alpha}$ and $\widehat{a_q, a_{q+1}}$, by y_2 the common origin of \hat{a}_q and $\widehat{a_q, a_{q+1}}$, by y_3 the common endpoint of \hat{a}_q and $\widehat{\alpha}_*$, and by y_4 the common endpoint of $\widehat{a_q, a_{q+1}}$ and $\hat{\alpha}$, we thus have by Assertion (1)

$$Q_\alpha = \frac{|\Gamma_{\hat{\alpha}_*}| |\Gamma_{\widehat{a_q, a_{q+1}}}|}{|\Gamma_{\hat{\alpha}}| |\Gamma_{\hat{a}_q}|} \frac{e^{\int_{y_1}^{y_4} (\tilde{F}-\delta)} e^{\int_{y_2}^{y_3} (\tilde{F}-\delta)}}{e^{\int_{y_1}^{y_3} (\tilde{F}-\delta)} e^{\int_{y_2}^{y_4} (\tilde{F}-\delta)}}.$$

Since y_1, y_2, y_3, y_4 are in this order on $[y_1, y_4]$, we have

$$Q_\alpha = \frac{|\Gamma_{\hat{\alpha}_*}| |\Gamma_{\widehat{a_q, a_{q+1}}}|}{|\Gamma_{\hat{\alpha}}| |\Gamma_{\hat{a}_q}|}.$$

Since every geodesic edge path of length $n + 1$ at least 3 in \mathbb{X}' defines an admissible sequence of length n at least 2 in \mathcal{A} , by Equation (5.6), we have $Q_\alpha = 1$ for every admissible sequence α in \mathcal{A} if and only if Γ is Markov-good.

(3) Let E be a finite subset of the alphabet \mathcal{A} , and let $w = (a_p, \dots, a_q)$ with $p \leq q$ in \mathbb{Z} and admissible sequence in \mathcal{A} such that $a_p, a_q \in E$. By Assertion (1), we have

$$\mathbb{P}([w]) = \frac{\mu_{\hat{w}_-}^-(\partial_{\hat{w}_+}^- \mathbb{X}') \mu_{\hat{w}_+}^+(\partial_{\hat{w}_+}^+ \mathbb{X}') e^{\int_{\hat{w}_-}^{\hat{w}_+} (\tilde{F}-\delta)}}{|\Gamma_{\hat{w}}| \|m_F\|}.$$

Since a_p, a_q are varying in the finite subset E of \mathcal{A} , the first and last edges of \hat{w} vary amongst the images under elements of Γ of finitely many edges of \mathbb{X} . Since w is admissible, the open sets $\partial_{\hat{w}}^{\pm} \mathbb{X}'$ are nonempty subsets of $\Lambda\Gamma$, hence they have positive Patterson measures. Furthermore, the quantities $\mu_{\hat{w}_{\pm}}^{\pm}(\partial_{\hat{w}}^{\pm} \mathbb{X}')$ are invariant under the action of Γ on the first/last edge of \hat{w} . Hence there exists $c_1 \geq 1$ depending only on E such that $1 \leq |\Gamma_{\hat{w}}| \leq |\Gamma_{\hat{w}_-}| \leq c_1$ and $\frac{1}{c_1} \leq \mu_{\hat{w}_{\pm}}^{\pm}(\partial_{\hat{w}}^{\pm} \mathbb{X}') \leq c_1$.

Note that the length of \hat{w} is equal to $q - p + 2$. Therefore

$$\frac{e^{-\delta}}{c_1^3 \|m_F\|} e^{-\delta(q-p+1) + \int_{\hat{w}_-}^{\hat{w}_+} \tilde{F}} \leq \mathbb{P}([w]) \leq \frac{e^{-\delta} c_1^2}{\|m_F\|} e^{-\delta(q-p+1) + \int_{\hat{w}_-}^{\hat{w}_+} \tilde{F}}.$$

If $\hat{w} = (f_{p-1}, f_p, \dots, f_q)$ and $x \in [w]$, we have by the definition of F_{symb}

$$\int_{\hat{w}_-}^{\hat{w}_+} \tilde{F} = \sum_{i=p-1}^q \int_{o(f_i)}^{t(f_i)} \tilde{F} = \int_{o(f_{p-1})}^{t(f_{p-1})} \tilde{F} + \sum_{i=p}^q F_{\text{symb}}(\sigma^i(x)).$$

Since F is continuous and $o(f_{p-1})$ remains in a finite subset of $V\mathbb{X}'$, there exists $c_2 > 0$ depending only on E such that $|\tilde{F}(v)| \leq c_2$ for every $v \in T^1X$ with $\pi(v) \in [o(f_{p-1}), t(f_{p-1})]$. Hence $|\int_{o(f_{p-1})}^{t(f_{p-1})} \tilde{F}| \leq c_2$, and Assertion (3) of Proposition 5.5 follows (see Equation (5.1) for the definition of the Gibbs property). \square

Again in order to consider the case when the length spectrum L_{Γ} of Γ is $2\mathbb{Z}$, we define

$$\mathbb{P}_{\text{even}} = \frac{1}{\|(m_F)_{|\Gamma \setminus \mathcal{G}_{\text{even}} \mathbb{X}'}\|} (\Theta_{\text{even}})_*(m_F)_{|\Gamma \setminus \mathcal{G}_{\text{even}} \mathbb{X}'},$$

and $(Z_{\text{even}, n})_{n \in \mathbb{Z}}$ the random process associated with the full shift σ_{even} on Σ_{even} , with $Z_{\text{even}, n} : \Sigma_{\text{even}} \rightarrow \mathcal{A}_{\text{even}}$ the n -th projection for every $n \in \mathbb{Z}$.

By a proof similar to the one of Proposition 5.5, we have the following result. We define a map $F_{\text{symb, even}} : \Sigma_{\text{even}} \rightarrow \mathbb{R}$ by

$$F_{\text{symb, even}}(x) = \int_{o(f_0^-)}^{t(f_0^+)} F \tag{5.8}$$

if $x = (x_i)_{i \in \mathbb{Z}}$ with $x_0 = (f_0^-, h_0^-, f_0^0, h_0^+, f_0^+)$. As previously, $F_{\text{symb, even}}$ is locally constant, hence continuous.

Proposition 5.6. *The measure \mathbb{P}_{even} on the topological Markov shift Σ_{even} satisfies the Gibbs property with Gibbs constant δ for the potential $F_{\text{symb, even}}$. \square*

Again, if Γ is a noncocompact geometrically finite lattice of \mathbb{X} and \mathbb{X}' has degrees at least 3, then $(Z_{\text{even}, n})_{n \in \mathbb{Z}}$ is not a Markov chain.

5.3 Coding continuous time geodesic flows on metric trees

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices, with $X = |\mathbb{X}|_{\lambda}$ its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$, and let $\tilde{F} : T^1X \rightarrow \mathbb{R}$ be a potential for Γ . Let $X' = \mathcal{C}\Lambda\Gamma$, which is the geometric realisation $|\mathbb{X}'|_{\lambda}$ of a metric

subtree (\mathbb{X}', λ) . Let $(\mu_x^\pm)_{x \in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty X$ for the pairs (Γ, F^\pm) , and assume that the associated Gibbs measure m_F is finite. We also assume in this Section that the lengths of the edges of (\mathbb{X}', λ) have a finite upper bound (which is in particular the case if (\mathbb{X}', λ) is uniform). They have a positive lower bound by definition (see Section 2.7).

In this Section, we prove that the continuous time geodesic flow on $\Gamma \backslash \mathcal{G}X'$ is isomorphic to a suspension of a transitive (two-sided) topological Markov shift on a countable alphabet, by an explicit construction that will be useful later on to study the variational principle (see Section 5.4) and rates of mixing (see Section 9.3). Since we are only interested in the support of the Gibbs measures, we will only give such a description for the geodesic flow on the non-wandering subset $\Gamma \backslash \mathcal{G}X'$ of $\Gamma \backslash \mathcal{G}X$. The same construction works with the full space $\Gamma \backslash \mathcal{G}X$, but the resulting Markov shift is then not necessarily transitive.

We start by recalling (see for instance [BrimS, §1.11]) the definitions of the suspension of a (invertible) discrete time dynamical system and of the first return map on a cross-section of a continuous time dynamical system, which allow to pass from transformations to flows and back, respectively.

Let (Z, μ, T) be a metric space Z endowed with an homeomorphism T and a T -invariant (Borel, positive) measure μ . Let $r : Z \rightarrow]0, +\infty[$ be a continuous map, such that for all $z \in Z$, the subset $\{r(T^n z) : n \in \mathbb{N}\} \cup \{-r(T^{-(n+1)}z) : n \in \mathbb{N}\}$ is discrete in \mathbb{R} . Then the *suspension* (or also *special flow*) *over* (Z, μ, T) *with roof function* r is the following continuous time dynamical system $(Z_r, \mu_r, (T_r^t)_{t \in \mathbb{R}})$:

- The space Z_r is the quotient topological space $(Z \times \mathbb{R}) / \sim$ where \sim is the equivalence relation on $Z \times \mathbb{R}$ generated by $(z, s + r(z)) \sim (Tz, s)$ for all $(z, s) \in Z \times \mathbb{R}$. We denote by $[z, s]$ the equivalence class of (z, s) . Note that

$$\mathcal{F} = \{(z, s) : z \in Z, 0 \leq s < r(z)\}$$

is a measurable strict fundamental domain for this equivalence relation. We endow Z_r with the *Bowen-Walters distance*, see [BowW] and particularly the appendix in [BarS].

- For every $t \in \mathbb{R}$, the map $T_r^t : Z_r \rightarrow Z_r$ is the map $[z, s] \mapsto [z, s + t]$. Equivalently, when $(z, s) \in \mathcal{F}$ and $t \geq 0$, then $T_r^t([z, s]) = [T^n z, s']$ where $n \in \mathbb{N}$ and $s' \in \mathbb{R}$ are such that

$$t + s = \sum_{i=0}^{n-1} r(T^i z) + s' \quad \text{and} \quad 0 \leq s' < r(T^n z).$$

- With ds the Lebesgue measure on \mathbb{R} , the measure μ_r is the pushforward of the restriction to \mathcal{F} of the product measure $d\mu ds$ by the restriction to \mathcal{F} of the canonical projection $(Z \times \mathbb{R}) \rightarrow Z_r$.

Note that $(T_r^t)_{t \in \mathbb{R}}$ is indeed a continuous 1-parameter group of homeomorphisms of Z_r , preserving the measure μ_r . The measure μ_r is finite if and only if $\int_Z r d\mu$ is finite, since

$$\|\mu_r\| = \int_Z r d\mu.$$

We will denote by $(Z, \mu, T)_r$ the continuous time dynamical system $(Z_r, \mu_r, (T_r^t)_{t \in \mathbb{R}})$ thus constructed.

Conversely, let $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ be a metric space Z endowed with a continuous 1-parameter group of homeomorphisms $(\phi_t)_{t \in \mathbb{R}}$, preserving a (Borel, positive) measure μ . Let Y be a

cross-section of $(\phi_t)_{t \in \mathbb{R}}$, that is a closed subspace of Z such that for every $z \in Z$, the set $\{t \in \mathbb{R} : \phi_t(z) \in Y\}$ is nonempty and discrete. Let $\tau : Y \rightarrow]0, +\infty[$ be the (continuous) *first return time* on the cross-section Y : for every $y \in Y$,

$$\tau(y) = \min\{t > 0 : \phi_t(y) \in Y\} .$$

Let $\phi_Y : Y \rightarrow Y$ be the (homeomorphic) *first return map* to (or *Poincaré map* of) the cross-section Y , defined by

$$\phi_Y : y \mapsto \phi_{\tau(y)}(y) .$$

By the invariance of μ under the flow $(\phi_t)_{t \in \mathbb{R}}$, the restriction of μ to

$$\{\phi_t(y) : y \in Y, 0 \leq t < \tau(y)\}$$

disintegrates⁸ by the (well-defined) map $\phi_t(y) \mapsto y$ over a measure μ_Y on Y , which is invariant under the first return map ϕ_Y :

$$d\mu(\phi_t(y)) = dt d\mu_Y(y) .$$

Note that if r has a positive lower bound and if μ is finite, then μ_Y is finite, since

$$\|\mu\| \geq \|\mu_Y\| \inf r .$$

Hence (Y, μ_Y, ϕ_Y) is a discrete time dynamical system.

Recall that an *isomorphism* from a continuous time dynamical system $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ to another one $(Z', \mu', (\phi'_t)_{t \in \mathbb{R}})$ is a homeomorphism between the underlying spaces preserving the underlying measures and commuting with the underlying flows.

Example 5.7. If (Z, μ, T) and (Z', μ', T') are (invertible) discrete time dynamical systems, endowed with roof functions $r : Z \rightarrow]0, +\infty[$ and $r' : Z' \rightarrow]0, +\infty[$ respectively, if $\theta : Z \rightarrow Z'$ is a measure preserving homeomorphism commuting with the transformations T and T' (that is, $\theta_*\mu = \mu'$, $\theta \circ T = T' \circ \theta$) and such that

$$r' \circ \theta = r ,$$

then the map $\check{\theta} : Z_r \rightarrow Z'_{r'}$ defined by $[z, s] \mapsto [\theta(z), s]$ is an isomorphism between the suspensions $(Z, \mu, T)_r$ and $(Z', \mu', T')_{r'}$.

It is well known (see for instance [BrinS, §1.11]) that the above two constructions are inverses one to another, up to isomorphism. In particular, we have the following result.

Proposition 5.8. *The suspension $(Y, \mu_Y, \phi_Y)_\tau$ over (Y, μ_Y, ϕ_Y) with roof function τ is isomorphic to $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ by the map $f_Y : [y, s] \mapsto \phi_s y$. \square*

In order to describe the continuous time dynamical system $(\Gamma \backslash \mathcal{G}X', \frac{m_F}{\|m_F\|}, (\mathbf{g}^t)_{t \in \mathbb{R}})$ as a suspension over a topological Markov shift, we will start by describing it as a suspension of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}'$. Note that the Patterson densities and Gibbs

⁸with conditional measure on the fiber $\{\phi_t(y) : 0 \leq t < \tau(y)\}$ over $y \in Y$ the image of the Lebesgue measure on $[0, \tau(y)[$ by $t \mapsto \phi_t(y)$

measures depend not only on the potential, but also on the lengths of the edges.⁹ We hence need to relate precisely the continuous time and discrete time situations, and we will use in this Section the left exponent \sharp to indicate a discrete time object whenever needed.

For instance, we set $\sharp X' = |X'|_1$ and we denote by $(\sharp g^t)_{t \in \mathbb{Z}}$ the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}X'$. Note that X' and $\sharp X'$ are equal as topological spaces (but not as metric spaces). The boundaries at infinity of X' and $\sharp X'$, which coincide with their spaces of ends as topological spaces (by the assumption on the lengths of the edges), are hence equal and denoted by $\partial_\infty X$.

We may assume by Section 3.5 that the potential $\tilde{F} : T^1 X \rightarrow \mathbb{R}$ is the potential \tilde{F}_c associated with a system of conductances \tilde{c} on the metric tree (X, λ) for Γ . Let $\delta_c = \delta_{F_c}$. We denote by $\sharp \tilde{c} : EX \rightarrow \mathbb{R}$ the Γ -invariant system of conductances

$$\sharp \tilde{c} : e \mapsto (c(e) - \delta_c)\lambda(e) \quad (5.9)$$

on the simplicial tree X for Γ , by $\tilde{F}_{\sharp c} : T^1(\sharp X) \rightarrow \mathbb{R}$ its associated potential, and by $\sharp c : \Gamma \backslash EX \rightarrow \mathbb{R}$ and $F_{\sharp c} : \Gamma \backslash T^1(\sharp X) \rightarrow \mathbb{R}$ their quotient maps.

Note that the inclusion morphism $\text{Aut}(X, \lambda) \rightarrow \text{Aut}(X)$ is a homeomorphism onto its image (for the compact-open topologies), by the assumption on the lengths of the edges, hence that Γ is also a nonelementary discrete subgroup of $\text{Aut}(X)$.

Now, let $(\Sigma, \sigma, \mathbb{P})$ be the (two-sided) topological Markov shift conjugated to the discrete time geodesic flow $(\Gamma \backslash \mathcal{G}X', \sharp \mathbf{g}^1, \frac{m_{F_{\sharp c}}}{\|m_{F_{\sharp c}}\|})$ by the bilipschitz homeomorphism $\Theta : \Gamma \backslash \mathcal{G}X' \rightarrow \Sigma$ of Theorem 5.1 (where the potential F is replaced by $F_{\sharp c}$). Let $r : \Sigma \rightarrow]0, +\infty[$ be the map

$$r : x \mapsto \lambda(e_0^+) \quad (5.10)$$

if $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma$ and $x_0 = (e_0^-, h_0, e_0^+) \in \mathcal{A}$. This map is locally constant, hence continuous on Σ , and has a positive lower bound, since the lengths of the edges of (X', λ) have a positive lower bound.

Theorem 5.9. *Assume that the lengths of the edges of (X', λ) have a finite upper bound, and that the Gibbs measure m_F is finite. Then there exists a > 0 such that the continuous time dynamical system $(\Gamma \backslash \mathcal{G}X', \frac{m_F}{\|m_F\|}, (\mathbf{g}^t)_{t \in \mathbb{R}})$ is isomorphic to the suspension $(\Sigma, \sigma, a\mathbb{P})_r$ over $(\Sigma, \sigma, a\mathbb{P})$ with roof function r , by a bilipschitz homeomorphism $\Theta_r : \Gamma \backslash \mathcal{G}X' \rightarrow \Sigma_r$.*

Proof. Let

$$Y = \{\ell \in \Gamma \backslash \mathcal{G}X' : \ell(0) \in \Gamma \backslash V X\}.$$

Then the (closed) subset Y of $\Gamma \backslash \mathcal{G}X'$ is a cross-section of the continuous time geodesic flow $(\mathbf{g}^t)_{t \in \mathbb{R}}$, since every orbit meets Y and since the lengths of the edges of (X', λ) have a positive lower bound. Let $\tau : Y \rightarrow]0, +\infty[$ be the first return time, let μ_Y be the measure on Y (obtained by disintegrating $\frac{m_F}{\|m_F\|}$), and let $\mathbf{g}_Y : Y \rightarrow Y$ be the first return map associated with this cross-section Y .

⁹The fact that the Patterson densities could be singular one with respect to another when the metric varies is a well known phenomenon, even when the potential vanishes. See for instance Kuusalo's theorem [Kuu] saying that the Patterson densities on the boundary at infinity of the real hyperbolic plane of two cocompact marked Fuchsian groups are absolutely continuous one with respect to the other if and only if the marked Fuchsian groups are conjugated, and the extension of this result in [HeP1]. See also the result of [KaN] which parametrises the Culler-Vogtmann space using Patterson densities for cocompact and free actions of free groups on metric trees.

We have a natural reparametrisation map $R : Y \rightarrow \Gamma \backslash \mathcal{G}\mathbb{X}'$, defined by $\ell \mapsto \# \ell$, where $\# \ell(n) = (\mathbf{g}_Y^n \ell)(0)$ is the n -th passage of ℓ in $V\mathbb{X}$, for every $n \in \mathbb{Z}$. Since there exists $m, M > 0$ such that $\lambda(E\mathbb{X}) \subset [m, M]$, the map R is a bilipschitz homeomorphism. It commutes with the first return map \mathbf{g}_Y and the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}'$:

$$R \circ \mathbf{g}_Y = \# \mathbf{g}^1 \circ R.$$

The main point of this proof is the following result relating the measures μ_Y and $m_{F_{\#c}}$.

Lemma 5.10. (1) *The family $(\mu_x^\pm)_{x \in V\mathbb{X}}$ is a Patterson density for $(\Gamma, F_{\#c})$ on the boundary at infinity of the simplicial tree \mathbb{X}' , and the critical exponent $\delta_{\#c}$ of $\#c$ is equal to 0.*

(2) *We have $R_* \frac{\mu_Y}{\|\mu_Y\|} = \frac{m_{F_{\#c}}}{\|m_{F_{\#c}}\|}$.*

Proof. (1) By the definition of the potential associated with a system of conductances (see Section 3.5), for every $x, y \in V\mathbb{X}'$, if (e_1, \dots, e_n) is the edge path in \mathbb{X} with $o(e_1) = x$ and $t(e_n) = y$, then (noting that the integrals along paths depend on the lengths of the edges, the first one below being in X' , the second one in $\#X'$)

$$\int_x^y (\widetilde{F}_c - \delta_c) = \sum_{i=1}^n (\widetilde{c}(e_i)\lambda(e_i) - \delta_c\lambda(e_i)) = \int_x^y \widetilde{F}_{\#c}. \quad (5.11)$$

Let us denote (see Section 3.3) by

$$\#Q(s) = Q_{\Gamma, F_{\#c}, x, y} = \sum_{\gamma \in \Gamma} e^{\int_x^y (\widetilde{F}_{\#c} - s)}$$

and

$$Q(s) = Q_{\Gamma, F_c - \delta_c, x, y}(s) = \sum_{\gamma \in \Gamma} e^{\int_x^y (\widetilde{F}_c - \delta_c - s)}$$

the Poincaré series for the simplicial tree with potential $\widetilde{F}_{\#c}$ and for the metric tree with normalised potential $\widetilde{F}_c - \delta_c$, respectively. We hence have $\#Q(s) \leq Q(\frac{s}{M}) < +\infty$ if $s > 0$ and $\#Q(s) \geq Q(\frac{s}{M}) = +\infty$ if $s < 0$. Thus the critical exponent $\delta_{\#c}$ of $(\Gamma, F_{\#c})$ for the simplicial tree \mathbb{X}' is equal to 0, hence $F_{\#c}$ is a normalised potential.

By the definition (see Section 3.4) of the Gibbs cocycles (which uses the normalised potential), Equation (5.11) also implies that the Gibbs cocycles C^\pm and $\#C^\pm$ for (Γ, F_c) and $(\Gamma, F_{\#c})$ respectively coincide on $\partial_\infty\mathbb{X} \times V\mathbb{X} \times V\mathbb{X}$. Thus by Equations (4.1) and (4.2), the family $(\mu_x^\pm)_{x \in V\mathbb{X}}$ is indeed a Patterson density for $(\Gamma, F_{\#c})$: for all $\gamma \in \Gamma$ and $x, y \in V\mathbb{X}$, and for (almost) all $\xi \in \partial_\infty\mathbb{X}$,

$$\gamma_* \mu_x^\pm = \mu_{\gamma x}^\pm \quad \text{and} \quad \frac{d\mu_x^\pm}{d\mu_y^\pm}(\xi) = e^{-\#C_\xi^\pm(x, y)}.$$

(2) We may hence choose these families $(\mu_x^\pm)_{x \in V\mathbb{X}}$ in order to define the Gibbs measure $m_{\#c}$ associated with the potential $F_{\#c}$ on $\Gamma \backslash \mathcal{G}\mathbb{X}$. Note that since we will prove that $m_{\#c}$ is finite, the normalised measure $\frac{m_{\#c}}{\|m_{\#c}\|}$ is independent of this choice (see Corollary 4.6).

Let $\widetilde{Y} = \{\ell \in \mathcal{G}X' : \ell(0) \in V\mathbb{X}'\}$ be the $(\Gamma$ -invariant) lift of the cross-section Y to $\mathcal{G}X'$, let $\widetilde{R} : \widetilde{Y} \rightarrow \mathcal{G}\mathbb{X}'$ be the lift of R , mapping a geodesic line $\ell \in \widetilde{Y}$ to a discrete geodesic line

$\sharp\ell$ obtained by reparametrisation, and let $\widetilde{\mu}_Y$ be the measure on \widetilde{Y} whose induced measure on $Y = \Gamma \backslash \widetilde{Y}$ is μ_Y . We have a partition of \widetilde{Y} into the closed-open subsets $\widetilde{Y}_x = \{\ell \in \mathcal{G}X' : \ell(0) = x\}$ as x varies in $V\mathbb{X}'$.

Let us fix $x \in V\mathbb{X}$. By the definition of μ_Y as a disintegration of $\frac{m_{F_c}}{\|m_{F_c}\|}$ with respect to the continuous time, by lifting to $\mathcal{G}X'$, by using Hopf's parametrisation with respect to x and Equation (4.3) with $x_0 = x$, we have for every $\ell \in \widetilde{Y}_x$,

$$d\widetilde{\mu}_Y(\ell) = \frac{1}{\|m_{F_c}\|} d\mu_x^-(\ell_-) d\mu_x^+(\ell_+).$$

Note that $\ell(0) = \sharp\ell(0)$, $\ell_- = \sharp\ell_-$, $\ell_+ = \sharp\ell_+$ since the reparametrisation does not change the origin nor the two points at infinity. Hence by Assertion (1), we have

$$\widetilde{R}_*(\widetilde{\mu}_Y) = \frac{1}{\|m_{F_c}\|} \widetilde{m}_{F_c}.$$

As μ_Y is a finite measure since τ has a positive lower bound, this implies that m_{F_c} is finite. By renormalizing as probability measures, this proves Assertion (2). \square

Let $a = \|\mu_Y\| > 0$, so that by Lemma 5.10 (2) we have $R_*\mu_Y = \frac{a m_{\sharp c}}{\|m_{\sharp c}\|}$. Let $\check{r} : \Gamma \backslash \mathcal{G}X' \rightarrow]0, +\infty]$ be the map

$$\check{r} : \Gamma\ell \mapsto \lambda(\ell([0, 1]))$$

given by the length for λ of the first edge followed by a discrete geodesic line $\ell \in \mathcal{G}X'$. Note that \check{r} is locally constant, hence continuous, and that \check{r} is a roof function for the discrete time dynamical system $(\Gamma \backslash \mathcal{G}X', \sharp\mathbf{g}^1)$. Also note that

$$\check{r} \circ R = \tau \quad \text{and} \quad r \circ \Theta = \check{r}$$

by the definitions of τ and r .

Let us finally define $\Theta_r : \Gamma \backslash \mathcal{G}X' \rightarrow \Sigma_r$ as the compositions of the following three maps

$$(\Gamma \backslash \mathcal{G}X', \frac{m_F}{\|m_F\|}, (\mathbf{g}^t)_{t \in \mathbb{R}}) \xrightarrow{f_Y^{-1}} (Y, \mu_Y, g_Y)_\tau \xrightarrow{\check{R}} (\Gamma \backslash \mathcal{G}X', \frac{a m_{\sharp c}}{\|m_{\sharp c}\|}, \sharp\mathbf{g}^1)_{\check{r}} \xrightarrow{\check{\Theta}} (\Sigma, a\mathbb{P}, \sigma)_r, \quad (5.12)$$

where the first one is the inverse of the tautological isomorphism given by Proposition 5.8 and the last two ones, given by Example 5.7, are the isomorphisms \check{R} and $\check{\Theta}$ of continuous time dynamical systems obtained by suspensions of the isomorphisms R and Θ of discrete time dynamical systems. It is easy to check that Θ_r is a bilipschitz homeomorphism, using the following description of the Bowen-Walters distance, see for instance [BarS, Appendix].

Proposition 5.11. *Let $(Z, \mu, T)_r$ be the suspension over an invertible dynamical system such that T is a bilipschitz homeomorphism, with roof function r having a positive lower bound and a finite upper bound. Let $d_{\text{BW}} : Z_r \times Z_r \rightarrow \mathbb{R}$ be the map¹⁰ defined (using the canonical representatives) by*

$$d_{\text{BW}}([x, s], [x', s']) = \min\{d(x, x') + |s - s'|, d(Tx, x') + r(x) - s + s', d(x, Tx') + r(y) + s - s'\}$$

¹⁰The map d_{BW} is actually not a distance, but may replace the Bowen-Walters true distance when working up to multiplicative constants or bilipschitz homeomorphisms.

Then there exists a constant $C_{\text{BW}} > 0$ such that the Bowen-Walters distance d on Σ_r satisfies

$$\frac{1}{C_{\text{BW}}} d_{\text{BW}} \leq d \leq C_{\text{BW}} d_{\text{BW}} . \quad \square$$

This concludes the proof of Theorem 5.9. □

5.4 The variational principle for metric and simplicial trees

In this Section, we assume that X is the geometric realisation of a locally finite metric tree without terminal vertices (\mathbb{X}, λ) (respectively of a locally finite simplicial tree \mathbb{X} without terminal vertices). Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$ (respectively $\text{Aut}(\mathbb{X})$).

We relate in this Section the Gibbs measures¹¹ to the equilibrium states (see the definitions below) for the continuous time geodesic flow on $\Gamma \backslash \mathcal{G}X$ (respectively for the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$).

When X is a Riemannian manifold with pinched negative curvature such that the derivatives of the sectional curvature are uniformly bounded, and when the potential is Hölder-continuous, the analogs of the results of this Section are due to [PauPS, Thm. 6.1]. Their proofs generalise the proofs of Theorems 1 and 2 of [OP], with ideas and techniques going back to [LedS]. When Y is a compact locally CAT(−1)-space, a complete statement about existence, uniqueness and Gibbs property of equilibrium states for any Hölder continuous potential is given in [ConLT].

The proof of the metric tree case will rely strongly (via the suspension process described in Section 5.3) on the proof of the simplicial tree case, hence we start by the latter.

The simplicial tree case.

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, with geometric realisation $X = |X|_1$. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$ and let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances for Γ on \mathbb{X} . Let $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ be its associated potential, and let δ_c be the critical exponent of c .

We define a map $\mathbb{F}_c : \mathcal{G}\mathbb{X} \rightarrow \mathbb{R}$ by

$$\tilde{\mathbb{F}}_c(\ell) = \tilde{c}(e_0^+(\ell)) = \int_{o(e_0^+(\ell))}^{t(e_0^+(\ell))} \tilde{F}_c$$

for all $\ell \in \mathcal{G}\mathbb{X}$, where $e_0^+(\ell)$ is the edge of \mathbb{X} in which ℓ enters at time $t = 0$. This map is locally constant, hence continuous, and it is Γ -invariant, hence it induces a continuous map $\mathbb{F}_c : \Gamma \backslash \mathcal{G}\mathbb{X} \rightarrow \mathbb{R}$.¹²

The following result proves that the Gibbs measure of (Γ, F_c) for the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ is an equilibrium state for the potential \mathbb{F}_c . We start by recalling the definition of an equilibrium state,¹³ see also [Bowe2, Rue3].

¹¹See the definition in Sections 4.2 and 4.3.

¹²See after the proof of Theorem 5.12 for a comment on cohomology classes.

¹³This definition is given for transformations and not flows, and for possibly unbounded potentials, contrarily to the one in the Introduction.

Let Z be a locally compact topological space, let $T : Z \rightarrow Z$ be a homeomorphism, and let $\phi : Z \rightarrow \mathbb{R}$ be a continuous map. Let \mathcal{M}_ϕ be the set of Borel probability measures m on Z , invariant under the transformation T , such that the negative part $\max\{0, -\phi\}$ of ϕ is m -integrable. Let $h_m(T)$ be the (metric) entropy of the transformation T with respect to $m \in \mathcal{M}_\phi$. The *metric pressure* for the potential ϕ of a measure $m \in \mathcal{M}_\phi$ is

$$P_\phi(m) = h_m(T) + \int_Z \phi \, dm .$$

The *pressure* of the potential ϕ is

$$P_\phi = \sup_{m \in \mathcal{M}_\phi} P_\phi(m) .$$

A measure $m_0 \in \mathcal{M}_\phi$ is an *equilibrium state* for the potential ϕ if $P_\phi(m_0) = P_\phi$.

Theorem 5.12 (The variational principle for simplicial trees). *Let $\mathbb{X}, \Gamma, \tilde{c}$ be as above. Assume that $\delta_c < +\infty$ and that there exists a finite Gibbs measure m_c for F_c such that the negative part of the potential \mathbb{F}_c is m_c -integrable. Then $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential \mathbb{F}_c under the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$, and the pressure of \mathbb{F}_c coincides with the critical exponent δ_c of c .*

In order to prove this result, using the coding of the discrete time geodesic flow given in Section 5.2, the main tool is the following result of J. Buzzi in symbolic dynamics, building on works of Sarig and Buzzi-Sarig, whose proof is given in the Appendix.

Let $\sigma : \Sigma \rightarrow \Sigma$ be a two-sided topological Markov shift¹⁴ with (countable) alphabet \mathcal{A} and transition matrix A , and let $\phi : \Sigma \rightarrow \mathbb{R}$ be a continuous map.

For every $n \in \mathbb{N}$, we denote by

$$\text{var}_n \phi = \sup_{\substack{x, y \in \Sigma \\ \forall i \in \{-n, \dots, n\}, x_i = y_i}} |\phi(x) - \phi(y)|$$

the n -variation of ϕ . For instance, if $\phi(x)$ depends only on x_0 , then $\text{var}_n \phi = 0$ (and hence $\sum_{n \in \mathbb{N}} n \text{var}_n \phi = 0$ converges).

A *weak Gibbs measure* for ϕ with Gibbs constant $C = C(m) \in \mathbb{R}$ is a σ -invariant Borel measure m on Σ such that for every $a \in \mathcal{A}$, there exists $c_a \geq 1$ such that for all $n \in \mathbb{N} - \{0\}$ and $x \in [a]$ such that $\sigma^n(x) = x$, we have

$$\frac{1}{c_a} \leq \frac{m([x_0, \dots, x_{n-1}])}{e^{-C(m)n + \sum_{i=0}^{n-1} \phi(\sigma^i x)}} \leq c_a . \quad (5.13)$$

Theorem 5.13 (J. Buzzi, see Corollary A.5). *Let (Σ, σ) be a two-sided transitive topological Markov shift on a countable alphabet \mathcal{A} and let $\phi : \Sigma \rightarrow \mathbb{R}$ be a continuous map such that $\sum_{n \in \mathbb{N}} n \text{var}_n \phi$ converges. Let m be a weak Gibbs measure for ϕ on Σ with Gibbs constant $C(m)$, such that $\int \phi^- \, dm < +\infty$. Then the pressure of ϕ is finite, equal to $C(m)$, and m is the unique equilibrium state. \square*

¹⁴See Section 5.1 for definitions.

Proof of Theorem 5.12. In Section 5.2, we constructed a transitive topological Markov shift (Σ, σ) on a countable alphabet \mathcal{A} and a homeomorphism $\Theta : \Gamma \backslash \mathcal{GX}' \rightarrow \Sigma$ which conjugates the time-one discrete geodesic flow \mathbf{g}^1 on the nonwandering subset $\Gamma \backslash \mathcal{GX}'$ of $\Gamma \backslash \mathcal{GX}$ and the shift σ on Σ (see Theorem 5.1). Let us define a potential $\mathbb{F}_{c, \text{symb}} : \Sigma \rightarrow \mathbb{R}$ by

$$\mathbb{F}_{c, \text{symb}}(x) = c(e_0^+) \quad (5.14)$$

if $x = (x_i)_{i \in \mathbb{Z}}$ with $x_0 = (e_0^-, h_0, e_0^+)$. Note that this potential is the one denoted by F_{symb} in Equation (5.5), when the potential F on T^1X is replaced by F_c . By the construction of Θ , we have

$$\mathbb{F}_{c, \text{symb}} \circ \Theta = \mathbb{F}_c . \quad (5.15)$$

Note that all probability measures on $\Gamma \backslash \mathcal{GX}$ invariant under the discrete time geodesic flow are supported on the nonwandering set $\Gamma \backslash \mathcal{GX}'$. The pushforward of measures Θ_* hence gives a bijection from the space $\mathcal{M}_{\mathbb{F}_c}$ of \mathbf{g}^1 -invariant probability measures on $\Gamma \backslash \mathcal{GX}$ for which the negative part of \mathbb{F}_c is integrable to the space $\mathcal{M}_{\mathbb{F}_{c, \text{symb}}}$ of σ -invariant probability measures on $\Gamma \backslash \mathcal{GX}$ for which the negative part of $\mathbb{F}_{c, \text{symb}}$ is integrable. This bijection induces a bijection between the subsets of equilibrium states.

Since $\mathbb{F}_{c, \text{symb}}(x)$ depends only on x_0 for every $x \in \Sigma$, the series $\sum_{n \in \mathbb{N}} n \text{var}_n \mathbb{F}_{c, \text{symb}}$ converges.

By definition (see Equation (5.4)), the measure \mathbb{P} is the pushforward of $\frac{m_{F_c}}{\|m_{F_c}\|}$ by Θ , hence is a σ -invariant probability measure on $\Gamma \backslash \mathcal{GX}$ for which the negative part of $\mathbb{F}_{c, \text{symb}}$ is integrable, by the assumption of Theorem 5.12. By Proposition 5.5 (3), the measure \mathbb{P} on Σ satisfies the Gibbs property with Gibbs constant δ_c for the potential $\mathbb{F}_{c, \text{symb}}$, hence¹⁵ satisfies the weak Gibbs property with Gibbs constant δ_c . Theorem 5.12 then follows from Theorem 5.13. \square

Remark. It follows from Equation (5.15), from the remark above Proposition 5.5 and from the fact that $\Theta \circ \mathbf{g}^1 = \sigma \circ \Theta$, that if $c, c' : \Gamma \backslash E\mathbb{X}' \rightarrow \mathbb{R}$ are cohomologous systems of conductances on $\Gamma \backslash E\mathbb{X}'$, then the corresponding maps $\mathbb{F}_c, \mathbb{F}_{c'} : \Gamma \backslash \mathcal{GX}' \rightarrow \mathbb{R}$ are cohomologous: there exists a continuous map $\mathbb{G} : \Gamma \backslash \mathcal{GX}' \rightarrow \mathbb{R}$ such that for every $\ell \in \Gamma \backslash \mathcal{GX}'$,

$$\mathbb{F}_{c'}(\ell) - \mathbb{F}_c(\ell) = \mathbb{G}(\mathbf{g}^1 \ell) - \mathbb{G}(\ell) .$$

The metric tree case.

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices with geometric realisation $X = |\mathbb{X}|_\lambda$, let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$ and let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances for Γ on \mathbb{X} . Let $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ be its associated potential (see Section 3.5), and let $\delta_c = \delta_{F_c}$ be the critical exponent of c .

Recall¹⁶ that we have a canonical projection $\mathcal{GX} \rightarrow T^1X$ which associates to a geodesic line ℓ its germ v_ℓ at its footprint $\ell(0)$. Let $\tilde{\mathbb{F}}_c^\natural : \mathcal{GX} \rightarrow \mathbb{R}$ be the Γ -invariant map obtained by precomposing the potential $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ with this canonical projection:

$$\tilde{\mathbb{F}}_c^\natural : \ell \mapsto \tilde{F}_c(v_\ell) .$$

¹⁵For every $a \in \mathcal{A}$, for the constant c_a required by the definition of the weak Gibbs property in Equation (5.13), take the constant C_E given by the definition (see Equation (5.1)) of the Gibbs property with $E = \{a\}$.

¹⁶See Section 2.4.

Let $\mathbb{F}_c^\natural : \Gamma \backslash \mathcal{G}X \rightarrow \mathbb{R}$ be its quotient map, which is continuous, as a composition of continuous maps.

The following result proves that the Gibbs measure of (Γ, F_c) for the continuous time geodesic flow on $\Gamma \backslash \mathcal{G}X$ is an equilibrium state for the potential \mathbb{F}_c^\natural . We start by recalling the definition of an equilibrium state for a possibly unbounded potential under a flow.¹⁷

Given $(Z, (\phi_t)_{t \in \mathbb{R}})$ a topological space endowed with a continuous one-parameter group of homeomorphisms and $\psi : Z \rightarrow \mathbb{R}$ a continuous map (called a *potential*), let \mathcal{M}_ψ be the set of Borel probability measures m on Z invariant under the flow $(\phi_t)_{t \in \mathbb{R}}$, such that the negative part of ψ is m -integrable. Let $h_m(\phi_1)$ be the (metric) entropy of the geodesic flow with respect to $m \in \mathcal{M}_\psi$. The *metric pressure* for ψ of a measure $m \in \mathcal{M}_\psi$ is

$$P_\psi(m) = h_m(\phi_1) + \int_Z \psi \, dm .$$

The *pressure* of the potential ψ is

$$P_\psi = \sup_{m \in \mathcal{M}_\psi} P_\psi(m) .$$

An element $m \in \mathcal{M}_\psi$ is an *equilibrium state* for ψ if the least upper bound defining P_ψ is attained on m .

Note that if ψ' is another potential *cohomologous* to ψ , that is, if there exists a continuous map $G : Z \rightarrow \mathbb{R}$, differentiable along every orbit of the flow, such that $\psi'(x) - \psi(x) = \frac{d}{dt}|_{t=0} G(g^t x)$, then $\mathcal{M}_{\psi'} = \mathcal{M}_\psi$, for every $m \in \mathcal{M}_\psi$, we have $P_{\psi'}(m) = P_\psi(m)$, $P_{\psi'} = P_\psi$ and the equilibrium states for ψ' are exactly the equilibrium states for ψ .

Theorem 5.14 (The variational principle for metric trees). *Let $(\mathbb{X}, \lambda), \Gamma, \tilde{c}$ be as above. Assume that the lengths of the edges of (\mathbb{X}, λ) have a finite upper bound.¹⁸ Assume that $\delta_c < +\infty$ and that there exists a finite Gibbs measure m_c for F_c such that the negative part of the potential \mathbb{F}_c^\natural is m_c -integrable. Then $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential \mathbb{F}_c^\natural under the continuous time geodesic flow on $\Gamma \backslash \mathcal{G}X$, and the pressure of \mathbb{F}_c^\natural coincides with the critical exponent δ_c of c .*

Using the description of the continuous time dynamical system $(\Gamma \backslash \mathcal{G}X', \frac{m_c}{\|m_c\|}, (\mathbf{g}^t)_{t \in \mathbb{R}})$ as a suspension over a topological Markov shift (see Theorem 5.9), this statement reduces to well-known techniques in the thermodynamic formalism of suspension flows, see for instance [JTT], as well as [BarI, Kemp, IJ, JKL]. Our situation is greatly simplified by the fact that our roof function has a positive lower bound and a finite upper bound, and that our symbolic potential is constant on the 1-cylinders $\{x \in \Sigma : x_0 = a\}$ for a in the alphabet.

Proof. Since finite measures invariant under the geodesic flow on $\Gamma \backslash \mathcal{G}X$ are supported on its nonwandering set, up to replacing X by $X' = \mathcal{C} \Lambda \Gamma$, we assume that $X = X'$.

Since equilibrium states are unchanged up to adding a constant to the potential, under the assumptions of Theorem 5.14, let us prove that $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential $\mathbb{F}_c^\natural - \delta_c$ under the continuous time geodesic flow on $\Gamma \backslash \mathcal{G}X$, and that the pressure of $\mathbb{F}_c^\natural - \delta_c$ vanishes. The last claim of Theorem 5.14 follows, since

$$P_{\mathbb{F}_c^\natural} - \delta_c = P_{\mathbb{F}_c^\natural - \delta_c} .$$

¹⁷This requires only minor modifications to the definition given in the Introduction for bounded potentials.

¹⁸They have a positive lower bound by definition, see Section 2.7.

We refer to the paragraphs before the statement of Theorem 5.9 for the definitions of

- the system of conductances $\# \tilde{c}$ for Γ on the simplicial tree \mathbb{X} ,
- the (two-sided) topological Markov shift $(\Sigma, \sigma, \mathbb{P})$ on the alphabet \mathcal{A} , conjugated to the discrete time geodesic flow $(\Gamma \backslash \mathcal{G}\mathbb{X}, \# \mathbf{g}^1, \frac{m_{F_c}}{\|m_{F_c}\|})$ by the homeomorphism $\Theta : \Gamma \backslash \mathcal{G}\mathbb{X} \rightarrow \Sigma$,
- the roof function $r : \Sigma \rightarrow]0, +\infty[$
- and the suspension $(\Sigma, \sigma, a\mathbb{P})_r = (\Sigma_r, (\sigma_r^t)_{t \in \mathbb{R}}, a\mathbb{P}_r)$ over $(\Sigma, \sigma, a\mathbb{P})$ with roof function r , conjugated to the continuous time geodesic flow $(\Gamma \backslash \mathcal{G}X, \frac{m_c}{\|m_c\|}, (\mathbf{g}^t)_{t \in \mathbb{R}})$ by the homeomorphism $\Theta_r : \Gamma \backslash \mathcal{G}X \rightarrow \Sigma_r$ defined at the end of the proof of Theorem 5.9. We will always (uniquely) represent the elements of Σ_r as $[x, s]$ with $x \in \Sigma$ and $0 \leq s < r(x)$.

We denote by $\mathbb{F}_{c, \text{symb}}^{\sharp} : \Sigma_r \rightarrow \mathbb{R}$ the potential defined by

$$\mathbb{F}_{c, \text{symb}}^{\sharp} = \mathbb{F}_c^{\sharp} \circ \Theta_r^{-1}, \quad (5.16)$$

which is continuous, as a composition of continuous maps. The key technical observation in this proof is the following one.

Lemma 5.15. *For every $x \in \Sigma$, we have $\mathbb{F}_{c, \text{symb}}^{\sharp}(x) = \int_0^{r(x)} (\mathbb{F}_{c, \text{symb}}^{\sharp} - \delta_c)([x, s]) ds$. For every $x \in \Sigma$, the sign of $\mathbb{F}_{c, \text{symb}}^{\sharp}([x, s])$ is constant on $s \in [0, r(x)]$.*

Proof. Let $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma$ and $x_0 = (e_0^-, h_0, e_0^+) \in \mathcal{A}$. By the definition of the first return time r in Equation (5.10), we have in particular

$$r(x) = \lambda(e_0^+).$$

By Equation (5.14) and by the definition of $\# \tilde{c}$ in Equation (5.9), we have

$$\mathbb{F}_{c, \text{symb}}^{\sharp}(x) = \# c(e_0^+) = (c(e_0^+) - \delta_c) \lambda(e_0^+).$$

Using in the following sequence of equalities respectively

- the definitions of the potential $\mathbb{F}_{c, \text{symb}}^{\sharp}$ in Equation (5.16) and of the suspension flow $(\sigma_r^t)_{t \in \mathbb{R}}$,
 - the fact that this suspension flow is conjugated to the continuous time geodesic flow by Θ_r ,
 - the definition of Θ_r using the reparametrisation map R of continuous time geodesic lines with origin on vertices to discrete geodesic lines,
 - the definition of the potential \mathbb{F}_c^{\sharp} ,
 - the fact that the first edge followed by the discrete geodesic line $\Theta^{-1}x$, hence by the geodesic line $R^{-1}\Theta^{-1}x$, is e_0^+ and the relation between the potential F_c associated with c and c (see Proposition 3.11),
- we have

$$\begin{aligned} \int_0^{r(x)} \mathbb{F}_{c, \text{symb}}^{\sharp}([x, s]) ds &= \int_0^{r(x)} \mathbb{F}_c^{\sharp}(\Theta_r^{-1} \sigma_r^s[x, 0]) ds = \int_0^{r(x)} \mathbb{F}_c^{\sharp}(\mathbf{g}^s \Theta_r^{-1}[x, 0]) ds \\ &= \int_0^{r(x)} \mathbb{F}_c^{\sharp}(\mathbf{g}^s R^{-1} \Theta^{-1}x) ds = \int_0^{\lambda(e_0^+)} F_c(v_{\mathbf{g}^s R^{-1} \Theta^{-1}x}) ds \\ &= c(e_0^+) \lambda(e_0^+). \end{aligned}$$

Since $\int_0^{r(x)} \delta_c ds = \delta_c \lambda(e_0^+)$, the first claim of Lemma 5.15 follows. The second claim follows by the definition of the potential F_c associated with c , see Equation (3.11). \square

By Equation (5.16), the pushforwards of measures by the homeomorphism Θ_r , which conjugates the flows $(\mathbf{g}^t)_{t \in \mathbb{R}}$ and $(\sigma_r^t)_{t \in \mathbb{R}}$, is a bijection from $\mathcal{M}_{\mathbb{F}_c^\sharp}$ to $\mathcal{M}_{\mathbb{F}_{c, \text{symp}}^\sharp}$, such that

$$P_{\mathbb{F}_{c, \text{symp}}^\sharp}((\Theta_r)_* m) = P_{\mathbb{F}_c^\sharp}(m)$$

for every $m \in \mathcal{M}_{\mathbb{F}_c^\sharp}$. In particular, we only have to prove that $(\Theta_r)_* \frac{m_c}{\|m_c\|} = a \mathbb{P}_r$ is the unique equilibrium state for the potential $\mathbb{F}_{c, \text{symp}}^\sharp - \delta_c$ under the suspension flow $(\sigma_r^t)_{t \in \mathbb{R}}$, and that the pressure of $\mathbb{F}_{c, \text{symp}}^\sharp - \delta_c$ vanishes.

The uniqueness follows for instance from [LJT, Theo. 3.5], since the roof function r is locally constant and the potential $g = \mathbb{F}_{c, \text{symp}}^\sharp$ is such that the map¹⁹ from Σ to \mathbb{R} defined by $x \mapsto \int_0^{r(x)} g([x, s]) ds$ is locally Hölder-continuous by Lemma 5.15 and since $\mathbb{F}_{c, \text{symp}}^\sharp$ is locally constant.

Let us now relate the σ -invariant measures on Σ with the $(\sigma_r^t)_{t \in \mathbb{R}}$ -invariant measures on Σ_r . Recall that we denote the Lebesgue measure on \mathbb{R} by ds and the points in Σ_r by $[x, s]$ with $x \in \Sigma$ and $0 \leq s \leq r(x)$.

Lemma 5.16. *The map $S : \mathcal{M}_{\mathbb{F}_{c, \text{symp}}^\sharp} \rightarrow \mathcal{M}_{\mathbb{F}_c^\sharp}$ defined by*

$$dS(m)([x, s]) = \frac{1}{\int_{\Sigma} r dm} d\mu(x) ds$$

for every $m \in \mathcal{M}_{\mathbb{F}_{c, \text{symp}}^\sharp}$ is a bijection, such that

$$P_{\mathbb{F}_{c, \text{symp}}^\sharp - \delta_c}(S(m)) = \frac{P_{\mathbb{F}_{c, \text{symp}}^\sharp}(m)}{\int_{\Sigma} r dm}.$$

Proof. Note that $\int_{\Sigma} r dm$ is the total mass of the measure $d\mu_r([x, s]) = d\mu(x) ds$ on Σ_r . In particular, $S(m)$ is indeed a probability measure.

Since r has a positive lower bound and a finite upper bound, it is well known since [AmK], see also [LJT, §2.4], that the map S defined above is a bijection from the set of σ -invariant probability measures m on Σ to the set of $(\sigma_r^t)_{t \in \mathbb{R}}$ -invariant probability measures on Σ_r .

Furthermore, for every σ -invariant probability measure m on Σ , we have the following *Kac formula*, by the definition of the probability measure $S(m)$ and by Lemma 5.15,

$$\begin{aligned} \int_{\Sigma_r} \mathbb{F}_{c, \text{symp}}^\sharp dS(m) - \delta_c &= \int_{\Sigma_r} (\mathbb{F}_{c, \text{symp}}^\sharp - \delta_c) dS(m) \\ &= \frac{1}{\int_{\Sigma} r dm} \int_{x \in \Sigma} \int_0^{r(x)} (\mathbb{F}_{c, \text{symp}}^\sharp - \delta_c)([x, s]) dm(x) ds \\ &= \frac{1}{\int_{\Sigma} r dm} \int_{\Sigma} \mathbb{F}_{c, \text{symp}}^\sharp dm. \end{aligned} \tag{5.17}$$

¹⁹denoted by Δ_g in loc. cit.

By the comment on the signs at the end of Lemma 5.15, this computation also proves that the negative part of $\mathbb{F}_{c, \text{sy mb}}^{\sharp}$ is integrable for $S(m)$ if and only if the negative part of $\mathbb{F}_{\sharp c, \text{sy mb}}$ is integrable for m . Hence S is indeed a bijection from $\mathcal{M}_{\mathbb{F}_{c, \text{sy mb}}^{\sharp}}$ to $\mathcal{M}_{\mathbb{F}_{\sharp c, \text{sy mb}}^{\sharp}}$.

By Abramov's formula [Abr], see also [LJT, Prop. 2.14], we have

$$h_{S(m)}(\sigma_r^1) = \frac{h_m(\sigma)}{\int_{\Sigma} r \, dm}. \quad (5.18)$$

The last claim of Lemma 5.16 follows by summation from Equations (5.17) and (5.18). \square

By the proof of Theorem 5.12 (replacing the potential c by $\sharp c$), the pressure of the potential $\mathbb{F}_{\sharp c, \text{sy mb}}$ is equal to the critical exponent $\delta_{\sharp c}$ of the potential $\sharp c$, and by Lemma 5.10 (1), we have $\delta_{\sharp c} = 0$. Hence for every $m \in \mathcal{M}_{\mathbb{F}_{\sharp c, \text{sy mb}}^{\sharp}}$, we have

$$P_{\mathbb{F}_{c, \text{sy mb}}^{\sharp} - \delta_c}(S(m)) = \frac{P_{\mathbb{F}_{\sharp c, \text{sy mb}}^{\sharp}}(m)}{\int_{\Sigma} r \, dm} \leq \frac{P_{\mathbb{F}_{\sharp c, \text{sy mb}}^{\sharp}}}{\int_{\Sigma} r \, dm} = \frac{\delta_{\sharp c}}{\int_{\Sigma} r \, dm} = 0.$$

In particular, the pressure of the potential $\mathbb{F}_{c, \text{sy mb}}^{\sharp} - \delta_c$ is at most 0, since S is a bijection. By the proof of Theorem 5.12 (replacing the potential c by $\sharp c$), we know that \mathbb{P} is an equilibrium state for the potential $\mathbb{F}_{\sharp c, \text{sy mb}}$. Hence

$$P_{\mathbb{F}_{c, \text{sy mb}}^{\sharp} - \delta_c}(S(\mathbb{P})) = \frac{P_{\mathbb{F}_{\sharp c, \text{sy mb}}^{\sharp}}(\mathbb{P})}{\int_{\Sigma} r \, d\mathbb{P}} = 0.$$

Therefore, $S(\mathbb{P})$ is an equilibrium state of the potential $\mathbb{F}_{c, \text{sy mb}}^{\sharp} - \delta_c$, with pressure 0. But $a\mathbb{P}_r$, which is equal to $\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$ since $a\mathbb{P}_r$ is a probability measure, is by construction equal to $S(\mathbb{P})$. The result follows. \square

With slightly different notation, this result implies Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. Any potential \tilde{F} for Γ on T^1X is cohomologous to a potential \tilde{F}_c associated with a system of conductances (see Proposition 3.12). If two potentials \tilde{F} and \tilde{F}' for Γ on T^1X are cohomologous²⁰ then the potentials $\ell \mapsto \tilde{F}(v_\ell)$ and $\ell \mapsto \tilde{F}'(v_\ell)$ for Γ on $\mathcal{G}X$ are cohomologous for the definition given before the statement of Theorem 5.14. Since the existence and uniqueness of an equilibrium state depends only on the cohomology class of the potential on $\mathcal{G}X$, the result follows. \square

²⁰See the definition at the end of Section 3.2.

Chapter 6

Random walks on weighted graphs of groups

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$.

In Section 6.1, we define an operator Δ_c on the functions defined on the set of vertices of the quotient graph of groups $\Gamma \backslash \mathbb{X}$ endowed with a system of conductances $c : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$. This operator is the infinitesimal generator of the random walk on $\Gamma \backslash \mathbb{X}$ associated with the (normalised) exponential of this system of conductances. When Γ is torsion free and the system of conductances vanishes, the construction recovers the standard Laplace operator on the graph $\Gamma \backslash \mathbb{X}$.

Under appropriate anti-reversibility assumptions on the system of conductances, using techniques of Sullivan and Coornaert-Papadopoulos, we prove that the total mass of the Patterson densities is a positive eigenvector for the operator Δ_c associated with the system of conductances.

In Section 6.2, we study the nonsymmetric nearest neighbour random walks on $V\mathbb{X}$ associated with anti-reversible systems of conductivities, and we show that the Patterson densities are the harmonic measures of these random walks.

6.1 Laplacian operators on weighted graphs of groups

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a (Γ -invariant) system of conductances for Γ .

We define $\tilde{c}^+ = \tilde{c}$ and $\tilde{c}^- : e \mapsto \tilde{c}(\bar{e})$, which is another system of conductances for Γ , and we denote by $c^\pm : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ the quotient maps. Recall (see Section 3.5) that \tilde{c} is reversible (respectively anti-reversible) if $\tilde{c}^- = \tilde{c}^+$ (respectively $\tilde{c}^- = -\tilde{c}^+$). For every $x \in V\mathbb{X}$, we define

$$\text{deg}_{\tilde{c}^\pm}(x) = \sum_{e \in E\mathbb{X}, o(e)=x} e^{\tilde{c}^\pm(e)}.$$

The quotient graph of groups $\Gamma \backslash \mathbb{X}$ is endowed with the quotient maps $c^\pm : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ of \tilde{c}^\pm . Also note that the quantity $\text{deg}_{\tilde{c}^\pm}(x)$ is constant on the Γ -orbit of x . Hence, it defines a map $\text{deg}_{c^\pm} : \Gamma \backslash V\mathbb{X} \rightarrow]0, +\infty[$.

On the vector space $\mathbb{C}^{V\mathbb{X}}$ of maps from $V\mathbb{X}$ to \mathbb{C} , we consider the operator Δ_{c^\pm} , called the (*weighted*) *Laplace operator* of (\mathbb{X}, c^\pm) ,¹ defined by setting, for all $f \in \mathbb{C}^{V\mathbb{X}}$ and $x \in V\mathbb{X}$,

$$\Delta_{c^\pm} f(x) = \frac{1}{\deg_{\tilde{c}^\pm}(x)} \sum_{e \in E\mathbb{X}, o(e)=x} e^{\tilde{c}^\pm(e)} (f(x) - f(t(e))). \quad (6.1)$$

This is the standard Laplace operator² of a weighted graph with the weight $e \mapsto e^{\tilde{c}^\pm(e)}$, except that usually one requires that $\tilde{c}(e) = \tilde{c}(\bar{e})$. Note that $p^\pm(e) = \frac{e^{\tilde{c}^\pm(e)}}{\deg_{c^\pm}(o(e))}$ is a Markov transition kernel on the tree \mathbb{X} , see Section 6.2.

The weighted Laplace operator Δ_{c^\pm} is invariant under Γ : for all $f \in \mathbb{C}^{V\mathbb{X}}$ and $\gamma \in \Gamma$, we have

$$\Delta_{c^\pm}(f \circ \gamma) = (\Delta_{c^\pm} f) \circ \gamma.$$

In particular, this operator induces an operator on functions defined on the quotient graph $\Gamma \backslash \mathbb{X}$, as follows.

Let (\mathbb{Y}, G_*) be a graph of finite groups and let $i : E\mathbb{Y} \rightarrow \mathbb{N} - \{0\}$ be the index map $i(e) = [G_{o(e)} : G_e]$. For every function $c : E\mathbb{Y} \rightarrow \mathbb{R}$, let $\deg_c : V\mathbb{Y} \rightarrow \mathbb{R}$ be the positive function defined by

$$\deg_c(x) = \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) e^{c(e)}.$$

The *Laplace operator*³ of (\mathbb{Y}, G_*, c) is the operator $\Delta_c = \Delta_{\mathbb{Y}, G_*, c}$ on $L^2(V\mathbb{Y}, \text{vol}_{\mathbb{Y}, G_*})$ defined by

$$\Delta_c f : x \mapsto \frac{1}{\deg_c(x)} \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) e^{c(e)} (f(x) - f(t(e))).$$

Remark 6.1. (1) Let $(\mathbb{Y}, G_*) = \Gamma \backslash \mathbb{X}$ be a graph of finite groups with $p : V\mathbb{X} \rightarrow V\mathbb{Y} = \Gamma \backslash V\mathbb{X}$ the canonical projection. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a potential and let $c : E\mathbb{Y} = \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ be the map induced by \tilde{c} . An easy computation shows that for all $f \in \mathbb{C}^{V\mathbb{Y}}$ and $x \in V\mathbb{Y}$, we have

$$\Delta_c f(x) = \Delta_{c^\pm} \tilde{f}(\tilde{x})$$

if $\tilde{f} = f \circ p : V\mathbb{X} \rightarrow \mathbb{C}$ and $\tilde{x} \in V\mathbb{X}$ satisfies $p(\tilde{x}) = x$.

(2) For every $x \in V\mathbb{Y}$, let

$$i(x) = \sum_{e \in E\mathbb{Y}, o(e)=x} i(e).$$

Then $i(x)$ is the degree of any vertex of any universal cover of (\mathbb{Y}, G_*) above x . In particular, the map $i : V\mathbb{Y} \rightarrow \mathbb{R}$ is bounded if and only if the universal cover of (\mathbb{Y}, G_*) has uniformly bounded degrees. When $c = 0$, we denote the Laplace operator by $\Delta = \Delta_{\mathbb{Y}, G_*}$ and for every $x \in V\mathbb{Y}$, we have

$$\Delta_0 f(x) = \frac{1}{i(x)} \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) (f(x) - f(t(e))).$$

We thus recover the Laplace operator of [Mor] on the edge-indexed graph (\mathbb{Y}, i) .

¹or on \mathbb{X} associated with the system of conductances \tilde{c}^\pm

²See for example [Car] with the opposite choice of the sign, or [ChGY].

³See for instance [Mor] when $c = 0$.

Returning to general graphs of finite groups, we denote by $\mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$ the Hilbert space of maps $f : V\mathbb{Y} \rightarrow \mathbb{C}$ with finite norm $\|f\|_{\text{vol}}$ for the following scalar product:

$$\langle f, g \rangle_{\text{vol}} = \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} f(x) \overline{g(x)}.$$

We denote by $\mathbb{L}^2(E\mathbb{Y}, \text{Tvol}_{(\mathbb{Y}, G_*)})$ the Hilbert space of maps $\phi : E\mathbb{Y} \rightarrow \mathbb{C}$ with finite norm $\|\phi\|_{\text{Tvol}}$ for the following scalar product:

$$\langle \phi, \psi \rangle_{\text{Tvol}} = \frac{1}{2} \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} \phi(e) \overline{\psi(e)}.$$

Proposition 6.2. *Let (\mathbb{Y}, G_*) be a graph of finite groups, whose map $i : V\mathbb{Y} \rightarrow \mathbb{R}$ is bounded. Let $c : E\mathbb{Y} \rightarrow \mathbb{R}$ be a system of conductances on \mathbb{Y} , and let*

$$p(e) = \frac{e^{c(e)}}{\text{deg}_c(o(e))}$$

for every $e \in E\mathbb{Y}$.

(1) *The Laplace operator $\Delta_c : \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)}) \rightarrow \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$ is linear and bounded.*

(2) *The map $d_c : \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)}) \rightarrow \mathbb{L}^2(E\mathbb{Y}, \text{Tvol}_{(\mathbb{Y}, G_*)})$ defined by*

$$d_c(f) : e \mapsto \sqrt{p(e)} (f(t(e)) - f(o(e)))$$

is linear and bounded, and its dual operator

$$d_c^* : \mathbb{L}^2(E\mathbb{Y}, \text{Tvol}_{(\mathbb{Y}, G_*)}) \rightarrow \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$$

is given by

$$d_c^*(\phi) : x \mapsto \sum_{e \in E\mathbb{Y}, o(e)=x} \frac{i(e)}{2} \left(\sqrt{p(\bar{e})} \phi(\bar{e}) - \sqrt{p(e)} \phi(e) \right)$$

(3) *Assume that c is reversible. Then*

$$\Delta_c = d_c^* d_c$$

In particular, Δ_c is self-adjoint and nonnegative.

Proof. By the assumptions, there exists $M \in \mathbb{N}$ such that $i(x) \leq M$ for every $x \in V\mathbb{Y}$, and hence $i(e) \leq M$ for every $e \in E\mathbb{Y}$. Note that $i(e) = \frac{|\Gamma_{o(e)}|}{|\Gamma_e|}$, that $\Gamma_e = \Gamma_{\bar{e}}$ and that $p(e) \leq 1$.

(1) For every $f \in \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$, using the Cauchy-Schwarz inequality and the change of

variable $e \mapsto \bar{e}$, we have

$$\begin{aligned}
\|\Delta_c f\|_{\text{vol}}^2 &= \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \left| \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) p(e) (f(x) - f(t(e))) \right|^2 \\
&\leq \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \left(\sum_{e \in E\mathbb{Y}, o(e)=x} i(e)^2 p(e)^2 \right) \left(\sum_{e \in E\mathbb{Y}, o(e)=x} |f(x) - f(t(e))|^2 \right) \\
&\leq 2M^2 \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \left(\sum_{e \in E\mathbb{Y}, o(e)=x} |f(x)|^2 + |f(t(e))|^2 \right) \\
&\leq 2M^2 \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} |f(o(e))|^2 + |f(t(e))|^2 \\
&= 4M^2 \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} |f(o(e))|^2 = 4M^2 \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) |f(x)|^2 \\
&\leq 4M^3 \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} |f(x)|^2 = 4M^3 \|f\|_{\text{vol}}^2.
\end{aligned}$$

Hence the linear operator Δ_c is bounded.

(2) For every $f \in \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$, we have

$$\begin{aligned}
\|d_c f\|_{\text{Tvol}}^2 &= \frac{1}{2} \sum_{e \in E\mathbb{Y}} \frac{p(e)}{|\Gamma_e|} |f(t(e)) - f(o(e))|^2 \\
&\leq \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} \left(|f(t(e))|^2 + |f(o(e))|^2 \right) = 2 \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} |f(o(e))|^2 \\
&= 2 \sum_{e \in E\mathbb{Y}} \frac{i(e)}{|\Gamma_{o(e)}} |f(o(e))|^2 = 2 \sum_{x \in V\mathbb{Y}} \frac{i(x)}{|\Gamma_x|} |f(x)|^2 \leq 2M \|f\|_{\text{vol}}^2.
\end{aligned}$$

Hence the linear operator d_c is bounded.

For all $f \in \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$ and $\phi \in \mathbb{L}^2(E\mathbb{Y}, \text{Tvol}_{(\mathbb{Y}, G_*)})$, using again the change of variable $e \mapsto \bar{e}$, we have

$$\begin{aligned}
\langle \phi, d_c f \rangle_{\text{Tvol}} &= \frac{1}{2} \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} \sqrt{p(e)} \phi(e) \overline{(f(t(e)) - f(o(e)))} \\
&= \frac{1}{2} \left(\sum_{e \in E\mathbb{Y}} \frac{\sqrt{p(\bar{e})}}{|\Gamma_e|} \phi(\bar{e}) \overline{f(o(e))} - \sum_{e \in E\mathbb{Y}} \frac{\sqrt{p(e)}}{|\Gamma_e|} \phi(e) \overline{f(o(e))} \right) \\
&= \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \sum_{e \in E\mathbb{Y}, o(e)=x} \frac{i(e)}{2} \left(\sqrt{p(\bar{e})} \phi(\bar{e}) - \sqrt{p(e)} \phi(e) \right) \overline{f(x)}.
\end{aligned}$$

This gives the formula for d_c^* .

(3) For all $f, g \in \mathbb{L}^2(V\mathbb{Y}, \text{vol}_{(\mathbb{Y}, G_*)})$, since $p(e) = p(\bar{e})$ by reversibility, by making the change

of variable $e \mapsto \bar{e}$ in half the value of the second line, we have

$$\begin{aligned}
\langle \Delta_c f, g \rangle_{\text{vol}} &= \sum_{x \in V\mathbb{Y}} \frac{1}{|\Gamma_x|} \sum_{e \in E\mathbb{Y}, o(e)=x} i(e) p(e) (f(x) - f(t(e))) \overline{g(x)} \\
&= \sum_{e \in E\mathbb{Y}} \frac{i(e)}{|\Gamma_{o(e)}|} p(e) (f(o(e)) \overline{g(o(e))} - f(t(e)) \overline{g(o(e))}) \\
&= \frac{1}{2} \sum_{e \in E\mathbb{Y}} \frac{1}{|\Gamma_e|} p(e) (f(t(e)) - f(o(e))) (\overline{g(t(e))} - \overline{g(o(e))}) \\
&= \langle d_c f, d_c g \rangle_{\text{Tvol}}.
\end{aligned}$$

This proves the last claim in Proposition 6.2. \square

The following result is an extension to anti-reversible systems of conductances of [CoP2, Prop. 3.3] (who treated the case of zero conductances), which is a discrete version of Sullivan's analogous result for hyperbolic manifolds (see [Sul1]). Let $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ be the potential for Γ associated with c , so that $(\tilde{F}_c)^\pm = \tilde{F}_{c^\pm}$, and let δ_c be their critical exponent. Let $C^\pm : \partial_\infty X \times V\mathbb{X} \times V\mathbb{X} \rightarrow \mathbb{R}$ be the associated Gibbs cocycles. Let $(\mu_x^\pm)_{x \in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty X$ for the pairs (Γ, F_{c^\pm}) .

Proposition 6.3. *Assume that \mathbb{X} is $(q+1)$ -regular, that the system of conductances c is anti-reversible and that the map $\text{deg}_{c^\pm} : V\mathbb{X} \rightarrow \mathbb{R}$ is constant with value κ^\pm . Then the total mass $\tilde{\phi}^\pm : x \mapsto \|\mu_x^\pm\|$ of the Patterson density is a positive eigenvector associated with the eigenvalue*

$$1 - \frac{e^{\delta_c} + qe^{-\delta_c}}{\kappa^\pm}.$$

for the Laplace operator Δ_{c^\pm} on $\mathbb{C}^{V\mathbb{X}}$.

Proof. Note that $c^\pm : V\mathbb{X} \rightarrow \mathbb{R}$ is bounded, hence $c^\pm : E\mathbb{X} \rightarrow \mathbb{R}$ is bounded, hence $(\tilde{F}_c)^\pm = \tilde{F}_{c^\pm}$ is bounded. As \mathbb{X} is $(q+1)$ -regular, the critical exponent δ_Γ is finite and hence also the critical exponent δ_c is finite by Assertions (6) and (7) of Lemma 3.7. Since

$$\tilde{\phi}^\pm(x) = \int_{\partial_\infty \mathbb{X}} d\mu_x^\pm = \int_{\partial_\infty \mathbb{X}} e^{-C_\xi^\pm(x, x_0)} d\mu_{x_0}^\pm,$$

by Equation (4.2) and by linearity, we only have to prove that for every fixed $\xi \in \partial_\infty \mathbb{X}$ the map

$$f : x \mapsto e^{-C_\xi^\pm(x, x_0)}$$

is an eigenvector with eigenvalue $1 - \frac{e^{\delta_c} + qe^{-\delta_c}}{\kappa^\pm}$ for Δ_{c^\pm} .

For every $e \in E\mathbb{X}$, recall⁴ that $\partial_e \mathbb{X}$ is the set of points at infinity of the geodesic rays whose initial edge is e . By Equation (3.8) and by the definition of the potential associated with a system of conductances⁵, for all $e \in E\mathbb{X}$ and $\eta \in \partial_e \mathbb{X}$, we have

$$C_\eta^\pm(t(e), o(e)) = \int_{o(e)}^{t(e)} (\tilde{F}_{c^\pm} - \delta_c) = c^\pm(e) - \delta_c.$$

⁴See Section 2.7.

⁵See Proposition 3.11 with the edge length map λ constant equal to 1

Thus

$$f(t(e)) = e^{-C_\xi^\pm(t(e), o(e)) - C_\xi^\pm(o(e), x_0)} = e^{-c^\pm(e) + \delta_c} f(o(e))$$

if $\xi \in \partial_e \mathbb{X}$, and otherwise

$$f(t(e)) = e^{C_\xi^\pm(t(\bar{e}), o(\bar{e})) - C_\xi^\pm(o(e), x_0)} = e^{c^\pm(\bar{e}) - \delta_c} f(o(e)) .$$

For every $x \in V\mathbb{X}$, let e_ξ be the unique edge of \mathbb{X} with origin x such that $\xi \in \partial_{e_\xi} \mathbb{X}$. Then,

$$\begin{aligned} \Delta_{c^\pm} f(x) &= f(x) - \frac{1}{c^\pm(x)} \sum_{o(e)=x} e^{c^\pm(e)} f(t(e)) \\ &= f(x) - \frac{1}{\kappa^\pm} e^{c^\pm(e_\xi)} f(t(e_\xi)) - \frac{1}{\kappa^\pm} \sum_{e \neq e_\xi, o(e)=x} e^{c^\pm(e)} f(t(e)) \\ &= \left(1 - \frac{e^{\delta_c}}{\kappa^\pm} - \frac{q e^{-\delta_c}}{\kappa^\pm}\right) f(x) . \end{aligned}$$

This proves the result. \square

Note that the anti-reversibility of the potential is used in an essential way in order to get the last equation in the proof of Proposition 6.3.

6.2 Patterson densities as harmonic measures for simplicial trees

In this Section, we define and study a Markov chain on the set of vertices of a simplicial tree endowed with a discrete group of automorphisms and with an appropriate system of conductances, such that the associated (nonsymmetric, nearest neighbour) random walk converges almost surely to points in the boundary of the tree, and we prove that the Patterson densities, once normalised, are the corresponding harmonic measures. We thereby generalise the zero potential case treated in [CoP2], which is also a special case of [CoM] when X is a tree under the additional restriction that the discrete group is cocompact. For other connections between harmonic measures and Patterson measures, we refer for instance to [CoM, BIHM, Tan, GouMM] and their references.

Let \mathbb{X} be a $(q+1)$ -regular simplicial tree, with $q \geq 2$. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be an anti-reversible system of conductances for Γ , such that the associated map $\tilde{c} : V\mathbb{X} \rightarrow \mathbb{R}$ on the vertices of \mathbb{X} is constant. Let $(\mu_x)_{x \in V\mathbb{X}}$ be a Patterson density for (Γ, F_c) , where F_c is the potential associated with c . We denote by $\phi_\mu : x \mapsto \|\mu_x\|$ the associated total mass function on $V\mathbb{X}$.

We start this Section by recalling a few facts about discrete Markov chains, for which we refer for instance to [Rev, Woe1]. A *state space* is a discrete and countable set I . A *transition kernel* on a I is a map $P : I \times I \rightarrow [0, 1]$ such that for every $x \in I$,

$$\sum_{y \in I} P(x, y) = 1 .$$

Let λ be a probability measure on I . A (discrete) *Markov chain* on a state space I with initial distribution λ and transition kernel P is a sequence $(Z_n)_{n \in \mathbb{N}}$ of random variables with values in I such that for all $n \in \mathbb{N}$ and $x_0, \dots, x_{n+1} \in I$, the probability of events \mathbb{P} satisfies

- (1) $\mathbb{P}[Z_0 = x_0] = \lambda(\{x_0\})$,
- (2) $\mathbb{P}[Z_{n+1} = x_{n+1} \mid Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n] = P(x_n, x_{n+1})$.

The associated *random walk* consists in choosing a point x_0 in I with law λ , and by induction, once x_n is constructed, in choosing x_{n+1} in I with probability $P(x_n, x_{n+1})$. Note that

$$\mathbb{P}[Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n] = \lambda(\{x_0\}) P(x_0, x_1) \dots P(x_{n-1}, x_n) .$$

When the initial distribution λ is the unit Dirac mass Δ_x at $x \in I$, the Markov chain is then uniquely determined by its transition kernel P and by x , and is denoted by $(Z_n^x)_{n \in \mathbb{N}}$.

For every $n \in \mathbb{N}$, we denote by $P^{(n)}$ the iterated matrix product of the transition kernel P : we have $P^{(1)} = P$, $P^{(0)}(x, y)$ is the Kronecker symbol $\delta_{x,y}$ for all $x, y \in I$, and by induction $P^{(n+1)} = P \cdot P^{(n)}$ where \cdot is the matrix product of $I \times I$ matrices, that is, for all $x, z \in I$,

$$P^{(n+1)}(x, z) = \sum_{y \in I} P(x, y) P^{(n)}(y, z) .$$

Note that

$$P^{(n)}(x, y) = \mathbb{P}[Z_n^x = y]$$

is the probability for the random walk starting at time 0 from x of being at time n at the point y . The *Green kernel* of P is the map G_P from $I \times I$ to $[0, +\infty]$ defined by

$$(x, y) \mapsto G_P(x, y) = \sum_{n \in \mathbb{N}} P^{(n)}(x, y) ,$$

and its *Green function* is the following power series in the complex variable z :

$$G_P(x, y \mid z) = \sum_{n \in \mathbb{N}} P^{(n)}(x, y) z^n .$$

Recall that if $G_P(x, y) \neq 0$ for all $x, y \in I$, then the random walk is *recurrent*⁶ if $G_P(x, y) = \infty$ for any (hence all) $(x, y) \in I \times I$, and *transient* otherwise. Note that, using again matrix products of $I \times I$ matrices,

$$G_P = \text{Id} + P \cdot G_P . \tag{6.2}$$

We will from now on consider as state space the set $V\mathbb{X}$ of vertices of \mathbb{X} . If a Markov chain $(Z_n^x)_{n \in \mathbb{N}}$ starting at time 0 from x converges almost surely in $V\mathbb{X} \cup \partial_\infty \mathbb{X}$ to a random variable Z_∞^x , the law of Z_∞^x is called the *harmonic measure* (or *hitting measure* on the boundary) associated with this Markov chain, and is denoted by

$$\nu_x = (Z_\infty^x)_*(\mathbb{P}) .$$

Note that ν_x is a probability measure on $\partial_\infty \mathbb{X}$.

For instance, the transition kernel of the simple nearest neighbour random walk on \mathbb{X} is defined by taking as transition kernel the map \underline{P} where

$$\underline{P}(x, y) = \frac{1}{q+1} A(x, y)$$

⁶that is, $\text{Card}\{n \in \mathbb{N} : Z_n^x = y\} = \infty$ for every $y \in I$ (or equivalently, there exists $y \in I$ such that $\text{Card}\{n \in \mathbb{N} : Z_n^x = y\} = \infty$)

for all $x, y \in V\mathbb{X}$, with $A : V\mathbb{X} \times V\mathbb{X} \rightarrow \{0, 1\}$ the *adjacency matrix* of the tree \mathbb{X} , defined by $A(x, y) = 1$ for any two vertices x, y of \mathbb{X} that are joined by an edge in \mathbb{X} and $A(x, y) = 0$ otherwise. We denote by

$$\underline{\mathfrak{G}}(x, y | z) = \sum_{k \in \mathbb{N}} \underline{P}^{(k)}(x, y) z^k.$$

the Green function of \underline{P} , whose radius of convergence is $\underline{r} = \frac{q+1}{2\sqrt{q}}$ and which diverges at $z = \underline{r}$, see for example [Woe1], [Woe2, Ex. 9.82], [LyP2, §6.3].

The antireversible system of conductances $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ defines a cocycle on the set of vertices of \mathbb{X} , as follows. For every u, v in $V\mathbb{X}$, let $c(u, v) = 0$ if $u = v$ and otherwise let

$$c(x, y) = \sum_{i=1}^n \tilde{c}(e_i),$$

where (e_1, e_2, \dots, e_n) is the geodesic edge path in \mathbb{X} from $u = o(e_1)$ to $v = t(e_n)$.

Lemma 6.4. (1) *For every edge path $(e'_1, e'_2, \dots, e'_{n'})$ from u to v , we have*

$$c(u, v) = \sum_{i=1}^{n'} \tilde{c}(e'_i).$$

(2) *The map $c : V\mathbb{X} \times V\mathbb{X} \rightarrow \mathbb{R}$ has the following cocycle property: for all $u, v, w \in V\mathbb{X}$,*

$$c(u, v) + c(v, w) = c(u, w) \quad \text{and hence} \quad c(v, u) = -c(u, v).$$

(3) *We have $c(u, v) = \int_u^v \tilde{F}_c$.*

(4) *For all $\xi \in \partial_\infty \mathbb{X}$ and $u, v \in V\mathbb{X}$, if $C^\xi(\cdot, \cdot)$ is the Gibbs cocycle associated with \tilde{F}_c , we have*

$$C_\xi^c(u, v) = c(v, u) + \delta_c \beta_\xi(u, v).$$

Proof. (1) Since \mathbb{X} is a simplicial tree, any nongeodesic edge path from u to v has a back-and-forth on some edge, which contributes to 0 to the sum defining $c(x, y)$ by the anti-reversibility assumption on the system of conductances. Therefore, by induction, the sum in Assertion (1) indeed does not depend on the choice of the edge path from u to v .

Assertion (2) is immediate from Assertion (1). Assertion (3) follows from the definition of $c(\cdot, \cdot)$ by Proposition 3.11.

(4) For every $\xi \in \partial_\infty \mathbb{X}$, if $p \in V\mathbb{X}$ is such that $[u, \xi[\cap [v, \xi[= [p, \xi[$, then using Equation (3.6) and Assertions (3) and (2), we have

$$\begin{aligned} C_\xi^c(u, v) &= \int_v^p (\tilde{F}_c - \delta_c) - \int_u^p (\tilde{F}_c - \delta_c) = c(v, p) - c(u, p) + \delta_c \beta_\xi(u, v) \\ &= c(v, u) + \delta_c \beta_\xi(u, v). \quad \square \end{aligned}$$

We now define the transition kernel P_c associated with⁷ the (logarithmic) system of conductances c by, for all $x, y \in V\mathbb{X}$,

$$P_c(x, y) = \kappa_c \frac{\phi_\mu(y)}{\phi_\mu(x)} e^{c(x, y)} \underline{P}(x, y).$$

⁷The transition kernel also depends on the choice of the Patterson density if Γ is not of divergence type.

From now on, we denote by $(Z_n^x)_{n \in \mathbb{N}}$ the Markov chain with initial distribution Δ_x and transition kernel P_c .

Let

$$\kappa_c = \frac{1+q}{e^{\delta_c} + q e^{-\delta_c}},$$

which belongs to $]0, \frac{q+1}{2\sqrt{q}}]$. Note that this constant κ_c is less than the radius of convergence \underline{r} of the Green function $\underline{\mathfrak{G}}(x, y \mid z)$ if and only if $\cosh(\delta_c - \frac{1}{2} \ln q) > 1$, that is, if and only if $\delta_c \neq \frac{1}{2} \ln q$. The computation (due to Kesten) of the Green function of \underline{P} is well known, and gives the following formula, see for instance [CoP2, Prop. 3.1]: If $\delta_c \neq \frac{1}{2} \ln q$, then there exists $\alpha > 0$ such that for all $x, y \in V\mathbb{X}$

$$\underline{\mathfrak{G}}(x, y \mid \kappa_c) = \alpha e^{-\delta_c d(x, y)}. \quad (6.3)$$

Lemma 6.5. (1) *The map P_c is indeed a transition kernel on $V\mathbb{X}$.*

(2) *The Green kernel $\mathfrak{G}_c = \mathfrak{G}_{P_c}$ of P_c is*

$$\mathfrak{G}_c(x, y) = e^{c(x, y)} \frac{\phi_\mu(y)}{\phi_\mu(x)} \underline{\mathfrak{G}}(x, y \mid \kappa_c). \quad (6.4)$$

In particular, the Green kernel of P_c is finite if $\delta_c \neq \frac{1}{2} \ln q$.

(3) *Assume that $\delta_c \neq \frac{1}{2} \ln q$. For all $x, y, z \in V\mathbb{X}$, we have*

$$\frac{\phi_\mu(y) \mathfrak{G}_c(y, z)}{\phi_\mu(x) \mathfrak{G}_c(x, z)} = e^{c(y, x) + \delta_c(d(x, z) - d(y, z))}.$$

If furthermore $z \notin [x, y[$, then, for every $\xi \in \mathcal{O}_x(z)$,

$$\frac{\phi_\mu(y) \mathfrak{G}_c(y, z)}{\phi_\mu(x) \mathfrak{G}_c(x, z)} = e^{C_\xi^c(x, y)}.$$

Proof. (1) By Proposition 6.3, the positive function ϕ_μ is an eigenvector with eigenvalue $e^{\delta_c} + q e^{-\delta_c}$ for the operator

$$f \mapsto \{x \mapsto \sum_{e \in E\mathbb{X}, o(e)=x} e^{\tilde{c}(e)} f(t(e))\}.$$

Since $\underline{P}(o(e), t(e)) = \frac{1}{q+1}$ for every $e \in E\mathbb{X}$, we hence have

$$\begin{aligned} \sum_{y \in V\mathbb{X}} P_c(x, y) &= \sum_{e \in E\mathbb{X}, o(e)=x} P_c(x, t(e)) \\ &= \frac{1+q}{(e^{\delta_c} + q e^{-\delta_c}) \phi_\mu(x)} \sum_{e \in E\mathbb{X}, o(e)=x} e^{\tilde{c}(e)} \phi_\mu(t(e)) \underline{P}(x, t(e)) = 1. \end{aligned}$$

(2) Let us first prove that for all $x, y \in V\mathbb{X}$ and $n \in \mathbb{N}$, we have

$$P_c^{(n)}(x, y) = (\kappa_c)^n \frac{\phi_\mu(y)}{\phi_\mu(x)} e^{c(x, y)} \underline{P}^{(n)}(x, y). \quad (6.5)$$

Indeed, by the cocycle property of $c(\cdot, \cdot)$ and by a telescopic cancellation argument, we have

$$\begin{aligned}
P_c^{(n)}(x, y) &= \sum_{x_1, \dots, x_{n-1} \in V\mathbb{X}} P_c(x, x_1) P_c(x_1, x_2) \dots P_c(x_{n-2}, x_{n-1}) P_c(x_{n-1}, y) \\
&= (\kappa_c)^n \frac{\phi_\mu(y)}{\phi_\mu(x)} e^{c(x, y)} \sum_{x_1, \dots, x_{n-1} \in V\mathbb{X}} \underline{P}(x, x_1) \underline{P}(x_1, x_2) \dots \underline{P}(x_{n-2}, x_{n-1}) \underline{P}(x_{n-1}, y) \\
&= (\kappa_c)^n \frac{\phi_\mu(y)}{\phi_\mu(x)} e^{c(x, y)} \underline{P}^{(n)}(x, y).
\end{aligned}$$

Equation (6.4) follows from Equation (6.5) by summation on n . As we have already seen, $\kappa_c < r$ if and only if $\delta_c \neq \frac{1}{2} \ln q$. The last claim of Assertion (2) follows.

(3) Let $x, y, z \in V\mathbb{X}$. Using (twice) Assertion (2), the cocycle property of c and (twice) Equation (6.3), we have

$$\begin{aligned}
\frac{\mathfrak{G}_c(y, z)}{\mathfrak{G}_c(x, z)} &= \frac{e^{c(y, z)} \phi_\mu(z) \phi_\mu(x) \underline{\mathfrak{G}}(y, z \mid \kappa_c)}{e^{c(x, z)} \phi_\mu(y) \phi_\mu(z) \underline{\mathfrak{G}}(x, z \mid \kappa_c)} = e^{c(y, x)} \frac{\phi_\mu(x)}{\phi_\mu(y)} \frac{\alpha e^{-\delta_c d(y, z)}}{\alpha e^{-\delta_c d(x, z)}} \\
&= \frac{\phi_\mu(x)}{\phi_\mu(y)} e^{c(y, x) + \delta_c(d(x, z) - d(y, z))}.
\end{aligned}$$

This proves the first claim of Assertion (3). Under the additional assumptions on x, y, z, ξ , we have

$$\beta_\xi(x, y) = d(x, z) - d(y, z).$$

The last claim of Assertion (3) hence follows from Lemma 6.4 (4). \square

Using the criterion that the random walk starting from a given vertex of \mathbb{X} with transition probabilities P_c is transient if and only if the Green kernel $\mathfrak{G}_c(x, y)$ of P_c is finite (for any, hence for all, $x, y \in \mathbb{X}$), Lemma 6.5 (2) implies that if $\delta_c \neq \frac{1}{2} \ln q$, then $(Z_n^x)_{n \in \mathbb{N}}$ almost surely leaves every finite subset of $V\mathbb{X}$. The following result strengthens this remark.

Proposition 6.6. *If $\delta_c \neq \frac{1}{2} \ln q$, then for every $x \in V\mathbb{X}$, the Markov chain $(Z_n^x)_{n \in \mathbb{N}}$ (with initial distribution Δ_x and transition kernel P_c) converges almost surely in $V\mathbb{X} \cup \partial_\infty \mathbb{X}$ to a random variable with values in $\partial_\infty \mathbb{X}$. In particular the harmonic measure ν_x of $(Z_n^x)_{n \in \mathbb{N}}$ is well defined if $\delta_c \neq \frac{1}{2} \ln q$.*

Proof. Since \mathbb{X} is a tree, if $(x_n)_{n \in \mathbb{N}}$ is a sequence in $V\mathbb{X}$ such that $d(x_n, x_{n+1}) = 1$ for every $n \in \mathbb{N}$ and which does not converge to a point in $\partial_\infty \mathbb{X}$, then there exists a point y such that this sequence passes infinitely often through y , that is, $\{n \in \mathbb{N} : x_n = y\}$ is infinite. The result then follows from the fact that the Markov chain $(Z_n^x)_{n \in \mathbb{N}}$ is transient since $\delta_c \neq \frac{1}{2} \ln q$. \square

The following result, generalising [CoP2] when $c = 0$, says that the Patterson measures associated with a system of conductances c , once renormalised to probability measures, are exactly the harmonic measures for the random walk with transition probabilities P_c .

Theorem 6.7. *Let $(\mathbb{X}, \Gamma, \tilde{c}, (\mu_x)_{x \in V\mathbb{X}})$ be as in the beginning of Section 6.2. If $\delta_c \neq \frac{1}{2} \ln q$, then for every $x \in \mathbb{X}$, the harmonic measure of the Markov chain $(Z_n^x)_{n \in \mathbb{N}}$ is*

$$\nu_x = \frac{\mu_x}{\|\mu_x\|}.$$

Proof. We fix $x \in \mathbb{X}$. Recall that given $z \in V\mathbb{X}$, the shadow $\mathcal{O}_x(z)$ of z seen from x is the set of points at infinity of the geodesic rays from x through z .

For every $n \in \mathbb{N}$, we denote by $S(x, n)$ and $B(x, n)$ the sphere and ball of centre x and radius n in $V\mathbb{X}$, and we define two maps $f_1, f_2 : V\mathbb{X} \rightarrow \mathbb{R}$ with finite support by

$$f_1(z) = \frac{\mu_x(\mathcal{O}_x(z))}{\|\mu_x\| \mathfrak{G}_c(x, z)} \quad \text{and} \quad f_2(z) = \frac{\nu_x(\mathcal{O}_x(z))}{\mathfrak{G}_c(x, z)}$$

if $z \in S(x, n)$, and $f_1(z) = f_2(z) = 0$ otherwise. Let us prove that $f_1 = f_2$ for every $n \in \mathbb{N}$. Since $\{\mathcal{O}_x(z) : z \in V\mathbb{X}\}$ generates the Borel σ -algebra of $\partial_\infty \mathbb{X}$, this proves that the Borel measures ν_x and $\frac{\mu_x}{\|\mu_x\|}$ coincide.

We will use the following criterion: For all maps $G : V\mathbb{X} \times V\mathbb{X} \rightarrow \mathbb{R}$ and $f : V\mathbb{X} \rightarrow \mathbb{R}$ such that f has finite support, let us again denote by $G \cdot f$ the matrix product of G and f : for every $y \in V\mathbb{X}$,

$$G \cdot f(y) = \sum_{z \in V\mathbb{X}} G(y, z) f(z).$$

Lemma 6.8. *For all $f, f' : V\mathbb{X} \rightarrow \mathbb{R}$ with finite support, if $\mathfrak{G}_c \cdot f = \mathfrak{G}_c \cdot f'$, then $f = f'$.*

Proof. By Equation (6.2), we have

$$f' = \mathfrak{G}_c \cdot f' - P_c \cdot \mathfrak{G}_c \cdot f' = \mathfrak{G}_c \cdot f - P_c \cdot \mathfrak{G}_c \cdot f = f. \quad \square$$

Let us hence fix $n \in \mathbb{N}$ and prove that $\mathfrak{G}_c \cdot f_1 = \mathfrak{G}_c \cdot f_2$. Theorem 6.7 then follows.

Step 1: For every $y \in B(x, n)$, since $\{\mathcal{O}_x(z) : z \in S(x, n)\}$ is a Borel partition of $\partial_\infty \mathbb{X}$, by Equation (4.2), since $z \notin [x, y[$ if $z \in S(x, n)$ and $y \in B(x, n)$, and by the second claim of Lemma 6.5 (3), we have

$$\begin{aligned} 1 &= \frac{1}{\phi_\mu(y)} \int_{\partial_\infty \mathbb{X}} d\mu_y = \frac{1}{\phi_\mu(y)} \sum_{z \in S(x, n)} \int_{\mathcal{O}_x(z)} e^{-C_\xi^c(y, x)} d\mu_x(\xi) \\ &= \frac{1}{\phi_\mu(y)} \sum_{z \in S(x, n)} \int_{\mathcal{O}_x(z)} \frac{\phi_\mu(y) \mathfrak{G}_c(y, z)}{\phi_\mu(x) \mathfrak{G}_c(x, z)} d\mu_x(\xi) \\ &= \sum_{z \in S(x, n)} \mathfrak{G}_c(y, z) \frac{\mu_x(\mathcal{O}_x(z))}{\|\mu_x\| \mathfrak{G}_c(x, z)} = \mathfrak{G}_c \cdot f_1(y). \end{aligned} \tag{6.6}$$

Step 2: For all $y, z \in V\mathbb{X}$ such that $z \notin [x, y[$, any random walk starting at time 0 from y and converging to a point in $\mathcal{O}_x(z)$ goes through z . Let us denote by $C_x(z)$ the set of vertices different from z on the geodesic rays from z to the points in $\mathcal{O}_x(z)$. Partitioning by the last time the random walk passes through z , using the Markov property saying that what happens before the random walk arrives at z and after it leaves z are independent, we have

$$\nu_y(\mathcal{O}_x(z)) = \mathbb{P}[Z_\infty^y \in \mathcal{O}_x(z)] = \mathfrak{G}_c(y, z) \mathbb{P}[\forall n > 0, Z_n^z \in C_x(z)],$$

so that

$$\frac{\nu_y(\mathcal{O}_x(z))}{\nu_x(\mathcal{O}_x(z))} = \frac{\mathfrak{G}_c(y, z)}{\mathfrak{G}_c(x, z)}. \tag{6.7}$$

Step 3: For every $y \in B(x, n)$, again since $\{\mathcal{O}_x(z) : z \in S(x, n)\}$ is a Borel partition of $\partial_\infty \mathbb{X}$, and by Equation (6.7), we have

$$1 = \|\nu_y\| = \sum_{z \in S(x, n)} \nu_y(\mathcal{O}_x(z)) = \sum_{z \in S(x, n)} \mathfrak{G}_c(y, z) \frac{\nu_x(\mathcal{O}_x(z))}{\mathfrak{G}_c(x, z)} = \mathfrak{G}_c \cdot f_2(y). \quad (6.8)$$

Step 4: By Steps 1 and 3, we have $\mathfrak{G}_c \cdot f_1(y) = \mathfrak{G}_c \cdot f_2(y)$ for every $y \in B(x, n)$. Let now $y \in V\mathbb{X} - B(x, n)$. Since $x \in B(x, n)$, we have as just said $\mathfrak{G}_c \cdot f_1(x) = \mathfrak{G}_c \cdot f_2(x)$. Hence by the first claim of Lemma 6.5 (3), we have

$$\begin{aligned} \mathfrak{G}_c \cdot f_1(y) &= \sum_{z \in S(x, n)} \mathfrak{G}_c(y, z) f_1(z) \\ &= \sum_{z \in S(x, n)} e^{c(y, x) + \delta_c(d(x, z) - d(y, z))} \frac{\phi_\mu(x)}{\phi_\mu(y)} \mathfrak{G}_c(x, z) f_1(z) \\ &= e^{c(y, x) + \delta_c(d(x, z) - d(y, z))} \frac{\phi_\mu(x)}{\phi_\mu(y)} \mathfrak{G}_c \cdot f_1(x) \\ &= e^{c(y, x) + \delta_c(d(x, z) - d(y, z))} \frac{\phi_\mu(x)}{\phi_\mu(y)} \mathfrak{G}_c \cdot f_2(x) = \mathfrak{G}_c \cdot f_2(y). \end{aligned}$$

This proves that $\mathfrak{G}_c \cdot f_1 = \mathfrak{G}_c \cdot f_2$, thereby concluding the proof of Theorem 6.7. \square

Chapter 7

Skinning measures with potential on CAT(-1) spaces

In this Chapter, we introduce skinning measures as weighted pushforwards of the Patterson-Sullivan densities associated with a potential to the unit normal bundles of convex subsets of a CAT(-1) space. The development follows [PaP14a] with modifications to fit the present context.

Let $X, x_0, \Gamma, \tilde{F}$ be as in the beginning of Chapter 4. Let $(\mu_x^\pm)_{x \in X}$ be Patterson densities on $\partial_\infty X$ for the pairs (Γ, F^\pm) .

7.1 Skinning measures

Let D be a nonempty proper closed convex subset of X . The *outer skinning measure* $\tilde{\sigma}_D^+$ on $\partial_+^1 D$ and the *inner skinning measure* $\tilde{\sigma}_D^-$ on $\partial_-^1 D$ associated with the Patterson densities $(\mu_x^\pm)_{x \in X}$ for (Γ, \tilde{F}^\pm) are the measures $\tilde{\sigma}_D^\pm = \tilde{\sigma}_{D, F^\pm}^\pm$ defined by

$$d\tilde{\sigma}_D^\pm(\rho) = e^{C_{\rho_\pm}^\pm(x_0, \rho(0))} d\mu_{x_0}^\pm(\rho_\pm), \quad (7.1)$$

where $\rho \in \partial_\pm^1 D$, using the endpoint homeomorphisms $\rho \mapsto \rho_\pm$ from $\partial_\pm^1 D$ to $\partial_\infty X - \partial_\infty D$, and noting that $\rho(0) = P_D(\rho_\pm)$ depends continuously on ρ_\pm .

When $\tilde{F} = 0$, the skinning measure has been defined by Oh and Shah [OhS2] for the outer unit normal bundles of spheres, horospheres and totally geodesic subspaces in real hyperbolic spaces. The definition was generalised in [PaP14a] to the outer unit normal bundles of nonempty proper closed convex sets in Riemannian manifolds with variable negative curvature.

Note that the Gibbs measure is defined on the space $\mathcal{G}X$ of geodesic lines, the potential is defined on the space T^1X of germs at time $t = 0$ of geodesic lines, and since $\partial_\pm^1 D$ is contained in $\mathcal{G}_{\pm, 0}X$ (see Section 2.5), the skinning measures are defined on the spaces $\mathcal{G}_{\pm, 0}X$ of (generalised) geodesic rays. In the manifold case, all the above spaces are canonically identified with the unit tangent bundle, but in general, the natural restriction maps $\mathcal{G}X \rightarrow T^1X$ and $\mathcal{G}X \rightarrow \mathcal{G}_{\pm, 0}X$ have infinite (though compact) fibers.

Remark 7.1. (1) If $D = \{x\}$ is a singleton, then

$$d\tilde{\sigma}_D^\pm(\rho) = d\mu_x^\pm(\rho_\pm) \quad (7.2)$$

where ρ is a geodesic ray starting (at time $t = 0$) from x .

(2) When the potential \tilde{F} is reversible (in particular when $F = 0$), we have $C^- = C^+$, we may (and we will) take $\mu_x^- = \mu_x^+$ for all $x \in X$, hence $\iota_* \tilde{m}_F = \tilde{m}_F$ and $\tilde{\sigma}_D^- = \iota_* \tilde{\sigma}_D^+$.

(3) The (normalised) Gibbs cocycle being unchanged when the potential F is replaced by the potential $F + \sigma$ for any constant σ , we may (and will) take the Patterson densities, hence the Gibbs measure and the skinning measures, to be unchanged by such a replacement.

When D is a horoball in X , let us now related the skinning measures of D with previously known measures on $\partial_\infty X$, constructed using techniques due to Hamenstädt.

Let \mathcal{H} be a horoball centred at a point $\xi \in \partial_\infty X$. Recall that $P_{\mathcal{H}} : \partial_\infty X - \{\xi\} \rightarrow \partial \mathcal{H}$ is the closest point map on \mathcal{H} , mapping $\eta \neq \xi$ to the intersection with the boundary of \mathcal{H} of the geodesic line from η to ξ . The following result is proved in [HeP3, §2.3] when $F = 0$.

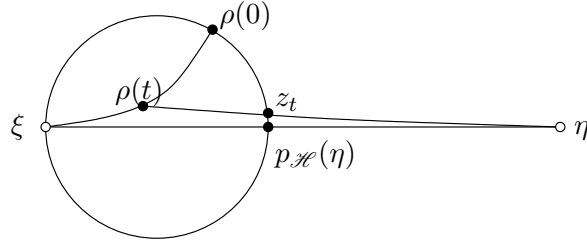
Proposition 7.2. *Let $\rho : [0, +\infty[\rightarrow X$ be the geodesic ray starting from any point of the boundary of \mathcal{H} and converging to ξ . The following weak-star limit of measures on $\partial_\infty X - \{\xi\}$*

$$d\mu_{\mathcal{H}}^\pm(\eta) = \lim_{t \rightarrow +\infty} e^{-\int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta)} d\mu_{\rho(t)}^\pm(\eta)$$

exists, and it does not depend on the choice of ρ . The measure $\mu_{\mathcal{H}}^\pm$ is invariant under the elements of Γ preserving \mathcal{H} , and it satisfies, for every $x \in X$ and (almost) every $\eta \in \partial_\infty X - \{\xi\}$,

$$\frac{d\mu_{\mathcal{H}}^\pm(\eta)}{d\mu_x^\pm(\eta)} = e^{-C_\eta^\pm(P_{\mathcal{H}}(\eta), x)} .$$

Proof. We prove all three assertions simultaneously. Let us fix $x \in X$. For all $t \geq 0$ and $\eta \in \partial_\infty X - \{\xi\}$, let z_t be the closest point to $P_{\mathcal{H}}(\eta)$ on the geodesic ray from $\rho(t)$ to η .



Using Equation (4.2) with x replaced by $\rho(t)$ and y by the present x , by the cocycle equation (3.7) and by Equation (3.8) as $z_t \in [\rho(t), \eta[$, we have

$$\begin{aligned} e^{-\int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta)} d\mu_{\rho(t)}^\pm(\eta) &= e^{-\int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta)} e^{-C_\eta^\pm(\rho(t), x)} d\mu_x^\pm(\eta) \\ &= e^{-\int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta)} e^{-C_\eta^\pm(\rho(t), z_t)} e^{-C_\eta^\pm(z_t, x)} d\mu_x^\pm(\eta) \\ &= e^{-\int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta) + \int_{\rho(t)}^{z_t} (\tilde{F}^\pm - \delta)} e^{-C_\eta^\pm(z_t, x)} d\mu_x^\pm(\eta) . \end{aligned}$$

As $t \rightarrow +\infty$, note that z_t converges to $P_{\mathcal{H}}(\eta)$ and that by the HC-property (and since \tilde{F} is bounded on any compact neighbourhood of $P_{\mathcal{H}}(\eta)$), we have

$$\left| \int_{\rho(t)}^{P_{\mathcal{H}}(\eta)} (\tilde{F}^\pm - \delta) - \int_{\rho(t)}^{z_t} (\tilde{F}^\pm - \delta) \right| \rightarrow 0 .$$

The result then follows by the continuity of the Gibbs cocycle (see Proposition 3.10 (3)). \square

Using this proposition and the cocycle property of C^\pm in the definition (4.3) of the Gibbs measure, we obtain, for every $\ell \in \mathcal{GX}$ such that $\ell_\pm \neq \xi$,

$$d\tilde{m}_F(\ell) = e^{C_{\ell_-}^-(P_{\mathcal{H}}(\ell_-), \ell(0)) + C_{\ell_+}^+(P_{\mathcal{H}}(\ell_+), \ell(0))} d\mu_{\mathcal{H}}^-(\ell_-) d\mu_{\mathcal{H}}^+(\ell_+) dt. \quad (7.3)$$

Note that it is easy to see that for every $\rho \in \partial_\pm^1 \mathcal{H}$, we have

$$d\tilde{\sigma}_{\mathcal{H}}^\pm(\rho) = d\mu_{\mathcal{H}}^\pm(\rho_\pm). \quad (7.4)$$

When $F = 0$, we obtain *Hamenstädt's measure*

$$\mu_{\mathcal{H}} = \lim_{t \rightarrow +\infty} e^{\delta_\Gamma t} \mu_{\rho(t)} \quad (7.5)$$

on $\partial_\infty X - \{\xi\}$ associated with the horoball \mathcal{H} , which is independent of the choice of the geodesic ray ρ starting from a point of the horosphere $\partial \mathcal{H}$ and converging to ξ . Note that for every $t \geq 0$, if $\mathcal{H}[t]$ is the horoball contained in \mathcal{H} whose boundary is at distance t from the boundary of \mathcal{H} , we then have

$$\mu_{\mathcal{H}[t]} = e^{-\delta_\Gamma t} \mu_{\mathcal{H}}. \quad (7.6)$$

The following results give the basic properties of the skinning measures analogous to those in [PaP14a, Sect. 3] when the potential is zero.

Proposition 7.3. *Let D be a nonempty proper closed convex subset of X , and let $\tilde{\sigma}_D^\pm$ be the skinning measures on $\partial_\pm^1 D$ for the potential \tilde{F} .*

- (i) *The skinning measures $\tilde{\sigma}_D^\pm$ are independent of x_0 .*
- (ii) *For all $\gamma \in \Gamma$, we have $\gamma_* \tilde{\sigma}_D^\pm = \tilde{\sigma}_{\gamma D}^\pm$. In particular, the measures $\tilde{\sigma}_D^\pm$ are invariant under the stabiliser of D in Γ .*
- (iii) *For all $s \geq 0$ and $w \in \partial_\pm^1 D$, denoting by $(\mathbf{g}^{\pm s} w)|_{\pm[0, +\infty[}$ the element of $\mathcal{G}_{\pm, 0}$ which coincides with $\mathbf{g}^{\pm s} w$ on $\pm[0, +\infty[$, we have*

$$d\tilde{\sigma}_{\mathcal{N}_s D}^\pm((\mathbf{g}^{\pm s} w)|_{\pm[0, +\infty[}) = e^{C_{\tilde{w}^\pm}^\pm(\pi(w), \pi(\mathbf{g}^{\pm s} w))} d\tilde{\sigma}_D^\pm(w) = e^{-\int_{\pi(w)}^{\pi(\mathbf{g}^{\pm s} w)} (\tilde{F}^\pm - \delta)} d\tilde{\sigma}_D^\pm(w).$$

- (iv) *The support of $\tilde{\sigma}_D^\pm$ is*

$$\{v \in \partial_\pm^1 D : v_\pm \in \Lambda\Gamma\} = P_D^\pm(\Lambda\Gamma - (\Lambda\Gamma \cap \partial_\infty D)).$$

In particular, $\tilde{\sigma}_D^\pm$ is the zero measure if and only if $\Lambda\Gamma$ is contained in $\partial_\infty D$.

For future use, the version¹ of Assertion (iii) when $F = 0$ is

$$\frac{d(\mathbf{g}^{\pm s})_* \tilde{\sigma}_D^\pm}{d\tilde{\sigma}_{\mathcal{N}_s D}^\pm}(\mathbf{g}^{\pm s} w) = e^{-\delta s}. \quad (7.7)$$

As another particular case of Assertion (iii) for future use, consider the case when $X = |\mathbb{X}|_\lambda$ is the geometric realisation of a metric tree (\mathbb{X}, λ) and when $\tilde{F} = \tilde{F}_c$ is the potential associated with a system of conductances c on \mathbb{X} (see Equation (3.11) and Proposition 3.12). Then for

¹contained in [PaP14a, Prop. 4]

all $w \in \partial_+^1 D$ (respectively $w \in \partial_-^1 D$), if e_w is the first (respectively the last) edge followed by w , with length $\lambda(e_w)$, then

$$\int_{\pi(w)}^{\pi(\mathbf{g}^{\pm\lambda(e_w)}w)} \tilde{F}^{\pm} = c(e_w)$$

by Proposition 3.11, so that

$$d\tilde{\sigma}_{\mathcal{N}_s D}^{\pm}((\mathbf{g}^{\pm\lambda(e_w)}w)|_{\pm[0,+\infty[}) = e^{-c(e_w)+\delta\lambda(e_w)} d\tilde{\sigma}_D^{\pm}(w). \quad (7.8)$$

Proof. We give details only for the proof of claim (iii) for the measure $\tilde{\sigma}_D^+$, the case of $\tilde{\sigma}_D^-$ being similar, and the proofs of the other claims being straightforward modifications of those in [PaP14a, Prop. 4]. Since $((\mathbf{g}^s w)|_{[0,+\infty[})_+ = w_+$ and since $w \in \partial_+^1 D$ if and only if $(\mathbf{g}^s w)|_{[0,+\infty[} \in \partial_+^1 \mathcal{N}_s D$, we have, using the definition of the skinning measure and the cocycle property (3.7), for all $s \geq 0$,

$$d\tilde{\sigma}_{\mathcal{N}_s D}^+((\mathbf{g}^s w)|_{[0,+\infty[}) = e^{C_{w_+}^+(x_0, \pi(\mathbf{g}^s w))} d\mu_{x_0}^+(w_+) = e^{C_{w_+}^+(\pi(w), \pi(\mathbf{g}^s w))} d\tilde{\sigma}_D^+(w).$$

This proves the claim (iii) for $\tilde{\sigma}_D^+$, using Equation (3.8). \square

Given two nonempty closed convex subsets D and D' of X , let

$$A_{D,D'} = \partial_\infty X - (\partial_\infty D \cup \partial_\infty D')$$

and let $h_{D,D'}^{\pm} : P_D^{\pm}(A_{D,D'}) \rightarrow P_{D'}^{\pm}(A_{D,D'})$ be the restriction of $P_{D'}^{\pm} \circ (P_D^{\pm})^{-1}$ to $P_D^{\pm}(A_{D,D'})$. It is a homeomorphism between open subsets of $\partial_{\pm}^1 D$ and $\partial_{\pm}^1 D'$, associating to the element w in the domain the unique element w' in the range with $w'_\pm = w_\pm$. The proof of Proposition 5 of [PaP14a] generalises immediately to give the following result.

Proposition 7.4. *Let D and D' be nonempty closed convex subsets of X and $h^{\pm} = h_{D,D'}^{\pm}$. The measures $(h^{\pm})_* \tilde{\sigma}_D^{\pm}$ and $\tilde{\sigma}_{D'}^{\pm}$ on $P_{D'}^{\pm}(A_{D,D'})$ are absolutely continuous one with respect to the other, with*

$$\frac{d(h^{\pm})_* \tilde{\sigma}_D^{\pm}}{d\tilde{\sigma}_{D'}^{\pm}}(w') = e^{-C_{w'_\pm}^{\pm}(\pi(w), \pi(w'))},$$

for (almost) all $w \in P_D^{\pm}(A_{D,D'})$ and $w' = h^{\pm}(w)$. \square

Let $w \in \mathcal{G}_{\pm} X$. With $N_w^{\pm} : W^{\pm}(w) \rightarrow \partial_{\mp}^1 HB_{\pm}(w)$ the canonical homeomorphism defined in Section 2.5, we define the *skinning measures* $\mu_{W^{\pm}(w)}$ on the strong stable or strong unstable leaves $W^{\pm}(w)$ by

$$\mu_{W^{\pm}(w)} = ((N_w^{\pm})^{-1})_* \tilde{\sigma}_{HB_{\pm}(w)}^{\mp},$$

so that

$$d\mu_{W^{\pm}(w)}(\ell) = e^{C_{\ell_{\mp}}^{\mp}(x_0, \ell(0))} d\mu_{x_0}^{\mp}(\ell_{\mp}) \quad (7.9)$$

for every $\ell \in W^{\pm}(w)$. By Proposition 7.3 (ii) and the naturality of N_w^{\pm} , for every $\gamma \in \Gamma$, we have

$$\gamma_* \mu_{W^{\pm}(w)} = \mu_{W^{\pm}(\gamma w)}. \quad (7.10)$$

By Proposition 7.3 (iv), the support of $\mu_{W^{\pm}(w)}$ is $\{\ell \in W^{\pm}(w) : \ell_{\mp} \in \Lambda\Gamma\}$. For all $t \in \mathbb{R}$ and $\ell \in W^{\pm}(w)$, we have, using Equations (3.7) and (3.8), and since $\ell_{\pm} = w_{\pm}$,

$$\frac{d(\mathbf{g}^{-t})_* \mu_{W^{\pm}(w)}}{d\mu_{W^{\pm}(\mathbf{g}^t w)}}(\mathbf{g}^t \ell) = e^{C_{\ell_{\mp}}^{\mp}(\ell(t), \ell(0))} = e^{C_{w_{\pm}}^{\pm}(\ell(0), \ell(t))}. \quad (7.11)$$

Let $w \in \mathcal{G}_\pm X$. The homeomorphisms $W^\pm(w) \times \mathbb{R} \rightarrow W^{0\pm}(w)$, defined by setting $(\ell, s) \mapsto \ell' = \mathbf{g}^s \ell$, conjugate the actions of \mathbb{R} by translation on the second factor of the domain and by the geodesic flow on the range, and the actions of Γ (trivial on the second factor of the domain). Let us consider the measures ν_w^\mp on $W^{0\pm}(w)$ given, using the above homeomorphism, by

$$d\nu_w^\mp(\ell') = e^{C_{w^\pm}^\pm(w(0), \ell(0))} d\mu_{W^\pm(w)}(\ell) ds. \quad (7.12)$$

They satisfy $(\mathbf{g}^t)_* \nu_w^\pm = \nu_w^\pm$ for all $t \in \mathbb{R}$ (since if $\ell' = \mathbf{g}^s \ell$, then $\mathbf{g}^{-t} \ell' = \mathbf{g}^{s-t} \ell$). Furthermore, $\gamma_* \nu_w^\pm = \nu_{\gamma w}^\pm$ for all $\gamma \in \Gamma$. In general, they depend on w , not only on $W^\pm(w)$. Furthermore, the support of ν_w^\pm is $\{\ell' \in W^{0\pm}(w) : \ell'_\mp \in \Lambda\Gamma\}$. These properties follow easily from the properties of the skinning measures on the strong stable or strong unstable leaves.

Lemma 7.5. (i) *For every nonempty proper closed convex subset D' in X , there exists $R_0 > 0$ such that for all $R \geq R_0$, $\eta > 0$, and $w \in \partial_\pm^1 D'$, we have $\nu_w^\mp(V_{w, \eta, R}^\pm) > 0$.*

(ii) *For all $w \in \mathcal{G}_\pm X$ and $t \in \mathbb{R}$, the measures $\nu_{\mathbf{g}^t w}^\mp$ and ν_w^\mp are proportional:*

$$\nu_{\mathbf{g}^t w}^\mp = e^{C_{w^\pm}^\pm(w(t), w(0))} \nu_w^\mp.$$

Proof. (i) By [PaP14a, Lem. 7],² there exists $R_0 > 0$ (depending only on D' and on the Patterson densities) such that for all $R \geq R_0$, $w \in \partial_+^1 D'$ and $w' \in \partial_-^1 D'$, we have $\mu_{W^+(w)}(B^+(w, R)) > 0$ and $\mu_{W^-(w')} (B^-(w', R)) > 0$. The result hence follows by the definitions of ν_w^\mp and $V_{w, \eta, R}^\pm$.

(ii) For all $w \in \mathcal{G}_\pm X$, $s, t \in \mathbb{R}$ and $\ell \in W^\pm(w)$, we have by Equations (7.12) and (7.11), and by the cocycle property of C^\pm ,

$$\begin{aligned} d\nu_{\mathbf{g}^t w}^\mp(\mathbf{g}^s \ell) &= d\nu_{\mathbf{g}^t w}^\mp(\mathbf{g}^{s-t} \mathbf{g}^t \ell) = e^{C_{(\mathbf{g}^t w)^\pm}^\pm(\mathbf{g}^t w(0), \mathbf{g}^t \ell(0))} d\mu_{W^\pm(\mathbf{g}^t w)}(\mathbf{g}^t \ell) d(s-t) \\ &= e^{C_{w^\pm}^\pm(w(t), \ell(t))} e^{-C_{w^\pm}^\pm(\ell(0), \ell(t))} d\mu_{W^\pm(w)}(\ell) ds \\ &= e^{C_{w^\pm}^\pm(w(t), \ell(0))} e^{-C_{w^\pm}^\pm(w(0), \ell(0))} d\nu_w^\mp(\mathbf{g}^s \ell) \\ &= e^{C_{w^\pm}^\pm(w(t), w(0))} d\nu_w^\mp(\mathbf{g}^s \ell). \quad \square \end{aligned}$$

The following disintegration result of the Gibbs measure over the skinning measures of any closed convex subset is a crucial tool for our equidistribution and counting results. Recall the definition in Equation (2.10) of the flow-invariant open sets \mathcal{U}_D^\pm and the definition of the fibrations $f_D^\pm : \mathcal{U}_D^\pm \rightarrow \partial_\pm^1 D$ from Section 2.5.

Proposition 7.6. *Let D be a nonempty proper closed convex subset of X . The restriction to \mathcal{U}_D^\pm of the Gibbs measure \tilde{m}_F disintegrates by the fibration $f_D^\pm : \mathcal{U}_D^\pm \rightarrow \partial_\pm^1 D$, over the skinning measure $\tilde{\sigma}_D^\pm$ of D , with conditional measure ν_ρ^\mp on the fiber $(f_D^\pm)^{-1}(\rho) = W^{0\pm}(\rho)$ of $\rho \in \partial_\pm^1 D$: when ℓ ranges over \mathcal{U}_D^\pm , we have*

$$d\tilde{m}_{F|\mathcal{U}_D^\pm}(\ell) = \int_{\rho \in \partial_\pm^1 D} d\nu_\rho^\mp(\ell) d\tilde{\sigma}_D^\pm(\rho).$$

Proof. In order to prove the claim for the fibration f_D^+ , let $\phi \in \mathcal{C}_c(\mathcal{U}_D^+)$. Using in the various steps below:

²whose proof extends to the present situation, although the notation is different.

- Hopf's parametrisation with time parameter t and the definitions of \tilde{m}_F (see Equation (4.3)) and of \mathcal{U}_D^+ (see Equation (2.10)),
 - the positive endpoint homeomorphism $w \mapsto w_+$ from $\partial_+^1 D$ to $\partial_\infty X - \partial_\infty D$, and the negative endpoint homeomorphism $v' \mapsto v'_-$ from $W^+(w)$ to $\partial_\infty X - \{w_+\}$, with $s \in \mathbb{R}$ the real parameter such that $v' = g^{-s}v \in W^+(w)$ where $v \in W^{0+}(w)$, noting that $t - s$ depends only on $v_+ = w_+$ and $v_- = v'_-$,
 - the definitions of the measures $\mu_{W^+(w)}$ (see Equation (7.9)) and $\tilde{\sigma}_D^+$ (see Equation (7.1)) and the cocycle property of C^\pm ,
 - Equation (3.8) and the cocycle properties of C^+ ,
- we have

$$\begin{aligned}
& \int_{v \in \mathcal{U}_D^+} \phi(v) d\tilde{m}_F(v) \\
&= \int_{v_+ \in \partial_\infty X - \partial_\infty D} \int_{v_- \in \partial_\infty X - \{v_+\}} \int_{t \in \mathbb{R}} \phi(v) e^{C_{v_-}^-(x_0, \pi(v)) + C_{v_+}^+(x_0, \pi(v))} dt d\mu_{x_0}^-(v_-) d\mu_{x_0}^+(v_+) \\
&= \int_{w \in \partial_+^1 D} \int_{v' \in W^+(w)} \int_{s \in \mathbb{R}} \phi(\mathbf{g}^s v') e^{C_{v'_-}^-(x_0, \pi(\mathbf{g}^s v')) + C_{w_+}^+(x_0, \pi(\mathbf{g}^s v'))} ds d\mu_{x_0}^-(v'_-) d\mu_{x_0}^+(w_+) \\
&= \int_{w \in \partial_+^1 D} \int_{v' \in W^+(w)} \int_{s \in \mathbb{R}} \phi(\mathbf{g}^s v') e^{C_{v'_-}^-(\pi(v'), \pi(\mathbf{g}^s v')) + C_{w_+}^+(\pi(w), \pi(\mathbf{g}^s v'))} ds d\mu_{W^+(w)}(v') d\tilde{\sigma}_D^+(w) \\
&= \int_{w \in \partial_+^1 D} \int_{v' \in W^+(w)} \int_{s \in \mathbb{R}} \phi(\mathbf{g}^s v') e^{C_{w_+}^+(\pi(w), \pi(v'))} ds d\mu_{W^+(w)}(v') d\tilde{\sigma}_D^+(w),
\end{aligned}$$

which implies the claim for the fibration f_D^+ . The proof for the fibration f_D^- is similar. \square

In particular, for every $u \in \mathcal{G}_- X$, applying the above proposition and a change of variable to $D = HB_-(u)$ for which $\partial_+^1 D = N_u^-(W^-(u))$ and

$$\mathcal{U}_D^+ = \mathcal{G}X - W^{0+}(lu) = \bigcup_{w \in W^-(u)} W^{0+}(w),$$

the restriction to $\mathcal{G}X - W^{0+}(lu)$ of the Gibbs measure \tilde{m}_F disintegrates over the strong unstable measure $\mu_{W^-(u)} = ((N_u^-)^{-1})_* \tilde{\sigma}_D^+$, with conditional measure on the fiber $W^{0+}(w)$ of $w \in W^-(u)$ the measure $\nu_w^- = \nu_{N_u^-(w)}^-$: for every $\phi \in \mathcal{C}_c(\mathcal{G}X - W^{0+}(lu))$, we have

$$\begin{aligned}
& \int_{\mathcal{G}X - W^{0+}(lu)} \phi(v) d\tilde{m}_F(v) = \\
& \int_{w \in W^-(u)} \int_{v' \in W^+(w)} \int_{s \in \mathbb{R}} \phi(\mathbf{g}^s v') e^{C_{w_+}^+(\pi(w), \pi(v'))} ds d\mu_{W^+(w)}^-(v') d\mu_{W^-(u)}(w). \quad (7.13)
\end{aligned}$$

Note that if the Patterson densities have no atoms, then the stable and unstable leaves have measure zero for the associated Gibbs measure. This happens for instance if the Gibbs measure m_F is finite, see Corollary 4.6 and Theorem 4.5.

7.2 Equivariant families of convex subsets and their skinning measures

Let I be an index set endowed with a left action of Γ . A family $\mathcal{D} = (D_i)_{i \in I}$ of subsets of X or of $\check{\mathcal{G}}X$ indexed by I is Γ -equivariant if $\gamma D_i = D_{\gamma i}$ for all $\gamma \in \Gamma$ and $i \in I$. We will denote

by $\sim = \sim_{\mathcal{D}}$ the equivalence relation on I defined by $i \sim j$ if and only if $D_i = D_j$ and there exists $\gamma \in \Gamma$ such that $j = \gamma i$. This equivalence relation is Γ -equivariant: for all $i, j \in I$ and $\gamma \in \Gamma$, we have $\gamma i \sim \gamma j$ if and only if $i \sim j$. We say that \mathcal{D} is *locally finite* if for every compact subset K in X or in $\check{\mathcal{G}}X$, the quotient set $\{i \in I : D_i \cap K \neq \emptyset\}/\sim$ is finite.

Examples. (1) Fixing a nonempty proper closed convex subset D of X , taking $I = \Gamma$ with the left action by translations $(\gamma, i) \mapsto \gamma i$, and setting $D_i = iD$ for every $i \in \Gamma$ gives a Γ -equivariant family $\mathcal{D} = (D_i)_{i \in \Gamma}$. In this case, we have $i \sim j$ if and only if $i^{-1}j$ belongs to the stabiliser Γ_D of D in Γ , and $I/\sim = \Gamma/\Gamma_D$. Note that γD depends only on the class $[\gamma]$ of γ in Γ/Γ_D . We could also take $I' = \Gamma/\Gamma_D$ with the left action by translations $(\gamma, [\gamma']) \mapsto [\gamma\gamma']$, and $\mathcal{D}' = (\gamma D)_{[\gamma] \in I'}$, so that for all $i, j \in I'$, we have $i \sim_{\mathcal{D}'} j$ if and only if $i = j$, and besides, \mathcal{D}' is locally finite if and only if \mathcal{D} is locally finite. The following choices of D yield equivariant families with different characteristics:

- (a) Let $\gamma_0 \in \Gamma$ be a loxodromic element with translation axis $D = \text{Ax}_{\gamma_0}$. The family $(\gamma D)_{\gamma \in \Gamma}$ is locally finite and Γ -equivariant. Indeed, by Lemma 2.1, only finitely many elements of the family $(\gamma D)_{\gamma \in \Gamma/\Gamma_D}$ meet any given bounded subset of X .
- (b) Let $\ell \in \mathcal{G}X$ be a geodesic line whose image under the canonical map $\mathcal{G}X \rightarrow \Gamma \backslash \mathcal{G}X$ has a dense orbit in $\Gamma \backslash \mathcal{G}X$ under the geodesic flow, and let $D = \ell(\mathbb{R})$ be its image. Then the Γ -equivariant family $(\gamma D)_{\gamma \in \Gamma}$ is not locally finite.
- (c) More generally, let D be a convex subset such that $\Gamma_D \backslash D$ is compact. Then the family $(\gamma D)_{\gamma \in \Gamma}$ is a locally finite Γ -equivariant family.
- (d) Let $\xi \in \partial_{\infty}X$ be a bounded parabolic limit point of Γ , and let \mathcal{H} be any horoball in X centred at ξ . Then the family $(\gamma \mathcal{H})_{\gamma \in \Gamma}$ is a locally finite Γ -equivariant family.

(2) More generally, let $(D^\alpha)_{\alpha \in A}$ be a finite family of nonempty proper closed convex subsets of X , and for every $\alpha \in A$, let F_α be a finite set. Define $I = \bigcup_{\alpha \in A} \Gamma \times \{\alpha\} \times F_\alpha$ with the action of Γ by left translation on the first factor, and for every $i = (\gamma, \alpha, x) \in I$, let $D_i = \gamma D^\alpha$. Then $I/\sim = \bigcup_{\alpha \in A} \Gamma/\Gamma_{D^\alpha} \times \{\alpha\} \times F_\alpha$ and the Γ -equivariant family $\mathcal{D} = (D_i)_{i \in I}$ is locally finite if and only if the family $(\gamma D^\alpha)_{\gamma \in \Gamma}$ is locally finite for every $\alpha \in A$. The cardinalities of F_α for $\alpha \in A$ contribute to the multiplicities (see Section 12.2).

Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper closed convex subsets of X . Let $\Omega = (\Omega_i)_{i \in I}$ be a Γ -equivariant family of subsets of $\check{\mathcal{G}}X$, where Ω_i is a measurable subset of $\partial_{\pm}^1 D_i$ for all $i \in I$ (the sign \pm being constant). Then

$$\tilde{\sigma}_{\Omega}^{\pm} = \sum_{i \in I/\sim} \tilde{\sigma}_{D_i}^{\pm} |_{\Omega_i},$$

is a well-defined Γ -invariant locally finite measure on $\check{\mathcal{G}}X$, whose support is contained in $\mathcal{G}_{\pm,0}X$. Hence, the measure $\tilde{\sigma}_{\Omega}^{\pm}$ induces a locally finite measure on $\Gamma \backslash \check{\mathcal{G}}X$, denoted by σ_{Ω}^{\pm} , see for example [PauPS, §2.6], in particular for warnings concerning the fact that Γ does not always act freely on $\mathcal{G}X$. When $\Omega = \partial_{\pm}^1 \mathcal{D} = (\partial_{\pm}^1 D_i)_{i \in I}$, the measure $\tilde{\sigma}_{\Omega}^{\pm}$ is denoted by

$$\tilde{\sigma}_{\mathcal{D}}^{\pm} = \sum_{i \in I/\sim} \tilde{\sigma}_{D_i}^{\pm}.$$

The measures $\tilde{\sigma}_{\mathcal{D}}^+$ and $\tilde{\sigma}_{\mathcal{D}}^-$ are respectively called the *outer and inner skinning measures* of \mathcal{D} on \check{X} , and their induced³ measures $\sigma_{\mathcal{D}}^+$ and $\sigma_{\mathcal{D}}^-$ on $\Gamma \backslash \check{X}$ are the *outer and inner skinning measures* of \mathcal{D} on $\Gamma \backslash \check{X}$.

Example. Consider the Γ -equivariant family $\mathcal{D} = (\gamma D)_{\gamma \in \Gamma/\Gamma_x}$ with $D = \{x\}$ a singleton in X . With $\pi^\pm = (P_D^\pm|_{\partial_\infty X})^{-1} : \partial_\pm^1 D \rightarrow \partial_\infty X$ the homeomorphism $\rho \mapsto \rho_\pm$, we have $(\pi^\pm)_* \tilde{\sigma}_D^\pm = \mu_x^\pm$ by Remark (1) in Section 7, and

$$\|\sigma_{\mathcal{D}}^\pm\| = \frac{\|\mu_x^\pm\|}{|\Gamma_x|}. \quad (7.14)$$

³See for instance [PauPS, §2.6] for details on the definition of the induced measure when Γ may have torsion.

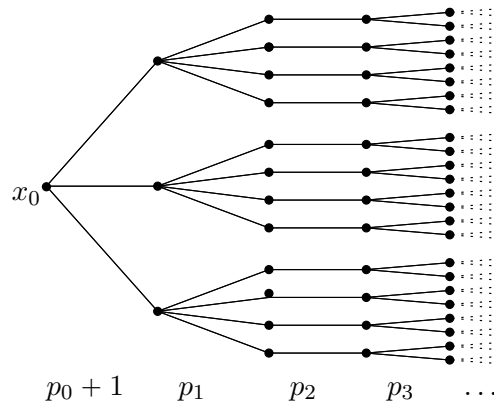
Chapter 8

Explicit measure computations for simplicial trees and graphs of groups

In this Chapter, we compute skinning measures and Bowen-Margulis measures for some highly symmetric simplicial trees \mathbb{X} endowed with a nonelementary discrete subgroup Γ of $\text{Aut}(\mathbb{X})$. The potentials F are supposed to be 0 in this Chapter, and we assume that the Patterson densities $(\mu_x^+)_{x \in V\mathbb{X}}$ and $(\mu_x^-)_{x \in V\mathbb{X}}$ of Γ are equal, denoted by $(\mu_x)_{x \in V\mathbb{X}}$. As the study of geometrically finite discrete subgroups of $\text{Aut}(\mathbb{X})$ mostly reduces to the study of particular (tree) lattices (see Remark 2.12), we will assume that Γ is a lattice in this Chapter.

The results of these computations will be useful when we state special cases of the equidistribution and counting results in regular and biregular trees and, in particular, in the arithmetic applications in Part III. The reader only interested in the continuous time case may skip directly to Chapter 9.

A rooted simplicial tree (\mathbb{X}, x_0) is *spherically symmetric* if \mathbb{X} is not reduced to x_0 and has no terminal vertex, and if the stabiliser of x_0 in $\text{Aut}(\mathbb{X})$ acts transitively on each sphere of centre x_0 . The set of isomorphism classes of spherically symmetric rooted simplicial trees (\mathbb{X}, x_0) is in bijection with the set of sequences $(p_n)_{n \in \mathbb{N}}$ in $\mathbb{N} - \{0\}$, where $p_n + 1$ is the degree of any vertex of \mathbb{X} at distance n from x_0 .



If (\mathbb{X}, x_0) is spherically symmetric, it is easy to check that the simplicial tree \mathbb{X} is uniform if and only if the sequence $(p_n)_{n \in \mathbb{N}}$ is periodic with palindromic period in the sense that there exists $N \in \mathbb{N} - \{0\}$ such that $p_{n+N} = p_n$ and $p_{N-n} = p_n$ for all $n \in \mathbb{N}$ (such that $n \leq N$ for

the second property). If $N = 1$, then $\mathbb{X} = \mathbb{X}_{p_0}$ is the regular tree of degree $p_0 + 1$, and if $N = 2$, then $\mathbb{X} = \mathbb{X}_{p_0, p_1}$ is the biregular tree of degrees $p_0 + 1$ and $p_1 + 1$.

The Hausdorff dimension $h_{\mathbb{X}}$ of $\partial_{\infty}\mathbb{X}$ for any visual distance is then

$$h_{\mathbb{X}} = \frac{1}{N} \ln(p_0 \dots p_{N-1}),$$

see for example [Lyo, p. 935].

8.1 Computations of Bowen-Margulis measures for simplicial trees

The next result gives examples of computations of the total mass of Bowen-Margulis measures for lattices of simplicial trees having some regularity properties.

Analogous computations can be performed for Riemannian manifolds having appropriate regularity properties. We refer for instance to [PaP16a, Prop. 10] and [PaP16c, Prop. 20 (1)] for computations of Bowen-Margulis measures for lattices in the isometry group of the real hyperbolic spaces, and to [PaP16b, Lem. 4.2 (iii)] for the computation in the complex hyperbolic case. In both cases, the main point is the computation of the proportionality constant between the Bowen-Margulis measure and Sasaki's Riemannian volume of the unit tangent bundle. When dealing now with simplicial trees, similar consequences of homogeneity properties will appear below.

We refer to Section 2.7 for the definitions of $T\pi$, Tvol , TVol appearing in the following result.

Proposition 8.1. *Let (\mathbb{X}, x_0) be a spherically symmetric rooted simplicial tree with associated sequence $(p_n)_{n \in \mathbb{N}}$ such that \mathbb{X} is uniform, and let Γ be a lattice of \mathbb{X} .*

(1) *For every $x \in V\mathbb{X}$, let $r_x = d(x, \text{Aut}(\mathbb{X})x_0)$, and let*

$$c_x = \frac{(p_{r_x} - 1)e^{2r_x h_{\mathbb{X}}}}{(p_0 + 1)^2 p_1^2 \dots p_{r_x-1}^2 p_{r_x}} + \frac{2p_0}{(p_0 + 1)^2}$$

if $r_x \neq 0$ and $c_x = \frac{p_0}{p_0+1}$ if $r_x = 0$. Then

$$\begin{aligned} \|m_{\text{BM}}\| &= \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} (\|\mu_x\|^2 - \sum_{e \in E\mathbb{X} : o(e)=x} \mu_x(\partial_e \mathbb{X})^2) \\ &= \|\mu_{x_0}\|^2 \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{c_x}{|\Gamma_x|}. \end{aligned} \tag{8.1}$$

(2) *If $\mathbb{X} = \mathbb{X}_{p,q}$ is the biregular tree of degrees $p + 1$ and $q + 1$, with $V\mathbb{X} = V_p\mathbb{X} \sqcup V_q\mathbb{X}$ the corresponding partition of the set of vertices of \mathbb{X} , if the Patterson density $(\mu_x)_{x \in V\mathbb{X}}$ of Γ is normalised so that $\|\mu_x\| = \frac{p+1}{\sqrt{p}}$ for all $x \in V_p\mathbb{X}$, then*

$$(T\pi)_* m_{\text{BM}} = \text{Tvol}_{\Gamma \backslash \mathbb{X}}$$

and

$$\|m_{\text{BM}}\| = \text{TVol}(\Gamma \backslash \mathbb{X}) = \sum_{[x] \in \Gamma \backslash V_p\mathbb{X}} \frac{p+1}{|\Gamma_x|} + \sum_{[x] \in \Gamma \backslash V_q\mathbb{X}} \frac{q+1}{|\Gamma_x|}. \tag{8.2}$$

(3) If $\mathbb{X} = \mathbb{X}_q$ is the regular tree of degree $q + 1$, if the Patterson density $(\mu_x)_{x \in V\mathbb{X}}$ of Γ is normalised to be a family of probability measures, then

$$\pi_* m_{\text{BM}} = \frac{q}{q+1} \text{vol}_{\Gamma \backslash \mathbb{X}}$$

and in particular

$$\|m_{\text{BM}}\| = \frac{q}{q+1} \text{Vol}(\Gamma \backslash \mathbb{X}). \quad (8.3)$$

Proof. Let us first prove the first equality of Assertion (1). For every $x \in V\mathbb{X}$, we may partition the set of geodesic lines $\ell \in \mathcal{G}\mathbb{X}$ with $\ell(0) = x$ according to the two edges starting from x contained in the image of ℓ . The only restriction for the edges is that they are required to be distinct.

For every $e \in E\mathbb{X}$, recall from Section 2.7 that $\partial_e \mathbb{X}$ is the set of points at infinity of the geodesic rays whose initial edge is e . For all $e \in E\mathbb{X}$ and $x \in V\mathbb{X}$, say that e *points away* from x if $o(e) \in [x, t(e)]$, and that e *points towards* x otherwise. In particular, all edges with origin x point away from x . Hence by Equation (4.11), and since $\mu_x = \mu_x^- = \mu_x^+$, we have

$$\pi_* m_{\text{BM}} = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \sum_{e, e' \in E\mathbb{X} : o(e)=o(e')=x, e \neq e'} \mu_x^-(\partial_e \mathbb{X}) \mu_x^+(\partial_{e'} \mathbb{X}) \Delta_{[x]} \quad (8.4)$$

$$= \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \left(\left(\sum_{e \in E\mathbb{X} : o(e)=x} \mu_x(\partial_e \mathbb{X}) \right)^2 - \sum_{e \in E\mathbb{X} : o(e)=x} \mu_x(\partial_e \mathbb{X})^2 \right) \Delta_{[x]}. \quad (8.5)$$

This gives the first equality of Assertion (1).

Let us prove the second equality of Assertion (1). By homogeneity, we assume that $\|\mu_{x_0}\| = 1$ and we will prove that

$$\|m_{\text{BM}}\| = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{c_x}{|\Gamma_x|}.$$

Let $N \in \mathbb{N} - \{0\}$ be such that $p_{n+N} = p_n$ and $p_{N-n} = p_n$ for all $n \in \mathbb{N}$, which exists since \mathbb{X} is assumed to be uniform. Then the automorphism group $\text{Aut}(\mathbb{X})$ of the simplicial tree \mathbb{X} acts transitively on the set of vertices at distance a multiple of N from x_0 . Hence for every $x \in V\mathbb{X}$, the distance $r_x = d(x, \text{Aut}(\mathbb{X})x_0)$ belongs to $\{0, 1, \dots, \lfloor \frac{N}{2} \rfloor\}$, and there exists $\gamma_x, \gamma'_x \in \text{Aut}(\mathbb{X})$ such that $d(x, \gamma_x x_0) = r_x$, $x \in [\gamma_x x_0, \gamma'_x x_0]$ and $d(\gamma_x x_0, \gamma'_x x_0) = N$. The map $x \mapsto r_x$ is constant on the orbits of Γ in $V\mathbb{X}$ (actually on the orbits of $\text{Aut}(\mathbb{X})$) and hence the right hand side of Equation (8.1) is well defined.

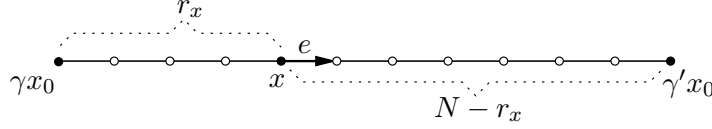
Since the family $(\mu_x^{\text{Haus}})_{x \in V\mathbb{X}}$ of Hausdorff measures of the visual distances $(\partial_\infty X, d_x)$ is invariant under any element of $\text{Aut}(\mathbb{X})$, since Γ is a lattice and by Proposition 4.14, we have $\delta_\Gamma = h_{\mathbb{X}}$ and $\gamma_* \mu_x = \mu_{\gamma x}$ for all $x \in V\mathbb{X}$ and $\gamma \in \text{Aut}(\mathbb{X})$.

Since (\mathbb{X}, x_0) is spherically symmetric, and since μ_{x_0} is a probability measure, we have by induction, for every $e \in E\mathbb{X}$ pointing away from x_0 with $d(x_0, o(e)) = n$,

$$\mu_{x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_0 + 1) p_1 \dots p_n} \quad (8.6)$$

if $n \neq 0$, and $\mu_{x_0}(\partial_e \mathbb{X}) = \frac{1}{p_0 + 1}$ otherwise.

For every fixed $x \in V\mathbb{X}$, let us now compute $\mu_x(\partial_e \mathbb{X})$ for every edge e of \mathbb{X} with origin x . Let $\gamma = \gamma_x, \gamma' = \gamma'_x \in \text{Aut}(\mathbb{X})$ be as above. By the spherical transitivity, we may assume that e or \bar{e} belongs to the edge path from γx_0 to $\gamma' x_0$.



There are two cases to consider.

Case 1: Assume first that e points away from γx_0 . There are $p_0 + 1$ such edges starting from x if $r_x = 0$, and p_{r_x} otherwise. By Equation (8.6) and by invariance under $\text{Aut}(\mathbb{X})$ of $(\mu_x^{\text{Haus}})_{x \in V\mathbb{X}}$, we have

$$\mu_{\gamma x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_0 + 1) p_1 \dots p_{r_x}},$$

with the convention that the denominator is $p_0 + 1$ if $r_x = 0$. Since the map $\xi \mapsto \beta_\xi(x, \gamma x_0)$ is constant with value $-r_x$ on $\partial_e \mathbb{X}$, and by the quasi-invariance property of the Patterson density (see Equation (4.2)), we have

$$\mu_x(\partial_e \mathbb{X}) = e^{-\delta_\Gamma(-r_x)} \mu_{\gamma x_0}(\partial_e \mathbb{X}) = \frac{e^{r_x h_{\mathbb{X}}}}{(p_0 + 1) p_1 \dots p_{r_x}},$$

with the same convention as above.

Case 2: Assume now that e points towards γx_0 . This implies that $r_x \geq 1$, and there is one and only one such edge starting from x . Then as above we have

$$\mu_{\gamma' x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_N + 1) p_{N-1} \dots p_{r_x}},$$

and

$$\mu_x(\partial_e \mathbb{X}) = e^{-\delta_\Gamma(-(N-r_x))} \mu_{\gamma' x_0}(\partial_e \mathbb{X}) = \frac{e^{(N-r_x) h_{\mathbb{X}}}}{(p_N + 1) p_{N-1} \dots p_{r_x}}.$$

Therefore, if we set for every $x \in V\mathbb{X}$,

$$C_x = \left(\sum_{e \in E\mathbb{X} : o(e)=x} \mu_x(\partial_e \mathbb{X}) \right)^2 - \sum_{e \in E\mathbb{X} : o(e)=x} \mu_x(\partial_e \mathbb{X})^2, \quad (8.7)$$

we have if $r_x \neq 0$, since $e^{N h_{\mathbb{X}}} = p_0 p_1 \dots p_{N-1}$ and $p_N = p_0$,

$$\begin{aligned} C_x &= \left(p_{r_x} \frac{e^{r_x h_{\mathbb{X}}}}{(p_0 + 1) p_1 \dots p_{r_x}} + \frac{e^{(N-r_x) h_{\mathbb{X}}}}{(p_N + 1) p_{N-1} \dots p_{r_x}} \right)^2 \\ &\quad - \left(p_{r_x} \left(\frac{e^{r_x h_{\mathbb{X}}}}{(p_0 + 1) p_1 \dots p_{r_x}} \right)^2 + \left(\frac{e^{(N-r_x) h_{\mathbb{X}}}}{(p_N + 1) p_{N-1} \dots p_{r_x}} \right)^2 \right) \\ &= \frac{(p_{r_x}^2 - p_{r_x}) e^{2 r_x h_{\mathbb{X}}}}{(p_0 + 1)^2 p_1^2 \dots p_{r_x}^2} + \frac{2 p_{r_x} e^{N h_{\mathbb{X}}}}{(p_0 + 1) p_1 \dots p_{r_x} p_{r_x} \dots p_{N-1} (p_N + 1)} \\ &= \frac{(p_{r_x} - 1) e^{2 r_x h_{\mathbb{X}}}}{(p_0 + 1)^2 p_1^2 \dots p_{r_x-1}^2 p_{r_x}} + \frac{2 p_0}{(p_0 + 1)^2} = c_x, \end{aligned}$$

and, if $r_x = 0$,

$$C_x = \left((p_0 + 1) \frac{1}{p_0 + 1} \right)^2 - (p_0 + 1) \left(\frac{1}{p_0 + 1} \right)^2 = \frac{p_0}{p_0 + 1} = c_x.$$

Assertion (1) of Proposition 8.1 now follows from Equation (8.5).

Let us prove Assertion (2) of Proposition 8.1. Note that $\mathbb{X} = \mathbb{X}_{p,q}$ is spherically symmetric with respect to any vertex of \mathbb{X} , and that

$$h_{\mathbb{X}} = \frac{1}{2} \ln(pq) .$$

Let e be an edge of \mathbb{X} , with $x = o(e) \in V_p\mathbb{X}$ and $y = t(e) \in V_q\mathbb{X}$. For every $z \in V\mathbb{X}$, we define C_z as in Equation (8.7).

Note that by homogeneity, we have $C_z = C_x$ and $\|\mu_z\| = \|\mu_x\|$ for all $z \in V_p\mathbb{X}$, as well as $C_z = C_y$ and $\|\mu_z\| = \|\mu_y\|$ for all $z \in V_q\mathbb{X}$. Hence the normalisation of the Patterson density as in the statement of Assertion (2) is possible. By the spherical symmetry at x , and the normalisation of the measure, we have $\mu_x(\partial_e\mathbb{X}) = \frac{1}{\sqrt{p}}$ and $\mu_x(\partial_{\bar{e}}\mathbb{X}) = \sqrt{p}$. Therefore

$$\begin{aligned} \|\mu_y\| &= \mu_y(\partial_e\mathbb{X}) + \mu_y(\partial_{\bar{e}}\mathbb{X}) = e^{h_{\mathbb{X}}} \mu_x(\partial_e\mathbb{X}) + e^{-h_{\mathbb{X}}} \mu_x(\partial_{\bar{e}}\mathbb{X}) \\ &= \sqrt{pq} \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{pq}} \sqrt{p} = \frac{q+1}{\sqrt{q}} . \end{aligned}$$

This symmetry in the values of $\|\mu_y\|$ and $\|\mu_x\|$ explains the choice of our normalisation. We have

$$C_x = \|\mu_x\|^2 - (p+1) \left(\frac{\|\mu_x\|}{p+1} \right)^2 = \frac{p}{p+1} \|\mu_x\|^2 = p+1$$

and similarly $C_y = \frac{q}{q+1} \|\mu_y\|^2 = q+1$. This proves the second equality in Equation (8.2), by the first equation of Assertion (1).

In order to prove that $(T\pi)_* m_{\text{BM}} = \text{Tvol}_{\Gamma \backslash \mathbb{X}}$, we now partition $\Gamma \backslash \mathcal{G}\mathbb{X}$ as

$$\bigcup_{[e] \in \Gamma \backslash E\mathbb{X}} \{ \ell \in \Gamma \backslash \mathcal{G}\mathbb{X} : \ell(0) = \pi(o(e)), \ell(1) = \pi(t(e)) \} .$$

Using on every element of this partition Hopf's decomposition with respect to the basepoint $o(e)$, we have, by a ramified covering argument already used in the proof of the second part of Proposition 4.13,

$$(T\pi)_* m_{\text{BM}} = \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{1}{|\Gamma_e|} \mu_{o(e)}(\partial_{\infty}\mathbb{X} - \partial_e\mathbb{X}) \mu_{o(e)}(\partial_e\mathbb{X}) \Delta_{[e]} .$$

Since $\mu_{o(e)}(\partial_e\mathbb{X}) = e^{-h_{\mathbb{X}}} \mu_{t(e)}(\partial_e\mathbb{X})$ and by homogeneity, we have

$$\begin{aligned} (T\pi)_* m_{\text{BM}} &= \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{1}{|\Gamma_e|} \frac{\deg o(e) - 1}{\deg o(e)} \|\mu_{o(e)}\| \frac{\deg t(e) - 1}{\deg t(e)} \|\mu_{t(e)}\| e^{-h_{\mathbb{X}}} \Delta_{[e]} \\ &= \sum_{[e] \in \Gamma \backslash E\mathbb{X}} \frac{1}{|\Gamma_e|} \frac{\|\mu_{o(e)}\| \|\mu_{t(e)}\| \sqrt{pq}}{(p+1)(q+1)} \Delta_{[e]} = \text{Tvol}_{\Gamma \backslash \mathbb{X}} . \end{aligned}$$

The first equality of Equation (8.2) follows.

Finally, the last claim of Assertion (3) of Proposition 8.1 follows from Equation (8.1), since $c_x = \frac{q}{q+1}$ for every $x \in V\mathbb{X}_q$ (or by taking $q = p$ in Equation (8.2) and by renormalising). The first claim of Assertion (3) follows from the first claim of Assertion (2), by using Equation (2.16). \square

Remark 8.2. (1) In particular, when $\mathbb{X} = \mathbb{X}_q$ is regular, the Patterson density is normalised to be a family of probability measures and Γ is torsion free, then $\pi_* m_{\text{BM}}$ is $\frac{q}{q+1}$ times the counting measure on $\Gamma \backslash V\mathbb{X}$. In this case, Equation (8.3) is given by [CoP4, Rem. 2].

(2) If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular with $p \neq q$, then $\pi_* m_{\text{BM}}$ is not proportional to $\text{vol}_{\Gamma \backslash \mathbb{X}}$. In particular, if Γ is torsion free and the Patterson density is normalised to be a family of probability measures, then $\pi_* m_{\text{BM}}$ is the sum of $\frac{p}{p+1}$ times the counting measure on $\Gamma \backslash V_p \mathbb{X}$ and $\frac{q}{q+1}$ times the counting measure on $\Gamma \backslash V_q \mathbb{X}$.

This statement is similar to the well-known fact that in pinched but variable curvature, the Bowen-Margulis measure is generally not absolutely continuous with respect to Sasaki's Riemannian measure on the unit tangent bundle (it would then be proportional by ergodicity of the geodesic flow in the lattice case).

8.2 Computations of skinning measures for simplicial trees

We now give examples of computations of the total mass of skinning measures (for zero potentials), after introducing some notation. Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, and let Γ be a discrete subgroup of $\text{Aut}(\mathbb{X})$.

For every simplicial subtree \mathbb{D} of \mathbb{X} , we define the *boundary* $\partial V\mathbb{D}$ of $V\mathbb{D}$ in \mathbb{X} as

$$\partial V\mathbb{D} = \{x \in V\mathbb{D} : \exists e \in E\mathbb{X}, o(e) = x, t(e) \notin V\mathbb{D}\}.$$

The *boundary* $\partial\mathbb{D}$ of \mathbb{D} is the maximal subgraph (which might not be connected) of \mathbb{X} with set of vertices $\partial V\mathbb{D}$. If Γ is a discrete subgroup of $\text{Aut}(\mathbb{X})$, then the stabiliser $\Gamma_{\mathbb{D}}$ of \mathbb{D} acts discretely on $\partial\mathbb{D}$.

For every $x \in V\mathbb{X}$, we define the *codegree* of x in \mathbb{D} as $\text{codeg}_{\mathbb{D}}(x) = 0$ if $x \notin \mathbb{D}$ and otherwise

$$\text{codeg}_{\mathbb{D}}(x) = \deg_{\mathbb{X}}(x) - \deg_{\mathbb{D}}(x).$$

Note that $\text{codeg}_{\mathbb{D}}(x) = 0$ if $x \notin \partial V\mathbb{D}$, and that the codegree $\text{codeg}_{\mathcal{N}_1\mathbb{D}}(x)$ of $x \in V\mathbb{X}$ is 0 unless x lies in the boundary of the 1-neighbourhood of \mathbb{D} , in which case it is constant equal to $\deg_{\mathbb{X}}(x) - 1$.

Let $\mathcal{D} = (\mathbb{D}_i)_{i \in I}$ be a locally finite Γ -equivariant family of simplicial subtrees of \mathbb{X} , and let $x \in V\mathbb{X}$. We define the *multiplicity*¹ of x in (the boundary of) \mathcal{D} as (see Section 7.2 for the definition of $\sim_{\mathcal{D}}$)

$$m_{\mathcal{D}}(x) = \frac{\text{Card}\{i \in I / \sim_{\mathcal{D}} : x \in \partial V\mathbb{D}_i\}}{|\Gamma_x|}.$$

The numerator and the denominator are finite, by the local finiteness of the family \mathcal{D} and the discreteness of Γ , and they depend only on the orbit of x under Γ . Note that if \mathbb{D} is a simplicial subtree of \mathbb{X} which is *precisely invariant* under Γ (that is, whenever $\gamma \in \Gamma$ is such that $\mathbb{D} \cap \gamma\mathbb{D}$ is nonempty, then γ belongs to the stabiliser $\Gamma_{\mathbb{D}}$ of \mathbb{D} in Γ), if $\mathcal{D} = (\gamma\mathbb{D})_{\gamma \in \Gamma/\Gamma_{\mathbb{D}}}$, and if $x \in \partial V\mathbb{D}$, then

$$m_{\mathcal{D}}(x) = \frac{1}{|\Gamma_x|}.$$

In particular, if furthermore Γ is torsion free, then $m_{\mathcal{D}}(x) = 1$ if $x \in \partial V\mathbb{D}$, and $m_{\mathcal{D}}(x) = 0$ otherwise.

¹See Section 12.2 for explanations on the terminology.

Example 8.3. Let \mathcal{G} be a connected graph without vertices of degree ≤ 2 and let \mathbb{X} be its universal cover, with covering group Γ . If C is a cycle in $\mathcal{G} = \Gamma \backslash \mathbb{X}$ and if \mathcal{D} is the family of geodesic lines in \mathbb{X} lifting C , then $m_{\mathcal{D}}(x) = 1$ for all $x \in V\mathbb{X}$ whose image in $\mathcal{G} = \Gamma \backslash \mathbb{X}$ belongs to C if C is a *simple* cycle (that is, if C passes through no vertex twice).

We define the *codegree* of x in \mathcal{D} as

$$\text{codeg}_{\mathcal{D}}(x) = \sum_{i \in I / \sim_{\mathcal{D}}} \text{codeg}_{\mathbb{D}_i}(x),$$

which is well defined as $\text{codeg}_{\mathbb{D}_i}(x)$ depends only on the class of $i \in I$ modulo $\sim_{\mathcal{D}}$. Note that

$$\text{codeg}_{\mathcal{D}}(x) = (\deg_{\mathbb{X}} x - k) |\Gamma_x| m_{\mathcal{D}}(x) \quad (8.8)$$

if $\deg_{\mathbb{D}_i}(x) = k$ for every $x \in \partial V\mathbb{D}_i$ and $i \in I$. If every vertex of \mathbb{X} has degree at least 3, this is in particular the case with $k = 2$ if \mathbb{D}_i is a line for all $i \in I$ and with $k = 1$ if \mathbb{D}_i is a horoball for all $i \in I$.

We will say that a simplicial subtree \mathbb{D} of \mathbb{X} , with stabiliser $\Gamma_{\mathbb{D}}$ in Γ , is *almost precisely invariant* if there exists $N \in \mathbb{N}$ such that for every $x \in \partial V\mathbb{D}$, the number of $\gamma \in \Gamma/\Gamma_{\mathbb{D}}$ such that $x \in \gamma \partial V\mathbb{D}$ is at most N . It follows from this property that if $\mathcal{D} = (\gamma\mathbb{D})_{\gamma \in \Gamma}$, then \mathcal{D} is locally finite and $\text{codeg}_{\mathcal{D}}(x) \leq N \text{codeg}_{\mathbb{D}}(x)$ for every $x \in \mathbb{X}$.

Proposition 8.4. *Assume that \mathbb{X} is a regular or biregular simplicial tree with degrees at least 3, and that Γ is a lattice of \mathbb{X} .*

(1) *For every simplicial subtree \mathbb{D} of \mathbb{X} , we have*

$$\pi_* \tilde{\sigma}_{\mathbb{D}}^{\pm} = \sum_{x \in V\mathbb{X}} \frac{\|\mu_x\| \text{codeg}_{\mathbb{D}}(x)}{\deg_{\mathbb{X}}(x)} \Delta_x.$$

(2) *If $\mathcal{D} = (\mathbb{D}_i)_{i \in I}$ is a locally finite Γ -equivariant family of simplicial subtrees of \mathbb{X} , then*

$$\pi_* \sigma_{\mathcal{D}}^{\pm} = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{\|\mu_x\| \text{codeg}_{\mathcal{D}}(x)}{|\Gamma_x| \deg_{\mathbb{X}}(x)} \Delta_{[x]}.$$

(3) *Let $k \in \mathbb{N}$ and let \mathbb{D} be a simplicial subtree of \mathbb{X} such that $\deg_{\mathbb{D}}(x) = k$ for every $x \in \partial V\mathbb{D}$ and the Γ -equivariant family $\mathcal{D} = (\gamma\mathbb{D})_{\gamma \in \Gamma}$ is locally finite. Then*

$$\pi_* \sigma_{\mathcal{D}}^{\pm} = \sum_{\Gamma_{\mathbb{D}} y \in \Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}} \frac{\|\mu_y\| (\deg_{\mathbb{X}}(y) - k)}{|\Gamma_{\mathbb{D}})_y| \deg_{\mathbb{X}}(y)} \Delta_{\Gamma y}.$$

(4) *If \mathbb{D} is a simplicial subtree of \mathbb{X} such that the Γ -equivariant family $\mathcal{D} = (\gamma\mathbb{D})_{\gamma \in \Gamma}$ is locally finite, then the skinning measure $\sigma_{\mathcal{D}}^{\pm}$ is finite if and only if the graph of groups $\Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}$ has finite volume.*

Before proving Proposition 8.4, let us give some immediate consequences of Assertion (3). If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular of degrees $p+1$ and $q+1$, let $V\mathbb{X} = V_p\mathbb{X} \sqcup V_q\mathbb{X}$ be the corresponding partition of the set of vertices of \mathbb{X} and, for $r \in \{p, q\}$, let $\partial_r \mathbb{D}$ be the edgeless graph with set of vertices $\partial V\mathbb{D} \cap V_r\mathbb{X}$.

Corollary 8.5. *Assume that (\mathbb{X}, Γ) is as in Proposition 8.4. Let \mathbb{D} be a simplicial subtree of \mathbb{X} such that the Γ -equivariant family $\mathcal{D} = (\gamma\mathbb{D})_{\Gamma/\Gamma_{\mathbb{D}}}$ is locally finite.*

(1) *If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular of degrees $p+1$ and $q+1$ and if the Patterson density $(\mu_x)_{x \in V\mathbb{X}}$ of Γ is normalised so that $\|\mu_x\| = \frac{\deg_{\mathbb{X}}(x)}{\sqrt{\deg_{\mathbb{X}}(x)-1}}$ for all $x \in V\mathbb{X}$, then*

- *if \mathbb{D} is a horoball,*

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \sqrt{p} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial_p \mathbb{D}) + \sqrt{q} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial_q \mathbb{D}),$$

- *if \mathbb{D} is a line,*

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \frac{p-1}{\sqrt{p}} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial_p \mathbb{D}) + \frac{q-1}{\sqrt{q}} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial_q \mathbb{D}). \quad (8.9)$$

(2) *If $\mathbb{X} = \mathbb{X}_q$ is the regular tree of degree $q+1$ and if the Patterson measures $(\mu_x)_{x \in V\mathbb{X}}$ are normalised to be probability measures, then*

- *if \mathbb{D} is a horoball,*

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \frac{q}{q+1} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial \mathbb{D}) \quad (8.10)$$

- *if \mathbb{D} is a line,*

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \frac{q-1}{q+1} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \mathbb{D}). \quad (8.11)$$

Proof of Proposition 8.4. (1) We may partition the outer/inner unit normal bundle $\partial_{\pm}^1 \mathbb{D}$ of \mathbb{D} according to the first/last edge of the elements in $\partial_{\pm}^1 \mathbb{D}$. On each of the elements of this partition, for the computation of the skinning measures using its definition and its independence of the basepoint (see Section 7.1), we take as basepoint the initial/terminal point of the corresponding edge. Since \mathbb{D} is a simplicial tree, note that for every $e \in E\mathbb{X}$ such that $o(e) \in V\mathbb{D}$, we have $e \in E\mathbb{D}$ if and only if $t(e) \in V\mathbb{D}$. Thus, we have

$$\begin{aligned} \pi_* \tilde{\sigma}_{\mathbb{D}}^+ &= \sum_{e \in E\mathbb{X} : o(e) \in V\mathbb{D}, t(e) \notin V\mathbb{D}} \mu_{o(e)}(\partial_e \mathbb{X}) \Delta_{o(e)} \\ &= \sum_{x \in \partial V\mathbb{D}} \left(\sum_{e \in E\mathbb{X} : o(e)=x, t(e) \notin V\mathbb{D}} \mu_x(\partial_e \mathbb{X}) \right) \Delta_x. \end{aligned}$$

and similarly

$$\begin{aligned} \pi_* \tilde{\sigma}_{\mathbb{D}}^- &= \sum_{e \in E\mathbb{X} : t(e) \in V\mathbb{D}, o(e) \notin V\mathbb{D}} \mu_{t(e)}(\partial_{\bar{e}} \mathbb{X}) \Delta_{t(e)} \\ &= \sum_{x \in \partial V\mathbb{D}} \left(\sum_{e \in E\mathbb{X} : o(e)=x, t(e) \in V\mathbb{D}} \mu_x(\partial_e \mathbb{X}) \right) \Delta_x. \end{aligned}$$

As in the proof of Proposition 8.1 (2), since \mathbb{X} is spherically homogeneous around each point and since Γ is a lattice (so that the Patterson density is $\operatorname{Aut}(\mathbb{X})$ -equivariant, see Proposition 4.14), we have $\mu_x(\partial_e \mathbb{X}) = \frac{\|\mu_x\|}{\deg_{\mathbb{X}}(x)}$ for all $x \in V\mathbb{X}$ and $e \in E\mathbb{X}$ with $o(e) = x$. Assertion (1) of Proposition 8.4 follows, since $\sum_{e \in E\mathbb{X} : o(e)=x, t(e) \notin V\mathbb{D}} 1 = \operatorname{codeg}_{\mathbb{D}}(x)$ if $x \in \partial V\mathbb{D}$ and $\operatorname{codeg}_{\mathbb{D}}(x) = 0$ otherwise.

(2) By the definition² of the skinning measures associated with Γ -equivariant families, we have $\tilde{\sigma}_{\mathcal{D}}^{\pm} = \sum_{i \in I/\sim} \tilde{\sigma}_{\mathbb{D}_i}^{\pm}$, where $\sim = \sim_{\mathcal{D}}$. Hence by Assertion (1)

$$\begin{aligned} \pi_* \tilde{\sigma}_{\mathcal{D}}^{\pm} &= \sum_{i \in I/\sim} \tilde{\pi}_* \sigma_{\mathbb{D}_i}^{\pm} = \sum_{i \in I/\sim} \sum_{x \in V\mathbb{X}} \frac{\|\mu_x\| \operatorname{codeg}_{\mathbb{D}_i}(x)}{\operatorname{deg}_{\mathbb{X}}(x)} \Delta_x \\ &= \sum_{x \in V\mathbb{X}} \left(\sum_{i \in I/\sim} \operatorname{codeg}_{\mathbb{D}_i}(x) \right) \frac{\|\mu_x\|}{\operatorname{deg}_{\mathbb{X}}(x)} \Delta_x = \sum_{x \in V\mathbb{X}} \frac{\|\mu_x\| \operatorname{codeg}_{\mathcal{D}}(x)}{\operatorname{deg}_{\mathbb{X}}(x)} \Delta_x . \end{aligned}$$

By the definition of the measure induced in $\Gamma \backslash V\mathbb{X}$ when Γ may have torsion (see for instance [PauPS, §2.6]), Assertion (2) follows.

(3) It follows from Assertion (2) and from Equation (8.8) that

$$\pi_* \sigma_{\mathcal{D}}^{\pm} = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{\operatorname{deg}_{\mathbb{X}}(x) - k}{\operatorname{deg}_{\mathbb{X}}(x)} \|\mu_x\| m_{\mathcal{D}}(x) \Delta_{[x]} .$$

For every $x \in V\mathbb{X}$, by the definition of $m_{\mathcal{D}}(x)$, we have, by partitioning $\partial V\mathbb{D}$ into its orbits under $\Gamma_{\mathbb{D}}$,

$$\begin{aligned} m_{\mathcal{D}}(x) &= \frac{1}{|\Gamma_x|} \operatorname{Card}\{\gamma \in \Gamma_{\mathbb{D}} \backslash \Gamma : \gamma x \in \partial V\mathbb{D}\} \\ &= \frac{1}{|\Gamma_x|} \sum_{\Gamma_{\mathbb{D}} y \in \Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}} \operatorname{Card}\{\gamma \in \Gamma_{\mathbb{D}} \backslash \Gamma : \Gamma_{\mathbb{D}} \gamma x = \Gamma_{\mathbb{D}} y\} \\ &= \frac{1}{|\Gamma_x|} \sum_{\Gamma_{\mathbb{D}} y \in \Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}, \Gamma x = \Gamma y} \operatorname{Card}\{\gamma \in \Gamma_{\mathbb{D}} \backslash \Gamma : \Gamma_{\mathbb{D}} \gamma y = \Gamma_{\mathbb{D}} y\} \\ &= \frac{1}{|\Gamma_x|} \sum_{\Gamma_{\mathbb{D}} y \in \Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}, \Gamma x = \Gamma y} [\Gamma_y : (\Gamma_{\mathbb{D}})_y] = \sum_{\Gamma_{\mathbb{D}} y \in \Gamma_{\mathbb{D}} \backslash \partial V\mathbb{D}, \Gamma x = \Gamma y} \frac{1}{|(\Gamma_{\mathbb{D}})_y|} . \end{aligned}$$

This proves Assertion (3), since $\sum_{[x] \in \Gamma \backslash V\mathbb{X}, \Gamma x = \Gamma y} \Delta_{[x]} = \Delta_{\Gamma y}$.

(4) It follows from Assertion (2) that

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{\|\mu_x\| \operatorname{codeg}_{\mathcal{D}}(x)}{|\Gamma_x| \operatorname{deg}_{\mathbb{X}}(x)} .$$

Note that for every $x \in \partial V\mathbb{D}$, we have

$$|\Gamma_x| m_{\mathcal{D}}(x) \leq \operatorname{codeg}_{\mathcal{D}}(x) \leq \operatorname{deg}_{\mathbb{X}}(x) |\Gamma_x| m_{\mathcal{D}}(x) .$$

Let $m = \min_{x \in V\mathbb{X}} \|\mu_x\|$ and $M = \max_{x \in V\mathbb{X}} \|\mu_x\|$, which are positive and finite, as the total mass of the Patterson measures takes at most two values, since Γ is a lattice and \mathbb{X} is biregular. By arguments similar to those in the proof of Assertion (3), we hence have

$$\frac{m}{\min_{x \in V\mathbb{X}} \operatorname{deg}_{\mathbb{X}}(x)} \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial \mathbb{D}) \leq \|\sigma_{\mathcal{D}}^{\pm}\| \leq M \operatorname{Vol}(\Gamma_{\mathbb{D}} \backslash \partial \mathbb{D})$$

The result follows. □

²See Section 7.2.

We now give a formula for the skinning measure (with zero potential) of a geodesic line in the simplicial tree \mathbb{X} , using Hamenstädt's distance $d_{\mathcal{H}}$ and measure³ $\mu_{\mathcal{H}}$ associated with a fixed horoball \mathcal{H} in \mathbb{X} . This expression for the skinning measure will be useful in Part III.

Lemma 8.6. *Let \mathcal{H} be a horoball in \mathbb{X} centred at a point $\xi \in \partial_{\infty}\mathbb{X}$. Let L be a geodesic line in \mathbb{X} with endpoints $L_{\pm} \in \partial_{\infty}\mathbb{X} - \{\xi\}$. Then for all $\rho \in \partial_+^1 L$ such that $\rho_+ \neq \xi$,*

$$d\tilde{\sigma}_L^+(\rho) = \frac{d_{\mathcal{H}}(L_+, L_-)^{\delta_{\Gamma}}}{d_{\mathcal{H}}(\rho_+, L_-)^{\delta_{\Gamma}} d_{\mathcal{H}}(\rho_+, L_+)^{\delta_{\Gamma}}} d\mu_{\mathcal{H}}(\rho_+).$$

Proof. By Equations (2.9) and (7.6), the power $d_{\mathcal{H}}^{\delta_{\Gamma}}$ of the distance and the measure $\mu_{\mathcal{H}}$ scale by the same factor when the horoball is replaced by another one centred at the same point. Thus, we can assume in the proof that L does not intersect the interior of \mathcal{H} .

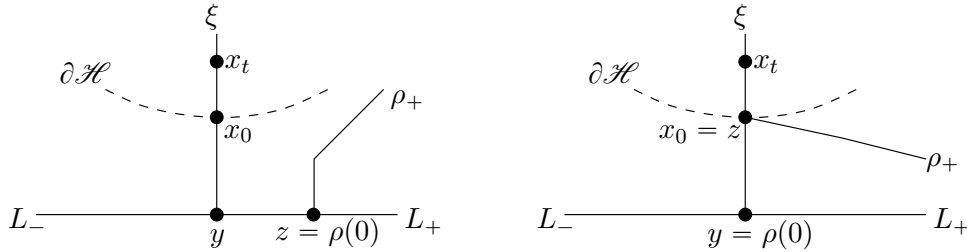
Fix $\rho \in \partial_+^1 L$ such that $\rho_+ \neq \xi$. Let y be the closest point to ξ on L , let x_0 be the closest point to L on \mathcal{H} , and let z be the closest point to ξ on $\rho([0, +\infty[)$. Let $t \mapsto x_t$ be the geodesic ray starting from x_0 at time $t = 0$ and converging to ξ . When t is big enough, the points ρ_+ , z , x_t and ξ are in this order on the geodesic line $] \rho_+, \xi [$.

We have, by the definition in Equation (7.1) of the skinning measure,

$$\begin{aligned} d\tilde{\sigma}_L^+(\rho) &= e^{\delta_{\Gamma} \beta_{\rho_+}(x_t, \rho(0))} d\mu_{x_t}(\rho_+) = e^{\delta_{\Gamma} \beta_{\rho_+}(x_t, z) + \delta_{\Gamma} \beta_{\rho_+}(z, \rho(0))} d\mu_{x_t}(\rho_+) \\ &= e^{-\delta_{\Gamma} \beta_{\xi}(x_t, z) - \delta_{\Gamma} d(z, \rho(0))} d\mu_{x_t}(\rho_+) = e^{\delta_{\Gamma} t - \delta_{\Gamma} \beta_{\xi}(x_0, z) - \delta_{\Gamma} d(z, \rho(0))} d\mu_{x_t}(\rho_+), \end{aligned}$$

and by the definition of Hamenstädt's measure $\mu_{\mathcal{H}}$ (see Equation (7.5))

$$d\mu_{\mathcal{H}}(\rho_+) = e^{\delta_{\Gamma} t} d\mu_{x_t}(\rho_+).$$



Case 1: Assume first that $\rho(0) \neq y$. We may assume that $\rho(0) \in [y, L_+[$. Then $z = \rho(0)$ and z is the closest point to \mathcal{H} on the geodesic line $]L_+, \rho_+[$. Thus $d_{\mathcal{H}}(L_-, L_+) = d_{\mathcal{H}}(L_-, \rho_+)$ and $d_{\mathcal{H}}(L_+, \rho_+) = e^{-d(z, x_0)} = e^{\beta_{\xi}(x_0, z)}$, and the claim follows.

Case 2: Assume now that $y = \rho(0)$. Then $[y, z] = [y, \xi[\cap [y, \rho_+[$, and we may assume that $x_0 = z$ up to adjusting the horoball \mathcal{H} while keeping its point at infinity. Thus $d_{\mathcal{H}}(L_-, L_+) = e^{-d(y, x_0)} = e^{-d(z, \rho(0))}$ and $d_{\mathcal{H}}(L_-, \rho_+) = d_{\mathcal{H}}(L_+, \rho_+) = 1$, and the claim follows. \square

³See the definitions of Hamenstädt's distance and measure in Sections 2.3 and 7.1 respectively.

Chapter 9

Rate of mixing for the geodesic flow

Let $X, x_0, \Gamma, \tilde{F}, (\mu_x^\pm)_{x \in X}$ be as in the beginning of Chapter 7. In this Chapter, we start by collecting in Section 9.1 known results on the rate of mixing of the geodesic flow for manifolds. The main part of the Chapter then consists in proving analogous bounds for the discrete time and continuous time geodesic flow for quotient spaces of simplicial and metric trees respectively.

We define $\overline{m}_F = \frac{m_F}{\|m_F\|}$ when the Gibbs measure is finite. Recall that this measure is nonzero since Γ is nonelementary.

Let $\alpha \in]0, 1]$.¹ We will say that the (continuous time) geodesic flow on $\Gamma \backslash \mathcal{G}X$ is *exponentially mixing for the α -Hölder regularity* or that it has *exponential decay of α -Hölder correlations* for the potential F if there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathcal{C}_b^\alpha(\Gamma \backslash \mathcal{G}X)$ and $t \in \mathbb{R}$, we have

$$\left| \int_{\Gamma \backslash \mathcal{G}X} \phi \circ \mathbf{g}^{-t} \psi \, d\overline{m}_F - \int_{\Gamma \backslash \mathcal{G}X} \phi \, d\overline{m}_F \int_{\Gamma \backslash \mathcal{G}X} \psi \, d\overline{m}_F \right| \leq C e^{-\kappa|t|} \|\phi\|_\alpha \|\psi\|_\alpha,$$

and that it is *polynomially mixing* or has *polynomial decay of α -Hölder correlations* if there exist $C > 0$ and $n \in \mathbb{N} - \{0\}$ such that for all $\phi, \psi \in \mathcal{C}_b^\alpha(\Gamma \backslash \mathcal{G}X)$ and $t \in \mathbb{R}$, we have

$$\left| \int_{\Gamma \backslash \mathcal{G}X} \phi \circ \mathbf{g}^{-t} \psi \, d\overline{m}_F - \int_{\Gamma \backslash \mathcal{G}X} \phi \, d\overline{m}_F \int_{\Gamma \backslash \mathcal{G}X} \psi \, d\overline{m}_F \right| \leq C (1 + |t|)^{-n} \|\phi\|_\alpha \|\psi\|_\alpha.$$

9.1 Rate of mixing for Riemannian manifolds

When $X = \tilde{M}$ is a complete Riemannian manifold with pinched negative sectional curvature with bounded derivatives, then the boundary at infinity of \tilde{M} , the strong unstable, unstable, stable, and strong stable foliations of $T^1\tilde{M}$ are only Hölder-smooth in general.² Hence Hölder regularity on functions on $T^1\tilde{M}$ is appropriate.

The geodesic flow is known to have exponential decay of Hölder correlations for compact manifolds M when

- M is two-dimensional and F is any Hölder potential by [Dol1],

¹We refer to Section 3.1 for the definition of the Banach space $\mathcal{C}_b^\alpha(Z)$ of bounded α -Hölder-continuous functions on a metric space Z .

²See for instance [Brin] when \tilde{M} has a compact quotient (a result first proved by Anosov), and [PauPS, Theo. 7.3].

- M is 1/9-pinned and $F = 0$ by [GLP, Coro. 2.7],
- m_F is the Liouville measure by [Live], see also [Tsu], [NZ, Coro. 5] who give more precise estimates,
- M is locally symmetric and F is any Hölder potential by [Sto], see also [MO].

When \widetilde{M} is a symmetric space, then the boundary at infinity of \widetilde{M} , the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are smooth. Hence talking about leafwise \mathcal{C}^ℓ -smooth functions on $T^1\widetilde{M}$ makes sense. We will denote by $\mathcal{C}_c^\ell(\mathbb{N})$ the vector space of real-valued \mathcal{C}^ℓ -smooth functions on the orbifold $T^1M = \Gamma \backslash T^1\widetilde{M}$ (that is, the maps induced on T^1M by the \mathcal{C}^ℓ -smooth Γ -invariant functions on $T^1\widetilde{M}$), with compact support in T^1M , and by $\|\psi\|_\ell$ the Sobolev $W^{\ell,2}$ -norm of any $\psi \in \mathcal{C}_c^\ell(T^1M)$.

Given $\ell \in \mathbb{N}$, we will say that the geodesic flow on T^1M is *exponentially mixing for the ℓ -Sobolev regularity* (or that it has *exponential decay of ℓ -Sobolev correlations*) for the potential F if there exist $c, \kappa > 0$ such that for all $\phi, \psi \in \mathcal{C}_c^\ell(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\left| \int_{T^1M} \phi \circ g^{-t} \psi \, d\overline{m}_F - \int_{T^1M} \phi \, d\overline{m}_F \int_{T^1M} \psi \, d\overline{m}_F \right| \leq c e^{-\kappa|t|} \|\psi\|_\ell \|\phi\|_\ell .$$

When $F = 0$ and Γ is an arithmetic lattice in the isometry group of \widetilde{M} (the Gibbs measure then coincides, up to a multiplicative constant, with the Liouville measure), this property, for some $\ell \in \mathbb{N}$, follows from [KM1, Theorem 2.4.5], with the help of [Clo, Theorem 3.1] to check its spectral gap property, and of [KM2, Lemma 3.1] to deal with finite cover problems.

9.2 Rate of mixing for simplicial trees

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, with geometric realisation $X = |X|_1$. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$ and let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances for Γ on \mathbb{X} .

In this Section, building on the end of Section 4.4 concerning the mixing properties themselves, we now study the rates of the mixing properties of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ for the Gibbs measure $m_c = m_{F_c}$, when it is mixing.

Let (Z, m, T) be a dynamical system with (Z, m) a probability space and $T : Z \rightarrow Z$ a (not necessarily invertible) measure preserving map. For all $n \in \mathbb{N}$ and $\phi, \psi \in \mathbb{L}^2(m)$, the (well-defined) n -th *correlation coefficient* of ϕ, ψ is

$$\text{cov}_{m,n}(\phi, \psi) = \int_Z \phi \circ T^n \psi \, dm - \int_Z \phi \, dm \int_Z \psi \, dm .$$

Let $\alpha \in]0, 1]$ and assume that Z is a metric space (endowed with its Borel σ -algebra). Similarly as for the case of flows in the beginning of Chapter 9, we will say that the dynamical system (Z, m, T) is *exponentially mixing for the α -Hölder regularity* or that it has *exponential decay of α -Hölder correlations* if there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathcal{C}_b^\alpha(Z)$ and $n \in \mathbb{N}$, we have

$$|\text{cov}_{m,n}(\phi, \psi)| \leq C e^{-\kappa n} \|\phi\|_\alpha \|\psi\|_\alpha .$$

Note that this property is invariant under measure preserving conjugations of dynamical systems by bilipschitz homeomorphisms.

The main result of this Section is a simple criterion for the exponential decay of correlation for the discrete time geodesic flow on $\Gamma \backslash \mathcal{GX}$.

We define $\overline{m_c} = \frac{m_c}{\|m_c\|}$ when the Gibbs measure m_c on $\Gamma \backslash \mathcal{GX}$ is finite, and we use the dynamical system $(\Gamma \backslash \mathcal{GX}, \overline{m_c}, \mathbf{g}^1)$ in the definition of the correlation coefficients.

Given a finite subset E of $\Gamma \backslash V\mathbb{X}$, we denote by $\tau_E : \Gamma \backslash \mathcal{GX} \rightarrow \mathbb{N} \cup \{+\infty\}$ the first return time to E of the discrete time geodesic flow:

$$\tau_E(\ell) = \inf\{n \in \mathbb{N} - \{0\} : \mathbf{g}^n \ell(0) \in E\},$$

with the usual convention that $\inf \emptyset = +\infty$.

Theorem 9.1. *Let $\mathbb{X}, \Gamma, \tilde{c}$ be as above, with δ_c finite. Assume that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow on $\Gamma \backslash \mathcal{GX}$. Assume moreover that there exist a finite subset E of $\Gamma \backslash V\mathbb{X}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have*

$$m_c(\{\ell \in \Gamma \backslash \mathcal{GX} : \ell(0) \in E \text{ and } \tau_E(\ell) \geq n\}) \leq C' e^{-\kappa' n}. \quad (9.1)$$

Then the discrete time geodesic flow on $\Gamma \backslash \mathcal{GX}$ has exponential decay of α -Hölder correlations for the system of conductances c .

A similar statement holds for the square of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}$ when m_c is finite, $\mathcal{C}\Lambda\Gamma$ is a uniform simplicial tree with degrees at least 3 and $L_\Gamma = 2\mathbb{Z}$.

Note that the crucial Hypothesis (9.1) of Theorem 9.1 is in particular satisfied if $\Gamma \backslash \mathbb{X}$ is finite, by taking $E = \Gamma \backslash V\mathbb{X}$. But the result is quite well-known in this case: when Γ is torsion free, it follows from Bowen's result [Bowe2, 1.26] that a mixing subshift of finite type is exponentially mixing.

Proof. Let $\mathbb{X}' = \mathcal{C}\Lambda\Gamma$. Using the coding introduced in Section 5.2, we first reduce this statement to a symbolic dynamics one.

Step 1 : Reduction to two-sided symbolic dynamics

Let (Σ, σ) be the (two-sided) topological Markov shift with alphabet \mathcal{A} and transition matrix A constructed in Section 5.2, conjugated to $(\Gamma \backslash \mathcal{GX}', \mathbf{g}^1)$ by the homeomorphism $\Theta : \Gamma \backslash \mathcal{GX}' \rightarrow \Sigma$ (see Theorem 5.1). Let $\mathbb{P} = \Theta_* \frac{m_c}{\|m_c\|}$, which is a mixing σ -invariant probability measure on Σ . Let

$$\mathcal{E} = \{(e^-, h, e^+) \in \mathcal{A} : t(e^-) = o(e^+) \in E\}.$$

The set \mathcal{E} is finite since the degrees and the vertex stabilisers of \mathbb{X} are finite. For all $x \in \Sigma$ and $k \in \mathbb{Z}$, we denote by x_k the k -th component of $x = (x_n)_{n \in \mathbb{Z}}$. Let

$$\tau_{\mathcal{E}}(x) = \inf\{n \in \mathbb{N} - \{0\} : x_n \in \mathcal{E}\}$$

be the first return time to \mathcal{E} of x under iteration of the shift σ .

Let $\pi_+ : \Sigma \rightarrow \mathcal{A}^{\mathbb{N}}$ be the *natural extension* $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n \in \mathbb{N}}$. Theorem 9.1 will follow from the following two-sided symbolic dynamics result.³

Theorem 9.2. *Let (Σ, σ) be a locally compact transitive two-sided topological Markov shift with alphabet \mathcal{A} and transition matrix A , and let \mathbb{P} be a mixing σ -invariant probability measure with full support on Σ . Assume that*

³Assumption (1) of Theorem 9.2 is far from being optimal, but will be sufficient for our purpose.

(1) for every A -admissible finite sequence (w_0, \dots, w_n) in \mathcal{A} , the Jacobian of the map from $\{(x_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) : x_0 = w_n\}$ to $\{(y_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) : y_0 = w_0, \dots, y_n = w_n\}$ defined by $(x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$, with respect to the restrictions of the pushforward measure $(\pi_+)_*\mathbb{P}$, is constant;

(2) there exist a finite subset \mathcal{E} of \mathcal{A} and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have

$$\mathbb{P}(\{x \in \Sigma : x_0 \in \mathcal{E} \text{ and } \tau_{\mathcal{E}}(x) \geq n\}) \leq C' e^{-\kappa' n}. \quad (9.2)$$

Then $(\Sigma, \mathbb{P}, \sigma)$ has exponential decay of α -Hölder correlations.

Proof that Theorem 9.2 implies Theorem 9.1. Since \tilde{m}_c is supported on $\mathcal{G}\mathbb{X}'$, up to replacing \mathbb{X} by \mathbb{X}' , we may assume that $\partial_{\infty}\Gamma = \Lambda\Gamma$.

By the construction of Θ just before the statement of Theorem 5.1, for every $\ell = \Gamma\tilde{\ell} \in \Gamma\backslash\mathcal{G}\mathbb{X}$, we have $(\Theta\ell)_0 = (e_0^-(\tilde{\ell}), h_0(\tilde{\ell}), e_0^+(\tilde{\ell}))$ with $e_0^+(\tilde{\ell}) = p(\tilde{\ell}([0, 1]))$ where $p : \mathbb{X} \rightarrow \Gamma\backslash\mathbb{X}'$ is the canonical projection, so that $o(e_0^+(\tilde{\ell})) = \ell(0)$. Since Θ conjugates \mathfrak{g}^1 to σ , we have

$$(\Theta\ell)_n = (\sigma^n(\Theta\ell))_0 = (\Theta(\mathfrak{g}^n\ell))_0 \in \mathcal{E}$$

if and only if $\mathfrak{g}^n\ell(0) \in E$, and

$$\tau_{\mathcal{E}}(\Theta\ell) = \tau_E(\ell).$$

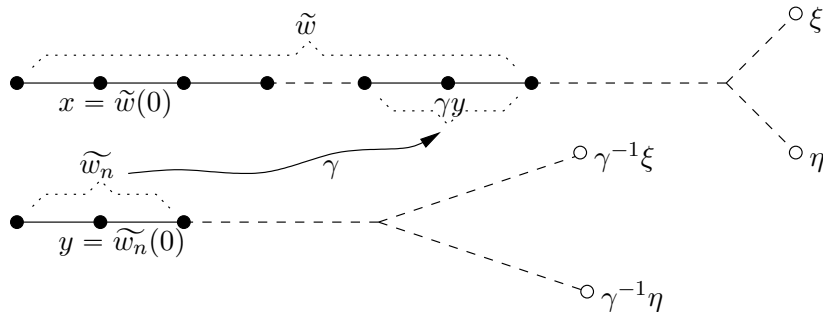
Therefore Theorem 9.1 will follow from Theorem 9.2 by conjugation since Θ is bilipschitz, once we have proved that Hypothesis (1) of Theorem 9.2 is satisfied for the two-sided topological Markov shift (Σ, σ) conjugated by Θ to $(\Gamma\backslash\mathcal{G}\mathbb{X}, \mathfrak{g}^1)$, which is the main point in this proof.

We hence fix an A -admissible finite sequence $w = (w_0, \dots, w_n)$ in \mathcal{A} . We denote by

$$[w_n] = \{(x_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) : x_0 = w_n\},$$

$$[w] = \{(y_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) : y_0 = w_0, \dots, y_n = w_n\}$$

and $f_w : (x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$ the sets and map appearing in Hypothesis (1). We denote by \tilde{w} and \tilde{w}_n the discrete generalised geodesic lines in \mathbb{X} associated with w and w_n (see the proof of Theorem 5.1 just after Equation (5.3)). Since w ends with w_n , by the construction of Θ , there exists $\gamma \in \Gamma$ sending the two consecutive edges of \tilde{w}_n to the last two consecutive edges of w . We denote by $x = \tilde{w}(0)$ and $y = \tilde{w}_n(0)$ the footpoints of \tilde{w} and \tilde{w}_n respectively.



For every discrete generalised geodesic line $\omega \in \check{\mathcal{G}}\mathbb{X}$ which is isometric exactly on an interval I containing 0 in its interior (as for $\omega = \tilde{w}, \tilde{w}_n$), let

$$\mathcal{G}_\omega\mathbb{X} = \{\ell \in \mathcal{G}\mathbb{X} : \ell|_I = \omega|_I\}$$

be the space of extensions of $\omega|_I$ to geodesic lines. With $\partial_\omega^\pm\mathbb{X} = \{\ell_\pm : \ell \in \mathcal{G}_\omega\mathbb{X}\}$ its set of points at $\pm\infty$, we have a homeomorphism $\mathcal{G}_\omega\mathbb{X} \rightarrow (\partial_\omega^-\mathbb{X} \times \partial_\omega^+\mathbb{X})$ defined by $\ell \mapsto (\ell_-, \ell_+)$, using Hopf's parametrisation with respect to the point $\omega(0)$, since all the geodesic lines in $\mathcal{G}_\omega\mathbb{X}$ are at the point $\omega(0)$ at time $t = 0$. Using as basepoint $x_0 = \omega(0)$ in the definition of the Gibbs measure (see Equation (4.10)), this homeomorphism sends the restriction to $\mathcal{G}_\omega\mathbb{X}$ of the Gibbs measure $d\tilde{m}_c(\ell)$ to the product measure $d\mu_{\omega(0)}^-(\ell_-) d\mu_{\omega(0)}^+(\ell_+)$. Hence the pushforward of $\tilde{m}_c|_{\mathcal{G}_\omega\mathbb{X}}$ by the positive endpoint map $e_+ : \ell \mapsto \ell_+$ is $\mu_{\omega(0)}^-(\partial_\omega^-\mathbb{X}) d\mu_{\omega(0)}^+(\ell_+)$, and note that $\mu_{\omega(0)}^-(\partial_\omega^-\mathbb{X})$ is a positive constant.

Since $\pi_+ : \Sigma \rightarrow \Sigma_+$ is the map which forgets about the past, there exist measurable maps $u_w : \partial_{\tilde{w}}^+\mathbb{X} \rightarrow [w]$ and $u_{w_n} : \partial_{\tilde{w}_n}^+\mathbb{X} \rightarrow [w_n]$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{G}_{\tilde{w}}\mathbb{X} & \xrightarrow{\pi_+ \circ \Theta \circ p} & [w] \\ e_+ \searrow & & \nearrow u_w \\ & \partial_{\tilde{w}}^+\mathbb{X} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{G}_{\tilde{w}_n}\mathbb{X} & \xrightarrow{\pi_+ \circ \Theta \circ p} & [w_n] \\ e_+ \searrow & & \nearrow u_{w_n} \\ & \partial_{\tilde{w}_n}^+\mathbb{X} & \end{array} .$$

Furthermore, the map u_w (respectively u_{w_n}) is surjective, and has constant finite order fibers given by the orbits of the finite stabiliser $\Gamma_{\tilde{w}}$ (respectively $\Gamma_{\tilde{w}_n}$). Since $\mathbb{P} = \Theta_* \frac{m_c}{\|m_c\|}$, the pushforward by the map u_w (respectively u_{w_n}) of the measure μ_x^+ (respectively μ_y^+) is a constant time the restriction of $(\pi_+)_*\mathbb{P}$ to $[w]$ (respectively $[w_n]$). Finally, by the construction of the (inverse of the) coding in the proof of Theorem 5.1, the following diagram is commutative:

$$\begin{array}{ccc} \partial_{\tilde{w}_n}^+\mathbb{X} & \xrightarrow{u_{w_n}} & [w_n] \\ \gamma \downarrow & & \downarrow f_w \\ \partial_{\tilde{w}}^+\mathbb{X} & \xrightarrow{u_w} & [w_n] . \end{array}$$

Recall that the pushforwards of measures μ, ν , which are absolutely continuous one with respect to the other, by a measurable map f are again absolutely continuous one with respect to the other, and satisfy (almost everywhere)

$$\frac{d f_* \mu}{d f_* \nu} \circ f = \frac{d \mu}{d \nu} .$$

Hence in order to prove that Hypothesis (1) in the statement of Theorem 9.2 is satisfied, we only have to prove that the map $\gamma : \partial_{\tilde{w}_n}^+\mathbb{X} \rightarrow \partial_{\tilde{w}}^+\mathbb{X}$ has a constant Jacobian for the measures μ_y^+ on $\partial_{\tilde{w}_n}^+\mathbb{X}$ and μ_x^+ on $\partial_{\tilde{w}}^+\mathbb{X}$ respectively.

For all $\xi, \eta \in \partial_{\tilde{w}}^+\mathbb{X}$, by the properties of the Patterson densities (see Equations (4.1) and (4.2)), and since γy belongs to the geodesic ray from x to ξ and η (see Equation (3.6) and the above picture), we have

$$\frac{\frac{d \gamma_* \mu_y^+}{d \mu_x^+}(\xi)}{\frac{d \gamma_* \mu_y^+}{d \mu_x^+}(\eta)} = \frac{\frac{d \mu_{\gamma y}^+}{d \mu_x^+}(\xi)}{\frac{d \mu_{\gamma y}^+}{d \mu_x^+}(\eta)} = \frac{e^{-C_\xi^+(\gamma y, x)}}{e^{-C_\eta^+(\gamma y, x)}} = \frac{e^{-\int_x^{\gamma y} (\tilde{F}_c^+ - \delta_c)}}{e^{-\int_x^{\gamma y} (\tilde{F}_c^+ - \delta_c)}} = 1 .$$

This proves that Hypothesis (1) in Theorem 9.2 is satisfied, and concludes the proof of Theorem 9.1. \square

We now indicate how to pass from a one-sided version of Theorem 9.2 to the two-sided one, as was communicated to us by J. Buzzi.

Step 2 : Reduction to one-sided symbolic dynamics

Let (Σ_+, σ_+) be the *one-sided topological Markov shift* with alphabet \mathcal{A} and transition matrix A , that is, Σ_+ is the closed subset of the topological product space $\mathcal{A}^{\mathbb{N}}$ defined by

$$\Sigma_+ = \{x = (x_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \forall n \in \mathbb{N}, A_{x_n, x_{n+1}} = 1\},$$

and $\sigma_+ : \Sigma_+ \rightarrow \Sigma_+$ is the (one-sided) *shift*⁴ defined by

$$(\sigma_+(x))_n = x_{n+1}$$

for all $x \in \Sigma_+$ and $n \in \mathbb{N}$. We endow Σ_+ with the distance

$$d(x, x') = e^{-\max\{n \in \mathbb{N} : \forall i \in \{0, \dots, n\}, x_i = x'_i\}}.$$

Note that the distances on Σ and Σ_+ are bounded by 1.

Let $\pi_+ : \Sigma \rightarrow \Sigma_+$ be the natural extension $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n \in \mathbb{N}}$, which satisfies $\pi_+ \circ \sigma = \sigma_+ \circ \pi_+$ and is 1-Lipschitz. Note that Σ is transitive (respectively locally compact) if and only if Σ_+ is transitive (respectively locally compact).

In the one-sided case, we always assume that the cylinders start at time $t = 0$: given an admissible sequence $w = (w_0, w_1, \dots, w_{n-1})$, the *cylinder* of length $|w| = n$ it defines is

$$[w] = [w_0, \dots, w_{n-1}] = \{(x_n)_{n \in \mathbb{N}} \in \Sigma_+ : \forall i \in \{0, \dots, n-1\}, x_i = w_i\}.$$

We first explain how to relate the decay of correlations for the two-sided and one-sided systems. This is well-known since the works of Sinai [Sin, §3] and Bowen [Bowe2, Lem. 1.6], see for instance [You1, §4], and the following proof has been communicated to us by J. Buzzi. We fix $\alpha \in]0, 1]$. For all metric space Z and bounded α -Hölder-continuous function $f : Z \rightarrow \mathbb{R}$, let

$$\|f\|'_\alpha = \sup_{\substack{x, y \in Z \\ 0 < d(x, y) \leq 1}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha},$$

so that⁵ $\|f\|_\alpha = \|f\|_\infty + \|f\|'_\alpha$.

Lemma 9.3. *For every $a \in \mathcal{A}$, let us fix $z^a \in \Sigma$ such that $(z^a)_0 = a$. Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a bounded α -Hölder-continuous map and $N \in \mathbb{N}$. Define $\phi^{(N)} : \Sigma_+ \rightarrow \mathbb{R}$ by:*

$$\forall x \in \Sigma_+, \quad \phi^{(N)}(x) = \phi(y) \text{ where } \begin{cases} y_i = x_{i+N} & \text{if } i \geq -N \\ y_i = z_{i+N}^{x_0} & \text{otherwise.} \end{cases}$$

Then $\phi^{(N)}$ is bounded and α -Hölder-continuous on Σ_+ , with

$$|\phi \circ \sigma^N - \phi^{(N)} \circ \pi_+| \leq \|\phi\|'_\alpha e^{-\alpha N}.$$

Moreover,

$$\|\phi^{(N)}\|'_\alpha \leq e^{\alpha N} \|\phi\|'_\alpha \quad \text{and} \quad \|\phi^{(N)}\|_\infty \leq \|\phi\|_\infty.$$

⁴Although it is standard to denote the one-sided shift by σ in the same way as the two-sided shift, we use σ_+ for readability.

⁵See Section 3.1 for the definition of the Hölder norm $\|\cdot\|_\alpha$.

Proof. For all $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma$, with y associated with $\pi_+(x)$ as in the statement, we have $(\sigma^N(x))_n = y_n$ if $|n| \leq N$, hence

$$|\phi \circ \sigma^N(x) - \phi^{(N)}(\pi_+(x))| = |\phi(\sigma^N(x)) - \phi(y)| \leq \|\phi\|'_\alpha d(\sigma^N(x), y)^\alpha \leq \|\phi\|'_\alpha e^{-\alpha N}.$$

Moreover, if $y, y' \in \Sigma$ are associated with $x = (x_n)_{n \in \mathbb{N}}, x' = (x'_n)_{n \in \mathbb{N}} \in \Sigma_+$ respectively, then $d(y, y') = e^N d(x, x')$ if $d(x, x') < e^{-N}$ and otherwise $d(y, y') \leq 1 \leq e^N d(x, x')$, so that

$$|\phi^{(N)}(x) - \phi^{(N)}(x')| = |\phi(y) - \phi(y')| \leq \|\phi\|'_\alpha d(y, y')^\alpha \leq \|\phi\|'_\alpha e^{\alpha N} d(x, x')^\alpha. \quad \square$$

Proposition 9.4. *Let μ be a σ -invariant probability measure on Σ . Assume that the dynamical system $(\Sigma_+, \sigma_+, (\pi_+)_*\mu)$ has exponential decay of α -Hölder correlations. Then (Σ, σ, μ) has exponential decay of α -Hölder correlations.*

Proof. Let $C, \kappa > 0$ be such that for all bounded α -Hölder-continuous maps $\phi', \psi' : \Sigma_+ \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$|\text{cov}_{(\pi_+)_*\mu, n}(\phi', \psi')| \leq C \|\phi\|_\alpha \|\psi\|_\alpha e^{-\kappa n}.$$

Let $\phi, \psi : \Sigma \rightarrow \mathbb{R}$ be bounded α -Hölder-continuous maps and $n \in \mathbb{N}$. Denoting by $\pm t$ any value in $[-t, t]$ for any $t \geq 0$, we have, by the first part of the above lemma and for any $N \in \mathbb{N}$ (to be chosen appropriately later on),

$$\begin{aligned} \int_\Sigma \phi \circ \sigma^n \psi \, d\mu &= \int_\Sigma \phi \circ \sigma^{n+N} \psi \circ \sigma^N \, d\mu \\ &= \int_\Sigma (\phi^{(N)} \circ \pi_+ \pm \|\phi\|'_\alpha e^{-\alpha N}) \circ \sigma^n (\psi^{(N)} \circ \pi_+ \pm \|\psi\|'_\alpha e^{-\alpha N}) \, d\mu \\ &= \int_{\Sigma_+} \phi^{(N)} \circ \sigma_+^n \psi^{(N)} \, d(\pi_+)_*\mu \pm \|\phi\|_\alpha \|\psi\|_\alpha e^{-\alpha N}. \end{aligned}$$

A similar estimate holds for the second term in the definition of the correlation coefficients. Hence by the second part of the above lemma

$$\begin{aligned} |\text{cov}_{\mu, n}(\phi, \psi)| &\leq |\text{cov}_{(\pi_+)_*\mu, n}(\phi^{(N)}, \psi^{(N)})| + 2 \|\phi\|_\alpha \|\psi\|_\alpha e^{-\alpha N} \\ &\leq C (\|\phi\|_\infty + \|\phi\|'_\alpha e^{\alpha N}) (\|\psi\|_\infty + \|\psi\|'_\alpha e^{\alpha N}) e^{-\kappa n} + 2 \|\phi\|_\alpha \|\psi\|_\alpha e^{-\alpha N} \\ &\leq \|\phi\|_\alpha \|\psi\|_\alpha (C e^{2\alpha N - \kappa n} + 2 e^{-\alpha N}). \end{aligned}$$

Taking $N = \lfloor \frac{\kappa n}{4\alpha} \rfloor$, $C' = C + 2 e^\alpha$ and $\kappa' = \frac{\kappa}{4}$, we have

$$|\text{cov}_{\mu, n}(\phi, \psi)| \leq C' \|\phi\|_\alpha \|\psi\|_\alpha e^{-\kappa' n},$$

and the result follows. \square

In order to conclude Step 2, we now state the one-sided version of Theorem 9.2 and prove how it implies Theorem 9.2.⁶

Theorem 9.5. *Let (Σ_+, σ_+) be a locally compact transitive one-sided topological Markov shift with alphabet \mathcal{A} and transition matrix A , and let \mathbb{P}_+ be a mixing σ_+ -invariant probability measure with full support on Σ_+ . Assume that*

⁶Assumption (1) of Theorem 9.5 is far from being optimal, but will be sufficient for our purpose.

(1) for every A -admissible finite sequence $w = (w_0, \dots, w_n)$ in \mathcal{A} , the Jacobian of the map from $[w_n]$ to $[w]$ defined by $(w_n, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$ with respect to the restrictions of the measure \mathbb{P}_+ is constant;

(2) there exist a finite subset \mathcal{E} of \mathcal{A} and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have

$$\mathbb{P}_+(\{x \in \Sigma_+ : x_0 \in \mathcal{E} \text{ and } \tau_{\mathcal{E}}(x) \geq n\}) \leq C' e^{-\kappa' n}. \quad (9.3)$$

Then $(\Sigma_+, \mathbb{P}_+, \sigma_+)$ has exponential decay of α -Hölder correlations.

Proof that Theorem 9.5 implies Theorem 9.2. Let $(\Sigma, \sigma, \mathbb{P}, \mathcal{E})$ be as in the statement of Theorem 9.2. Let $\mathbb{P}_+ = (\pi_+)_* \mathbb{P}$, which is a mixing σ_+ -invariant probability measure on Σ_+ . Note that Hypothesis (1) in Theorem 9.5 follows from Hypothesis (1) of Theorem 9.2. Similarly, Equation (9.3) follows from Equation (9.2). Hence Theorem 9.2 follows from Theorem 9.5 and Proposition 9.4. \square

Let us now consider Theorem 9.5. The scheme of its proof, using inducing and Young tower arguments, was communicated to us by O. Sarig.

Step 3 : Proof of Theorem 9.5

In this final Step, using inducing of the dynamical system (Σ_+, σ_+) on the subspace $\{x \in \Sigma_+ : x_0 \in \mathcal{E}\} = \bigcup_{a \in \mathcal{E}} [a]$ (a finite union of 1-cylinders), we present (Σ_+, σ_+) as a Young tower to which we will apply the results of [You2].

Note that since σ_+ is mixing with full measure, there exists a σ_+ -invariant measurable subset Δ of Σ_+ such that the orbit under σ_+ of every element of Δ passes infinitely many times inside the nonempty open subset $\bigcup_{a \in \mathcal{E}} [a]$. We again denote by $\tau_{\mathcal{E}} : \Delta \rightarrow \mathbb{N} - \{0\}$ the restriction to Δ of the first return time in $\bigcup_{a \in \mathcal{E}} [a]$, so that if

$$\Delta_0 = \{x \in \Delta : x_0 \in \mathcal{E}\} = \bigcup_{a \in \mathcal{E}} \Delta \cap [a],$$

then $\tau_{\mathcal{E}}(x) = \min\{n \in \mathbb{N} - \{0\} : \sigma_+^n x \in \Delta_0\}$ for all $x \in \Delta$. We denote by $F : \Delta \rightarrow \Delta_0$ the first return map to Δ_0 under iteration of the one-sided shift, that is

$$F : x \mapsto \sigma_+^{\tau_{\mathcal{E}}(x)}(x).$$

Let W be the set of admissible sequences w of length $|w|$ at least 2 such that if $w = (w_0, \dots, w_n)$ then

$$w_0, w_n \in \mathcal{E} \quad \text{and} \quad w_1, \dots, w_{n-1} \notin \mathcal{E}.$$

We have the following properties:

- the sets $\Delta_a = \Delta \cap [a]$ for $a \in \mathcal{E}$ form a finite measurable partition of Δ_0 and for every $a \in \mathcal{E}$, the sets $\Delta_w = \Delta \cap [w]$ for $w \in W$ and $w_0 = a$ form a countable measurable partition of Δ_a ;
- for every $w \in W$, the first return time $\tau_{\mathcal{E}}$ is constant (equal to $|w| - 1$) on each Δ_w , and if $w_{|w|-1} = b$, then the first return map F is a bijection from Δ_w to Δ_b ;
- for all $w \in W$ and $x, y \in \Delta_w$, since x, y have the same $|w|$ first components, we have

$$d(F(x), F(y)) = d(\sigma_+^{|w|-1} x, \sigma_+^{|w|-1} y) = e^{|w|-1} d(x, y) \geq e d(x, y);$$

- for all $w \in W$, $n \in \{0, \dots, |w| - 2\}$ and $x, y \in \Delta_w$, we have

$$d(\sigma_+^n x, \sigma_+^n y) = e^n d(x, y) \leq e^{|w|-2} d(x, y) < d(F(x), F(y)) ;$$

- for every $w \in W$, the Jacobian of the first return map $F : \Delta_w \rightarrow \Delta_{w|_{|w|-1}}$ for the restrictions to Δ_w and $\Delta_{w|_{|w|-1}}$ of \mathbb{P}_+ is constant.⁷

By an easy adaptation of [You2, Theo. 3] (see also [Mell, §2.1]) which considers the case when \mathcal{E} is a singleton, we have the following noneffective⁸ exponential decay of correlation: there exists $\kappa > 0$ such that for every $\phi, \psi \in \mathcal{C}_b^\alpha(\Sigma_+)$, there exists a constant $C_{\phi, \psi} > 0$ such that

$$|\text{cov}_{\mathbb{P}_+, n}(\phi, \psi)| \leq C_{\phi, \psi} e^{-\kappa n}$$

By an elegant argument using the Principle of Uniform Boundedness, it is proved in [ChCS, Appendix B] that this implies that there exists $C, \kappa > 0$ such that for every $\phi, \psi \in \mathcal{C}_b^\alpha(\Sigma_+)$, we have

$$|\text{cov}_{\mathbb{P}_+, n}(\phi, \psi)| \leq C \|\phi\|_\alpha \|\psi\|_\alpha e^{-\kappa n} .$$

This concludes the proof of Theorem 9.5, hence the proof of Theorem 9.1. \square \square

The next result gives examples of applications of Theorem 9.1 when $\Gamma \backslash \mathbb{X}$ is infinite. It strengthens [AtGP, Theo. 2.1] that applies only to arithmetic lattices and only for the locally constant regularity (see Section 15.4), see also [BekL] for an approach using spectral gaps. It was claimed in [Kwo], but was retracted by the author.

Corollary 9.6. *Let \mathbb{X} be a locally finite simplicial tree without terminal vertices. Let Γ be a geometrically finite subgroup of $\text{Aut}(\mathbb{X})$ such that the smallest nonempty Γ -invariant subtree of \mathbb{X} is uniform without vertices of degree 2. Let $\alpha \in]0, 1]$.*

(1) *If $L_\Gamma = \mathbb{Z}$, then the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ has exponential decay of α -Hölder correlations for the system of conductances $c = 0$.*

(2) *If $L_\Gamma = 2\mathbb{Z}$, then the square of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}$ has exponential decay of α -Hölder correlations for the 0 system of conductances, that is, there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathcal{C}_b^\alpha(\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X})$ and $n \in \mathbb{Z}$, we have*

$$\left| \int_{\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}} \phi \circ \mathbf{g}^{-2n} \psi \, d m_{\text{BM}} - \frac{1}{m_{\text{BM}}(\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X})} \int_{\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}} \phi \, d m_{\text{BM}} \int_{\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}} \psi \, d m_{\text{BM}} \right| \leq C e^{-\kappa|n|} \|\phi\|_\alpha \|\psi\|_\alpha .$$

The main point of this corollary is to prove the exponential decay of volumes of geodesic lines going high in the cuspidal rays of $\Gamma \backslash \mathbb{X}$, stated as Assumption (9.1) in Theorem 9.1. There is a long history of similar results, starting from the exponential decay of volumes of small cusp neighbourhoods in noncompact finite volume hyperbolic manifolds (based on the description of their ends) used by Sullivan to deduce Diophantine approximation results (see [Sul3, §9]).⁹ These results were extended to the case of locally symmetric Riemannian

⁷Actually, only a much weaker assumption is required, such as a Hölder-continuity property of this Jacobian, see [You2].

⁸Actually, there is in [You2] (see also [CyS]) a control on the constant in terms of some norms of the test functions, but these norms are not the ones we are interested in.

⁹and by probabilists in order to study the statistics of cusp excursions (see for instance [EF])

manifolds by Kleinbock-Margulis [KM2] (based on the description of their ends using Siegel sets). Note that the geometrically finite lattice assumption on Γ is here in order to obtain similar descriptions of the ends of $\Gamma \backslash \mathbb{X}$.

Proof. Up to replacing \mathbb{X} by $\mathcal{C}\Lambda\Gamma$, we assume that \mathbb{X} is a uniform simplicial tree with degrees at least 3 and that Γ is a geometrically finite lattice of \mathbb{X} . We use the 0 system of conductances.

(1) By [Pau2] and as recalled in Section 2.7, the graph $\Gamma \backslash \mathbb{X}$ is the union of a finite graph \mathbb{Y} and finitely many geodesic rays R_i for $i \in \{1, \dots, k\}$, such that if $(x_{i,n})_{n \in \mathbb{N}}$ is the sequence of vertices in increasing order along R_i for $i = 1, \dots, k$, then the vertex group $G_{x_{i,n}}$ of $x_{i,n}$ in the quotient graph of groups $\Gamma \backslash \mathbb{X}$ satisfies $G_{x_{i,n}} \subset G_{x_{i,n+1}}$ for all $n \in \mathbb{N}$, and the edge group of the edge $e_{i,n}$ with origin $x_{i,n}$ and endpoint $x_{i,n+1}$ is equal to $G_{x_{i,n}}$.¹⁰ Note that since the degrees of \mathbb{X} are at least 3, we have $[G_{x_{i,n+1}} : G_{x_{i,n}}] \geq 2$ and $|G_{x_{i,0}}| \geq 1$, so that

$$|G_{x_{i,n}}| \geq 2^n. \quad (9.4)$$

Let E be the (finite) set of vertices $V\mathbb{Y}$ of \mathbb{Y} . Note that for all $n \in \mathbb{N} - \{0\}$ and $\ell \in \Gamma \backslash \mathcal{G}\mathbb{X}$, if $\ell(0) \in E$ and $\tau_E(\ell) \geq 2n$, then ℓ needs to leave \mathbb{Y} after time 0 and it travels (geodesically) inside some cuspidal ray for a time at least n , so that there exists $i \in \{1, \dots, k\}$ such that $\ell(n) = x_{i,n}$. Hence for all $n \in \mathbb{N}$, using

- the invariance of m_{BM} under the (discrete time) geodesic flow in order to get the third term,
 - Equation (8.4) where $\tilde{x}_{i,n}$ is a fixed lift of $x_{i,n}$ in $V\mathbb{X}$ for the fifth term, and
 - Equation (9.4) since $|\Gamma_{\tilde{x}_{i,n}}| = |G_{x_{i,n}}|$ and the fact that the degrees of the uniform simplicial tree \mathbb{X} are uniformly bounded and that the total mass of the Patterson measures of the lattice Γ are uniformly bounded (see Proposition 4.14) for the last term,
- we have

$$\begin{aligned} & m_{\text{BM}}(\{\ell \in \Gamma \backslash \mathcal{G}\mathbb{X} : \ell(0) \in E \text{ and } \tau_E(\ell) \geq 2n\}) \\ & \leq \sum_{i=1}^k m_{\text{BM}}(\{\ell \in \Gamma \backslash \mathcal{G}\mathbb{X} : \ell(n) = x_{i,n}\}) \\ & = \sum_{i=1}^k m_{\text{BM}}(\{\ell \in \Gamma \backslash \mathcal{G}\mathbb{X} : \ell(0) = x_{i,n}\}) = \sum_{i=1}^k \pi_* m_{\text{BM}}(\{x_{i,n}\}) \\ & = \sum_{i=1}^k \frac{1}{|\Gamma_{\tilde{x}_{i,n}}|} \sum_{e, e' \in E\mathbb{X} : o(e)=o(e')=\tilde{x}_{i,n}, e \neq e'} \mu_{\tilde{x}_{i,n}}(\partial_e \mathbb{X}) \mu_{\tilde{x}_{i,n}}(\partial_{e'} \mathbb{X}) \\ & \leq k \frac{1}{2^n} \max_{x \in V\mathbb{X}} \deg(x)^2 \max_{x \in V\mathbb{X}} \|\mu_x\|^2. \end{aligned}$$

The result then follows from Theorem 9.1 using the above finite set E which satisfies Assumption (9.1) as we just proved, and using Propositions 4.14 and 4.15 in order to check that under the assumption that $L_\Gamma = \mathbb{Z}$, the Bowen-Margulis measure m_{BM} of Γ is finite and mixing under the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$.

(2) The proof of Assertion (2) of Corollary 9.6 is similar to the one of Assertion (1). \square

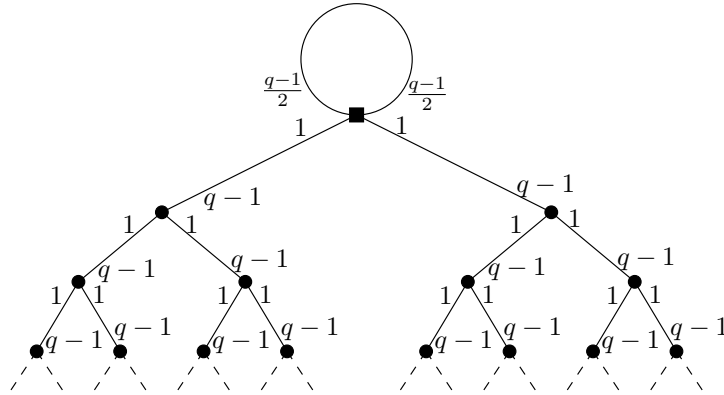
¹⁰identifying the edge group of an edge e with its image by the structural map $G_e \mapsto G_{o(e)}$

Remark. The techniques introduced in the above proof in order to check the main hypothesis of Theorem 9.1 may be applied to numerous other examples. For instance, let \mathbb{X} be a locally finite simplicial tree without terminal vertices. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$ such that the smallest nonempty Γ -invariant subtree of \mathbb{X} is uniform without vertices of degree 2, and such that $L_\Gamma = \mathbb{Z}$. Let $\alpha \in]0, 1]$. Assume that $\Gamma \backslash \mathbb{X}$ is the union of a finite graph A and finitely many trees $\mathbb{T}_1, \dots, \mathbb{T}_k$ meeting A in one and exactly one vertex $*_1, \dots, *_n$ such that for every edge e in T_i pointing away from the root $*_i$ of \mathbb{T}_i , the canonical morphism $G_e \rightarrow G_{o(e)}$ between edge and vertex groups of the quotient graph of groups $\Gamma \backslash \mathbb{X}$ is an isomorphism. Assume that there exists $C, \kappa > 0$ such that for all $n \in \mathbb{N}$,

$$\sum_{i=1, \dots, k, x \in V\mathbb{T}_i : d(x, *_i) = n} \frac{1}{|G_x|} \leq C e^{-\kappa n}.$$

Then the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ has exponential decay of α -Hölder correlations for the 0 system of conductances.

This is in particular the case for every $k, q \in \mathbb{N}$ such that $k \geq 2$, $q > 2k + 1$ and $q - k$ is odd, when the quotient graph of groups $\Gamma \backslash \mathbb{X}$ has underlying edge-indexed graph¹¹ a loop-edge with both indices equal to $\frac{q-k+1}{2}$ glued to the root of a regular k -ary rooted tree, with indices 1 for the edges pointing towards the root and $q - k + 1$ for the edges pointing away from the root (see the picture below with $k = 2$). Note that \mathbb{X} is then the $(q + 1)$ -regular tree, and that the loop edge is here in order to ensure that $L_\Gamma = \mathbb{Z}$. For instance, the vertex group of a point at distance n from the root may be chosen to be $\mathbb{Z}/(\frac{q-k+1}{2})\mathbb{Z} \times (\mathbb{Z}/(q - k + 1)\mathbb{Z})^n$.



9.3 Rate of mixing for metric trees

Let (\mathbb{X}, λ) , X , Γ , \tilde{F} , $(\mu_x^\pm)_{x \in V\mathbb{X}}$ and m_F be as in the beginning of Section 4.4. The aim of this Section is to study the problem of finding conditions on these data under which the (continuous time) geodesic flow on $\Gamma \backslash \mathcal{G}X$ is polynomially mixing for the Gibbs measure m_F .

We will actually prove a stronger property, though it applies only to observables which are smooth enough along the flow. Let us fix $\alpha \in]0, 1]$. Let $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ be a topological space Z endowed with a continuous one-parameter group $(\phi_t)_{t \in \mathbb{R}}$ of homeomorphisms preserving a (Borel) probability measure μ on Z . For all $k \in \mathbb{N}$, let $\mathcal{C}_b^{k, \alpha}(Z)$ be the real vector space of maps $f : Z \rightarrow \mathbb{R}$ such that for all $z \in Z$, the map $t \mapsto f(\phi_t z)$ is C^k -smooth, and such

¹¹See definition in Section 2.7.

that the maps $\partial_t^i f : Z \rightarrow \mathbb{R}$ defined by $x \mapsto \frac{d^i}{dt^i}|_{t=0} f(\phi_t x)$ for $0 \leq i \leq k$ are bounded and α -Hölder-continuous. It is a Banach space when endowed with the norm

$$\|f\|_{k,\alpha} = \sum_{i=0}^k \|\partial_t^i f\|_\alpha,$$

and it is contained in $\mathbb{L}^2(Z, \mu)$ by the finiteness of μ . We denote by $\mathcal{C}_c^{k,\alpha}(Z)$ the vector subspace of elements of $\mathcal{C}_b^{k,\alpha}(Z)$ with compact support.

For all $\psi, \psi' \in \mathbb{L}^2(Z, \mu)$ and $t \in \mathbb{R}$, let

$$\text{cov}_{\mu,t}(\psi, \psi') = \int_Z \psi \circ \phi_t \psi' d\mu - \int_Z \psi d\mu \int_Z \psi' d\mu$$

be the *correlation coefficient* of the observables ψ, ψ' at time t under the flow $(\phi_t)_{t \in \mathbb{R}}$ for the measure μ . We say¹² that the (continuous time) dynamical system $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ has *superpolynomial decay of α -Hölder correlations* if for every $n \in \mathbb{N}$ there exist $C = C_n > 0$ and $k = k_n \in \mathbb{N}$ such that for all $\psi, \psi' \in \mathcal{C}_b^{k,\alpha}(Z)$ and $t \in \mathbb{R}$, we have

$$|\text{cov}_{\mu,t}(\psi, \psi')| \leq C (1 + |t|)^{-n} \|\psi\|_{k,\alpha} \|\psi'\|_{k,\alpha}.$$

Following Dolgopyat, we say that the dynamical system $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ is *rapidly mixing* if there exists $\alpha > 0$ such that $(Z, \mu, (\phi_t)_{t \in \mathbb{R}})$ has superpolynomial decay of α -Hölder correlations

The two assumptions on our data that we will use are the following ones, introduced respectively in [Dol2] and [Mel1]. Recall that the Gibbs measure m_F , when finite, is mixing if and only the length spectrum L_Γ is dense in \mathbb{R} (see Theorem 4.8). The rapid mixing property will require stronger assumptions on L_Γ .

We say that the length spectrum L_Γ of Γ is *2-Diophantine* if there exists a ratio of two translation lengths of elements of Γ which is Diophantine. Recall that a real number x is Diophantine if there exist $\alpha, \beta > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \alpha q^{-\beta}$$

for all $p, q \in \mathbb{Z}$ with $q > 0$.

Let E be a finite subset of vertices of $\Gamma \backslash \mathbb{X}$, and let \tilde{E} be the set of vertices of \mathbb{X} mapping to E . We denote by T_E the set of triples $(\lambda(\gamma), d(\gamma), q(\gamma, p))$ where $\gamma \in \Gamma$ has translation length $\lambda(\gamma) > 0$, has $d(\gamma)$ vertices on its translation axis $\text{Ax}(\gamma)$ modulo $\gamma\mathbb{Z}$ and if the first return time of a vertex p in $\tilde{E} \cap \text{Ax}(\gamma)$ in $\tilde{E} \cap \text{Ax}(\gamma)$ under the discrete time geodesic flow along the translation axis has period $q(\gamma, p)$. We say that the length spectrum L_Γ of Γ is *4-Diophantine* with respect to E if for all sequences $(b_k)_{k \in \mathbb{N}}$ in $[1 + \infty[$ converging to $+\infty$ and $(\omega_k)_{k \in \mathbb{N}}, (\varphi_k)_{k \in \mathbb{N}}$ in $[0, 2\pi[$, there exists $N \in \mathbb{N}$ such that for all $a \geq N$ and $C, \beta \geq 1$, there exist $k \geq 1$ and $(\tau, d, q) \in T_E$ such that

$$d((b_k \tau + \omega_k d) [\beta \ln b_k] + q \varphi_k, 2\pi\mathbb{Z}) \geq C q b_k^{-a}.$$

We define the first return time after time ϵ on a finite subset E of vertices of $\Gamma \backslash \mathbb{X}$ as the map $\tau_E^{\geq \epsilon} : \Gamma \backslash \mathcal{G}X \rightarrow [0, +\infty]$ defined by $\tau_E^{\geq \epsilon}(\ell) = \inf\{t > \epsilon : \ell(t) \in E\}$.

Theorem 9.7. *Assume that the critical exponent δ_F is finite, that the Gibbs measure m_F is finite and mixing, and that the lengths of the edges of (\mathbb{X}, λ) have a finite upper bound.¹³*

¹²See [Dol2], and more precisely [Mel1, Def. 2.2] whose definition is slightly different but implies the one given in this paper by the Principle of Uniform Boundedness argument of [ChCS, Appendix B] already used in Section 9.2.

¹³They have a positive lower bound by definition, see Section 2.7.

Furthermore assume that

- (a) either $\Gamma \backslash X$ is compact and the length spectrum of $\Gamma \backslash X$ is 2-Diophantine,
- (b) or there exists a finite subset E of vertices of $\Gamma \backslash \mathbb{X}$ such that

(1) there exist $C, \kappa > 0$ and $\epsilon \in]0, \min \lambda[$ such that for all $t \geq 0$,

$$m_c(\{\ell \in \Gamma \backslash \mathcal{G}X : d(\ell(0), E) \leq \epsilon \text{ and } \tau_E^{>\epsilon}(\ell) \geq t\}) \leq C e^{-\kappa t},$$

(2) the length spectrum of Γ is 4-Diophantine with respect to E .

Then the (continuous time) geodesic flow on $\Gamma \backslash \mathcal{G}X$ has superpolynomial decay of α -Hölder correlations for the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$.

Note that the existence of E satisfying the exponentially small tail Hypothesis (1) is in particular satisfied if Γ is geometrically finite with E the set of vertices of a finite subgraph of $\Gamma \backslash \mathbb{X}$ whose complement in $\Gamma \backslash \mathbb{X}$ is the underlying graph of a union of cuspidal rays in $\Gamma \backslash \mathbb{X}$: see the proof of Corollary 9.6 and use the hypothesis on the lengths of edges.

Note that the exponentially small tail Hypothesis (1) might be weakened to a superpolynomially small tail hypothesis while keeping the same conclusion, see [Mel2]. Since the former is easier to check than the latter, we prefer to state Theorem 9.7 as it is.

We will follow a scheme of proof analogous to the one in Section 9.2 for simplicial trees, by reducing the study to a problem of suspensions of Young towers, and then apply results of [Dol2] and [Mel1] for the rapid mixing property of suspensions of hyperbolic and nonuniformly hyperbolic dynamical systems.

Proof. Since the Gibbs measure normalised to be a probability measure depends only on the cohomology class of the potential (see Equation (4.9)), we may assume by Proposition 3.12 that $F = F_c$ is the potential on $\Gamma \backslash T^1 X$ associated with a system of conductances $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ for Γ . We denote by δ_c the critical exponent of (Γ, F_c) , and by m_c the Gibbs measure m_{F_c} .

Step 1 : Reduction to a suspension of a two-sided symbolic dynamics

We refer to the paragraphs before the statement of Theorem 5.9 for the definitions of

- the system of conductances \sharp_c for Γ on the simplicial tree \mathbb{X} ,
- the (two-sided) topological Markov shift $(\Sigma, \sigma, \mathbb{P})$ on the alphabet \mathcal{A} , conjugated to the discrete time geodesic flow $(\Gamma \backslash \mathcal{G}\mathbb{X}, \sharp_{\mathbf{g}^1}, \frac{m_{\sharp_c}}{\|m_{\sharp_c}\|})$ by the homeomorphism $\Theta : \Gamma \backslash \mathcal{G}\mathbb{X} \rightarrow \Sigma$,
- the roof function $r : \Sigma \rightarrow]0, +\infty[$
- and the suspension $(\Sigma, \sigma, a\mathbb{P})_r = (\Sigma_r, (\sigma_r^t)_{t \in \mathbb{R}}, a\mathbb{P}_r)$ over $(\Sigma, \sigma, a\mathbb{P})$ with roof function r , where $a = \frac{1}{\|\mathbb{P}_r\|}$. This suspension is conjugated to the continuous time geodesic flow $(\Gamma \backslash \mathcal{G}X, \frac{m_c}{\|m_c\|}, (\mathbf{g}^t)_{t \in \mathbb{R}})$ by the bilipschitz homeomorphism $\Theta_r : \Gamma \backslash \mathcal{G}X \rightarrow \Sigma_r$ defined at the end of the proof of Theorem 5.9. We will always (uniquely) represent the elements of Σ_r as $[x, s]$ with $x \in \Sigma$ and $0 \leq s < r(x)$.

Note that since Θ_r^{-1} conjugates $(\sigma_r^t)_{t \in \mathbb{R}}$ and $(\mathbf{g}^t)_{t \in \mathbb{R}}$, we have for all $f : \Gamma \backslash \mathcal{G}X \rightarrow \mathbb{R}$ and $x \in \Sigma_r$, when defined,

$$\partial_t^i (f \circ \Theta_r^{-1})(x) = \frac{d^i}{dt^i} \Big|_{t=0} f \circ \Theta_r^{-1}(\sigma_r^t x) = \frac{d^i}{dt^i} \Big|_{t=0} f(\mathbf{g}^t \Theta_r^{-1}(x)) = (\partial_t^i f) \circ \Theta_r^{-1}(x).$$

Hence if $f : \Gamma \backslash \mathcal{G}X \rightarrow \mathbb{R}$ is C^k -smooth along the orbits of $(\mathbf{g}^t)_{t \in \mathbb{R}}$, then $f \circ \Theta_r^{-1}$ is C^k -smooth along the orbits of $(\sigma_r^t)_{t \in \mathbb{R}}$. Furthermore, since Θ_r is bilipschitz, the precomposition map by Θ_r^{-1} is a continuous linear isomorphism from $\mathcal{C}_b^{k, \alpha}(\Gamma \backslash \mathcal{G}X)$ to $\mathcal{C}_b^{k, \alpha}(\Sigma_r)$.

Note that since Θ_r conjugates $(\mathbf{g}^t)_{t \in \mathbb{R}}$ and $(\sigma_r^t)_{t \in \mathbb{R}}$, and sends $\frac{m_c}{\|m_c\|}$ to $\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$, we have, for all $\psi, \psi' \in \mathbb{L}^2(\Gamma \backslash \mathcal{G}X)$ and $t \in \mathbb{R}$,

$$\text{cov}_{\frac{m_c}{\|m_c\|}, t}(\psi, \psi') = \text{cov}_{\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}, t}(\psi \circ \Theta_r^{-1}, \psi' \circ \Theta_r^{-1}).$$

Therefore we only have to prove that the suspension $(\Sigma_r, (\sigma_r^t)_{t \in \mathbb{R}}, \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|})$ is rapid mixing.

Step 2 : Reduction to a suspension of a one-sided symbolic dynamics

In this Step, we explain the rather standard reduction concerning mixing rates from suspensions of two-sided topological Markov shifts to suspensions of one-sided topological Markov shifts. We use the obvious modifications of the notation and constructions concerning the suspension of a noninvertible transformation to a semiflow, given for invertible transformations at the beginning of Section 5.3.

We consider the one-sided topological Markov shift $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ over the alphabet \mathcal{A} constructed at the beginning of Step 2 of the proof of Theorem 9.1, with the system of conductances c now replaced by $\sharp c$. Let $\pi_+ : \Sigma \rightarrow \Sigma_+$ be the natural extension so that $\mathbb{P}_+ = (\pi_+)_* \mathbb{P}$ and $\pi_+ \circ \sigma = \sigma_+ \circ \pi_+$.

We are going to construct in Step 2, as the suspension of $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ with an appropriate roof function r_+ , a semiflow $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \geq 0}, (\mathbb{P}_+)_{r_+})$, and prove that the flow $(\Sigma_r, (\sigma_r^t)_{t \in \mathbb{R}}, \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|})$ is rapid mixing if the semiflow $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \geq 0}, \frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|})$ is rapid mixing.

We start by introducing the notation that will be used in Step 2.

Let $r_+ : \Sigma_+ \rightarrow]0, +\infty[$ be the map

$$r_+ : x \mapsto \lambda(e_0^+) \tag{9.5}$$

if $x = (x_n)_{n \in \mathbb{N}} \in \Sigma_+$ and $x_0 = (e_0^-, h_0, e_0^+) \in \mathcal{A}$. Note that this map has a positive lower bound, and a finite upper bound, and that it is locally constant (and even constant on the 1-cylinders of Σ_+). By Equation (5.10), we have

$$r_+ \circ \pi_+ = r. \tag{9.6}$$

We denote by $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \geq 0}, (\mathbb{P}_+)_{r_+})$ the suspension semiflow over $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ with roof function r_+ . We (uniquely) represent the points of the suspension space $(\Sigma_+)_{r_+}$ as $[x, s]$ for $x \in \Sigma_+$ and $0 \leq s < r_+(x)$. For all $t \geq 0$, we have $(\sigma_+)_{r_+}^t([x, s]) = [\sigma_+^n x, s']$ where $n \in \mathbb{N}$ and $s' \in \mathbb{R}$ are such that $t + s = \sum_{i=0}^{n-1} r_+(\sigma_+^i x) + s'$ and $0 \leq s' \leq r_+(\sigma_+^n x)$.

We define the *suspended natural extension* as the map $\pi_+^r : \Sigma_r \rightarrow (\Sigma_+)_{r_+}$ by

$$\pi_+^r : [x, s] \mapsto [\pi_+(x), s],$$

which is well defined by Equation (9.6). Note that π_+^r is 1-Lipschitz for the Bowen-Walters distance on Σ_r and $(\Sigma_+)_{r_+}$ (see Proposition 5.11).

For all $\psi : \Sigma_r \rightarrow \mathbb{R}$ and $T \geq 0$, let us construct a function $\psi^{(T)} : (\Sigma_+)_{r_+} \rightarrow \mathbb{R}$ as follows. For every $[x, s] \in (\Sigma_+)_{r_+}$, let $N \in \mathbb{N}$ and $s' \geq 0$ be such that $(\sigma_+)_{r_+}^T [x, s] = [\sigma_+^N x, s']$, with

$$0 \leq s' < r_+(\sigma_+^N x) \quad \text{and} \quad s + T = \sum_{i=0}^{N-1} r_+(\sigma_+^i x) + s'.$$

Let

$$\psi^{(T)}([x, s]) = \psi([y, s'])$$

where $y = (y_n)_{n \in \mathbb{Z}}$ is such that $y_i = x_{i+N}$ if $i \geq -N$ and $y_i = z_{i+N}^{x_0}$ otherwise. Note that $y_0 = x_N$, hence $r(y) = r_+(\sigma_+^N(x))$, and the above map is well defined.

Finally, for every $\psi \in \mathcal{C}_b^{k, \alpha}(\Sigma_r)$ or $\psi \in \mathcal{C}_b^{k, \alpha}((\Sigma_+)_{r_+})$, let

$$\|\psi\|_{k, \infty} = \sum_{i=0}^k \|\partial_t^i \psi\|_{\infty}$$

and

$$\|\psi\|'_{k, \alpha} = \sum_{i=0}^k \|\partial_t^i \psi\|'_{\alpha},$$

so that

$$\|\psi\|_{k, \alpha} = \|\psi\|_{k, \infty} + \|\psi\|'_{k, \alpha}.$$

Lemma 9.8. *Let $T \geq 0$ and $\psi \in \mathcal{C}_b^{k, \alpha}(\Sigma_r)$.*

- (1) *For all $t \geq 0$, we have $(\sigma_+)_{r_+}^t \circ \pi_+^r = \pi_+^r \circ \sigma_r^t$.*
- (2) *With $\alpha' = \frac{\alpha}{\sup \lambda}$, there exists a constant $C_1 > 0$ (independent of k, T and ψ) such that $|\psi \circ \sigma_r^T - \psi^{(T)} \circ \pi_+^r| \leq C_1 \|\psi\|'_{\alpha} e^{-\alpha' T}$.*
- (3) *We have $\psi^{(T)} \in \mathcal{C}_b^{k, \alpha}((\Sigma_+)_{r_+})$ and $\|\psi^{(T)}\|_{\infty} \leq \|\psi\|_{\infty}$. With $\alpha'' = \frac{\alpha}{\inf \lambda}$, there exists a constant $C_2 > 0$ (independent of k, T and ψ) such that*

$$\|\psi^{(T)}\|'_{k, \alpha} \leq C_2 e^{\alpha'' T} \|\psi\|'_{k, \alpha}. \quad (9.7)$$

Proof. (1) For every $[x, s] \in \Sigma_r$, let $n \in \mathbb{N}$ and $s' \geq 0$ be such that

$$t + s = \sum_{i=0}^{n-1} r_+(\sigma_+^i \pi_+(x)) + s' \quad \text{and} \quad 0 \leq s' \leq r_+(\sigma_+^n \pi_+(x)).$$

Since $r_+ \circ \sigma_+^i \circ \pi_+ = r \circ \sigma^i$ for all $i \in \mathbb{N}$, these two equalities are equivalent to

$$t + s = \sum_{i=0}^{n-1} r(\sigma^i x) + s' \quad \text{and} \quad 0 \leq s' \leq r(\sigma^n x).$$

Hence

$$(\sigma_+)_{r_+}^t \circ \pi_+^r([x, s]) = (\sigma_+)_{r_+}^t([\pi_+(x), s]) = [\sigma_+^n \pi_+(x), s']$$

and

$$\pi_+^r \circ \sigma_r^t([x, s]) = \pi_+^r([\sigma^n x, s']) = [\pi_+(\sigma^n x), s'].$$

This proves Assertion (1) since $\pi_+ \circ \sigma = \sigma_+ \circ \pi_+$.

(2) By Proposition 5.11, we may assume that, in the formula of the Hölder norms, the Bowen-Walters distance is replaced by the function d_{BW} , as this will only change C_1 by $C_{\text{BW}}^\alpha C_1$.

For every $[x, s] \in \Sigma_r$, with $[y, s']$ and N associated with $\pi_+^r([x, s]) = [\pi_+(x), s]$ as in the definition of $\psi^{(T)}([\pi_+(x), s])$, we have

$$d_{\text{BW}}(\sigma_r^T[x, s], [y, s']) = d_{\text{BW}}([\sigma^N x, s'], [y, s']) \leq d(\sigma^N x, y) \leq e^{-N}.$$

Since the positive roof function r is bounded from above by the least upper bound $\sup \lambda$ of the lengths of the edges, we have

$$N \geq \sum_{i=0}^{N-1} \frac{r(\sigma^i x)}{\sup \lambda} = \frac{1}{\sup \lambda} (s + T - s') \geq \frac{T}{\sup \lambda} - 1.$$

Hence

$$\begin{aligned} |\psi \circ \sigma_r^T([x, s]) - \psi^{(T)}(\pi_+^r(x))| &= |\psi(\sigma_r^T[x, s]) - \psi([y, s'])| \\ &\leq \|\psi\|'_\alpha d_{\text{BW}}(\sigma_r^T[x, s], [y, s'])^\alpha \leq \|\psi\|'_\alpha e^{-\frac{\alpha}{\sup \lambda} T + \alpha}. \end{aligned}$$

(3) The inequality $\|\psi^{(T)}\|_\infty \leq \|\psi\|_\infty$ is immediate by construction. Let us prove that $\psi^{(T)}$ is C^k along semiflow lines. Fix $[x, s] \in (\Sigma_+)_r$. Let us consider $\epsilon > 0$ small enough, so that $\epsilon < r_+(x) - s$ and $\epsilon < r_+(\sigma_+^N x) - s'$, with $[y, s']$ and N as in the construction of $\psi^{(T)}([x, s])$. Then

$$\psi^{(T)} \circ (\sigma_+)_r^\epsilon([x, s]) = \psi^{(T)}([x, s + \epsilon]) = \psi([y, s' + \epsilon]) = \psi \circ \sigma_r^\epsilon([y, s']).$$

Therefore by taking derivatives with respect to ϵ in this formula, $\psi^{(T)}$ is indeed C^k along semiflow lines, and, for $i = 0, \dots, k$, we have

$$\partial_t^i(\psi^{(T)}) = (\partial_t^i \psi)^{(T)}. \quad (9.8)$$

Let us prove that there exists a constant $C_2 > 0$ (independent of T and ψ) such that

$$\|\psi^{(T)}\|'_\alpha \leq C_2 e^{\alpha' T} \|\psi\|'_\alpha. \quad (9.9)$$

Let $[x, s] \in \Sigma_r$, with $[y, s']$ and N as in the definition of $\psi^{(T)}([x, s])$. Let $[\underline{x}, \underline{s}] \in \Sigma_r$, with $[\underline{y}, \underline{s}']$ and \underline{N} as in the definition of $\psi^{(T)}([\underline{x}, \underline{s}])$. Up to exchanging $[x, s]$ and $[\underline{x}, \underline{s}]$, we assume that $\underline{N} \leq N$.

By Proposition 5.11, we may assume that, in the Hölder norms formulas, the Bowen-Walters distance is replaced by the function d_{BW} , as this will only change C_2 by $C_{\text{BW}}^{2\alpha} C_2$. Let

$$C_3 = \min\{e^{-1}, \inf \lambda\}.$$

Note that the map d_{BW} on $\Sigma_r \times \Sigma_r$ is bounded from above by $1 + \sup \lambda$, since the distance on Σ is at most 1 and since the roof function r is bounded from above by $\sup \lambda$.

We have

$$|\psi^{(T)}([x, s]) - \psi^{(T)}([\underline{x}, \underline{s}])| = |\psi([y, s']) - \psi([\underline{y}, \underline{s}'])| \leq \|\psi\|'_\alpha d_{\text{BW}}([y, s'], [\underline{y}, \underline{s}'])^\alpha. \quad (9.10)$$

If $d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]) \geq C_3 e^{-\frac{T}{\inf \lambda}}$, then

$$d_{\text{BW}}([y, s'], [\underline{y}, \underline{s}']) \leq 1 + \sup \lambda \leq \frac{1 + \sup \lambda}{C_3} e^{\frac{T}{\inf \lambda}} d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]).$$

Therefore Equation (9.9) follows from Equation (9.10) whenever $C_2 \geq \frac{(1 + \sup \lambda)^\alpha}{C_3^\alpha}$.

Conversely, suppose that $d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]) < C_3 e^{-\frac{T}{\inf \lambda}}$. Assume that

$$d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]) = d(x, \underline{x}) + |s - \underline{s}|,$$

the other possibilities are treated similarly. Since

$$s + T = \sum_{i=0}^{N-1} r_+(\sigma_+^i x) + s'$$

and since the roof function r_+ is bounded from below by $\inf \lambda$, we have $T \geq N \inf \lambda - \inf \lambda$, or equivalently $N \leq \frac{T}{\inf \lambda} + 1$. Hence $d(x, \underline{x}) < C_3 e^{-\frac{T}{\inf \lambda}} \leq e^{-N}$ by the definition of C_3 . In particular the sequences x and \underline{x} indexed by \mathbb{N} have the same $N + 1$ first coefficients. Since $r_+(z)$ depends only on z_0 for all $z \in \Sigma_+$, we thus have $r_+(\sigma_+^i x) = r_+(\sigma_+^i \underline{x})$ for $i = 0, \dots, N$. Note that we have

$$\underline{s} + T = \sum_{i=0}^{N-1} r_+(\sigma_+^i \underline{x}) + \underline{s}'.$$

If $\underline{N} = N$, then by taking the difference of the last two centred equations, we have $\underline{s} - s = \underline{s}' - s'$, and by construction, the sequences y and \underline{y} indexed by \mathbb{Z} satisfy $y_i = (\underline{y})_i$ if $i \leq 0$ and if $0 \leq i \leq -\ln d(x, \underline{x}) - N$. Therefore $d(y, \underline{y}) \leq e^N d(x, \underline{x})$ and

$$\begin{aligned} d_{\text{BW}}([y, s'], [\underline{y}, \underline{s}']) &\leq d(y, \underline{y}) + |s' - \underline{s}'| \leq e^N d(x, \underline{x}) + |\underline{s} - s| \\ &\leq e^N d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]) \leq e^{\frac{T}{\inf \lambda} + 1} d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]). \end{aligned}$$

Therefore Equation (9.9) follows from Equation (9.10) whenever $C_2 \geq e^\alpha$.

If on the contrary $\underline{N} > N$, then again by difference

$$\underline{s} - s = \sum_{i=N+1}^{\underline{N}-1} r_+(\sigma_+^i \underline{x}) + r_+(\sigma_+^N x) - s' + \underline{s}'.$$

Note that $\underline{s}' \geq 0$, that $r_+(\sigma_+^N x) - s' \geq 0$, and that $|\underline{s} - s| < C_3 e^{-\frac{T}{\inf \lambda}} \leq \inf \lambda$ by the definition of C_3 . Hence we have $\underline{N} = N + 1$ and $\underline{s} - s = r_+(\sigma_+^N x) - s' + \underline{s}'$. By construction, the sequences σy and \underline{y} indexed by \mathbb{Z} satisfy $(\sigma y)_i = (\underline{y})_i$ if $i \leq 0$ and if $0 \leq i \leq -\ln d(x, \underline{x}) - N - 1$. Hence by the definition of d_{BW} and since $r(y) = r_+(\sigma_+^N x)$, we have

$$\begin{aligned} d_{\text{BW}}([y, s'], [\underline{y}, \underline{s}']) &\leq d(\sigma y, \underline{y}) + r(y) - s' + \underline{s}' \leq e^{N+1} d(x, \underline{x}) + |\underline{s} - s| \\ &\leq e^{N+1} d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]) \leq e^{\frac{T}{\inf \lambda} + 2} d_{\text{BW}}([x, s], [\underline{x}, \underline{s}]). \end{aligned}$$

Therefore Equation (9.9) follows from Equation (9.10) whenever $C_2 \geq e^{2\alpha}$. This ends the proof of Equation (9.9).

Now note that Equations (9.8) and (9.9) imply Equation (9.7) by summation (using the independence of C_2 on ψ), thus concluding the proof of Lemma 9.8. \square

Proposition 9.9. *Let μ be a $(\sigma_r^t)_{t \in \mathbb{R}}$ -invariant probability measure on Σ_r . Assume that the dynamical system $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \in \mathbb{R}}, (\pi_+^r)_*\mu)$ has superpolynomial decay of α -Hölder correlations. Then $(\Sigma_r, (\sigma_r^t)_{t \in \mathbb{R}}, \mu)$ has superpolynomial decay of α -Hölder correlations.*

Proof. We fix $n \in \mathbb{N}$. Let $N = 1 + 2\lceil \frac{\sup \lambda}{\inf \lambda} \rceil$. Let $k \in \mathbb{N}$ and $C_4 > 0$ (depending on n) be such that for all $\psi, \psi' \in \mathcal{C}_b^{k, \alpha}((\Sigma_+)_{r_+})$, we have for all $t \geq 1$

$$|\text{cov}_{(\pi_+^r)_*\mu, t}(\psi, \psi')| \leq C_4 \|\psi\|_{k, \alpha} \|\psi'\|_{k, \alpha} t^{Nn}. \quad (9.11)$$

Now let $\bar{\psi}, \bar{\psi}' \in \mathcal{C}_b^{k, \alpha}(\Sigma_r)$. We again denote by $\pm t$ any value in $[-t, t]$ for any $t \geq 0$. By invariance of μ under $(\sigma_r^t)_{t \in \mathbb{R}}$, by Lemma 9.8 (2) and by Lemma 9.8 (1), we have, for any $T \geq 0$ (to be chosen appropriately later on),

$$\begin{aligned} & \int_{\Sigma_r} \bar{\psi} \circ \sigma_r^t \bar{\psi}' \, d\mu = \int_{\Sigma_r} \bar{\psi} \circ \sigma_r^{T+t} \bar{\psi}' \circ \sigma_r^T \, d\mu \\ &= \int_{\Sigma_r} (\bar{\psi}^{(T)} \circ \pi_+^r \pm C_1 \|\bar{\psi}\|'_\alpha e^{-\alpha' T}) \circ \sigma_r^t (\bar{\psi}'^{(T)} \circ \pi_+^r \pm C_1 \|\bar{\psi}'\|'_\alpha e^{-\alpha' T}) \, d\mu \\ &= \int_{(\Sigma_+)_{r_+}} \bar{\psi}^{(T)} \circ (\sigma_+)_{r_+}^t \bar{\psi}'^{(T)} \, d(\pi_+^r)_*\mu \pm C_1^2 \|\bar{\psi}\|_\alpha \|\bar{\psi}'\|_\alpha e^{-\alpha' T}. \end{aligned}$$

A similar estimate holds for the second term in the definition of the correlation coefficients. Hence, applying Equation (9.11) to the observables $\bar{\psi}^{(T)}$ and $\bar{\psi}'^{(T)}$, by Lemma 9.8 (3), we have

$$\begin{aligned} & |\text{cov}_{\mu, t}(\bar{\psi}, \bar{\psi}')| \leq |\text{cov}_{(\pi_+^r)_*\mu, t}(\bar{\psi}^{(T)}, \bar{\psi}'^{(T)})| + 2 C_1^2 \|\bar{\psi}\|_\alpha \|\bar{\psi}'\|_\alpha e^{-\alpha' T} \\ & \leq C_4 (\|\bar{\psi}\|_{k, \infty} + C_2 \|\bar{\psi}\|'_{k, \alpha} e^{\alpha'' T}) (\|\bar{\psi}'\|_{k, \infty} + C_2 \|\bar{\psi}'\|'_{k, \alpha} e^{\alpha'' T}) t^{-Nn} \\ & \quad + 2 C_1^2 \|\bar{\psi}\|_\alpha \|\bar{\psi}'\|_\alpha e^{-\alpha' T} \\ & \leq \|\bar{\psi}\|_{k, \alpha} \|\bar{\psi}'\|_{k, \alpha} (C_4 C_2^2 e^{2\alpha'' T} t^{-Nn} + 2 C_1^2 e^{-\alpha' T}). \end{aligned}$$

Take $T = \frac{n}{\alpha'} \ln t \geq 0$. Since $N = 1 + 2\lceil \frac{\alpha''}{\alpha'} \rceil$, we have $2\alpha'' \frac{n}{\alpha'} - Nn \leq -n$. Hence with $C_5 = C_4 C_2^2 + 2 C_1^2$, we have for all $t \geq 1$

$$|\text{cov}_{\mu, t}(\bar{\psi}, \bar{\psi}')| \leq C_5 \|\bar{\psi}\|_{k, \alpha} \|\bar{\psi}'\|_{k, \alpha} t^{-n}.$$

This concludes the proof of Proposition 9.9. □

Step 3 : Conclusion of the proof of Theorem 9.7

In this Step, we prove that the semiflow $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \geq 0}, \frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|})$ is rapid mixing, which concludes the proof of Theorem 9.7, using Proposition 9.9 with $\mu = \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$.

Recall (see the proof of Theorem 5.9) that $Y = \{\ell \in \Gamma \backslash \mathcal{G}X : \ell(0) \in V\mathbb{X}\}$ is a cross-section of the geodesic flow on $\Gamma \backslash \mathcal{G}X$, and that if $R : Y \rightarrow \Gamma \backslash \mathcal{G}\mathbb{X}$ is the reparametrisation map of $\ell \in Y$ to a discrete geodesic line $\sharp \ell \in \Gamma \backslash \mathcal{G}\mathbb{X}$ with the same origin, then the measure μ_Y , induced by the Gibbs measure m_c on the cross-section Y by disintegration along the flow, maps by R_* to a constant multiple of $m_{\sharp c}$ (see Lemma 5.10 (2)). Hence for all $n \in \mathbb{N}$, by

Assumption (b) (1) in the statement of Theorem 9.7, we have

$$\begin{aligned}
& m_{\sharp c} \left(\left\{ \sharp \ell \in \Gamma \backslash \mathcal{G} \mathbb{X} : \begin{array}{l} \sharp \ell(0) \in E \\ \forall k \in \{1, \dots, n-1\}, \sharp \ell(k) \notin E \end{array} \right\} \right) \\
& \leq \frac{\|m_{\sharp c}\|}{\|\mu_Y\|} \mu_Y \left(\left\{ R^{-1}(\sharp \ell) \in Y : \begin{array}{l} R^{-1}(\sharp \ell)(0) \in E \\ \forall t \in]0, n \inf \lambda[, R^{-1}(\sharp \ell)(t) \notin E \end{array} \right\} \right) \\
& \leq \frac{\|m_{\sharp c}\|}{\epsilon \|\mu_Y\|} m_c \left(\left\{ \mathbf{g}^s R^{-1}(\sharp \ell) \in \Gamma \backslash \mathcal{G} X : \begin{array}{l} 0 \leq s \leq \epsilon, d(\mathbf{g}^s R^{-1}(\sharp \ell)(0), E) \leq \epsilon \\ \forall t \in]\epsilon, n \inf \lambda - \epsilon[, \mathbf{g}^s R^{-1}(\sharp \ell)(t) \notin E \end{array} \right\} \right) \\
& \leq \frac{\|m_{\sharp c}\|}{\epsilon \|\mu_Y\|} C e^{-\kappa (\inf \lambda) n + \kappa \epsilon}.
\end{aligned}$$

Therefore Equation (9.1) (where c is replaced by $\sharp c$) is satisfied, with $C' = \frac{\|m_{\sharp c}\| C e^{\kappa \epsilon}}{\epsilon \|\mu_Y\|}$ and $\kappa' = \kappa \inf \lambda$. As seen in the proof of Theorem 9.1, this implies that there exists a finite subset \mathcal{E} of the alphabet \mathcal{A} such that Equation (9.3) is satisfied.

We now apply [Mel1, Theo. 2.3] with the dynamical system $(X, m_0, T) = (\Sigma_+, \mathbb{P}_+, \sigma_+)$ (using the system of conductances $\sharp c$) and the roof function $h = r_+$. This dynamical system is presented as a Young tower in Step 3 of the proof of Theorem 9.1. Equation (9.3) for the first return map $\tau_{\mathcal{E}}$ and the 4-Diophantine hypothesis are exactly the hypothesis needed in order to apply [Mel1, Theo. 2.3]. Thus the semiflow $((\Sigma_+)_{r_+}, ((\sigma_+)_{r_+}^t)_{t \geq 0}, \frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|})$ has superpolynomial decay of α -Hölder correlations.

When $\Gamma \backslash X$ is compact, the alphabet \mathcal{A} is finite and $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ is a (one-sided) subshift of finite type, hence we do not need the exponentially small tail assumption, and only the 2-Diophantine hypothesis, and we may apply [Dol2]. \square

Corollary 9.10. *Assume that the critical exponent δ_F is finite, that the Gibbs measure m_F is finite and mixing, that the lengths of the edges of (\mathbb{X}, λ) have a finite upper bound, and that $\Gamma \backslash X$ is geometrically finite. There exists a full measure subset A of \mathbb{R}^4 (for the Lebesgue measure) such that if Γ has a quadruple of translation lengths in A , then the (continuous time) geodesic flow on $\Gamma \backslash \mathcal{G} X$ has superpolynomial decay of α -Hölder correlations for the Bowen-Margulis measure m_{BM} .*

Proof. The exponentially small tail Assumption (b) (1) is checked as in the proof of Corollary 9.6. The deduction of Corollary 9.10 from Theorem 9.7 then proceeds, by an argument going back in part to Dolgopyat, as for the deduction of Corollary 2.4 from Theorem 2.3 in [Mel1]. \square

Note that under the general assumptions of Theorem 4.8, the geodesic flow on $\Gamma \backslash \mathcal{G} X$ might not be exponentially mixing, see for instance [Pol1, page 162] or [Rue2] for analogous behaviour.

Part II

Geometric equidistribution and counting

Chapter 10

Equidistribution of equidistant level sets to Gibbs measures

Let X be a geodesically complete proper $\text{CAT}(-1)$ space, let Γ be a nonelementary discrete group of isometries of X , let \tilde{F} be a continuous Γ -invariant map on T^1X such that $\delta = \delta_{\Gamma, F^\pm}$ is finite and positive and that the triple (X, Γ, \tilde{F}) satisfies the **HC**-property,¹ and let $(\mu_x^\pm)_{x \in X}$ be Patterson densities for the pairs (Γ, F^\pm) .

In this Chapter, we prove that the skinning measure on (any nontrivial piece of) the outer unit normal bundle of any properly immersed nonempty proper closed convex subset of X , pushed a long time by the geodesic flow, equidistributes towards the Gibbs measure, under finiteness and mixing assumptions. This result gives four important extensions of [PaP14a, Thm. 1], one for general $\text{CAT}(-1)$ spaces with constant potentials, one for Riemannian manifolds with pinched negative curvature and Hölder potentials, one for \mathbb{R} -trees with general potentials, and one for simplicial trees.

10.1 A general equidistribution result

Before stating this equidistribution result, we start by a technical construction which will also be useful in the following Chapter 11. We refer to Section 2.5 for the notation concerning the dynamical neighbourhoods (including $V_{w, \eta', R}^\mp$) and to Chapter 7 for the notation concerning the skinning measures (including ν_w^\pm).

Technical construction of bump functions. Let D^\pm be nonempty proper closed convex subsets of X , and let $R > 0$ be such that $\nu_w^\pm(V_{w, \eta', R}^\mp) > 0$ for all $\eta' > 0$ and $w \in \partial_\mp^1 D^\pm$. Let $\eta > 0$ and let Ω^\pm be measurable subsets of $\partial_\mp^1 D^\pm$. We now construct functions $\phi_{\eta, R, \Omega^\pm}^\mp : \mathcal{G}X \rightarrow [0, +\infty[$ whose supports are contained in dynamical neighbourhoods of Ω^\pm . If $X = \tilde{M}$ is a manifold and $\tilde{F} = 0$, we recover the same bump functions after the standard identifications.

For all $\eta' > 0$, let $h_{\eta, \eta'}^\pm : \mathcal{G}_\mp X \rightarrow [0, +\infty[$ be the Γ -invariant measurable maps defined by

$$h_{\eta, \eta'}^\pm(w) = \frac{1}{\nu_w^\pm(V_{w, \eta, \eta'}^\mp)} \quad (10.1)$$

if $\nu_w^\pm(V_{w, \eta, \eta'}^\mp) > 0$ (which is for instance satisfied if $w_\pm \in \Lambda\Gamma$) and $h_{\eta, \eta'}^\pm(w) = 0$ otherwise.

¹See Definition 3.4.

These functions $h_{\eta, \eta'}^\pm$ have the following behaviour under precomposition by the geodesic flow. By Lemma 7.5 (2), by Equation (2.13), and by the invariance of ν_w^\pm under the geodesic flow, we have, for all $t \in \mathbb{R}$ and $w \in \mathcal{G}_\pm X$,

$$h_{\eta, \eta'}^\mp(\mathbf{g}^{\mp t} w) = e^{C_{\tilde{w}^\pm}^\pm(w(0), w(\mp t))} h_{\eta, e^{-t}\eta'}^\mp(w). \quad (10.2)$$

Let us also describe the behaviour of $h_{\eta, \eta'}^\pm$ when η' is small. Let $w \in \mathcal{G}_\pm X$ be such that w is isometric at least on $\pm[0, +\infty[$, which is for instance the case if $w \in \partial_\pm^1 D^\mp$. For all $\eta' > 0$ and $\ell \in B^\pm(w, \eta')$, let \hat{w} be an extension of w such that $d_{W^\pm(w)}(\ell, \hat{w}) < \eta'$. Then $\hat{w}(0) = w(0)$ by the assumption on w , and using [PauPS, Lem. 2.4],² we have

$$d(\ell(0), w(0)) = d(\ell(0), \hat{w}(0)) \leq d_{W^\pm(w)}(\ell, \hat{w}) < \eta'.$$

Hence, with κ_1 and κ_2 the constants in the Definition 3.4, if

$$c_1 = \kappa_1 + \delta + \sup_{\pi^{-1}(B(w(0), \eta'))} |\tilde{F}|,$$

we have, by Proposition 3.10 (2),

$$|C_{w^\mp}^\mp(w(0), \ell(0))| \leq c_1 (\eta')^{\kappa_2}.$$

Using the defining Equation (7.12) of ν_w^\mp , for all $s \in \mathbb{R}$, $\eta' > 0$ and $\ell \in B^\pm(w, \eta')$, we have

$$e^{-c_1(\eta')^{\kappa_2}} ds d\mu_{W^\pm(w)}(\ell) \leq d\nu_w^\mp(g^s \ell) \leq e^{c_1(\eta')^{\kappa_2}} ds d\mu_{W^\pm(w)}(\ell).$$

It follows that for all $\eta' \in]0, 1]$ and $w \in \partial_\pm^1 D^\mp$ such that $w_\pm \in \Lambda\Gamma$, we have the following control of $h_{\eta, \eta'}^\mp(w)$:

$$\frac{e^{-c_1(\eta')^{\kappa_2}}}{2\eta \mu_{W^\pm(w)}(B^\pm(w, \eta'))} \leq h_{\eta, \eta'}^\mp(w) \leq \frac{e^{c_1(\eta')^{\kappa_2}}}{2\eta \mu_{W^\pm(w)}(B^\pm(w, \eta'))}. \quad (10.3)$$

Note that when X is an \mathbb{R} -tree, we may take $\kappa_2 = 1$ and $c_1 = \|\tilde{F} - \delta\|_\infty$ in this equation, as we saw in the proof of Proposition 3.5.

Recall that $\mathbb{1}_A$ denotes the characteristic function of a subset A . We now define the test functions $\phi_{\eta, R, \Omega^\pm}^\mp : \mathcal{G}X \rightarrow [0, +\infty[$ by

$$\phi_{\eta, R, \Omega^\mp}^\mp = h_{\eta, R}^\mp \circ f_{D^\mp}^\pm \mathbb{1}_{\mathcal{V}_{\eta, R}^\pm(\Omega^\mp)}, \quad (10.4)$$

where $\mathcal{V}_{\eta, R}^\pm(\Omega^\mp)$ and $f_{D^\mp}^\pm$ are as in Section 2.5. Note that if $\ell \in \mathcal{V}_{\eta, R}^\pm(\Omega^\mp)$, then $\ell_\pm \notin \partial_\infty D^\mp$ by convexity. Thus, ℓ belongs to the domain of definition $\mathcal{U}_{D^\mp}^\pm$ of $f_{D^\mp}^\pm$; hence $\phi_{\eta, R, \Omega^\mp}^\mp(\ell) = h_{\eta, R}^\mp \circ f_{D^\mp}^\pm(\ell)$ is well defined. By convention, $\phi_{\eta, R, \Omega^\mp}^\mp(\ell) = 0$ if $\ell \notin \mathcal{V}_{\eta, R}^\pm(\Omega^\mp)$.

The following property of the bump functions is proved as in [PaP14a, Prop. 18], using the disintegration result of Proposition 7.6.

Lemma 10.1. *For every $\eta > 0$, the functions $\phi_{\eta, R, \Omega^\pm}^\mp$ are measurable, nonnegative and satisfy*

$$\int_{\mathcal{G}X} \phi_{\eta, R, \Omega^\pm}^\mp d\tilde{m}_F = \tilde{\sigma}^\pm(\Omega^\mp). \quad \square$$

²Although it is stated for Riemannian manifolds, the argument is valid in general CAT(-1) spaces.

We now state and prove the aforementioned equidistribution result. Note that as elements of the outer unit normal bundles are now only geodesic rays on $[0, +\infty[$, their pushforwards by the geodesic flow at time t are geodesic rays on $[-t, +\infty[$ and the convergence towards geodesic lines (defined on $] - \infty, +\infty[$) does take place in the full space of generalised geodesic lines $\check{\mathcal{G}}X$. This explains why it is important not to forget to consider the negative times – skinning measures, supported on geodesic rays, pushed by the geodesic flow, have a chance to weak-star converge to Gibbs measures, supported on geodesic lines, up to renormalisation.

The proof of the following result has similarities with that of [PaP14a, Theo. 1], but the computations do not apply in the present context because the proof in loc. cit. does not keep track of the past: here we can no longer reduce our study to the outer unit normal bundle of the t -neighbourhood of the elements of \mathcal{D} .

Theorem 10.2. *Let (X, Γ, \tilde{F}) be as in the beginning of Chapter 10. Assume that the Gibbs measure m_F on $\Gamma \backslash \mathcal{G}X$ is finite and mixing for the geodesic flow. Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper closed convex subsets of X with $\Gamma \backslash I$ finite. Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}X$, with $\Omega_i \subset \partial_{\pm}^1 D_i$ for all $i \in I$. Assume that σ_{Ω}^{\pm} is finite and nonzero. Then, as $t \rightarrow +\infty$, for the weak-star convergence of measures on $\Gamma \backslash \check{\mathcal{G}}X$,*

$$\frac{1}{\|(\mathbf{g}^{\pm t})_* \sigma_{\Omega}^{\pm}\|} (\mathbf{g}^{\pm t})_* \sigma_{\Omega}^{\pm} \xrightarrow{*} \frac{1}{\|m_F\|} m_F .$$

Proof. We only give the proof when $\pm = +$, the other case is treated similarly. Given three numbers a, b, c (depending on some parameters), we write $a = b \pm c$ if $|a - b| \leq c$.

Let $\eta \in]0, 1]$. As in the proof of [PaP14a, Thm. 19], we may assume that $\Gamma \backslash I$ is finite. Hence, using Lemma 7.5 (i), we may fix $R > 0$ such that $\nu_w^-(V_{w, \eta, R}^+) > 0$ for all $w \in \partial_{\pm}^1 D_i$ and $i \in I$.

Using the notation introduced in the above construction of the bump functions $\phi_{\eta, R, \Omega_i}^-$, we may hence consider the global test functions $\tilde{\Phi}_{\eta} : \mathcal{G}X \rightarrow [0, +\infty[$,

$$\tilde{\Phi}_{\eta}(v) = \sum_{i \in I / \sim} \phi_{\eta, R, \Omega_i}^- = \sum_{i \in I / \sim} h_{\eta, R}^- \circ f_{D_i}^+ \mathbb{1}_{\mathcal{V}_{\eta, R}^+(\Omega_i)} .$$

As in loc. cit., the function $\tilde{\Phi}_{\eta} : \mathcal{G}X \rightarrow [0, +\infty[$ is well defined (independent of the representatives of i), measurable and Γ -equivariant. Hence it defines, by passing to the quotient, a measurable function $\Phi_{\eta} : \Gamma \backslash \mathcal{G}X \rightarrow [0, +\infty[$. By Lemma 10.1, the function Φ_{η} is integrable and satisfies

$$\int_{\Gamma \backslash \mathcal{G}X} \Phi_{\eta} dm_F = \|\sigma_{\Omega}^+\| . \tag{10.5}$$

Fix $\psi \in \mathcal{C}_c(\Gamma \backslash \check{\mathcal{G}}X)$. Let us prove that

$$\lim_{t \rightarrow +\infty} \frac{1}{\|(\mathbf{g}^t)_* \sigma_{\Omega}^+\|} \int_{\Gamma \backslash \mathcal{G}X} \psi d(\mathbf{g}^t)_* \sigma_{\Omega}^+ = \frac{1}{\|m_F\|} \int_{\Gamma \backslash \mathcal{G}X} \psi dm_F .$$

Consider a fundamental domain Δ_{Γ} for the action of Γ on $\check{\mathcal{G}}X$, such that the boundary of Δ_{Γ} has zero measure, the interiors of its translates are disjoint and any compact subset of $\check{\mathcal{G}}X$ meets only finitely many translates of Δ_{Γ} : see [Rob2, p. 13] (or the proof of [PaP14a,

Prop. 18]), using the fact that \tilde{m}_F has no atoms, since m_F is finite and according to Corollary 4.6(1) and to Theorem 4.5.

By a standard argument of finite partition of unity and up to modifying Δ_Γ , we may assume that there exists a function $\tilde{\psi} : \check{\mathcal{G}}X \rightarrow \mathbb{R}$ whose support has a small neighbourhood contained in Δ_Γ such that $\tilde{\psi} = \psi \circ p$, where $p : \check{\mathcal{G}}X \rightarrow \Gamma \backslash \check{\mathcal{G}}X$ is the canonical projection (which is Lipschitz). Fix $\epsilon > 0$. Since $\tilde{\psi}$ is uniformly continuous, for every $\eta > 0$ small enough and for every $t \geq 0$ large enough, for all $w \in \mathcal{G}_+X$ and $\ell \in V_{w,\eta,e^{-t}R}^+$, we have

$$\tilde{\psi}(\ell) = \tilde{\psi}(w) \pm \frac{\epsilon}{2}. \quad (10.6)$$

If t is big enough and η small enough, we have, using respectively

- the definition of the global test function $\tilde{\Phi}_\eta$, since the support of $\tilde{\psi}$ is contained in Δ_Γ and the support of ϕ_{η,R,Ω_i}^- is contained in $\mathcal{Z}_{D_i}^+$, for the second equality,
- the disintegration property of $f_{D_i}^+$ in Proposition 7.6 for the third equality,
- the fact that if ℓ is in the support of ν_ρ^- , then $f_{D_i}^+(\mathbf{g}^{-t}\ell) = f_{D_i}^+(\ell) = \rho$ and the change of variables by the geodesic flow $w = g^t\rho$ for the fourth equality,
- the fact that the support of $\nu_{\mathbf{g}^{-t}w}^-$ is contained in $W^{0+}(\mathbf{g}^{-t}w)$, and that

$$W^{0+}(\mathbf{g}^{-t}w) \cap g^t V_{\eta,R}^+(\Omega_i) = g^t(W^{0+}(\mathbf{g}^{-t}w) \cap V_{\eta,R}^+(\Omega_i)) = g^t V_{\mathbf{g}^{-t}w,\eta,R}^+ = V_{w,\eta,e^{-t}R}^+$$

for the fifth equality,

- Equation (10.6) for the sixth equality, and
- the definition of h^- , the invariance of the measure $\nu_{\mathbf{g}^{-t}w}^-$ and the Gibbs measure \tilde{m}_F under the geodesic flow, and the definition of the measure σ_Ω^+ for the last two equalities.

$$\begin{aligned} & \int_{\Gamma \backslash \mathcal{G}X} \psi \Phi_\eta \circ \mathbf{g}^{-t} dm_F = \int_{\Delta_\Gamma \cap \mathcal{G}X} \tilde{\psi} \tilde{\Phi}_\eta \circ \mathbf{g}^{-t} d\tilde{m}_F \\ &= \sum_{i \in I/\sim} \int_{\ell \in \mathcal{Z}_{D_i}^+} \tilde{\psi}(\ell) \phi_{\eta,R,\Omega_i}^-(\mathbf{g}^{-t}\ell) d\tilde{m}_F(\ell) \\ &= \sum_{i \in I/\sim} \int_{\rho \in \partial_+^1 D_i} \int_{\ell \in \mathcal{Z}_{D_i}^+} \tilde{\psi}(\ell) h_{\eta,R}^-(f_{D_i}^+(\mathbf{g}^{-t}\ell)) \mathbb{1}_{\mathcal{V}_{\eta,R}^+(\Omega_i)}(\mathbf{g}^{-t}\ell) d\nu_\rho^-(\ell) d\tilde{\sigma}_{D_i}^+(\rho) \\ &= \sum_{i \in I/\sim} \int_{w \in \mathbf{g}^t \partial_+^1 D_i} \int_{\ell \in \mathbf{g}^t \mathcal{V}_{\eta,R}^+(\Omega_i)} \tilde{\psi}(\ell) h_{\eta,R}^-(\mathbf{g}^{-t}w) d\nu_{\mathbf{g}^{-t}w}^-(\ell) d(\mathbf{g}^t)_* \tilde{\sigma}_{D_i}^+(w) \\ &= \sum_{i \in I/\sim} \int_{w \in \mathbf{g}^t \partial_+^1 D_i} \int_{\ell \in V_{w,\eta,e^{-t}R}^+} \tilde{\psi}(\ell) h_{\eta,R}^-(\mathbf{g}^{-t}w) d\nu_{\mathbf{g}^{-t}w}^-(\ell) d(\mathbf{g}^t)_* \tilde{\sigma}_{D_i}^+(w) \\ &= \sum_{i \in I/\sim} \int_{w \in \mathbf{g}^t \partial_+^1 D_i} \tilde{\psi}(w) h_{\eta,R}^-(\mathbf{g}^{-t}w) \nu_{\mathbf{g}^{-t}w}^-(\mathbf{g}^t V_{\mathbf{g}^{-t}w,\eta,R}^+) d(\mathbf{g}^t)_* \tilde{\sigma}_{D_i}^+(w) \\ &\quad \pm \frac{\epsilon}{2} \int_{\Delta_\Gamma \cap \mathcal{G}X} \tilde{\Phi}_\eta \circ \mathbf{g}^{-t} d\tilde{m}_F \\ &= \sum_{i \in I/\sim} \int_{\check{\mathcal{G}}X} \tilde{\psi} d(\mathbf{g}^t)_* \tilde{\sigma}_{D_i}^+ \pm \frac{\epsilon}{2} \int_{\Gamma \backslash \mathcal{G}X} \Phi_\eta \circ \mathbf{g}^{-t} dm_F \\ &= \int_{\Gamma \backslash \check{\mathcal{G}}X} \psi d(\mathbf{g}^t)_* \sigma_\Omega^+ \pm \frac{\epsilon}{2} \int_{\Gamma \backslash \mathcal{G}X} \Phi_\eta dm_F \end{aligned} \quad (10.7)$$

We then conclude as in the end of the proof of [PaP14a, Thm. 19]. By Equation (10.5), we have $\|(\mathbf{g}^t)_*\sigma_\Omega^+\| = \|\sigma_\Omega^+\| = \int_{\Gamma \backslash \mathcal{G}X} \Phi_\eta \, dm_F$. By the mixing property of the geodesic flow on $\Gamma \backslash \mathcal{G}X$ for the Gibbs measure m_F , for $t \geq 0$ large enough (while η is small but fixed), we hence have

$$\frac{\int_{\Gamma \backslash \check{\mathcal{G}}X} \psi \, d(\mathbf{g}^t)_*\sigma_\Omega^+}{\|(\mathbf{g}^t)_*\sigma_\Omega^+\|} = \frac{\int_{\Gamma \backslash \mathcal{G}X} \Phi_\eta \circ \mathbf{g}^{-t} \psi \, dm_F}{\int_{\Gamma \backslash \mathcal{G}X} \Phi_\eta \, dm_F} \pm \frac{\epsilon}{2} = \frac{\int_{\Gamma \backslash \mathcal{G}X} \psi \, dm_F}{\|m_F\|} \pm \epsilon.$$

This proves the result. \square

Recall that by Proposition 3.5, Theorem 10.2 applies to Riemannian manifolds with pinched negative curvature and for \mathbb{R} -trees for which the geodesic flow is mixing and which satisfy the finiteness requirements of the Theorem.

Since pushforwards of measures are weak-star continuous and preserve total mass, we have, under the assumptions of Theorem 10.2, the following equidistribution result in X of the immersed t -neighbourhood of a properly immersed nonempty proper closed convex subset of X : as $t \rightarrow +\infty$,

$$\frac{1}{\|\sigma_\Omega^+\|} \pi_*(\mathbf{g}^t)_*\sigma_\Omega^+ \xrightarrow{*} \frac{1}{\|m_F\|} \pi_*m_F. \quad (10.8)$$

10.2 Rate of equidistribution of equidistant level sets for manifolds

If $X = \widetilde{M}$ is a simply connected Riemannian manifold of pinched negative curvature and if the geodesic flow of $\Gamma \backslash \widetilde{M}$ is mixing with exponentially decaying correlations, we get a version of Theorem 10.2 with error bounds. See Section 9.1 for conditions on Γ and \widetilde{F} that imply exponential mixing.

Theorem 10.3. *Let \widetilde{M} be a complete simply connected Riemannian manifold with negative sectional curvature. Let Γ be a nonelementary discrete group of isometries of \widetilde{M} . Let $\widetilde{F} : T^1\widetilde{M} \rightarrow \mathbb{R}$ be a bounded Γ -invariant Hölder-continuous function with positive critical exponent $\delta = \delta_{\Gamma, F}$. Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper closed convex subsets of \widetilde{M} , with finite nonzero skinning measure $\sigma_{\mathcal{D}}$. Let $M = \Gamma \backslash \widetilde{M}$ and let $F : T^1M \rightarrow \mathbb{R}$ be the potential induced by \widetilde{F} .*

(i) *If M is compact and if the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F , then there exist $\alpha \in]0, 1]$ and $\kappa'' > 0$ such that for all $\psi \in \mathcal{C}_c^\alpha(T^1M)$, we have, as $t \rightarrow +\infty$,*

$$\frac{1}{\|\sigma_{\mathcal{D}}\|} \int \psi \, d(g^t)_*\sigma_{\mathcal{D}} = \frac{1}{\|m_F\|} \int \psi \, dm_F + O(e^{-\kappa''t} \|\psi\|_\alpha).$$

(ii) *If \widetilde{M} is a symmetric space, if D_i has smooth boundary for every $i \in I$, if m_F is finite and smooth, and if the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F , then there exists $\ell \in \mathbb{N}$ and $\kappa'' > 0$ such that for all $\psi \in \mathcal{C}_c^\ell(T^1M)$, we have, as $t \rightarrow +\infty$,*

$$\frac{1}{\|\sigma_{\mathcal{D}}\|} \int \psi \, d(g^t)_*\sigma_{\mathcal{D}} = \frac{1}{\|m_F\|} \int \psi \, dm_F + O(e^{-\kappa''t} \|\psi\|_\ell).$$

Note that if \widetilde{M} is a symmetric space and M has finite volume, then M is geometrically finite. Theorem 4.7 implies that m_F is finite if F is small enough. The maps $O(\cdot)$ depend on $\widetilde{M}, \Gamma, F, \mathcal{D}$, and the speeds of mixing.

Proof. Up to rescaling, we may assume that the sectional curvature is bounded from above by -1 . The critical exponent δ and the Gibbs measure m_F are finite in all the cases we consider here.

The deduction of this result from the proof of Theorem 10.2 by regularisations of the global test function Φ_η introduced in the proof of Theorem 10.2 is analogous to the deduction of [PaP14a, Theo. 20] from [PaP14a, Theo. 19] when $F = 0$. The doubling property of the Patterson densities and the Gibbs measure for general F , required by this deduction in the Hölder regularity case, is given by [PauPS, Prop. 3.12]. For the assertion (ii), the required smoothness of m_F (that is, the fact that m_F is absolutely continuous with respect to the Lebesgue measure with smooth Radon-Nikodym derivative) allows to use the convolution approximation. \square

10.3 Equidistribution of equidistant level sets on simplicial graphs and random walks on graphs of groups

Let $\mathbb{X}, X, \Gamma, \tilde{c}, \widetilde{F}_c, \delta_c$ be as in the beginning of Section 9.2. Let $\tilde{F}_c : T^1X \rightarrow \mathbb{R}$ be its associated potential, and let $\delta = \delta_c$ be the critical exponent of c . Let $(\mu_x^\pm)_{x \in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty X$ for the pairs (Γ, F_c^\pm) , and let $m_c = m_{F_c}$ be the associated Gibbs measure on $\Gamma \backslash \mathcal{G}\mathbb{X}$.

In this Section, we state an equidistribution result analogous to Theorem 10.2, which now holds in the space of generalised discrete geodesic lines $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$, but whose proof is completely analogous.

Theorem 10.4. *Let $\mathbb{X}, \Gamma, \tilde{c}, (\mu_x^\pm)_{x \in V\mathbb{X}}$ be as above, with δ_c finite. Assume that the Gibbs measure m_c on $\Gamma \backslash \mathcal{G}\mathbb{X}$ is finite and mixing for the discrete time geodesic flow. Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper simplicial subtrees of \mathbb{X} . Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}\mathbb{X}$, with $\Omega_i \subset \partial_+^1 D_i$ for all $i \in I$. Assume that σ_Ω^+ is finite and nonzero. Then, as $n \rightarrow +\infty$, for the weak-star convergence of measures on $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$,*

$$\frac{1}{\|(\mathbf{g}^n)_* \sigma_\Omega^+\|} \pi_* (\mathbf{g}^n)_* \sigma_\Omega^+ \xrightarrow{*} \frac{1}{\|m_F\|} m_F. \quad \square$$

We leave to the reader the analog of this result when the restriction to $\Gamma \backslash \mathcal{G}_{\text{even}}\mathbb{X}$ of the Gibbs measure is finite and mixing for the square of the discrete time geodesic flow.

Using Propositions 4.14 and 4.15 in order to check that the Bowen-Margulis measure m_{BM} on $\Gamma \backslash \mathcal{G}\mathbb{X}$ is finite and mixing, we have the following consequence of Theorem 10.4, using the system of conductances $\tilde{c} = 0$.

Corollary 10.5. *Let \mathbb{X} be a uniform simplicial tree. Let Γ be a lattice of \mathbb{X} such that the graph $\Gamma \backslash \mathbb{X}$ is not bipartite. Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper simplicial subtrees of \mathbb{X} . Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}\mathbb{X}$, with $\Omega_i \subset \partial_+^1 D_i$ for all $i \in I$. Assume that the skinning measure*

σ_Ω^+ (with vanishing potential) is finite and nonzero. Then, as $n \rightarrow +\infty$, for the weak-star convergence of measures on $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$,

$$\frac{1}{\|(\mathbf{g}^n)_* \sigma_\Omega^+\|} \pi_*(\mathbf{g}^n)_* \sigma_\Omega^+ \xrightarrow{*} \frac{1}{\|m_{\text{BM}}\|} m_{\text{BM}}. \quad \square$$

When furthermore \mathbb{X} is regular, we have the following corollary, using Proposition 8.1 (3).

Corollary 10.6. *Let \mathbb{X} be a regular simplicial tree of degree at least 3. Let Γ be a lattice of \mathbb{X} such that the graph $\Gamma \backslash \mathbb{X}$ is not bipartite. Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper simplicial subtrees of \mathbb{X} . Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}\mathbb{X}$, with $\Omega_i \subset \partial_+^1 D_i$ for all $i \in I$. Assume that the skinning measure σ_Ω^+ (with vanishing potential) is finite and nonzero. Then, as $n \rightarrow +\infty$, for the weak-star convergence of measures on $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$,*

$$\frac{1}{\|(\mathbf{g}^n)_* \sigma_\Omega^+\|} \pi_*(\mathbf{g}^n)_* \sigma_\Omega^+ \xrightarrow{*} \frac{1}{\text{Vol}(\Gamma \backslash \mathbb{X})} \text{vol}_{\Gamma \backslash \mathbb{X}}. \quad \square$$

Let us give an application of Corollary 10.6 in terms of random walks on graphs of groups, which might also be deduced from general result on random walks, as indicated by M. Burger and S. Mozes.

Let (\mathbb{Y}, G_*) be a connected graph of finite groups with finite volume, and let (\mathbb{Y}', G'_*) be a connected subgraph of subgroups.³ Note that (\mathbb{Y}', G'_*) also has finite volume, less than or equal to the volume of (\mathbb{Y}, G_*) . We say that (\mathbb{Y}, G_*) is *homogeneous* if $\sum_{e \in E\mathbb{Y}, o(e)=x} \frac{|G_x|}{|G_e|}$ is constant at least 3 for all $x \in V\mathbb{Y}$. We say that a connected graph of groups is *2-acylindrical* if the action of its fundamental group on its Bass-Serre tree is 2-acylindrical (see Remark 5.4). In particular, this action is faithful if the graph has at least two edges.

The *non-backtracking simple random walk on (\mathbb{Y}, G_*) starting transversally to (\mathbb{Y}', G'_*)* is the following Markovian random process $(X_n = (f_n, \gamma_n))_{n \in \mathbb{N}}$ where $f_n \in E\mathbb{Y}$ and $\gamma_n \in G_{o(f_n)}$ for all $n \in \mathbb{N}$. Choose at random a vertex y_0 of \mathbb{Y}' for the probability measure $\frac{1}{\text{Vol}(\mathbb{Y}', G'_*)} \text{vol}_{\mathbb{Y}', G'_*}$ (we will call y_0 the *origin* of the random path). Then choose uniformly at random $X_0 = (f_0, \gamma_0)$ where $f_0 \in E\mathbb{Y}$ is such that $o(f_0) = y_0$ and γ_0 is a double coset in $G'_{y_0} \backslash G_{y_0} / \rho_{\overline{f_0}}(G_{f_0})$ such that if $f_0 \in E\mathbb{Y}'$ then $\gamma_0 \notin G'_{y_0} \rho_{\overline{f_0}}(G_{f_0})$.⁴ Assuming $X_n = (f_n, \gamma_n)$ constructed, choose uniformly at random $X_{n+1} = (f_{n+1}, \gamma_{n+1})$ where $f_{n+1} \in E\mathbb{Y}$ is such that $o(f_{n+1}) = t(f_n)$ and $\gamma_{n+1} \in G_{o(f_{n+1})} / \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$ is such that if $f_{n+1} = \overline{f_n}$ then $\gamma_{n+1} \notin \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$. The n -th vertex of $(X_n = (f_n, \gamma_n))_{n \in \mathbb{N}}$ is $o(f_n)$.

Corollary 10.7. *Let (\mathbb{Y}, G_*) be a homogeneous 2-acylindrical nonbipartite connected graph of finite groups with finite volume, and let (\mathbb{Y}', G'_*) be a homogeneous nonempty proper connected subgraph of subgroups. Then the n -th vertex of the non-backtracking simple random walk on (\mathbb{Y}, G_*) starting transversally to (\mathbb{Y}', G'_*) converges in distribution to $\frac{1}{\text{Vol}(\mathbb{Y}, G_*)} \text{vol}_{\mathbb{Y}, G_*}$ as $n \rightarrow +\infty$.*

Proof. Let Γ be the fundamental group of (\mathbb{Y}, G_*) (with respect to a choice of basepoint in $V\mathbb{Y}'$), which is a lattice of the Bass-Serre tree \mathbb{X} of (\mathbb{Y}, G_*) , since Γ acts faithfully on \mathbb{X}

³See Section 2.7 for definitions and background.

⁴This last condition says that γ_0 is not the double coset of the trivial element.

and (\mathbb{Y}, G_*) has finite volume. Note that \mathbb{X} is regular since (\mathbb{Y}, G_*) is homogeneous. Let $p : \mathbb{X} \rightarrow \mathbb{Y} = \Gamma \backslash \mathbb{X}$ be the canonical projection.

Let Γ' be the fundamental group of (\mathbb{Y}', G'_*) (with respect to the same choice of basepoint). By [Bass, 2.15], there exists a simplicial subtree \mathbb{X}' whose stabiliser in Γ is Γ' , such that the quotient graph of groups $\Gamma' \backslash \mathbb{X}'$ identifies with (\mathbb{Y}', G'_*) . Similarly, \mathbb{X}' is regular since (\mathbb{Y}', G'_*) is homogeneous. Let $\mathcal{D} = (\gamma \mathbb{X}')_{\gamma \in \Gamma/\Gamma'}$, which is a locally finite family of nonempty proper simplicial subtrees of \mathbb{X} .

Using the notation of Example 2.10 for the graph of groups $\Gamma \backslash \mathbb{X}$ (which identifies with (\mathbb{Y}, G_*)), we fix lifts \tilde{f} and \tilde{y} in \mathbb{X} by p of every edge f and vertex y of \mathbb{Y} such that $\tilde{f} = \tilde{f}$, and elements $g_f \in \Gamma$ such that $g_f t(\tilde{f}) = t(\tilde{f})$. We may assume that $\tilde{f} \in E\mathbb{X}'$ if $f \in E\mathbb{Y}'$, that $\tilde{y} \in V\mathbb{X}'$ if $y \in V\mathbb{Y}'$, and that $g_f \in \Gamma'$ if $f \in E\mathbb{Y}'$.

Let (Ω, \mathbb{P}) be the (canonically constructed) probability space of the random walk $(X_n = (f_n, \gamma_n))_{n \in \mathbb{N}}$. For all $n \in \mathbb{N}$, let $y_n = o(f_n)$ be the random variable (with values in the discrete space $\mathbb{Y} = \Gamma \backslash \mathbb{X}$) of the n -th vertex of the random walk $(X_n)_{n \in \mathbb{N}}$.

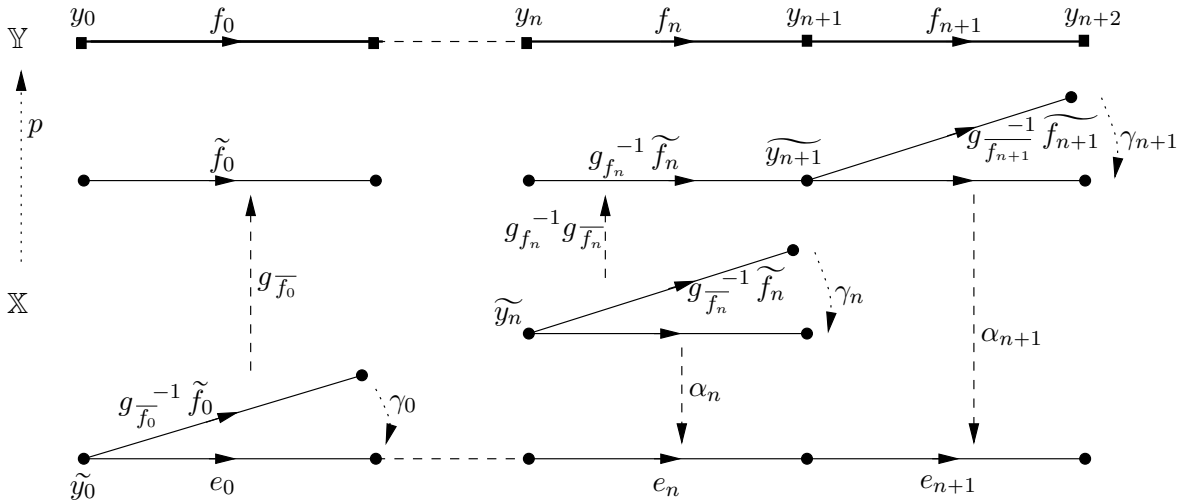
Let us define a measurable map $\Theta : \Omega \rightarrow \Gamma \backslash \check{\mathcal{G}}\mathbb{X}$, with image contained in the image of $\partial_+^1 \mathbb{X}'$ by the canonical projection $\check{\mathcal{G}}\mathbb{X} \rightarrow \Gamma \backslash \check{\mathcal{G}}\mathbb{X}$, such that $\Theta_* \mathbb{P}$ is the normalised skinning measure $\frac{\sigma_{\mathcal{D}}^+}{\|\sigma_{\mathcal{D}}^+\|}$ and that the following diagram commutes for all $n \in \mathbb{N}$:

$$\begin{array}{ccc} \Omega & \xrightarrow{\Theta} & \Gamma \backslash \check{\mathcal{G}}\mathbb{X} \\ y_n \searrow & & \swarrow \pi \circ g^n \\ \mathbb{Y} = \Gamma \backslash \mathbb{X} & & \end{array} . \quad (10.9)$$

Assuming that we have such a map, we have

$$(y_n)_* \mathbb{P} = (\pi_* \circ (g^n)_* \circ \Theta_*) \mathbb{P} = \pi_*(g^n)_* \frac{\sigma_{\mathcal{D}}^+}{\|\sigma_{\mathcal{D}}^+\|} = \frac{1}{\|(g^n)_* \sigma_{\mathcal{D}}^+\|} \pi_*(g^n)_* \sigma_{\mathcal{D}}^+$$

so that the convergence of the law of y_n to $\frac{1}{\text{Vol}(\mathbb{Y}, G_*)} \text{vol}_{\mathbb{Y}, G_*}$ follows from Corollary 10.6 applied to $\partial_+^1 \mathcal{D}$.



Let $(X_n = (f_n, \gamma_n))_{n \in \mathbb{N}}$ be a random path with origin $y_0 \in \mathbb{Y}'$, corresponding to $\omega \in \Omega$. Fix a representative of γ_n in its right class for every $n \geq 1$, and a representative of γ_0 in its double class, that we still denote by γ_n and γ_0 respectively. Using ideas used for the coding introduced in Section 5.2, let us construct by induction an infinite geodesic edge path $(e_n)_{n \in \mathbb{N}}$ with origin $o(e_0) = \tilde{y}_0$ and a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in Γ such that

$$e_n = \alpha_n \gamma_n g_{\overline{f_n}}^{-1} \widetilde{f_n}. \quad (10.10)$$

Let $\alpha_0 = \text{id}$ and $e_0 = \gamma_0 g_{\overline{f_0}}^{-1} \widetilde{f_0}$. Since $o(\widetilde{f_0}) = g_{\overline{f_0}} \tilde{y}_0$ by the construction of the lifts and since $\gamma_0 \in G_{y_0} = \Gamma_{\tilde{y}_0}$, we have $o(e_0) = \tilde{y}_0$. Since the stabiliser of $g_{\overline{f_0}}^{-1} \widetilde{f_0}$ is $\rho_{\overline{f_0}}(G_{f_0})$, the edge e_0 does not depend on the choice of the representative γ_0 modulo $\rho_{\overline{f_0}}(G_{f_0})$ on the right, but depends on the choice of the representative γ_0 modulo $G'_{y_0} = \Gamma'_{\tilde{y}_0}$ on the left.

The hypothesis that if $f_0 \in E\mathbb{Y}'$ then $\gamma_0 \notin G'_{y_0} \rho_{\overline{f_0}}(G_{f_0})$ ensures that the edge e_0 does not belong to $E\mathbb{X}'$. Indeed, assume otherwise that e_0 belongs to $E\mathbb{X}'$. Then $f_0 = p(e_0) \in E\mathbb{Y}'$, and by the assumptions on the choice of lifts, the edges $g_{\overline{f_0}}^{-1} \widetilde{f_0}$ and e_0 both belong to $E\mathbb{X}'$. Since they are both mapped to f_0 by the map $\mathbb{X}' \rightarrow \mathbb{Y}' = \Gamma \backslash \mathbb{X}'$, they are mapped one to the other by an element of $\Gamma'_{\tilde{y}_0} = G'_{y_0}$. Let $\gamma'_0 \in \Gamma'_{\tilde{y}_0}$ be such that $\gamma'_0 e_0 = g_{\overline{f_0}}^{-1} \widetilde{f_0}$. Then $\gamma'_0{}^{-1} \gamma_0$ belongs to the stabiliser in Γ of the edge $g_{\overline{f_0}}^{-1} \widetilde{f_0}$, which is equal to $g_{\overline{f_0}}^{-1} \Gamma_{\tilde{y}_0} g_{\overline{f_0}} = \rho_{\overline{f_0}}(G_{f_0})$. Therefore $\gamma_0 \in G'_{y_0} \rho_{\overline{f_0}}(G_{f_0})$, a contradiction.

Assume by induction that e_n and α_n are constructed. Define

$$\alpha_{n+1} = \alpha_n \gamma_n g_{\overline{f_n}}^{-1} g_{f_n}$$

and

$$e_{n+1} = \alpha_{n+1} \gamma_{n+1} g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}},$$

so that the induction formula (10.10) at rank $n + 1$ is satisfied. By the construction of the lifts, since $y_{n+1} = t(f_n) = o(f_{n+1})$, we have

$$\widetilde{y_{n+1}} = g_{f_n}^{-1} t(\widetilde{f_n}) = g_{\overline{f_{n+1}}}^{-1} o(\widetilde{f_{n+1}}).$$

Hence, since $\gamma_{n+1} \in G_{y_{n+1}}$ fixes $\widetilde{y_{n+1}}$, using the induction formula (10.10) at rank n for the last equality,

$$\begin{aligned} o(e_{n+1}) &= \alpha_{n+1} \gamma_{n+1} g_{\overline{f_{n+1}}}^{-1} o(\widetilde{f_{n+1}}) = \alpha_{n+1} \gamma_{n+1} \widetilde{y_{n+1}} = \alpha_{n+1} \widetilde{y_{n+1}} \\ &= \alpha_{n+1} g_{f_n}^{-1} t(\widetilde{f_n}) = \alpha_n \gamma_n g_{\overline{f_n}}^{-1} t(\widetilde{f_n}) = t(e_n). \end{aligned}$$

In particular, the sequence $(e_n)_{n \in \mathbb{N}}$ is an edge path in \mathbb{X} .

Since the stabiliser of $g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}}$ is $\rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$, the edge $\gamma_{n+1} g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}}$ does not depend on the choice of the representative of the right coset γ_{n+1} . Let us prove that the length 2 edge path $(g_{f_n}^{-1} \widetilde{f_n}, \gamma_{n+1} g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}})$ is geodesic. Otherwise, the two edges of this path are opposite one to another, hence $f_{n+1} = \overline{f_n}$ by using the projection $p : \mathbb{X} \rightarrow \mathbb{Y}$, therefore $g_{\overline{f_{n+1}}} = g_{f_n}$. Thus γ_{n+1} maps $g_{f_n}^{-1} \widetilde{f_n}$ to $g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}}$, hence belongs to $\rho_{f_n}(G_{f_n}) = \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$, a contradiction by the assumptions on the random walk.

By construction, the element α_{n+1} of Γ sends the above length 2 geodesic edge path $(g_{f_n}^{-1} \widetilde{f_n}, \gamma_{n+1} g_{\overline{f_{n+1}}}^{-1} \widetilde{f_{n+1}})$ to (e_n, e_{n+1}) . This implies on the one hand that the edge path

(e_n, e_{n+1}) is geodesic, and on the other hand that α_{n+1} is uniquely defined, since the action of Γ on \mathbb{X} is 2-acylindrical.

In particular, $(e_n)_{n \in \mathbb{N}}$ is the sequence of edges followed by a (discrete) geodesic ray in \mathbb{X} , starting from a point of \mathbb{X}' but not by an edge of \mathbb{X}' , that is, an element of $\partial_+^1 \mathbb{X}'$. Furthermore, this ray is well defined up to the action of $\Gamma'_{\tilde{y}_0}$, hence its image, that we denote by $\Theta(\omega)$, is well defined in $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$. Since $p(o(e_n)) = p(\tilde{y}_n) = y_n$ for all $n \in \mathbb{N}$, the commutativity of the diagram (10.9) is immediate.

For every $x \in V\mathbb{X}'$, let $\partial_+^1 \mathbb{X}'(x)$ be the subset of $\partial_+^1 \mathbb{X}'$ consisting of the elements w with $w(0) = x$. By construction, the above map from the subset of random paths in Ω starting from y_0 to $\Gamma'_{\tilde{y}_0} \backslash \partial_+^1 \mathbb{X}'(\tilde{y}_0)$, which associates to $(X_n)_{n \in \mathbb{N}}$ the $\Gamma'_{\tilde{y}_0}$ -orbit of the geodesic ray with consecutive edges $(e_n)_{n \in \mathbb{N}}$, is clearly a bijection. This bijection maps the measure \mathbb{P} to the normalised skinning measure $\frac{\sigma_{\mathcal{D}}^+}{\|\sigma_{\mathcal{D}}^+\|}$, since by homogeneity, the restriction to $\partial_+^1 \mathbb{X}'(\tilde{y}_0)$ of $\tilde{\sigma}_{\mathbb{X}'}^+$ to $\partial_+^1 \mathbb{X}'(\tilde{y}_0)$, normalized to be a probability measure, is the restriction to $\partial_+^1 \mathbb{X}'(\tilde{y}_0)$ of the $\text{Aut}(\mathbb{X})_x$ -homogeneous probability measure on the space of geodesic rays with origin x in the regular tree \mathbb{X} . This proves the result. \square

When \mathbb{Y} is finite, all the groups G_y for $y \in V\mathbb{Y}$ are trivial and \mathbb{Y}' is reduced to a vertex,⁵ the above random walk is the non-backtracking simple random walk on the nonbipartite regular finite graph \mathbb{Y} , and $\frac{1}{\text{Vol}(\mathbb{Y}, G_*)} \text{vol}_{\mathbb{Y}, G_*}$ is the uniform distribution on $V\mathbb{Y}$. Hence this result (stated as Corollary 1.3 in the Introduction) is classical. See for instance [OW, Thm. 1.2] and [ABLS], which under further assumptions on the spectral properties of \mathbb{Y} gives precise rates of convergence, and also the book [LyP2], including its Section 6.3 and its references.

10.4 Rate of equidistribution for metric and simplicial trees

In this Section, we give error terms for the equidistribution results stated in Theorem 10.2 for metric trees, and in Theorem 10.4 for simplicial trees, under additional assumptions required in order to get the error terms for the mixing property discussed in Chapter 9.

We first consider the simplicial case, for the discrete time geodesic flow. Let \mathbb{X} , X , Γ , \tilde{c} , \tilde{F}_c , δ_c , $(\mu_x^\pm)_{x \in V\mathbb{X}}$, $m_c = m_{F_c}$ be as in Section 10.3.

Theorem 10.8. *Assume that δ_c is finite and that the Gibbs measure m_c on $\Gamma \backslash \check{\mathcal{G}}\mathbb{X}$ is finite. Assume furthermore that*

- (1) *the families $(\Lambda\Gamma, \mu_x^-, d_x)_{x \in V\mathcal{C}\Lambda\Gamma}$ and $(\Lambda\Gamma, \mu_x^+, d_x)_{x \in V\mathcal{C}\Lambda\Gamma}$ of metric measure spaces are uniformly doubling,⁶*
- (2) *there exists $\alpha \in]0, 1]$ such that the discrete time geodesic flow on $(\Gamma \backslash \check{\mathcal{G}}\mathbb{X}, m_c)$ is exponentially mixing for the α -Hölder regularity.*

Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper simplicial subtrees of \mathbb{X} with $\Gamma \backslash I$ finite. Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}\mathbb{X}$, with $\Omega_i \subset \partial_\pm^1 D_i$ for all $i \in I$. Assume that σ_Ω^\pm is finite and nonzero. Then there exists $\kappa' > 0$ such that for all $\psi \in \mathcal{C}_c^\alpha(\Gamma \backslash \check{\mathcal{G}}\mathbb{X})$, we have, as $n \rightarrow +\infty$,

$$\frac{1}{\|(\mathbf{g}^{\pm n})_* \sigma_\Omega^\pm\|} \int \psi d\pi_*(\mathbf{g}^{\pm n})_* \sigma_\Omega^\pm = \frac{1}{\|m_c\|} \int \psi dm_c + O(\|\psi\|_\alpha e^{-\kappa' n}).$$

⁵The result for general \mathbb{Y}' follows by averaging.

⁶See Section 4.1 for definitions.

Remarks. (1) If $\tilde{c} = 0$, if the simplicial tree \mathbb{X}' with $|\mathbb{X}'|_1 = \mathcal{C}\Lambda\Gamma$ is uniform, if $L_\Gamma = \mathbb{Z}$ and if Γ is a lattice of \mathbb{X}' , then we claim that $\delta_c = \delta_\Gamma$ is finite, $m_c = m_{\text{BM}}$ is finite and mixing, and $(\Lambda\Gamma, \mu_x = \mu_x^\pm, d_x)_{x \in V\mathcal{C}\Lambda\Gamma}$ is uniformly doubling.

Indeed, the above finiteness and mixing properties follow from the results of Section 4.4. Since \mathbb{X}' is uniform, it has a cocompact discrete subgroup Γ' whose Patterson density (for the vanishing potential) is uniformly doubling on $\Lambda\Gamma' = \Lambda\Gamma$, by Lemma 4.2 (4). Since $\tilde{c} = 0$ and Γ is a lattice, the Patterson densities of Γ and of Γ' coincide (up a scalar multiple) by Proposition 4.14 (2).

(2) Assume that $\tilde{c} = 0$, that the simplicial tree \mathbb{X}' with $|\mathbb{X}'|_1 = \mathcal{C}\lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma = \mathbb{Z}$ and that Γ is a geometrically finite lattice of \mathbb{X}' . Then all assumptions of Theorem 10.8 are satisfied by the first remark and by Corollary 9.6. Therefore we have an exponentially small error term in the equidistribution of the equidistant levels sets.

Proof. We only give the proof when $\pm = +$, the other case is treated similarly. We follow the proof of Theorem 10.2, concentrating on the new features. We now have $\Gamma \setminus I$ finite by assumption. Let $\eta \in]0, 1]$ and $\psi \in \mathcal{C}_c^\alpha(\Gamma \setminus \check{\mathcal{G}}X)$. We consider the constant $R > 0$, the test function Φ_η , the fundamental domain Δ_Γ and the lift $\tilde{\psi} = \psi \circ p$ as in the proof of Theorem 10.2.

For all $n \in \mathbb{N}$, all $w \in \mathcal{G}_+X$ isometric on $[-n, +\infty[$ and all $\ell \in V_{w, \eta, e^{-n}R}^+ = B^+(w, e^{-n}R)$,⁷ by Lemma 2.7 where we can take $\eta = 0$, we have $d(\ell, w) = O(e^{-n})$. Since p is Lipschitz, the map $\tilde{\psi}$ is α -Hölder with α -Hölder norm at most $\|\psi\|_\alpha$. Hence for all $n \in \mathbb{N}$, all $w \in \mathcal{G}_+X$ isometric on $[-n, +\infty[$ and all $\ell \in V_{w, \eta, e^{-n}R}^+$, we have

$$\tilde{\psi}(\ell) = \tilde{\psi}(w) + O(e^{-n\alpha} \|\psi\|_\alpha). \quad (10.11)$$

As in the proof of Theorem 10.2 with t replaced by n , using Equation (10.11) instead of Equation (10.6) in the series of equations (10.7), since the symbols w that appear in them are indeed generalised geodesic lines isometric on $[-n, +\infty[$, we have

$$\frac{\int_{\Gamma \setminus \check{\mathcal{G}}X} \psi d(\mathbf{g}^n)_* \sigma_\Omega^+}{\|(\mathbf{g}^n)_* \sigma_\Omega^+\|} = \frac{\int_{\Gamma \setminus \mathcal{G}X} \psi \Phi_\eta \circ \mathbf{g}^{-n} dm_c}{\int_{\Gamma \setminus \mathcal{G}X} \Phi_\eta dm_c} + O(e^{-n\alpha} \|\psi\|_\alpha). \quad (10.12)$$

Let us now apply the assumption on the decay of correlations. In order to do that, we need to regularise our test functions Φ_η . We start by an independent lemma.

Lemma 10.9. *There are universal constants $\epsilon_0 > 0, c_0 \geq 1$ such that for all $\epsilon \in]0, \epsilon_0[$ and $\ell \in \mathcal{G}X$, the ball $B_d(\ell, \epsilon)$ is contained in*

$$\{\ell' \in \mathcal{G}X : \ell'(0) = \ell(0), \ell'(\pm\infty) \in B_{d_{\ell(0)}}(\ell(\pm\infty), c_0 \sqrt{\epsilon})\}$$

and contains $\{\ell' \in \mathcal{G}X : \ell'(0) = \ell(0), \ell'(\pm\infty) \in B_{d_{\ell(0)}}(\ell(\pm\infty), \frac{1}{c_0} \sqrt{\epsilon})\}$.

Proof. If $\ell, \ell' \in \mathcal{G}X$ have distinct footpoints, then $d(\ell(0), \ell'(0)) \geq 1$, so that $d(\ell(t), \ell'(t)) \geq \frac{1}{4}$ if $|t| \leq \frac{1}{4}$, so that $d(\ell, \ell') \geq \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{4} e^{-2|t|} = \epsilon_0$.

⁷We have $V_{w, \eta, s}^+ = B^+(w, s)$ for every $s > 0$ since X is simplicial and $\eta < 1$.

Conversely, assume that $\ell, \ell' \in \mathcal{GX}$ have equal footpoints, so that they coincide on $[-N, N']$ for some $N, N' \in \mathbb{N}$. By the definition of the visual distances (see Equation (2.1)), we have

$$d_{\ell(0)}(\ell(+\infty), \ell'(+\infty)) = e^{-N'}$$

and similarly $d_{\ell(0)}(\ell(-\infty), \ell'(-\infty)) = e^{-N}$. By the definition of the distance on $\check{\mathcal{G}}X$ (see Equation (2.4)), we have, by an easy change of variables,

$$\begin{aligned} d(\ell, \ell') &= \int_{N'}^{+\infty} 2|t - N'| e^{-2t} dt + \int_{-\infty}^{-N} 2|-N - t| e^{2t} dt \\ &= (e^{-2N'} + e^{-2N}) \int_0^{+\infty} 2u e^{-2u} du = \frac{1}{2}(e^{-2N'} + e^{-2N}). \end{aligned}$$

The result follows. \square

By the definition of the Gibbs measures,⁸ this lemma implies that

$$\begin{aligned} &\mu_{\ell(0)}^-(B_{d_{\ell(0)}}(\ell(-\infty), \frac{1}{c_0} \sqrt{\epsilon})) \mu_{\ell(0)}^+(B_{d_{\ell(0)}}(\ell(+\infty), \frac{1}{c_0} \sqrt{\epsilon})) \\ &\leq \tilde{m}_c(B_d(\ell, \epsilon)) \\ &\leq \mu_{\ell(0)}^-(B_{d_{\ell(0)}}(\ell(-\infty), c_0 \sqrt{\epsilon})) \mu_{\ell(0)}^+(B_{d_{\ell(0)}}(\ell(+\infty), c_0 \sqrt{\epsilon})). \end{aligned}$$

Since the Patterson densities are uniformly doubling for basepoints in $\mathcal{C}\lambda\Gamma$, since the footpoints of the geodesic lines in the support of m_c belong to $\mathcal{C}\lambda\Gamma$, the Gibbs measure m_c is hence doubling on its support.

As in the proof of [PaP14a, Thm. 20], using discrete convolution approximation (see for instance [Sem, p. 290-292] or [KinKST]), there exists $\kappa'' > 0$ and, for every $\eta > 0$, a nonnegative function $\mathbf{R}\Phi_\eta \in C_c^\alpha(\Gamma \backslash \mathcal{GX})$ such that

- (1) $\int_{\Gamma \backslash \mathcal{GX}} \mathbf{R}\Phi_\eta d\bar{m}_c = \int_{\Gamma \backslash \mathcal{GX}} \Phi_\eta d\bar{m}_c,$
- (2) $\int_{\Gamma \backslash \mathcal{GX}} |\mathbf{R}\Phi_\eta - \Phi_\eta| d\bar{m}_c = O(\eta \int_{\Gamma \backslash \mathcal{GX}} \Phi_\eta d\bar{m}_c),$
- (3) $\|\mathbf{R}\Phi_\eta\|_\alpha = O(\eta^{-\kappa''} \int_{\Gamma \backslash \mathcal{GX}} \Phi_\eta d\bar{m}_c).$

By Equation (10.5), the integral $\int_{\Gamma \backslash \mathcal{GX}} \Phi_\eta dm_c = \|\sigma_\Omega^+\|$ is constant (in particular independent of η). Let $\bar{m}_c = \frac{m_c}{\|m_c\|}$. All integrals below besides the first one being over $\Gamma \backslash \mathcal{GX}$, and using

- Equation (10.12) and the above property (2) of the regularised map $\mathbf{R}\Phi_\eta$ for the first equality,
- the assumption of exponential decay of correlations for the second one, involving some constant $\kappa > 0$, for the second equality,
- the above properties (1) and (3) of the regularised map $\mathbf{R}\Phi_\eta$ for the last equality,

⁸See Equation (4.3)

we hence have

$$\begin{aligned}
\frac{\int_{\Gamma \backslash \check{\mathcal{G}}X} \psi d(\mathbf{g}^n)_* \sigma_\Omega^+}{\|(\mathbf{g}^n)_* \sigma_\Omega^+\|} &= \frac{\int \psi \mathbf{R}\Phi_\eta \circ \mathbf{g}^{-n} d\bar{m}_c}{\int \Phi_\eta d\bar{m}_c} + O(e^{-n\alpha} \|\psi\|_\alpha + \eta \|\psi\|_\infty) \\
&= \frac{\int \mathbf{R}\Phi_\eta d\bar{m}_c}{\int \Phi_\eta d\bar{m}_c} \frac{\int \psi d\bar{m}_c}{\int \Phi_\eta d\bar{m}_c} + O(e^{-n\alpha} \|\psi\|_\alpha + \eta \|\psi\|_\infty + e^{-\kappa n} \|\mathbf{R}\Phi_\eta\|_\alpha \|\psi\|_\alpha) \\
&= \int \psi d\bar{m}_c + O((e^{-n\alpha} + \eta + e^{-\kappa n} \eta^{-\kappa''}) \|\psi\|_\alpha).
\end{aligned}$$

Taking $\eta = e^{-\lambda n}$ with $\lambda = \frac{\kappa}{2\kappa''}$, the result follows with $\kappa' = \min\{\alpha, \frac{\kappa}{2\kappa''}, \frac{\kappa}{2}\}$. \square

Let us now consider the metric tree case, for the continuous time geodesic flow, where the main change is to assume a superpolynomial decay of correlations and hence get a superpolynomial error term, for observables which are smooth enough along the flow lines. Let (\mathbb{X}, λ) , $X, \Gamma, \tilde{F}, \delta_F (\mu_x^\pm)_{x \in X}$ and m_F be as in the beginning of Section 4.4.

Theorem 10.10. *Assume that δ_F is finite and that the Gibbs measure m_F on $\Gamma \backslash \mathcal{G}X$ is finite. Assume furthermore that*

- (1) *the families $(\Lambda\Gamma, \mu_x^-, d_x)_{x \in \mathcal{C}\Lambda\Gamma}$ and $(\Lambda\Gamma, \mu_x^+, d_x)_{x \in \mathcal{C}\Lambda\Gamma}$ of metric measure spaces are uniformly doubling,⁹ and F is bounded on $T^1\mathcal{C}\Lambda\Gamma$,*
- (2) *there exists $\alpha \in]0, 1]$ such that the (continuous time) geodesic flow on $(\Gamma \backslash \mathcal{G}X, m_F)$ has superpolynomial decay of α -Hölder correlations.*

Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper closed convex subsets of X with $\Gamma \backslash I$ finite. Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ -equivariant family of measurable subsets of $\check{\mathcal{G}}X$, with $\Omega_i \subset \partial_\pm^1 D_i$ for all $i \in I$. Assume that σ_Ω^\pm is finite and nonzero. Then for every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that for all $\psi \in \mathcal{C}_c^{k, \alpha}(\Gamma \backslash \check{\mathcal{G}}X)$, we have, as $t \rightarrow +\infty$,

$$\frac{1}{\|(\mathbf{g}^{\pm t})_* \sigma_\Omega^\pm\|} \int \psi d\pi_*(\mathbf{g}^{\pm t})_* \sigma_\Omega^\pm = \frac{1}{\|m_F\|} \int \psi dm_F + O(\|\psi\|_{k, \alpha} t^{-n}).$$

Remarks. (1) If $F = 0$, if the metric subtree $X' = \mathcal{C}\Lambda\Gamma$ of X is uniform, if the length spectrum of Γ on X is not contained in a discrete subgroup of \mathbb{R} and if Γ is a lattice of X' , then we claim that $\delta_F = \delta_\Gamma$ is finite, $m_F = m_{\text{BM}}$ is finite and mixing, and $(\Lambda\Gamma, \mu_x = \mu_x^\pm, d_x)_{x \in \mathcal{C}\Lambda\Gamma}$ is uniformly doubling.

Indeed, the above finiteness and mixing properties follow from Proposition 4.14 and Theorem 4.8. Since X' is uniform, it has a cocompact discrete subgroup of isometries Γ' whose Patterson density (for the vanishing potential) is uniformly doubling on $\Lambda\Gamma' = \Lambda\Gamma$, by Lemma 4.2 (4). Since $F = 0$ and Γ is a lattice, the Patterson densities of Γ and of Γ' coincide (up to a scalar multiple) by Proposition 4.14.

(2) Assume that $F = 0$, that the metric subtree $X' = \mathcal{C}\Lambda\Gamma$ of X is uniform, that the length spectrum of Γ on X is 4-Diophantine and that Γ is a geometrically finite lattice of \mathbb{X}' . Then all assumptions of Theorem 10.10 are satisfied by the first remark and by Corollary

⁹See Section 4.1 for definitions.

9.10. Therefore we have a superpolynomially small error term in the equidistribution of the equidistant levels sets.

Proof. The proof is similar to the one of Theorem 10.8, except for the doubling property of the Gibbs measure on its support and the conclusion of the proof. Let $X' = \mathcal{C}\Lambda\Gamma$. The modification of Lemma 10.9 is the third assertion of the following result, which will also be useful in Section 12.6. Its second claim strengthens for \mathbb{R} -trees the Hölder-continuity of the footpoint projection stated in Section 2.3. If a and b are positive functions of some parameters, we write $a \asymp b$ if there exists a universal constant $C > 0$ such that $\frac{1}{C} b \leq a \leq C b$.

Lemma 10.11. *Let Y be an \mathbb{R} -tree.*

- (1) *There exists a universal constant $c_1 > 0$ such that for all $\ell, \ell' \in \mathcal{G}Y$, if $d(\ell, \ell') \leq c_1$, then $\ell'(0) \in \ell(\mathbb{R})$, the intersection $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is not reduced to a point, the orientations of ℓ and ℓ' coincide on this intersection, and*

$$d(\ell, \ell') \asymp d_{\ell(0)}(\ell(-\infty), \ell'(-\infty))^2 + d_{\ell(0)}(\ell(+\infty), \ell'(+\infty))^2 + d(\ell(0), \ell'(0)).$$

- (2) *The footpoint map $\pi : \mathcal{G}Y \rightarrow Y$ defined by $\ell \mapsto \ell(0)$ is (uniformly locally) Lipschitz.*
- (3) *There are universal constants $\epsilon_0 > 0, c_0 \geq 1$ such that for all $\epsilon \in]0, \epsilon_0[$ and $\ell \in \mathcal{G}Y$, the ball $B_d(\ell, \epsilon)$ in $\mathcal{G}Y$ is contained in*

$$\{\ell' \in \mathcal{G}Y : \ell'(0) \in \ell(\mathbb{R}), d(\ell'(0), \ell(0)) \leq c_0 \epsilon, \ell'(\pm\infty) \in B_{d_{\ell(0)}}(\ell(\pm\infty), c_0 \sqrt{\epsilon})\}$$

and contains

$$\{\ell' \in \mathcal{G}Y : \ell'(0) \in \ell(\mathbb{R}), d(\ell'(0), \ell(0)) \leq \frac{1}{c_0} \epsilon, \ell'(\pm\infty) \in B_{d_{\ell(0)}}(\ell(\pm\infty), \frac{1}{c_0} \sqrt{\epsilon})\}.$$

Proof. (1) Let $\ell, \ell' \in \mathcal{G}Y$. If $\ell'(0) \notin \ell(\mathbb{R})$, then $\ell'(t) \notin \ell(\mathbb{R})$ for all $t \geq 0$ or $\ell'(t) \notin \ell(\mathbb{R})$ for all $t \leq 0$, since Y is an \mathbb{R} -tree. In the first case, we hence have $d(\ell(t), \ell'(t)) \geq t$ for all $t \geq 0$, thus $d(\ell, \ell')$ is at least $c_2 = \int_0^{+\infty} t e^{-2t} dt = \frac{1}{4} > 0$. The same estimate holds in the second case. This argument furthermore shows that if the geodesic segment (or ray or line) $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is reduced to a point, then $d(\ell, \ell')$ is at least $c_2 > 0$.

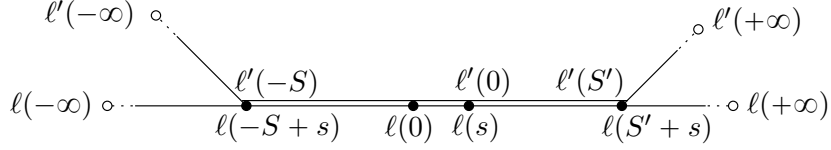
If $d(\ell'(0), \ell(0)) \geq 1$, then $d(\ell(t), \ell'(t)) \geq \frac{1}{4}$ for $|t| \leq \frac{1}{4}$, thus

$$d(\ell, \ell') \geq \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{4} e^{-2|t|} dt,$$

which is a positive universal constant.

If $d(\ell'(0), \ell(0)) \leq 1$, if $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ contains $\ell'(0)$ and is not reduced to a point, but if the orientations of ℓ and ℓ' do not coincide on this intersection, then $d(\ell(t), \ell'(t)) \geq 2t - d(\ell(0), \ell'(0)) \geq 2t - 1$ for all $t \geq 1$, so that $d(\ell, \ell')$ is at least $\int_1^{+\infty} (2t - 1) e^{-2t} dt$, which is a positive constant.

Assume now that $\ell'(0) \in \ell(\mathbb{R})$, that $d(\ell'(0), \ell(0)) \leq 1$, that $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is not reduced to a point and that the orientations of ℓ and ℓ' coincide on this intersection. Then there exists $s \in \mathbb{R}$ such that $\ell'(0) = \ell(s)$, so that $|s| = d(\ell(0), \ell'(0))$. Assume for instance that $s \geq 0$, the other case being treated similarly. Then there exist $S, S' \geq 0$ maximal such that $\ell'(t) = \ell(t + s)$ for all $t \in [-S, S']$, with the convention that $S = +\infty$ if $\ell'(-\infty) = \ell(-\infty)$, that $S' = +\infty$ if $\ell'(+\infty) = \ell(+\infty)$, and that $e^{-\infty} = 0$.



By the definition of the visual distances (see Equation (2.1)), we have, for t big enough,

$$d_{\ell(0)}(\ell(+\infty), \ell'(+\infty)) = e^{\frac{1}{2}[(t-S')+(t-S'-s)]-t} \asymp e^{-S'} .$$

Similarly $d_{\ell(0)}(\ell(-\infty), \ell'(-\infty)) \asymp e^{-S}$.

As can be seen in the above picture, we have

$$d(\ell(t), \ell'(t)) = \begin{cases} -2t - 2S + s & \text{if } t \leq -S \\ s & \text{if } -S \leq t \leq S' + s \\ 2t - 2S' - s & \text{if } t \geq S' + s . \end{cases}$$

By the definition of the distance on \check{Y} (see Equation (2.4)), by easy changes of variables, assuming that at least one of S, S' is larger than some positive constant for the last line, we have

$$\begin{aligned} d(\ell, \ell') &= \int_{-\infty}^{-S} (-2t - 2S + s) e^{-2|t|} dt + \int_{-S}^{S'+s} s e^{-2|t|} dt + \int_{S'+s}^{+\infty} (2t - 2S' - s) e^{-2t} dt \\ &= e^{-2S+s} \int_s^{+\infty} u e^{-u} du + s \int_{-S}^{S'+s} e^{-2|t|} dt + e^{-2S'-s} \int_s^{+\infty} u e^{-u} du \\ &\asymp e^{-2S} + e^{-2S'} + s . \end{aligned}$$

Assertion (1) of Lemma 10.11 follows.

(2) The second assertion follows immediately from the first one.

(3) The first inclusion in the third assertion follows easily from the first one, and the second one follows from the argument of its proof, and the fact that if $d_{\ell(0)}(\ell(+\infty), \ell'(+\infty))$, $d_{\ell(0)}(\ell(+\infty), \ell'(0))$ and $d(\ell(0), \ell'(0))$ are at most some small positive constant, then $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ contains $\ell'(0)$ and is not reduced to a point. \square

If the footpoints of $\ell, \ell' \in \mathcal{G}X'$ are at distance bounded by $c_0 \epsilon_0$, then by Proposition 3.10 (2), since $|\tilde{F}|$ is bounded on T^1X' by assumption, the quantities $|C_\xi^\pm(\ell(0), \ell'(0))|$ for $\xi \in \Lambda F$ are bounded by the constant $c'_0 = c_0 \epsilon_0 (\max_{T^1X'} |\tilde{F} - \delta_F|)$. By the definition of the Gibbs measures (see Equation (4.3)), Assertion (3) of the above lemma hence implies that if $\epsilon \leq \epsilon_0$ then

$$\begin{aligned} &e^{-2c'_0} \epsilon \mu_{\ell(0)}^-(B_{d_{\ell(0)}}(\ell(-\infty), \frac{1}{c_0} \sqrt{\epsilon})) \mu_{\ell(0)}^+(B_{d_{\ell(0)}}(\ell(+\infty), \frac{1}{c_0} \sqrt{\epsilon})) \\ &\leq \tilde{m}_c(B_d(\ell, \epsilon)) \\ &\leq e^{2c'_0} \epsilon \mu_{\ell(0)}^-(B_{d_{\ell(0)}}(\ell(-\infty), c_0 \sqrt{\epsilon})) \mu_{\ell(0)}^+(B_{d_{\ell(0)}}(\ell(+\infty), c_0 \sqrt{\epsilon})) . \end{aligned}$$

As in the simplicial case, since the Patterson densities are uniformly doubling for basepoints in X' , the Gibbs measure m_c is hence doubling on its support.

Fix $n \in \mathbb{N}$. As in the end of the proof of Theorem 10.8, using the assumption of superpolynomial decay of correlations, involving some degree of regularity k in order to have polynomial decay in t^{-Nn} where $N = \lceil \kappa'' \rceil + 1$, instead of the exponential one, we have for all $t \geq 1$

$$\frac{\int_{\Gamma \setminus \check{\mathcal{G}}_{\mathbb{X}}} \psi d(\mathbf{g}^t)_* \sigma_{\Omega}^+}{\|(\mathbf{g}^t)_* \sigma_{\Omega}^+\|} = \int \psi d\bar{m}_c + O((e^{-t\alpha} + \eta + t^{-Nn} \eta^{-\kappa''}) \|\psi\|_{k, \alpha}).$$

Taking $\eta = t^{-n}$, by the definition of N , we hence have

$$\frac{\int_{\Gamma \setminus \check{\mathcal{G}}_{\mathbb{X}}} \psi d(\mathbf{g}^t)_* \sigma_{\Omega}^+}{\|(\mathbf{g}^t)_* \sigma_{\Omega}^+\|} = \int \psi d\bar{m}_c + O(t^{-n} \|\psi\|_{k, \alpha}).$$

This proves Theorem 10.10. □

Chapter 11

Equidistribution of common perpendicular arcs

In this Chapter, we prove the equidistribution of the initial and terminal vectors of common perpendiculars of convex subsets in the universal covering space level for Riemannian manifolds and for metric and simplicial trees. The results generalise [PaP16c, Thm. 8].

In Sections 11.1 to 11.3, we consider the continuous time situation where the $\text{CAT}(-1)$ -space X is either a proper \mathbb{R} -tree without terminal point or a complete Riemannian manifold with pinched negative curvature at most -1 , and x_0 is any basepoint in X . In Section 11.4, X is the geometric realisation of a simplicial tree \mathbb{X} , with the discrete time geodesic flow, and x_0 is any vertex of \mathbb{X} .

Let Γ be a nonelementary discrete group of isometries of X . Let \tilde{F} be a continuous Γ -invariant potential on T^1X , which is Hölder-continuous if X is a manifold. Assume that $\delta = \delta_{\Gamma, F^\pm}$ is positive and let $(\mu_x^\pm)_{x \in X}$ be Patterson densities for the pairs (Γ, F^\pm) , with associated Gibbs measure m_F . Let $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ and $\mathcal{D}^+ = (D_j^+)_{j \in I^+}$ be locally finite Γ -equivariant families of nonempty proper closed convex subsets of X .

For every (i, j) in $I^- \times I^+$ such that the closures $\overline{D_i^-}$ and $\overline{D_j^+}$ of D_i^- and D_j^+ in $X \cap \partial_\infty X$ have empty intersection, let $\lambda_{i,j} = d(D_i^-, D_j^+)$ be the length of the common perpendicular from D_i^- to D_j^+ , and $\alpha_{i,j}^- \in \tilde{\mathcal{G}}X$ its parametrisation: it is the unique map from \mathbb{R} to X such that $\alpha_{i,j}^-(t) = \alpha_{i,j}^-(0) \in D_i^-$ if $t \leq 0$, $\alpha_{i,j}^-(t) = \alpha_{i,j}^-(\lambda_{i,j}) \in D_j^+$ if $t \geq \lambda_{i,j}$, and $\alpha_{i,j}^-|_{[0, \lambda_{i,j}]} = \alpha_{i,j}$ is the shortest geodesic arc starting from a point of D_i^- and ending at a point of D_j^+ . Let $\alpha_{i,j}^+ = g^{\lambda_{i,j}} \alpha_{i,j}^-$. In particular, we have $g^{\frac{\lambda_{i,j}}{2}} \alpha_{i,j}^- = g^{\frac{-\lambda_{i,j}}{2}} \alpha_{i,j}^+$.

We now state our main equidistribution result of common perpendiculars between convex subsets in the continuous time and upstairs settings. We will give the discrete time version in Section 11.4, and the downstairs version in Chapter 12.

Theorem 11.1. *Let X be a proper \mathbb{R} -tree without terminal points or a complete Riemannian manifold with pinched negative curvature at most -1 . Let Γ be a nonelementary discrete group of isometries of X and let \tilde{F} be a bounded Γ -invariant potential on X as above. Assume that the critical exponent δ is finite and positive, and that the Gibbs measure m_F is finite and*

mixing for the geodesic flow on $\Gamma \backslash \mathcal{G}X$. Then

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_{i, \gamma j} \leq t}} e^{\int \alpha_{i, \gamma j} \tilde{F}} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^-$$

for the weak-star and narrow convergences of measures on the locally compact space $\check{\mathcal{G}}X \times \check{\mathcal{G}}X$.

Recall that the *narrow topology*¹ on the set $\mathcal{M}_f(Y)$ of finite measures on a Polish space Y is the smallest topology such that the map from $\mathcal{M}_f(Y)$ to \mathbb{R} defined by $\mu \mapsto \mu(g)$ is continuous for every bounded continuous function $g : Y \rightarrow \mathbb{R}$.

The proof of Theorem 11.1 follows that of [PaP16c, Thm.8], which proves this result when X is a Riemannian manifold with pinched negative curvature at most -1 and $F = 0$. The first two and a half steps work for both trees and manifolds and are given in Section 11.1. The differences begin in Step 3T. After this, the steps for trees are called 3T and 4T and are given in Section 11.2 and the corresponding steps for manifolds are 3M and 4M, given in Section 11.3.

In the special case of $\mathcal{D}^- = (\gamma x)_{\gamma \in \Gamma}$ and $\mathcal{D}^+ = (\gamma y)_{\gamma \in \Gamma}$ for some $x, y \in X$, this statement gives the following version with potentials of Roblin's double equidistribution theorem [Rob2, Theo. 4.1.1] when $F = 0$, see [PauPS, Theo. 9.1] for general F when X is a Riemannian manifold with pinched sectional curvature at most -1 .

Corollary 11.2. *Assume that \tilde{F} is bounded and that the Gibbs measure m_F is finite and mixing for the geodesic flow on $\mathcal{G}X$. Then*

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{\gamma \in \Gamma : d(x, \gamma y) \leq t} e^{\int_x^{\gamma y} \tilde{F}} \Delta_{\gamma y} \otimes \Delta_{\gamma^{-1}x} = \mu_x^+ \otimes \mu_y^-$$

for the weak-star convergence of measures on the compact space $(X \cup \partial_\infty X) \times (X \cup \partial_\infty X)$. \square

Here is a version of Theorem 11.1 without the assumption that the critical exponent of $\delta = \delta_{\Gamma, F}$ is positive.

Theorem 11.3. *Assume that \tilde{F} is bounded and that the Gibbs measure m_F is finite and mixing for the geodesic flow on $\mathcal{G}X$. Then for every $\tau > 0$, we have*

$$\lim_{t \rightarrow +\infty} \frac{\delta \|m_F\|}{1 - e^{-\tau \delta}} e^{-\delta t} \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, t - \tau < \lambda_{i, \gamma j} \leq t}} e^{\int \alpha_{i, \gamma j} \tilde{F}} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^-$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}X \times \check{\mathcal{G}}X$.

Proof. The claim follows from Theorem 11.1 by replacing F by $F + \kappa$ for κ large enough so that $\delta_{\Gamma, F + \kappa} = \delta_{\Gamma, F} + \kappa > 0$, and by using a classical geometric series type of arguments, see for instance [PauPS, Lem. 9.5] for more details. \square

¹also called *weak topology* see for instance [DM, p. 71-III] or [Bil, Part]

11.1 Part I of the proof of Theorem 11.1: the common part

Step 1: Reduction. By additivity, by the local finiteness of the families \mathcal{D}^\pm , and by the definition of $\tilde{\sigma}_{\mathcal{D}^\pm}^\pm = \sum_{k \in I^\mp / \sim} \tilde{\sigma}_{D_k^\pm}^\pm$, we only have to prove, for all fixed $i \in I^-$ and $j \in I^+$, that

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{\gamma \in \Gamma: 0 < \lambda_{i, \gamma j} \leq t} e^{\int \alpha_{i, \gamma j} \tilde{F}} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} = \tilde{\sigma}_{D_i^-}^+ \otimes \tilde{\sigma}_{D_j^+}^- \quad (11.1)$$

for the weak-star convergence of measures on $\check{\mathcal{G}}X \times \check{\mathcal{G}}X$.

Let Ω^- be a Borel subset of $\partial_+^1 D_i^-$ and let Ω^+ be a Borel subset of $\partial_-^1 D_j^+$. To simplify the notation, let

$$D^- = D_i^-, \quad D^+ = D_j^+, \quad \alpha_\gamma^- = \alpha_{i, \gamma j}^-, \quad \alpha_\gamma^+ = \alpha_{\gamma^{-1}i, j}^+, \quad \lambda_\gamma = \lambda_{i, \gamma j}, \quad \tilde{\sigma}^\pm = \tilde{\sigma}_{D^\pm}^\pm. \quad (11.2)$$

Assume that Ω^- and Ω^+ have positive finite skinning measures and that their boundaries in $\partial_+^1 D^-$ and $\partial_-^1 D^+$ have zero skinning measures (for $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ respectively). Let

$$I_{\Omega^-, \Omega^+}(t) = \delta \|m_F\| e^{-\delta t} \sum_{\substack{\gamma \in \Gamma: 0 < \lambda_\gamma \leq t \\ \alpha_\gamma^- |_{]0, \lambda_\gamma] \in \Omega^-, \alpha_\gamma^+ |_{] -\lambda_\gamma, 0] \in \Omega^+}} e^{\int \alpha_\gamma^- \tilde{F}}. \quad (11.3)$$

We will prove the stronger statement that, for every such Ω^\pm , we have

$$\lim_{t \rightarrow +\infty} I_{\Omega^-, \Omega^+}(t) = \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+). \quad (11.4)$$

Step 2: First upper and lower bounds. Using Lemma 7.5 (1), we may fix $R > 0$ such that $\nu_w^\pm(V_{w, \eta, R}^\pm) > 0$ for all $\eta \in]0, 1]$ and $w \in \partial_\mp^1 D^\pm$. Let $\phi_\eta^\mp = \phi_{\eta, R, \Omega^\pm}^\mp$ be the test functions defined in Equation (10.4).

For all $t \geq 0$, let

$$a_\eta(t) = \sum_{\gamma \in \Gamma} \int_{\ell \in \mathcal{G}X} \phi_\eta^-(\mathbf{g}^{-t/2} \ell) \phi_\eta^+(\mathbf{g}^{t/2} \gamma^{-1} \ell) d\tilde{m}_F(\ell). \quad (11.5)$$

As in [PaP16c], the heart of the proof is to give two pairs of upper and lower bounds, as $T \geq 0$ is big enough and $\eta \in]0, 1]$ is small enough, of the (Césaro-type) quantity

$$i_\eta(T) = \int_0^T e^{\delta t} a_\eta(t) dt. \quad (11.6)$$

By passing to the universal cover, the mixing property of the geodesic flow on $\Gamma \backslash \mathcal{G}X$ for the Gibbs measure m_F gives that, for every $\epsilon > 0$, there exists $T_\epsilon = T_{\epsilon, \eta} \geq 0$ such that for all $t \geq T_\epsilon$, we have

$$\frac{e^{-\epsilon}}{\|m_F\|} \int_{\mathcal{G}X} \phi_\eta^- d\tilde{m}_F \int_{\mathcal{G}X} \phi_\eta^+ d\tilde{m}_F \leq a_\eta(t) \leq \frac{e^\epsilon}{\|m_F\|} \int_{\mathcal{G}X} \phi_\eta^- d\tilde{m}_F \int_{\mathcal{G}X} \phi_\eta^+ d\tilde{m}_F.$$

Hence by Lemma 10.1, for all $\epsilon > 0$ and $\eta \in]0, 1]$, there exists $c_\epsilon = c_{\epsilon, \eta} > 0$ such that for every $T \geq 0$, we have

$$e^{-\epsilon} \frac{e^{\delta T}}{\delta \|m_F\|} \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+) - c_\epsilon \leq i_\eta(T) \leq e^\epsilon \frac{e^{\delta T}}{\delta \|m_F\|} \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+) + c_\epsilon. \quad (11.7)$$

Step 3: Second upper and lower bounds. Let $T \geq 0$ and $\eta \in]0, 1]$. By Fubini's theorem for nonnegative measurable maps, the definition² of the test functions ϕ_η^\pm and the flow-invariance³ of the fibrations $f_{D^\mp}^\pm$, we have

$$i_\eta(T) = \sum_{\gamma \in \Gamma} \int_0^T e^{\delta t} \int_{\mathcal{G}X} h_{\eta,R}^- \circ f_{D^-}^+(\ell) h_{\eta,R}^+ \circ f_{D^+}^-(\gamma^{-1}\ell) \mathbb{1}_{\mathfrak{g}^{t/2}\gamma_{\eta,R}^+(\Omega^-)}(\ell) \mathbb{1}_{\mathfrak{g}^{-t/2}\gamma_{\eta,R}^-(\gamma\Omega^+)}(\ell) d\tilde{m}_F dt. \quad (11.8)$$

We start the computations by rewriting the product term involving the functions $h_{\eta,R}^\pm$. For all $\gamma \in \Gamma$ and $\ell \in \mathcal{U}_{D^-}^+ \cap \mathcal{U}_{\gamma D^+}^-$, define (using Equation (2.11))

$$w^- = f_{D^-}^+(\ell) \in \mathcal{G}_{+,0}X \quad \text{and} \quad w^+ = f_{\gamma D^+}^-(\ell) = \gamma f_{D^+}^-(\gamma^{-1}\ell) \in \mathcal{G}_{-,0}X. \quad (11.9)$$

This notation is ambiguous (w^- depends on ℓ , and w^+ depends on ℓ and γ), but it makes the computations less heavy. By Equations (10.2) and (3.8), we have, for every $t \geq 0$,

$$h_{\eta,R}^-(w^-) = h_{\eta,R}^- \circ \mathfrak{g}^{-t/2}(\mathfrak{g}^{t/2}w^-) = e^{\int_{w^-(0)}^{w^-(t/2)} (\tilde{F} - \delta)} h_{\eta, e^{-t/2}R}^-(\mathfrak{g}^{t/2}w^-).$$

Similarly,

$$h_{\eta,R}^+(\gamma^{-1}w^+) = e^{\int_{w^+(-t/2)}^{w^+(0)} (\tilde{F} - \delta)} h_{\eta, e^{-t/2}R}^+(\mathfrak{g}^{-t/2}w^+).$$

Hence,

$$\begin{aligned} & h_{\eta,R}^- \circ f_{D^-}^+(\ell) h_{\eta,R}^+ \circ f_{D^+}^-(\gamma^{-1}\ell) \\ &= e^{-\delta t} e^{\int_{w^-(0)}^{w^-(t/2)} \tilde{F} + \int_{w^+(-t/2)}^{w^+(0)} \tilde{F}} h_{\eta, e^{-t/2}R}^-(\mathfrak{g}^{t/2}w^-) h_{\eta, e^{-t/2}R}^+(\mathfrak{g}^{-t/2}w^+). \end{aligned} \quad (11.10)$$

11.2 Part II of the proof of Theorem 11.1: the metric tree case

In this Section, we assume that X is an \mathbb{R} -tree and we will consider the manifold case separately in Section 11.3.

Step 3T. Consider the product term in Equation (11.8) involving the characteristic functions. By Lemma 2.8 (applied by replacing D^+ by γD^+), there exists $t_0 \geq 2 \ln R + 4$ such that for all $\eta \in]0, 1]$ and $t \geq t_0$, for all $\ell \in \mathcal{G}X$, if $\mathbb{1}_{\mathfrak{g}^{t/2}\gamma_{\eta,R}^+(\Omega^-)}(\ell) \mathbb{1}_{\mathfrak{g}^{-t/2}\gamma_{\eta,R}^-(\gamma\Omega^+)}(\ell) \neq 0$, then the following facts hold.

- (i) By the convexity of D^\pm , we have $\ell \in \mathcal{U}_{D^-}^+ \cap \mathcal{U}_{\gamma D^+}^-$.
- (ii) By the definition⁴ of w^\pm , we have $w^- \in \Omega^-$ and $w^+ \in \gamma\Omega^+$. The notation (w^-, w^+) here coincides with the notation (w^-, w^+) in Lemma 2.8.

²See Equation (10.4).

³See Equation (2.11).

⁴See Equation (11.9).

(iii) There exists a common perpendicular α_γ from D^- to γD^+ , whose length λ_γ satisfies

$$|\lambda_\gamma - t| \leq 2\eta,$$

whose origin is $\alpha_\gamma^-(0) = w^-(0)$, whose endpoint is $\alpha_\gamma^+(0) = w^+(0)$, such that the points $w^-(\frac{t}{2})$ and $w^+(\frac{t}{2})$ are at distance at most η from $\ell(0) \in \alpha_\gamma$.

Hence, by Lemma 3.3 and since \tilde{F} is bounded,

$$e^{-2\eta\|F\|_\infty} e^{\int_{\alpha_\gamma} \tilde{F}} \leq e^{\int_{w^-(0)}^{w^-(t/2)} \tilde{F} + \int_{w^+(-t/2)}^{w^+(0)} \tilde{F}} \leq e^{2\eta\|F\|_\infty} e^{\int_{\alpha_\gamma} \tilde{F}}. \quad (11.11)$$

For all $\eta \in]0, 1]$, $\gamma \in \Gamma$ and $T \geq t_0$, let

$$\mathcal{A}_{\eta,\gamma}(T) = \{(t, \ell) \in [t_0, T] \times \mathcal{G}X : \ell \in \mathcal{V}_{\eta, e^{-t/2}R}^+(\mathbf{g}^{t/2}\Omega^-) \cap \mathcal{V}_{\eta, e^{-t/2}R}^-(\gamma\mathbf{g}^{-t/2}\Omega^+)\}$$

and

$$j_{\eta,\gamma}(T) = \iint_{(t, \ell) \in \mathcal{A}_{\eta,\gamma}(T)} h_{\eta, e^{-t/2}R}^-(\mathbf{g}^{t/2}w^-) h_{\eta, e^{-t/2}R}^+(\mathbf{g}^{-t/2}w^+) dt d\tilde{m}_F(\ell).$$

By the above, since the integral of a function is equal to the integral on any Borel set containing its support, and since the integral of a nonnegative function is nondecreasing in the integration domain, there hence exists $c_4 > 0$ such that for all $T \geq 0$ and $\eta \in]0, 1]$, we have

$$i_\eta(T) \geq -c_4 + e^{-2\eta\|F\|_\infty} \sum_{\substack{\gamma \in \Gamma : t_0 + 2 \leq \lambda_\gamma \leq T - 2\eta \\ \alpha_\gamma^-|_{[0, \lambda_\gamma]} \in \Omega^-|_{[0, \lambda_\gamma]}, \alpha_\gamma^+|_{[-\lambda_\gamma, 0]} \in \gamma\Omega^+|_{[-\lambda_\gamma, 0]}} e^{\int_{\alpha_\gamma} \tilde{F}} j_{\eta,\gamma}(T),$$

and similarly, for every $T' \geq T$ (later on, we will take T' to be $T + 4\eta$),

$$i_\eta(T) \leq c_4 + e^{2\eta\|F\|_\infty} \sum_{\substack{\gamma \in \Gamma : t_0 + 2 \leq \lambda_\gamma \leq T + 2\eta \\ \alpha_\gamma^-|_{[0, \lambda_\gamma]} \in \Omega^-|_{[0, \lambda_\gamma]}, \alpha_\gamma^+|_{[-\lambda_\gamma, 0]} \in \gamma\Omega^+|_{[-\lambda_\gamma, 0]}} e^{\int_{\alpha_\gamma} \tilde{F}} j_{\eta,\gamma}(T').$$

Step 4T: Conclusion. Let $\epsilon > 0$. Let $\gamma \in \Gamma$ be such that D^- and γD^+ do not intersect and the length of their common perpendicular satisfies $\lambda_\gamma \geq t_0 + 2$. Let us prove that if η is small enough and λ_γ is large enough (with the enough's independent of γ), then for every $T \geq \lambda_\gamma + 2\eta$, we have

$$1 - \epsilon \leq j_{\eta,\gamma}(T) \leq 1 + \epsilon. \quad (11.12)$$

This estimate proves the claim (11.4), as follows. For every $\epsilon > 0$, if $\eta > 0$ is small enough, we have

$$i_\eta(T + 2\eta) \geq -c_4 + e^{-2\eta\|F\|_\infty} (1 - \epsilon) \left(\frac{I_{\Omega^-, \Omega^+}(T)}{\delta \|m_F\| e^{-\delta T}} - \frac{I_{\Omega^-, \Omega^+}(t_0 + 2)}{\delta \|m_F\| e^{-\delta(t_0 + 2)}} \right)$$

and by Equation (11.7)

$$i_\eta(T + 2\eta) \leq c_\epsilon + \frac{e^\epsilon \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+)}{\delta \|m_F\| e^{-\delta(T + 2\eta)}}.$$

Thus, for η small enough,

$$\tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+) \geq \frac{e^{2\epsilon}}{1-\epsilon} I_{\Omega^-, \Omega^+}(T) + o(1)$$

as $T \rightarrow +\infty$, which gives

$$\limsup_{T \rightarrow +\infty} I_{\Omega^-, \Omega^+}(T) \leq \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+).$$

The similar estimate for the lower limit proves the claim (11.4).

To prove the claim (11.12), let $\eta \in]0, 1]$ and $T \geq \lambda_\gamma + 2\eta$. To simplify the notation, let

$$r_t = e^{-t/2} R, \quad w_t^- = \mathbf{g}^{t/2} w^-, \quad \text{and} \quad w_t^+ = \mathbf{g}^{-t/2} w^+.$$

By the definition of $j_{\eta, \gamma}$, using the inequalities (10.3) (and the comment following them), we hence have

$$\begin{aligned} j_{\eta, \gamma}(T) &= \iint_{(t, \ell) \in \mathcal{A}_{\eta, \gamma}(T)} h_{\eta, r_t}^-(w_t^-) h_{\eta, r_t}^+(w_t^+) dt d\tilde{m}_F(\ell) \\ &= \frac{e^{O(e^{-\ell_\gamma/2})}}{(2\eta)^2} \iint_{(t, \ell) \in \mathcal{A}_{\eta, \gamma}(T)} \frac{dt d\tilde{m}_F(\ell)}{\mu_{W^+(w_t^-)}(B^+(w_t^-, r_t)) \mu_{W^-(w_t^+)}(B^-(w_t^+, r_t))}. \end{aligned} \quad (11.13)$$

Let x_γ be the midpoint of the common perpendicular α_γ . Let us use the Hopf parametrisation of $\mathcal{G}X$ with basepoint x_γ , denoting by s its time parameter. When $(t, \ell) \in \mathcal{A}_{\eta, \gamma}(T)$, we have, by Proposition 3.10 where we may take $\kappa_2 = 1$,

$$\begin{aligned} d\tilde{m}_F(\ell) &= e^{C_{\ell_-}^-(x_\gamma, \ell(0)) + C_{\ell_+}^+(x_\gamma, \ell(0))} d\mu_{x_\gamma}^-(\ell_-) d\mu_{x_\gamma}^+(\ell_+) ds \\ &= e^{O(\eta)} d\mu_{x_\gamma}^-(\ell_-) d\mu_{x_\gamma}^+(\ell_+) ds. \end{aligned} \quad (11.14)$$

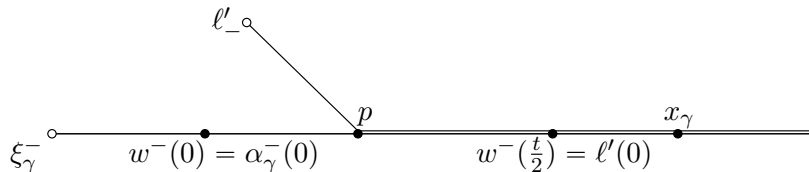
Let P_γ be the plane domain of $(t, s) \in \mathbb{R}^2$ such that there exist $s^\pm \in]-\eta, \eta[$ with $s^\mp = \frac{\lambda_\gamma - t}{2} \pm s$. It is easy to see that P_γ is a rhombus centred at $(\lambda_\gamma, 0)$ whose area is $(2\eta)^2$.

Let ξ_γ^\pm be the point at infinity of any fixed geodesic ray from x_γ through $\alpha_\gamma^\pm(0)$. If A is a subset of $\mathcal{G}X$, we denote by A_\pm the subset $\{\ell_\pm : \ell \in A\}$ of $\partial_\infty X$.

Lemma 11.4. *For every $t \geq t_0$, we have*

$$(B^\pm(w_t^\mp, r_t))_\mp = B_{d_{x_\gamma}}(\xi_\gamma^\mp, R e^{-\frac{\lambda_\gamma}{2}}).$$

Proof. We prove the statement for the negative endpoints, the proof of the claim for positive endpoints is similar. Let $\ell' \in B^\pm(w_t^-, r_t)$, with $\ell'_- \neq \xi_\gamma^-$. Let $p \in X$ be such that $[\ell'(0), \xi_\gamma^-] \cap [\ell'(0), \ell'_-] = [\ell'(0), p]$.



Since $t \geq t_0 > 2 \ln R$, we have $r_t < 1$, hence $\ell'(0) = w_t^-(0) = w^-(t/2)$ and $p \in]\ell'(0), \xi_\gamma^-[$. Since $t \geq t_0 \geq 1 \geq \eta$, we have $p \in [x_\gamma, \xi_\gamma^-]$. Hence

$$d_{x_\gamma}(\ell'_-, \xi_\gamma^-) = e^{-d(p, x_\gamma)} = e^{-d(p, w^-(\frac{t}{2})) - (\frac{\lambda_\gamma}{2} - \frac{t}{2})} < r_t e^{-\frac{\lambda_\gamma}{2} + \frac{t}{2}} = R e^{-\frac{\lambda_\gamma}{2}}.$$

Conversely, if $\xi \in B_{d_{x_\gamma}}(\xi_\gamma^\mp, R e^{-\frac{\lambda_\gamma}{2}})$ with $\xi \neq \xi_\gamma^\mp$, let $\ell' \in \mathcal{G}X$ be such that $\ell'(0) = w^-(t/2)$ and $\ell'_- = \xi$. Let \widehat{w}_t^- be the extension of w_t^- such that $(\widehat{w}_t^-)_- = \xi_\gamma^-$. Then as above, we have $d_{W^-(\widehat{w}_t^-)}(\widehat{w}_t^-, \ell') < r_t$. \square

It follows from this lemma that, for all $t \geq t_0$, $s^\pm \in]-\eta, \eta[$ and $\ell \in \mathcal{G}X$, we have $\mathbf{g}^\mp s^\mp \ell \in B^\pm(w_t^\mp, r_t)$ if and only if $d(\ell(0), \alpha_\gamma^\pm(0)) = s^\pm + \frac{t}{2}$ (or equivalently by the definition of the time parameter s in Hopf's parametrisation, $s^\pm + \frac{t}{2} = \frac{\lambda_\gamma}{2} \pm s$), and $\ell_\pm \in B_{d_{x_\gamma}}(\xi_\gamma^\pm, R e^{-\frac{\lambda_\gamma}{2}})$. Thus,

$$\mathcal{A}_{\eta, \gamma}(T) = P_\gamma \times B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\frac{\lambda_\gamma}{2}}) \times B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\frac{\lambda_\gamma}{2}}).$$

To finish Step 4T and the proof of the theorem for \mathbb{R} -trees, note that by the definition of the skinning measure (using again the Hopf parametrisation with basepoint x_γ), by the above Lemma 11.4 and by Proposition 3.10 where we may take $\kappa_2 = 1$, we have

$$\mu_{W^\pm(w_t^\mp)}(B^\pm(w_t^\mp, r_t)) = e^{O(\eta)} \mu_{x_\gamma}^\mp(B_{d_{x_\gamma}}(\xi_\gamma^\mp, R e^{-\frac{\lambda_\gamma}{2}})). \quad (11.15)$$

Thus, by the above and by Equations (11.13) and (11.14), (and noting that $O(\eta) + O(\eta) = O(\eta)$)

$$\begin{aligned} j_{\eta, \gamma}(T) &= \frac{e^{O(e^{-\frac{\lambda_\gamma}{2}})}}{4\eta^2} e^{O(\eta)} (2\eta)^2 \frac{\mu_{x_\gamma}^-(B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\frac{\lambda_\gamma}{2}})) \mu_{x_\gamma}^+(B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\frac{\lambda_\gamma}{2}}))}{\mu_{x_\gamma}^-(B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\frac{\lambda_\gamma}{2}})) \mu_{x_\gamma}^+(B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\frac{\lambda_\gamma}{2}}))} \\ &= e^{O(\eta + e^{-\frac{\lambda_\gamma}{2}})}, \end{aligned} \quad (11.16)$$

which gives the inequalities (11.12). \square

The effective control on $j_{\eta, \gamma}(T)$ given by Equation (11.16) is stronger than what is needed in order to prove Equation (11.12) in Step 4T. We will use it in Section 12.6 in order to obtain error terms.

11.3 Part III of the proof of Theorem 11.1: the manifold case

The proof of Theorem 11.1 for manifolds is the same as for trees until Equation (11.10). The rest of the proof that we give below is more technical than for trees but the structure of the proof is the same. In this Section, $X = \widetilde{M}$ is a Riemannian manifold, and we identify $\mathcal{G}X$ with $T^1\widetilde{M}$.

Step 3M. Consider the product term in Equation (11.8) involving the characteristic functions. The quantity $\mathbb{1}_{\mathcal{V}_{\eta, R}^+(\Omega^-)}(\mathbf{g}^{-t/2}v) \mathbb{1}_{\mathcal{V}_{\eta, R}^-(\Omega^+)}(\gamma^{-1}\mathbf{g}^{t/2}v)$ is different from 0 (hence equal to 1) if and only if

$$v \in \mathbf{g}^{t/2} \mathcal{V}_{\eta, R}^+(\Omega^-) \cap \gamma \mathbf{g}^{-t/2} \mathcal{V}_{\eta, R}^-(\Omega^+) = \mathcal{V}_{\eta, e^{-t/2}R}^+(\mathbf{g}^{t/2}\Omega^-) \cap \mathcal{V}_{\eta, e^{-t/2}R}^-(\gamma \mathbf{g}^{-t/2}\Omega^+),$$

see Section 2.5 and in particular Equation (2.15). By [PaP16c, Lem. 7] (applied by replacing D^+ by γD^+ and w by v), there exists $t_0, c_0 > 0$ such that for all $\eta \in]0, 1]$ and $t \geq t_0$, for all $v \in T^1 \widetilde{M}$, if $\mathbb{1}_{\mathcal{V}_{\eta, R}^+(\Omega^-)}(\mathbf{g}^{-t/2}v) \mathbb{1}_{\mathcal{V}_{\eta, R}^-(\Omega^+)}(\gamma^{-1}\mathbf{g}^{t/2}v) \neq 0$, then the following facts hold:

- (i) by the convexity of D^\pm , we have $v \in \mathcal{U}_{D^-}^+ \cap \mathcal{U}_{\gamma D^+}^-$,
- (ii) by the definition of w^\pm (see Equation (11.9)), we have $w^- \in \Omega^-$ and $w^+ \in \gamma\Omega^+$. The notation (w^-, w^+) here coincides with the notation (w^-, w^+) in Lemma 2.8,
- (iii) there exists a common perpendicular α_γ from D^- to γD^+ , whose length λ_γ satisfies

$$|\lambda_\gamma - t| \leq 2\eta + c_0 e^{-t/2},$$

whose origin $\pi(v_\gamma^-)$ is at distance at most $c_0 e^{-t/2}$ from $\pi(w^-)$, whose endpoint $\pi(v_\gamma^+)$ is at distance at most $c_0 e^{-t/2}$ from $\pi(w^+)$, such that both points $\pi(\mathbf{g}^{t/2}w^-)$ and $\pi(\mathbf{g}^{-t/2}w^+)$ are at distance at most $\eta + c_0 e^{-t/2}$ from $\pi(v)$, which is at distance at most $c_0 e^{-t/2}$ from some point p_v of α_γ .

In particular, using (iii) and the uniform continuity property of the \widetilde{F} -weighted length (see Proposition 3.10 (3) which introduces a constant $c_2 \in]0, 1]$), and since \widetilde{F} is bounded, for all $\eta \in]0, 1]$, $t \geq t_0$ and $v \in T^1 \widetilde{M}$ for which $\mathbb{1}_{\mathcal{V}_{\eta, R}^+(\Omega^-)}(\mathbf{g}^{-t/2}v) \mathbb{1}_{\mathcal{V}_{\eta, R}^-(\Omega^+)}(\gamma^{-1}\mathbf{g}^{t/2}v) \neq 0$, we have

$$\begin{aligned} e^{\int_{\pi(w^-)}^{\pi(\mathbf{g}^{t/2}w^-)} \widetilde{F} + \int_{\pi(\mathbf{g}^{-t/2}w^+)}^{\pi(w^+)} \widetilde{F}} &= e^{\int_{\pi(v_\gamma^-)}^{p_v} \widetilde{F} + \int_{p_v}^{\pi(v_\gamma^+)} \widetilde{F} + \mathcal{O}((\eta + e^{-t/2})c_2)} \\ &= e^{\int_{\alpha_\gamma} \widetilde{F}} e^{\mathcal{O}((\eta + e^{-\lambda_\gamma/2})c_2)}. \end{aligned} \quad (11.17)$$

For all $\eta \in]0, 1]$, $\gamma \in \Gamma$ and $T \geq t_0$, define $\mathcal{A}_{\eta, \gamma}(T)$ as the set of $(t, v) \in [t_0, T] \times T^1 \widetilde{M}$ such that $v \in \mathcal{V}_{\eta, e^{-t/2}R}^+(\mathbf{g}^{t/2}\Omega^-) \cap \mathcal{V}_{\eta, e^{-t/2}R}^-(\gamma\mathbf{g}^{-t/2}\Omega^+)$, and

$$j_{\eta, \gamma}(T) = \iint_{(t, v) \in \mathcal{A}_{\eta, \gamma}(T)} h_{\eta, e^{-t/2}R}^-(\mathbf{g}^{t/2}w^-) h_{\eta, e^{-t/2}R}^+(\mathbf{g}^{-t/2}w^+) dt d\widetilde{m}_F(v).$$

By the above, since the integral of a function is equal to the integral on any Borel set containing its support, and since the integral of a nonnegative function is nondecreasing in the integration domain, there hence exists $c_4 > 0$ such that for all $T \geq 0$ and $\eta \in]0, 1]$, we have

$$i_\eta(T) \geq -c_4 + \sum_{\substack{\gamma \in \Gamma: t_0 + 2 + c_0 \leq \lambda_\gamma \leq T - \mathcal{O}(\eta + e^{-\lambda_\gamma/2}) \\ v_\gamma^- \in \mathcal{N}_{-\mathcal{O}(\eta + e^{-\lambda_\gamma/2})}^-\Omega^-, v_\gamma^+ \in \gamma \mathcal{N}_{-\mathcal{O}(\eta + e^{-\lambda_\gamma/2})}^+\Omega^+}} e^{\int_{\alpha_\gamma} \widetilde{F}} j_{\eta, \gamma}(T) e^{-\mathcal{O}((\eta + e^{-\lambda_\gamma/2})c_2)},$$

and similarly, for every $T' \geq T$,

$$i_\eta(T) \leq c_4 + \sum_{\substack{\gamma \in \Gamma: t_0 + 2 + c_0 \leq \lambda_\gamma \leq T + \mathcal{O}(\eta + e^{-\lambda_\gamma/2}) \\ v_\gamma^- \in \mathcal{N}_{\mathcal{O}(\eta + e^{-\lambda_\gamma/2})}^-\Omega^-, v_\gamma^+ \in \gamma \mathcal{N}_{\mathcal{O}(\eta + e^{-\lambda_\gamma/2})}^+\Omega^+}} e^{\int_{\alpha_\gamma} \widetilde{F}} j_{\eta, \gamma}(T') e^{\mathcal{O}((\eta + e^{-\lambda_\gamma/2})c_2)}.$$

We will take T' to be of the form $T + \mathcal{O}(\eta + e^{-\lambda_\gamma/2})$, for a bigger $\mathcal{O}(\cdot)$ than the one appearing in the index of the above summation.

Step 4M: Conclusion. Let $\gamma \in \Gamma$ be such that D^- and γD^+ have a common perpendicular with length $\lambda_\gamma \geq t_0 + 2 + c_0$. Let us prove that for all $\epsilon > 0$, if η is small enough and λ_γ is large enough, then for every $T \geq \lambda_\gamma + O(\eta + e^{-\lambda_\gamma/2})$ (with the enough's and $O(\cdot)$ independent of γ), we have

$$1 - \epsilon \leq j_{\eta, \gamma}(T) \leq 1 + \epsilon. \quad (11.18)$$

Note that $\tilde{\sigma}^\pm(\mathcal{N}_\epsilon(\Omega^\mp))$ and $\tilde{\sigma}^\pm(\mathcal{N}_{-\epsilon}(\Omega^\mp))$ tend to $\tilde{\sigma}^\pm(\Omega^\mp)$ as $\epsilon \rightarrow 0$ (since $\tilde{\sigma}^\pm(\partial\Omega^\mp) = 0$ as required in Step 1). Using Steps 2, 3M and 4M, this will prove Equation (11.4), hence will complete the proof of Theorem 11.1.

We say that $(\widetilde{M}, \Gamma, \widetilde{F})$ has *radius-continuous strong stable/unstable ball masses* if for every $\epsilon > 0$, if $r > 1$ is close enough to 1, then for every $v \in T^1\widetilde{M}$, if $B^-(v, 1)$ meets the support of $\mu_{W^-(v)}^+$, then

$$\mu_{W^-(v)}^+(B^-(v, r)) \leq e^\epsilon \mu_{W^-(v)}^+(B^-(v, 1))$$

and if $B^+(v, 1)$ meets the support of $\mu_{W^+(v)}^-$, then

$$\mu_{W^+(v)}^-(B^+(v, r)) \leq e^\epsilon \mu_{W^+(v)}^-(B^+(v, 1)).$$

We say that $(\widetilde{M}, \Gamma, \widetilde{F})$ has *radius-Hölder-continuous strong stable/unstable ball masses* if there exists $c \in]0, 1]$ and $c' > 0$ such that for every $\epsilon \in]0, 1]$, if $B^-(v, 1)$ meets the support of $\mu_{W^-(v)}^+$, then

$$\mu_{W^-(v)}^+(B^-(v, r)) \leq e^{c'\epsilon^c} \mu_{W^-(v)}^+(B^-(v, 1))$$

and if $B^+(v, 1)$ meets the support of $\mu_{W^+(v)}^-$, then

$$\mu_{W^+(v)}^-(B^+(v, r)) \leq e^{c'\epsilon^c} \mu_{W^+(v)}^-(B^+(v, 1)).$$

Note that when $F = 0$ and M has locally symmetric with finite volume, the conditional measures on the strong stable/unstable leaves are homogeneous. Hence $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses.

When the sectional curvature has bounded derivatives and when $(\widetilde{M}, \Gamma, \widetilde{F})$ has Hölder strong stable/unstable ball masses, we will prove a stronger statement: with a constant $c_\gamma > 0$ and functions $O(\cdot)$ independent of γ , for all $\eta \in]0, 1]$ and $T \geq \lambda_\gamma + O(\eta + e^{-\lambda_\gamma/2})$, we have

$$j_{\eta, \gamma}(T) = \left(1 + O\left(\frac{e^{-\lambda_\gamma/2}}{2\eta}\right)\right)^2 e^{O((\eta + e^{-\lambda_\gamma/2})^{c_\gamma})}. \quad (11.19)$$

This stronger version will be needed for the error term estimate in Section 12.3. In order to obtain Theorem 11.1, only the fact that $j_{\eta, \gamma}(T)$ tends to 1 as firstly λ_γ tends to $+\infty$, secondly η tends to 0 is needed. A reader not interested in the error term may skip many technical details below.

Given $a, b > 0$ and a point x in a metric space X (with a, b, x depending on parameters), we will denote by $B(x, a e^{O(b)})$ any subset Y of X such that there exists a constant $c > 0$ (independent of the parameters) with

$$B(x, a e^{-cb}) \subset Y \subset B(x, a e^{cb}).$$

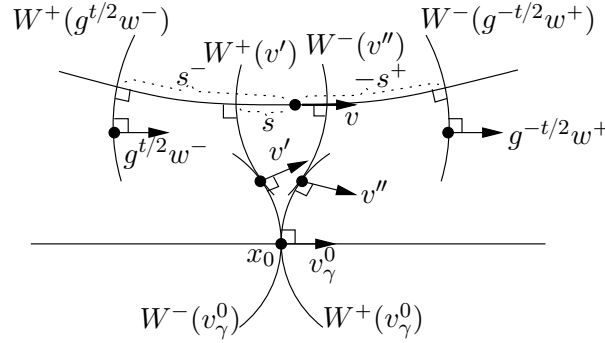
Let $\eta \in]0, 1]$ and $T \geq \lambda_\gamma + O(\eta + e^{-\lambda_\gamma/2})$. To simplify the notation, let

$$r_t = e^{-t/2}R, \quad w_t^- = \mathbf{g}^{t/2}w^- \quad \text{and} \quad w_t^+ = \mathbf{g}^{-t/2}w^+.$$

By the definition of $j_{\eta,\gamma}$, using the inequalities (10.3), we hence have

$$\begin{aligned} j_{\eta,\gamma}(T) &= \iint_{(t,v) \in \mathcal{A}_{\eta,\gamma}(T)} h_{\eta,r_t}^-(w_t^-) h_{\eta,r_t}^+(w_t^+) dt d\tilde{m}_F(v) \\ &= \frac{e^{O(e^{-c_2\lambda\gamma/2})}}{(2\eta)^2} \iint_{(t,v) \in \mathcal{A}_{\eta,\gamma}(T)} \frac{dt d\tilde{m}_F(v)}{\mu_{W^+(w_t^-)}^-(B^+(w_t^-, r_t)) \mu_{W^-(w_t^+)}^+(B^-(w_t^+, r_t))}. \end{aligned} \quad (11.20)$$

We start the proof of Equation (11.18) by defining parameters s^+, s^-, s, v', v'' associated with $(t, v) \in \mathcal{A}_{\eta,\gamma}(T)$.



We have $(t, v) \in \mathcal{A}_{\eta,\gamma}(T)$ if and only if there exist $s^\pm \in]-\eta, \eta[$ such that

$$g^{\mp s^\pm} v \in B^\pm(g^{\pm t/2} w^\mp, e^{-t/2} R).$$

The notation s^\pm coincides with the one in the proof of Lemma 2.8 (where (D^+, w) has been replaced by $(\gamma D^+, v)$).

In order to define the parameters s, v', v'' , we use the well known local product structure of the unit tangent bundle in negative curvature. If $v \in T^1 M$ is close enough to v_γ^0 (in particular, $v_- \neq (v_\gamma^0)_+$ and $v_+ \neq (v_\gamma^0)_-$), then let $v' = f_{HB_-(v_\gamma^0)}^+(v)$ be the unique element of $W^-(v_\gamma^0)$ such that $v'_+ = v_+$, let $v'' = f_{HB_+(v_\gamma^0)}^-(v)$ be the unique element of $W^+(v_\gamma^0)$ such that $v''_- = v_-$, and let s be the unique element of \mathbb{R} such that $g^{-s} v \in W^+(v')$. The map $v \mapsto (s, v', v'')$ is a homeomorphism from a neighbourhood of v_γ^0 in $T^1 \tilde{M}$ to a neighbourhood of $(0, v_\gamma^0, v_\gamma^0)$ in $\mathbb{R} \times W^-(v_\gamma^0) \times W^+(v_\gamma^0)$. Note that if $v = g^r v_\gamma^0$ for some $r \in \mathbb{R}$ close to 0, then

$$w^- = v_\gamma^-, w^+ = v_\gamma^+, s = r, v' = v'' = v_\gamma^0, s^- = \frac{\lambda_\gamma - t}{2} + s, s^+ = \frac{\lambda_\gamma - t}{2} - s.$$

Up to increasing t_0 (which does not change Step 4, up to increasing c_4), we may assume that for every $(t, v) \in \mathcal{A}_{\eta,\gamma}(T)$, the vector v belongs to the domain of this local product structure of $T^1 \tilde{M}$ at v_γ^0 .

The vectors v, v', v'' are close to v_γ^0 if t is large and η small, as the following result shows. We denote (also) by d the Riemannian distance induced by Sasaki's metric on $T^1 \tilde{M}$.

Lemma 11.5. *For every $(t, v) \in \mathcal{A}_{\eta,\gamma}(T)$, we have $d(v, v_\gamma^0), d(v', v_\gamma^0), d(v'', v_\gamma^0) = O(\eta + e^{-t/2})$.*

Proof. Consider the distance d' on $T^1\widetilde{M}$, defined by

$$\forall v_1, v_2 \in T^1\widetilde{M}, \quad d'(v_1, v_2) = \max_{r \in [-1, 0]} d(\pi(\mathbf{g}^r v_1), \pi(\mathbf{g}^r v_2)) .$$

As seen in (iii) of Step 3M, we have $d(\pi(w^\pm), \pi(v_\gamma^\pm)), d(\pi(v), \alpha_\gamma) = O(e^{-t/2})$, and furthermore, $d(\pi(\mathbf{g}^{t/2} w^-), \pi(v)), \frac{\lambda_\gamma}{2} - \frac{t}{2} = O(\eta + e^{-t/2})$. Hence $d(\pi(v), \pi(v_\gamma^0)) = O(\eta + e^{-t/2})$. By [PauPS, Lem. 2.4], we have

$$d(\pi(\mathbf{g}^{-\frac{t}{2}-s^-} v), \pi(v_\gamma^-)) \leq d(\pi(\mathbf{g}^{-\frac{t}{2}-s^-} v), \pi(w^-)) + d(\pi(w^-), \pi(v_\gamma^-)) \leq R + c_0 e^{-t/2} .$$

By an exponential pinching argument, we hence have $d'(v, v_\gamma^0) = O(\eta + e^{-\lambda_\gamma/2})$. Since d and d' are equivalent (see [Bal, page 70]), we therefore have $d(v, v_\gamma^0) = O(\eta + e^{-\lambda_\gamma/2})$.

For all $w \in T^1\widetilde{M}$ and $V \in T_w T^1\widetilde{M}$, we may uniquely write $V = V^- + V^0 + V^+$ with $V^- \in T_w W^-(w)$, $V^0 \in \mathbb{R} \frac{d}{dt}|_{t_0} \mathbf{g}^t w$ and $V^+ \in T_w W^+(w)$. By [PauPS, §7.2] (building on [Brin] whose compactness assumption on M and torsion free assumption on Γ are not necessary for this, the pinched negative curvature assumption is sufficient), Sasaki's metric (with norm $\|\cdot\|$) is equivalent to the Riemannian metric with (product) norm

$$\|V\|' = \sqrt{\|V^-\|^2 + \|V^0\|^2 + \|V^+\|^2} .$$

By the dynamical local product structure of $T^1\widetilde{M}$ in the neighbourhood of v_γ^0 and by the definition of v', v'' , the result follows, since the exponential map of $T^1\widetilde{M}$ at v_γ^0 is almost isometric close to 0 and the projection to a factor of a product norm is Lipschitz. \square

We now use the local product structure of the Gibbs measure to prove the following result.

Lemma 11.6. *For every $(t, v) \in \mathcal{A}_{\eta, \gamma}(T)$, we have*

$$dt d\widetilde{m}_F(v) = e^{O((\eta + e^{-\lambda_\gamma/2})^{c_2})} dt ds d\mu_{W^-(v_\gamma^0)}(v') d\mu_{W^+(v_\gamma^0)}(v'') .$$

Proof. By the definition of the measures (see Equations (4.3) and (7.9)), since the above parameter s differs, when v_-, v_+ are fixed, only up to a constant from the time parameter in Hopf's parametrisation, we have

$$\begin{aligned} d\widetilde{m}_F(v) &= e^{C_{v_-}^-(x_0, \pi(v)) + C_{v_+}^+(x_0, \pi(v))} d\mu_{x_0}^-(v_-) d\mu_{x_0}^+(v_+) ds \\ d\mu_{W^-(v_\gamma^0)}(v') &= e^{C_{v_+}^+(x_0, \pi(v'))} d\mu_{x_0}^+(v_+) , \\ d\mu_{W^+(v_\gamma^0)}(v'') &= e^{C_{v_-}^-(x_0, \pi(v''))} d\mu_{x_0}^-(v_-) . \end{aligned}$$

By Proposition 3.10 (2) since F is bounded, we have $|C_\xi^\pm(z, z')| = O(d(z, z')^{c_2})$ for all $\xi \in \partial_\infty \widetilde{M}$ and $z, z' \in \widetilde{M}$ with $d(z, z')$ bounded. Since the map $\pi : T^1\widetilde{M} \rightarrow \widetilde{M}$ is Lipschitz, and since $v_+ = v_+'$ and $v_- = v_-''$, the result follows from Lemma 11.5 and the cocycle property (3.7). \square

When λ_γ is large, the submanifold $g^{\lambda_\gamma/2}\Omega^-$ has a second order contact at v_γ^0 with $W^-(v_\gamma^0)$ and similarly, $g^{-\lambda_\gamma/2}\Omega^+$ has a second order contact at v_γ^0 with $W^+(v_\gamma^0)$. Let P_γ be the plane domain of $(t, s) \in \mathbb{R}^2$ such that there exist $s^\pm \in]-\eta, \eta[$ with $s^\mp = \frac{\lambda_\gamma - t}{2} \pm s + O(e^{-\lambda_\gamma/2})$.

Note that its area is $(2\eta + O(e^{-\lambda\gamma/2}))^2$. By the above, we have (with the obvious meaning of a double inclusion)

$$\mathcal{A}_{\eta,\gamma}(T) = P_\gamma \times B^-(v_\gamma^0, r_{\lambda_\gamma} e^{O(\eta+e^{-\lambda\gamma/2})}) \times B^+(v_\gamma^0, r_{\lambda_\gamma} e^{O(\eta+e^{-\lambda\gamma/2})}).$$

By Lemma 11.6, we hence have

$$\begin{aligned} \int_{\mathcal{A}_{\eta,\gamma}(T)} dt d\tilde{m}_F(v) &= e^{O((\eta+e^{-\lambda\gamma/2})c_2)} (2\eta + O(e^{-\lambda\gamma/2}))^2 \times \\ &\mu_{W^-(v_\gamma^0)}(B^-(v_\gamma^0, r_{\lambda_\gamma} e^{O(\eta+e^{-\lambda\gamma/2})})) \mu_{W^+(v_\gamma^0)}(B^+(v_\gamma^0, r_{\lambda_\gamma} e^{O(\eta+e^{-\lambda\gamma/2})})). \end{aligned} \quad (11.21)$$

The last ingredient of the proof of Step 4M is the following continuity property of strong stable and strong unstable ball volumes as their centre varies. See [Rob2, Lem. 1.16], [PauPS, Prop. 10.16] for related properties, although we need a more precise control for the error term in Section 12.3.

Lemma 11.7. *Assume that $(\tilde{M}, \Gamma, \tilde{F})$ has radius-continuous strong stable/unstable ball masses. There exists $c_5 > 0$ such that for every $\epsilon > 0$, if η is small enough and λ_γ large enough, then for every $(t, v) \in \mathcal{A}_{\eta,\gamma}(T)$, we have*

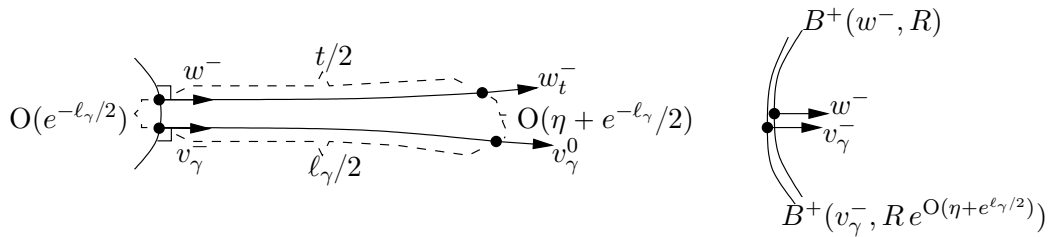
$$\mu_{W^-(w_t^+)}(B^-(w_t^+, r_t)) = e^{O(\epsilon^{c_5})} \mu_{W^-(v_\gamma^0)}(B^-(v_\gamma^0, r_{\lambda_\gamma}))$$

and

$$\mu_{W^+(w_t^-)}(B^+(w_t^-, r_t)) = e^{O(\epsilon^{c_5})} \mu_{W^+(v_\gamma^0)}(B^+(v_\gamma^0, r_{\lambda_\gamma})).$$

If we furthermore assume that the sectional curvature of \tilde{M} has bounded derivatives and that $(\tilde{M}, \Gamma, \tilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses, then we may replace ϵ by $(\eta + e^{-\lambda\gamma/2})^{c_6}$ for some constant $c_6 > 0$.

Proof. We prove the (second) claim for \mathcal{W}^+ , the (first) one for \mathcal{W}^- follows similarly. The final statement is only used for the error estimates in Section 12.3.



Using respectively Equation (2.13) since $w_t^- = \mathbf{g}^{t/2}w^-$ and $r_t = e^{-t/2}R$, Equation (7.11) where (v, t, w) is replaced by $(\mathbf{g}^{t/2}v, t/2, \mathbf{g}^{t/2}w^-)$, and Equation (3.8), we have

$$\begin{aligned} \mu_{W^+(w_t^-)}(B^+(w_t^-, r_t)) &= \int_{v \in B^+(w^-, R)} d\mu_{W^+(\mathbf{g}^{t/2}w^-)}(\mathbf{g}^{t/2}v) \\ &= \int_{v \in B^+(w^-, R)} e^{C_{v^-}(\pi(v), \pi(\mathbf{g}^{t/2}v))} d\mu_{W^+(w^-)}(v) \\ &= \int_{v \in B^+(w^-, R)} e^{\int_{\pi(v)}^{\pi(\mathbf{g}^{t/2}v)} (\tilde{F} - \delta)} d\mu_{W^+(w^-)}(v). \end{aligned} \quad (11.22)$$

Similarly, for every $a > 0$, we have

$$\mu_{W^+(v_\gamma^0)}(B^+(v_\gamma^0, ar_t)) = \int_{v \in B^+(v_\gamma^0, aR)} e^{\int_{\pi(v)}^{\pi(\mathbf{g}^{t/2}v)} (\tilde{F} - \delta)} d\mu_{W^+(v_\gamma^0)}(v). \quad (11.23)$$

Let $h^- : B^+(w^-, R) \rightarrow W^+(v_\gamma^-)$ be the map such that $(h^-(v))_- = v_-$, which is well defined and a homeomorphism onto its image if λ_γ is large enough (since R is fixed). By Proposition 7.4 applied with $D = HB_+(w^-)$ and $D' = HB_+(v_\gamma^-)$, we have, for every $v \in B^+(w^-, R)$,

$$d\mu_{W^+(w^-)}(v) = e^{-C_{v^-}(\pi(v), \pi(h^-(v)))} d\mu_{W^+(v_\gamma^-)}(h^-(v)).$$

Let us fix $\epsilon > 0$. The strong stable balls of radius R centred at w^- and v_γ^- are very close (see the picture in the beginning of the proof). More precisely, recall that R is fixed, and that $d(\pi(w^-), \pi(v_\gamma^-)) = O(e^{-\lambda_\gamma/2})$ and $d(\pi(\mathbf{g}^{t/2}w^-), \pi(\mathbf{g}^{\lambda_\gamma/2}v_\gamma^-)) = O(\eta + e^{-\lambda_\gamma/2})$. Therefore we have $d(\pi(v), \pi(h^-(v))) \leq \epsilon$ for every $v \in B^+(w^-, R)$ if η is small enough and λ_γ is large enough. If we furthermore assume that the sectional curvature has bounded derivatives, then by Anosov's arguments, the strong stable foliation is Hölder-continuous, see for instance [PauPS, Theo. 7.3]. Hence we have $d(\pi(v), \pi(h^-(v))) = O((\eta + e^{-\lambda_\gamma/2})^{c_5})$ for every $v \in B^+(w^-, R)$, for some constant $c_5 > 0$, under the additional regularity assumption on the curvature. We also have $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R e^{O(\epsilon)})$ and, under the additional hypothesis on the curvature, $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R e^{O((\eta + e^{-\lambda_\gamma/2})^{c_5})})$.

In what follows, we assume that $\epsilon = (\eta + e^{-\lambda_\gamma/2})^{c_5}$ under the additional assumption on the curvature. By Proposition 3.10 (2), we hence have, for every $v \in B^+(w^-, R)$,

$$d\mu_{W^+(w^-)}(v) = e^{O(\epsilon^{c_2})} d\mu_{W^+(v_\gamma^-)}(h^-(v))$$

and, using Proposition 3.10 (3),

$$\int_{\pi(v)}^{\pi(\mathbf{g}^{t/2}v)} (\tilde{F} - \delta) - \int_{\pi(h^-(v))}^{\pi(\mathbf{g}^{t/2}h^-(v))} (\tilde{F} - \delta) = O(\epsilon^{c_2}).$$

The result follows, by Equation (11.22) and (11.23) and the continuity property in the radius. \square

Now Lemma 11.7 (with ϵ as in its statement, and when its hypotheses are satisfied) implies that

$$\begin{aligned} & \iint_{(t,v) \in \mathcal{A}_{\eta,\gamma}(T)} \frac{dt d\tilde{m}_F(v)}{\mu_{W^+(w_t^-)}^-(B^+(w_t^-, r_t)) \mu_{W^-(w_t^+)}^+(B^-(w_t^+, r_t))} \\ &= \frac{e^{O(\epsilon^{c_5})} \iint_{(t,v) \in \mathcal{A}_{\eta,\gamma}(T)} dt d\tilde{m}_F(v)}{\mu_{W^+(v_\gamma^0)}^-(B^+(v_\gamma^0, r_t)) \mu_{W^-(v_\gamma^0)}^+(B^-(v_\gamma^0, r_t))}. \end{aligned}$$

By Equation (11.20) and Equation (11.21), we hence have

$$j_{\eta,\gamma}(T) = e^{O((\eta + e^{-\lambda_\gamma/2})^{c_2})} e^{\epsilon^{c_5}} \frac{(2\eta + O(e^{-\lambda_\gamma/2}))^2}{(2\eta)^2}$$

under the technical assumptions of Lemma 11.7. The assumption on radius-continuity of strong stable/unstable ball masses can be bypassed using bump functions, as explained in [Rob2, page 81]. \square

11.4 Equidistribution of common perpendiculars in simplicial trees

In this Section, we prove a version of Theorem 11.1 for the discrete time geodesic flow on quotients of simplicial trees (and we leave to the reader the version without the assumption that the critical exponent of the system of conductances is positive).

Let \mathbb{X} be a locally finite simplicial tree without terminal vertices. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant system of conductances on \mathbb{X} , let \tilde{F}_c be its associated potential (see Section 3.5), and let $m_c = m_{F_c}$. Let $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ and $\mathcal{D}^+ = (D_j^+)_{j \in I^+}$ be locally finite Γ -equivariant families of nonempty proper simplicial subtrees of \mathbb{X} .

For every edge path $\alpha = (e_1, \dots, e_n)$ in \mathbb{X} , we set

$$c(\alpha) = \sum_{i=1}^n \tilde{c}(e_i).$$

Theorem 11.8. *Assume that the critical exponent δ_c of \tilde{c} is positive and that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$. Then*

$$\lim_{t \rightarrow +\infty} \frac{e^{\delta_c} - 1}{e^{\delta_c}} \|m_c\| e^{-\delta_c t} \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_{i, \gamma j} \leq t}} e^{c(\alpha_{i, \gamma j})} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^-$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$.

Proof. The proof is a modification of the continuous time proof for metric trees in Sections 11.1 and 11.2. Here, we indicate the changes to adapt the proof to the discrete time. We use the conventions for the discrete time geodesic flow described in Section 2.7.

Note that for all $i \in I^-$, $j \in I^+$, $\gamma \in \Gamma$, the common perpendicular $\alpha_{i, \gamma j}$ is now an edge path from D_i^- to $D_{\gamma j}^+$, and that by Proposition 3.11, we have $\int_{\alpha_{i, \gamma j}} \tilde{F}_c = c(\alpha_{i, \gamma j})$.

In the definition of the bump functions in Section 10.1, we assume (as we may) that $\eta < 1$, so that for all $\eta' \in]0, 1[$ and $w \in \partial_{\mp}^1 D^\pm$ such that $w_\mp \in \Lambda\Gamma$, we have

$$V_{w, \eta, \eta'}^\pm = B^\pm(w, \eta'),$$

see Equation (2.12) and recall that we are only considering discrete geodesic lines. As $\ell(0) = w(0)$ for every $\ell \in B^\pm(w, \eta)$ since $\eta < 1$, and as the time is now discrete, Equations (10.1) and (7.12) give

$$h_{\eta, \eta'}^\pm(w) = \frac{1}{\mu_{W^\mp(w)}(B^\pm(w, \eta'))}. \quad (11.24)$$

This is a considerable simplification compared with the inequalities of Equation (10.3).

In the whole proof, we restrict to $t = n \in \mathbb{N}$, $T = N \in \mathbb{N}$. In Steps 1 and 2, we define instead of Equation (11.3)

$$I_{\Omega^-, \Omega^+}(N) = (e^{\delta_c} - 1) \|m_c\| e^{-\delta_c(N+1)} \sum_{\substack{\gamma \in \Gamma : 0 < \lambda_\gamma \leq N \\ \alpha_\gamma^- |_{]0, \lambda_\gamma]} \in \Omega^-, \alpha_\gamma^+ |_{] -\lambda_\gamma, 0]} \in \Omega^+ |_{] -\lambda_\gamma, 0]}} e^{\int_{\alpha_\gamma^-} \tilde{F}_c},$$

and instead of Equation (11.5)

$$a_\eta(n) = \sum_{\gamma \in \Gamma} \int_{\ell \in \mathcal{GX}} \phi_\eta^-(\mathbf{g}^{-\lfloor n/2 \rfloor} \ell) \phi_\eta^+(\mathbf{g}^{\lfloor n/2 \rfloor} \gamma^{-1} \ell) d\tilde{m}_c(\ell).$$

Equation (11.6) is replaced by

$$i_\eta(N) = \sum_{n=0}^N e^{\delta_c n} a_\eta(n),$$

so that by a geometric sum argument, the pair of inequalities (11.7) becomes

$$e^{-\epsilon} \frac{e^{\delta_c(N+1)} \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+)}{(e^{\delta_c} - 1) \|m_c\|} - c_\epsilon \leq i_\eta(N) \leq e^\epsilon \frac{e^{\delta_c(N+1)} \tilde{\sigma}^+(\Omega^-) \tilde{\sigma}^-(\Omega^+)}{(e^{\delta_c} - 1) \|m_c\|} + c_\epsilon.$$

Step 3 is unchanged up to replacing \int_0^T by $\sum_{n=0}^N$, \tilde{F} by \tilde{F}_c , δ by δ_c and $t/2$ by either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$, so that Equation (11.10) becomes, since $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$,

$$\begin{aligned} & h_{\eta,R}^- \circ f_{D^-}^+ (\ell) h_{\eta,R}^+ \circ f_{D^+}^-(\gamma^{-1} \ell) \\ &= e^{-\delta_c n} e^{\int_{w^-(0)}^{w^-(\lfloor n/2 \rfloor)} \tilde{F}_c + \int_{w^+(\lceil n/2 \rceil)}^{w^+(0)} \tilde{F}_c} h_{\eta, e^{-\lfloor n/2 \rfloor} R}^-(\mathbf{g}^{\lfloor n/2 \rfloor} w^-) h_{\eta, e^{-\lceil n/2 \rceil} R}^+(\mathbf{g}^{-\lceil n/2 \rceil} w^+). \end{aligned}$$

The proof then follows similarly as in Section 11.2, with the simplifications in the point (iii) that, taking $\eta < 1/2$, we have λ_γ equal to $t = n$, and the points $w^-(\lfloor \frac{n}{2} \rfloor)$, $w^+(\lceil \frac{n}{2} \rceil)$ and $\ell(0)$ are equal. In particular, Equation (11.11) simplifies as

$$e^{\int_{w^-(0)}^{w^-(\lfloor \frac{1}{2} \rfloor)} \tilde{F}_c + \int_{w^+(\lceil \frac{1}{2} \rceil)}^{w^+(0)} \tilde{F}_c} = e^{\int_{\alpha_\gamma} \tilde{F}_c},$$

thus avoiding the assumption that F_c (or equivalently c , see Section 3.2) is bounded. The end of Step 3T simplifies as

$$-c_4 \leq i_\eta(N) - \sum_{\substack{\gamma \in \Gamma: t_0 + 2 \leq \lambda_\gamma \leq N \\ \alpha_\gamma^- |_{[0, \lambda_\gamma]} \in \Omega^- |_{[0, \lambda_\gamma]}, \alpha_\gamma^+ |_{[-\lambda_\gamma, 0]} \in \gamma \Omega^+ |_{[-\lambda_\gamma, 0]}}} e^{\int_{\alpha_\gamma} \tilde{F}_c} j_{\eta, \gamma}(N) \leq c_4.$$

The statement of Step 4T now simplifies as

$$j_{\eta, \gamma}(N) = 1,$$

if $\eta < \frac{1}{2}$, and if $\gamma \in \Gamma$ is such that D^- and γD^+ do not intersect and λ_γ is large enough. We introduce in its proof the slightly modified notation

$$r_n^- = e^{-\lfloor \frac{n}{2} \rfloor} R, \quad r_n^+ = e^{-\lceil \frac{n}{2} \rceil} R, \quad w_n^- = \mathbf{g}^{\lfloor \frac{n}{2} \rfloor} w^- \quad \text{and} \quad w_n^+ = \mathbf{g}^{-\lceil \frac{n}{2} \rceil} w^+.$$

and we now take as x_γ the point at distance $\lfloor \frac{n}{2} \rfloor$ from its origin on the common perpendicular α_γ . Equation (11.13) becomes (using Equation (11.24) instead of Equation (10.3))

$$j_{\eta, \gamma}(N) = \iint_{(n, \ell) \in \mathcal{A}_{\eta, \gamma}(N)} \frac{dn d\tilde{m}_c(\ell)}{\mu_{W^+(w_n^-)}(B^+(w_n^-, r_n^-)) \mu_{W^-(w_n^+)}(B^-(w_n^+, r_n^+))}.$$

Since $\ell(0) = x_\gamma$ if $(n, \ell) \in \mathcal{A}_{\eta, \gamma}(N)$, Equation (11.14) simplifies as

$$d\tilde{m}_c(\ell) = d\mu_{x_\gamma}^-(\ell_-) d\mu_{x_\gamma}^+(\ell_+) ds ,$$

with ds the counting measure on the Hopf parameter $s \in \mathbb{Z}$ of ℓ . Replacing P_γ with its intersection with \mathbb{Z}^2 reduces it to one point $(\lambda_\gamma, 0)$, and now $s = s^\pm = 0$. Lemma 11.4 becomes

$$(B^+(w_n^-, r_n^-))_- = B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}), \quad (B^-(w_n^+, r_n^+))_+ = B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}),$$

so that

$$\mathcal{A}_{\eta, \gamma}(N) = \{(\lambda_\gamma, 0)\} \times B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}) \times B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}).$$

Finally, since $\ell(0) = x_\gamma$ if $(n, \ell) \in \mathcal{A}_{\eta, \gamma}(N)$, Equation (11.15) becomes

$$\begin{aligned} \mu_{W^+(w_n^-)}(B^+(w_n^-, r_n^-)) &= \mu_{x_\gamma}^-(B_{d_{x_\gamma}}(\xi_\gamma^-, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor})), \\ \mu_{W^-(w_n^+)}(B^\pm(w_n^+, r_n)) &= \mu_{x_\gamma}^+(B_{d_{x_\gamma}}(\xi_\gamma^+, R e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor})). \end{aligned}$$

The last centred equation in Step 4T now reduces to $j_{\eta, \gamma}(T) = 1$. □

For lattices in regular trees, we get more explicit expressions.

Corollary 11.9. *Let \mathbb{X} be a $(q+1)$ -regular simplicial tree (with $q \geq 2$) and let Γ be a lattice of \mathbb{X} such that $\Gamma \backslash \mathbb{X}$ is not bipartite. Assume that the Patterson density is normalised to be a family of probability measures. Let \mathbb{D}^\pm be nonempty proper simplicial subtrees of \mathbb{X} with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ , such that $\mathcal{D}^\pm = (\gamma \mathbb{D}^\pm)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^\pm}}$ is locally finite. Then*

$$\lim_{t \rightarrow +\infty} \frac{q-1}{q+1} \text{Vol}(\Gamma \backslash \mathbb{X}) q^{-t} \sum_{\substack{(\alpha, \beta, \gamma) \in \Gamma / \Gamma_{\mathbb{D}^-} \times \Gamma / \Gamma_{\mathbb{D}^+} \times \Gamma \\ 0 < d(\alpha \mathbb{D}^-, \gamma \beta \mathbb{D}^+) \leq t}} \Delta_{\alpha_{\alpha, \gamma \beta}^-} \otimes \Delta_{\alpha_{\gamma^{-1} \alpha, \beta}^+} = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^- ,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$.

If the measure $\sigma_{\mathcal{D}^+}^-$ is nonzero and finite, then

$$\lim_{t \rightarrow +\infty} \frac{q-1}{q+1} \frac{\text{Vol}(\Gamma \backslash \mathbb{X})}{\|\sigma_{\mathcal{D}^+}^-\|} q^{-t} \sum_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+}, 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq t} \Delta_{\alpha_{e, \gamma}^-} = \tilde{\sigma}_{\mathbb{D}^-}^+ ,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X}$.

Proof. In order to prove the first claim, we apply Theorem 11.8 with $\tilde{c} \equiv 0$, so that by Propositions 4.14, 4.15, and 8.1 (3), we have $\delta_c = \ln q > 0$, $m_c = m_{\text{BM}}$ is finite and mixing, and $\|m_{\text{BM}}\| = \frac{q}{q+1} \text{Vol}(\Gamma \backslash \mathbb{X})$.

The second claim follows by restricting to $\alpha = \beta = e$ and integrating on an appropriate fundamental domain (note that Equation 11.4 does not require Ω^+ to be relatively compact, just to have finite measure for $\tilde{\sigma}^-$). □

The mixing assumption in Theorem 11.8 implies that the length spectrum L_Γ of Γ is equal to \mathbb{Z} .⁵ The next result considers the other case, when only the square of the geodesic flow is mixing, when appropriately restricted. Note that the smallest nonempty Γ -invariant simplicial subtree of \mathbb{X} is uniform, without vertices of degree 2, for instance in the case when \mathbb{X} is $(p+1, q+1)$ -biregular with $p, q \geq 2$ and Γ is a lattice of \mathbb{X} .

⁵In fact, ergodicity is sufficient to have this.

Theorem 11.10. *Assume that the smallest nonempty Γ -invariant simplicial subtree of \mathbb{X} is uniform, without vertices of degree 2, and that the length spectrum L_Γ of Γ is $2\mathbb{Z}$. Assume that the critical exponent δ_c of \tilde{c} is positive, that the Gibbs measure m_c is finite and that its restriction to $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$ is mixing for the square of the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$. Then*

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{e^{2\delta_c} - 1}{2 e^{2\delta_c}} \|m_c\| e^{-\delta_c t} \sum_{\substack{i \in I^-/\sim, j \in I^+/\sim, \gamma \in \Gamma \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_{i, \gamma j} \leq t}} e^{c(\alpha_{i, \gamma j})} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} \\ = \tilde{\sigma}_{\mathcal{G}^-}^+ \otimes \tilde{\sigma}_{\mathcal{G}^+}^- \end{aligned}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}} \mathbb{X} \times \check{\mathcal{G}} \mathbb{X}$.

Proof. We denote by $\tilde{\sigma}_{\mathcal{G}^\pm, \text{even}}^\pm$ the restriction of $\tilde{\sigma}_{\mathcal{G}^\pm}^\pm$ to $\check{\mathcal{G}}_{\text{even}} \mathbb{X}$, and by $\tilde{\sigma}_{\mathcal{G}^\pm, \text{odd}}^\pm$ the restriction of $\tilde{\sigma}_{\mathcal{G}^\pm}^\pm$ to $\check{\mathcal{G}}_{\text{odd}} \mathbb{X} = \check{\mathcal{G}} \mathbb{X} - \check{\mathcal{G}}_{\text{even}} \mathbb{X}$. We denote by $V_{\text{even}} \mathbb{X}$ the subset of $V \mathbb{X}$ consisting of the vertices at even distance from x_0 , and by $V_{\text{odd}} \mathbb{X} = V \mathbb{X} - V_{\text{even}} \mathbb{X}$ its complement. The subsets $V_{\text{even}} \mathbb{X}$ and $V_{\text{odd}} \mathbb{X}$ are Γ -invariant if $L_\Gamma = 2\mathbb{Z}$ by Equation (4.13).

Let us first prove that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{e^{2\delta_c} - 1}{2 e^{2\delta_c}} \|m_c\| e^{-\delta_c t} \sum_{\substack{i \in I^-/\sim, j \in I^+/\sim, \gamma \in \Gamma \\ \pi(\alpha_{i, \gamma j}^-), \pi(\alpha_{\gamma^{-1}i, j}^+) \in V_{\text{even}} \mathbb{X} \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_{i, \gamma j} \leq t}} e^{c(\alpha_{i, \gamma j})} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i, j}^+} \\ = \tilde{\sigma}_{\mathcal{G}^-, \text{even}}^+ \otimes \tilde{\sigma}_{\mathcal{G}^+, \text{even}}^- \quad (11.25) \end{aligned}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}_{\text{even}} \mathbb{X} \times \check{\mathcal{G}}_{\text{even}} \mathbb{X}$.

The proof of this Equation (11.25) is a modification of the proof of the previous Theorem 11.8. We now restrict to $t = 2n \in \mathbb{N}$, $T = 2N \in \mathbb{N}$, and we replace \tilde{m}_c by $(\tilde{m}_c)|_{\mathcal{G}_{\text{even}} \mathbb{X}}$ and $(g^t)_{t \in \mathbb{Z}}$ by $(g^{2t})_{t \in \mathbb{Z}}$. Note that since \tilde{m}_c is invariant under the time 1 of the geodesic flow, which maps $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$ to $\Gamma \backslash \mathcal{G} \mathbb{X} - \Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$, we have

$$\|(\tilde{m}_c)|_{\mathcal{G}_{\text{even}} \mathbb{X}}\| = \frac{1}{2} \|\tilde{m}_c\|. \quad (11.26)$$

Note that for all $i \in I^-$, $j \in I^+$ and $\gamma \in \Gamma$, if $\pi(\alpha_{i, \gamma j}^-)$ and $\pi(\alpha_{\gamma^{-1}i, j}^+)$ belong to $V_{\text{even}} \mathbb{X}$, then the distance between D_i^- and γD_j^+ is even (since for all x, y, z in a simplicial tree, if p is the closest point from x to $[y, z]$, then $d(y, z) = d(y, x) + d(x, z) - 2d(x, p)$).

In Steps 1 and 2, we now consider Ω^\pm two Borel subsets of $\partial_{\mp}^1 D^\pm \cap \check{\mathcal{G}}_{\text{even}} \mathbb{X}$, and we define instead of Equation (11.3)

$$\begin{aligned} I_{\Omega^-, \Omega^+}(2N) = (e^{2\delta_c} - 1) \frac{\|m_c\|}{2} e^{-2\delta_c(N+1)} \\ \sum_{\substack{\gamma \in \Gamma : 0 < \lambda_\gamma \leq 2N, \pi(\alpha_\gamma^-), \pi(\alpha_\gamma^+) \in V_{\text{even}} \mathbb{X} \\ \alpha_\gamma^- |]0, \lambda_\gamma] \in \Omega^-, \alpha_\gamma^+ |]-\lambda_\gamma, 0] \in \Omega^+ |]-\lambda_\gamma, 0]} e^{\int_{\alpha_\gamma^-} \tilde{F}_c}, \end{aligned}$$

and instead of Equation (11.5)

$$a_\eta(2n) = \sum_{\gamma \in \Gamma} \int_{\ell \in \mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^-(\mathbf{g}^{-2\lfloor n/2 \rfloor} \ell) \phi_\eta^+(\mathbf{g}^{2\lfloor n/2 \rfloor} \gamma^{-1} \ell) d\tilde{m}_c(\ell).$$

Equation (11.6) is replaced by

$$i_\eta(2N) = \sum_{n=0}^N e^{\delta_c 2n} a_\eta(2n).$$

The mixing property of the square of the geodesic flow on $\Gamma \backslash \mathcal{G}_{\text{even}} X$ for the restriction of the Gibbs measure m_c gives that, for every $\epsilon > 0$, there exists $T_\epsilon = T_{\epsilon, \eta} \geq 0$ such that for all $n \geq T_\epsilon$, we have

$$\begin{aligned} & \frac{e^{-\epsilon} \int_{\mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^- d\tilde{m}_c \int_{\mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^+ d\tilde{m}_c}{\|(\tilde{m}_c)|_{\mathcal{G}_{\text{even}} \mathbb{X}}\|} \\ & \leq a_\eta(2n) \\ & \leq \frac{e^\epsilon \int_{\mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^- d\tilde{m}_c \int_{\mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^+ d\tilde{m}_c}{\|(\tilde{m}_c)|_{\mathcal{G}_{\text{even}} \mathbb{X}}\|}. \end{aligned}$$

Note that $\mathcal{G}_{\text{even}} \mathbb{X}$ is saturated by the strong stable and strong unstable leaves, since two points x, y on a given horosphere of centre $\xi \in \partial_\infty X$ are at even distance one from another (equal to $2d(x, p)$ where $[x, \xi[\cap [y, \xi[= [p, \xi[$). By the disintegration proposition 7.6, when ℓ ranges over $\mathcal{U}_D^\pm \cap \mathcal{G}_{\text{even}} \mathbb{X}$, we have

$$d\tilde{m}_{F|_{\mathcal{U}_D^\pm \cap \mathcal{G}_{\text{even}} \mathbb{X}}}(\ell) = \int_{\rho \in \partial_\pm^1 D \cap \check{\mathcal{G}}_{\text{even}} \mathbb{X}} d\nu_\rho^\mp(\ell) d\check{\sigma}_D^\pm(\rho).$$

Hence the proof of Lemma 10.1 extends to give

$$\int_{\mathcal{G}_{\text{even}} \mathbb{X}} \phi_\eta^\mp d\tilde{m}_c = \tilde{\sigma}_{\text{even}}^\pm(\Omega^\mp), \quad (11.27)$$

where in order to simplify notation $\tilde{\sigma}_{\text{even}}^\pm = \tilde{\sigma}_{D^\mp, \text{even}}^\pm$.

Therefore, by Equations (11.26) and (11.27), and by a geometric sum argument, the pair of inequalities (11.7) becomes

$$\begin{aligned} & \frac{2 e^{-\epsilon} e^{2\delta_c (N+1)} \tilde{\sigma}_{\text{even}}^+(\Omega^-) \tilde{\sigma}_{\text{even}}^-(\Omega^+)}{(e^{2\delta_c} - 1) \|m_c\|} - c_\epsilon \\ & \leq i_\eta(2N) \\ & \leq \frac{2 e^\epsilon e^{2\delta_c (N+1)} \tilde{\sigma}_{\text{even}}^+(\Omega^-) \tilde{\sigma}_{\text{even}}^-(\Omega^+)}{(e^{2\delta_c} - 1) \|m_c\|} + c_\epsilon. \end{aligned}$$

Up to replacing the summations from $n = 0$ to N to summations on even numbers between 0 to $2N$, and replacing $\lfloor n/2 \rfloor$ by $2\lfloor n/2 \rfloor$ as well as $\lceil n/2 \rceil$ by $2\lfloor n/2 \rfloor$, the rest of the proof applies and gives the result, noting that in claim (iii) of Step 3T, we furthermore have that the origin and endpoint of the constructed common perpendicular α_γ are in $V_{\text{even}} \mathbb{X}$. This concludes the proof of Equation (11.25).

The remainder of the proof of Theorem 11.10 consists in proving versions of the equidistribution result Equation (11.25) respectively in $\check{\mathcal{G}}_{\text{odd}} \times \check{\mathcal{G}}_{\text{odd}}$, $\check{\mathcal{G}}_{\text{even}} \times \check{\mathcal{G}}_{\text{odd}}$, $\check{\mathcal{G}}_{\text{odd}} \times \check{\mathcal{G}}_{\text{even}}$, and in summing these four contributions.

By applying Equation (11.25) by replacing x_0 by a vertex x'_0 in $V_{\text{odd}}\mathbb{X}$, which exchanges $V_{\text{even}}\mathbb{X}$ and $V_{\text{odd}}\mathbb{X}$, $\check{\mathcal{G}}_{\text{even}}$ and $\check{\mathcal{G}}_{\text{odd}}$, as well as $\tilde{\sigma}_{\mathcal{D}^\mp, \text{even}}^\pm$ and $\tilde{\sigma}_{\mathcal{D}^\mp, \text{odd}}^\pm$, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{e^{2\delta_c} - 1}{2e^{2\delta_c}} \|m_c\| e^{-\delta_c t} & \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ \pi(\alpha_{i, \gamma j}^-), \pi(\alpha_{\gamma-1, i, j}^+) \in V_{\text{odd}}\mathbb{X} \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_i, \gamma j \leq t}} e^{c(\alpha_{i, \gamma j})} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma-1, i, j}^+} \\ & = \tilde{\sigma}_{\mathcal{D}^-, \text{odd}}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+, \text{odd}}^- \end{aligned} \quad (11.28)$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}_{\text{odd}}\mathbb{X} \times \check{\mathcal{G}}_{\text{odd}}\mathbb{X}$.

Let us now apply Equation (11.25) by replacing $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ by $\mathcal{N}_1\mathcal{D}^- = (\mathcal{N}_1 D_i^-)_{i \in I^-}$. Let us consider the map $\varphi_+ : \check{\mathcal{G}}\mathbb{X} \rightarrow \check{\mathcal{G}}\mathbb{X}$, which maps a generalised geodesic line ℓ to the generalised geodesic line which coincides with $g^{+1}\ell$ on $[0, +\infty[$ and is constant (with value $\ell(1)$) on $] -\infty, 0[$. Note that this map is continuous and Γ -equivariant, and that it maps $\check{\mathcal{G}}_{\text{even}}\mathbb{X}$ in $\check{\mathcal{G}}_{\text{odd}}\mathbb{X}$ and $\check{\mathcal{G}}_{\text{odd}}\mathbb{X}$ in $\check{\mathcal{G}}_{\text{even}}\mathbb{X}$.

Furthermore, by convexity, φ_+ induces for every $i \in I^-$ an homeomorphism from $\partial_+^1 D_i^-$ to $\partial_+^1 \mathcal{N}_1 D_i^-$, which sends $\partial_+^1 D_i^- \cap \check{\mathcal{G}}_{\text{odd}}\mathbb{X}$ to $\partial_+^1 \mathcal{N}_1 D_i^- \cap \check{\mathcal{G}}_{\text{even}}\mathbb{X}$, such that, by Equation (7.8), for all $w \in \partial_+^1 D_i^- \cap \check{\mathcal{G}}_{\text{odd}}\mathbb{X}$, if e_w is the first (respectively the last) edge followed by w

$$d\tilde{\sigma}_{D_i^-, \text{odd}}^+(w) = e^{c(e_w) - \delta_c} d\tilde{\sigma}_{\mathcal{N}_1 D_i^-, \text{even}}^+(\varphi_+(w)).$$

Note that for all $\ell > 0$, there is a one-to-one correspondance between the set of common perpendiculars of length ℓ , with origin and endpoint both in V_{even} , between $\mathcal{N}_1 D_i^-$ and γD_j^+ for all $i \in I^-$, $j \in I^+$ and $\gamma \in \Gamma$, and the set of common perpendiculars of length $\ell + 1$, with origin in V_{odd} and endpoint in V_{even} , between D_i^- and γD_j^+ for all $i \in I^-$, $j \in I^+$ and $\gamma \in \Gamma$. In particular, $\varphi_+(\alpha_{i, \gamma j}^-)$ is the common perpendicular between $\mathcal{N}_1 D_i^-$ and γD_j^+ , starting at time $t = 0$ from $\mathcal{N}_1 D_i^-$.

Therefore Equation (11.25) applied by replacing $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ by $\mathcal{N}_1\mathcal{D}^- = (\mathcal{N}_1 D_i^-)_{i \in I^-}$ gives

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{e^{2\delta_c} - 1}{2e^{2\delta_c}} \|m_c\| e^{-\delta_c t} & \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ \pi(\alpha_{i, \gamma j}^-) \in V_{\text{odd}}\mathbb{X}, \pi(\alpha_{\gamma-1, i, j}^+) \in V_{\text{even}}\mathbb{X} \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_i, \gamma j \leq t+1}} e^{c(e_{\alpha_{i, \gamma j}^-}) + c(\varphi_+(\alpha_{i, \gamma j}^-))} \Delta_{\alpha_{i, \gamma j}^-} \otimes \Delta_{\alpha_{\gamma-1, i, j}^+} \\ & = e^{\delta_c} \tilde{\sigma}_{\mathcal{D}^-, \text{odd}}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+, \text{even}}^- \end{aligned}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}_{\text{odd}}\mathbb{X} \times \check{\mathcal{G}}_{\text{even}}\mathbb{X}$.

Since $c(e_{\alpha_{i,\gamma j}^-}) + c(\varphi_+(\alpha_{i,\gamma j}^-)) = c(\alpha_{i,\gamma j}^-)$, replacing t by $t - 1$ and simplifying by e^{δ_c} , we get

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \frac{e^{2\delta_c} - 1}{2 e^{2\delta_c}} \|m_c\| e^{-\delta_c t} \\ & \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ \pi(\alpha_{i,\gamma j}^-) \in V_{\text{odd}} \mathbb{X}, \pi(\alpha_{\gamma^{-1}i,j}^+) \in V_{\text{even}} \mathbb{X} \\ D_i^- \cap D_{\gamma j}^+ = \emptyset, \lambda_{i,\gamma j} \leq t}} e^{c(\alpha_{i,\gamma j}^-)} \Delta_{\alpha_{i,\gamma j}^-} \otimes \Delta_{\alpha_{\gamma^{-1}i,j}^+} \\ & = \tilde{\sigma}_{\mathcal{D}^-, \text{odd}}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+, \text{even}}^- \end{aligned} \quad (11.29)$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}_{\text{odd}} \mathbb{X} \times \check{\mathcal{G}}_{\text{even}} \mathbb{X}$.

Now Theorem 11.10 follows by

summing Equation (11.25), Equation (11.28), Equation (11.29) and the formula, proven similarly, obtained from Equation (11.25) by replacing $\mathcal{D}^+ = (D_j^+)_{j \in I^+}$ by $(\mathcal{N}_1 D_j^+)_{j \in I^+}$. \square

The following result for bipartite graphs (of groups) is used in the arithmetic applications in Part III (see Section 15.4).

Corollary 11.11. *Let \mathbb{X} be a $(p+1, q+1)$ -biregular simplicial tree (with $p, q \geq 2$, possibly with $p = q$), with corresponding partition $V\mathbb{X} = V_p\mathbb{X} \sqcup V_q\mathbb{X}$. Let Γ be a lattice of \mathbb{X} such that this partition is Γ -invariant. Assume that the Patterson density is normalised so that $\|\mu_x\| = \frac{p+1}{\sqrt{p}}$ for every $x \in V_p\mathbb{X}$. Let \mathbb{D}^\pm be nonempty proper simplicial subtrees of \mathbb{X} with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ , such that the families $\mathcal{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite. Then*

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \frac{pq - 1}{2} \text{TVol}(\Gamma \backslash \mathbb{X}) \sqrt{pq}^{-t-2} \\ & \sum_{\substack{(\alpha, \beta, \gamma) \in \Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma \\ 0 < d(\alpha\mathbb{D}^-, \gamma\beta\mathbb{D}^+) \leq t}} \Delta_{\alpha_{\alpha, \gamma\beta}^-} \otimes \Delta_{\alpha_{\gamma^{-1}\alpha, \beta}^+} \\ & = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^- \end{aligned}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$.

If the measure $\sigma_{\mathcal{D}^+}^-$ is nonzero and finite, then

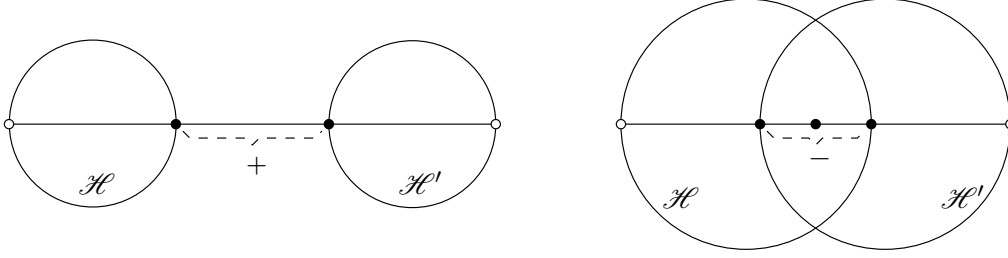
$$\lim_{t \rightarrow +\infty} \frac{pq - 1}{2} \frac{\text{TVol}(\Gamma \backslash \mathbb{X})}{\|\sigma_{\mathcal{D}^+}^-\|} \sqrt{pq}^{-t-2} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq t}} \Delta_{\alpha_{e, \gamma}^-} = \tilde{\sigma}_{\mathbb{D}^-}^+,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X}$.

Proof. In order to prove the first result, we apply Theorem 11.10 with $\tilde{c} \equiv 0$, so that by Propositions 4.14, 4.15, and 8.1 (2), we have $\delta_c = \frac{1}{2} \ln(pq) > 0$, $m_c = m_{\text{BM}}$ is finite and its restriction to $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$ is mixing under the square of the geodesic flow, and $\|m_{\text{BM}}\| = \text{TVol}(\Gamma \backslash \mathbb{X})$.

The second claim follows as in the proof of Corollary 11.9. \square

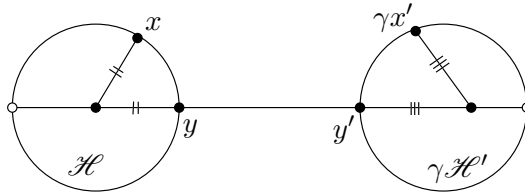
Remark. In some special occasions, the measures involved in the statements of Theorem 11.10 and Corollary 11.11 (whether skinning measures or Dirac masses) are actually all supported on $\check{\mathcal{G}}_{\text{even}} \mathbb{X}$ (up to choosing appropriately x_0). This is in particular the case if $X = [\mathbb{X}]_1$ is a simplicial tree and if $\mathcal{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ with $\mathbb{D}^-, \mathbb{D}^+$ at even signed distance (see below), as the following proposition shows.



The *signed distance* between horoballs \mathcal{H} and \mathcal{H}' in an \mathbb{R} -tree that are not centred at the same point at infinity is the distance between them (that is, the length of their common perpendicular) if they are disjoint, or the opposite of the diameter of their intersection otherwise. Note that if nonempty, the intersection of \mathcal{H} and \mathcal{H}' is a ball centred at the midpoint of the segment contained in the geodesic line between the two points at infinity of the horoballs, which lies in both horoballs.

Lemma 11.12. *Let \mathbb{X} be a simplicial tree, Γ a subgroup of $\text{Aut}(\mathbb{X})$ and $\mathcal{H}, \mathcal{H}'$ two horoballs in \mathbb{X} (whose boundaries are contained in $V\mathbb{X}$), which either are equal or have distinct points at infinity. If $\Lambda\Gamma \subset 2\mathbb{Z}$ and $\mathcal{H}, \mathcal{H}'$ are at even signed distance, then the signed distance between \mathcal{H} and $\gamma\mathcal{H}'$ is even for every $\gamma \in \Gamma$ such that \mathcal{H} and $\gamma\mathcal{H}'$ do not have the same point at infinity.*

Proof. For every horoball \mathcal{H}'' and for all $s \in \mathbb{N}$, let $\mathcal{H}''[s]$ be the horoball contained in \mathcal{H}'' , whose boundary is at distance s from the boundary of \mathcal{H}'' . Shrinking the horoballs \mathcal{H} and \mathcal{H}' , by replacing them by the horoballs $\mathcal{H}[s]$ and $\mathcal{H}'[s]$ for any $s \in \mathbb{N}$, only changes by $\pm 2s$ the considered signed distances. Hence, taking s big enough, we may assume that \mathcal{H} and $\gamma\mathcal{H}'$ are disjoint, and that \mathcal{H} and \mathcal{H}' are disjoint or equal. Let $[x, x']$ be the common perpendicular between \mathcal{H} and \mathcal{H}' with $x \in \partial\mathcal{H}$, $x' \in \partial\mathcal{H}'$, and let $[y, y']$ be the one between \mathcal{H} and $\gamma\mathcal{H}'$, with $y \in \partial\mathcal{H}$, $y' \in \partial(\gamma\mathcal{H}')$. Note that $\gamma x' \in \partial(\gamma\mathcal{H}')$.



The distance between two points x, y of a horosphere is always even (equal to twice the distance from x to the geodesic ray from y to the point at infinity of the horosphere). Since geodesic triangles in trees are tripod, for all a, b, c in a simplicial tree, since

$$d(a, c) = d(a, b) + d(b, c) - 2d(b, [a, c]) ,$$

if $d(a, b)$ and $d(b, c)$ are even, so is $d(a, c)$.

Since $\Lambda\Gamma \subset 2\mathbb{Z}$, the distance between x' and $\gamma x'$ is even by Equation (4.13). Since $d(x, x')$ is even by assumption, we hence have that $d(x, \gamma x')$ is even. Therefore

$$d(y, y') = d(x, \gamma x') - d(x, y) - d(y', \gamma x')$$

is even. □

Chapter 12

Equidistribution and counting of common perpendiculars in quotient spaces

In this Chapter, we use the results of Chapter 11 to prove equidistribution and counting results in Riemannian manifolds (or good orbifolds) and in metric and simplicial graphs (of groups).

Let X , x_0 , Γ and \tilde{F} be as in the beginning of Chapter 11.

12.1 Multiplicities and counting functions in Riemannian orbifolds

In this Section, we assume that $X = \tilde{M}$ is a Riemannian manifold. We denote its quotient Riemannian orbifold under Γ by $M = \Gamma \backslash \tilde{M}$, and the quotient Riemannian orbifold under Γ of its unit tangent bundle by $T^1M = \Gamma \backslash T^1\tilde{M}$. We use the identifications $\mathcal{G}X = \mathcal{G}_{\pm,0}X = T^1X = T^1\tilde{M}$ explained in Chapter 2.

Let $\mathcal{D} = (D_i)_{i \in I}$ be a locally finite Γ -equivariant family of nonempty proper closed convex subsets of \tilde{M} . Let $\Omega = (\Omega_i)_{i \in I}$ be a Γ -equivariant family of subsets of $T^1\tilde{M}$, where Ω_i is a measurable subset of $\partial_{\pm}^1 D_i$ for all $i \in I$ (the sign \pm being constant). The *multiplicity* of an element $v \in T^1M$ with respect to Ω is

$$m_{\Omega}(v) = \frac{\text{Card} \{i \in I/\sim : \tilde{v} \in \Omega_i\}}{\text{Card}(\text{Stab}_{\Gamma} \tilde{v})},$$

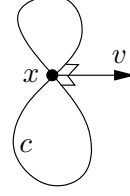
for any preimage \tilde{v} of v in $T^1\tilde{M}$. The numerator and the denominator are finite by the local finiteness of the family \mathcal{D} and the discreteness of Γ , and they depend only on the orbit of \tilde{v} under Γ .

The numerator takes into account the multiplicities of the images of the elements of Ω in T^1M . The denominator of this multiplicity is also natural, as any counting problem of objects possibly having symmetries, the appropriate counting function consists in taking as the multiplicity of an object the inverse of the cardinality of its symmetry group.

Examples 12.1. The following examples illustrate the behaviour of the multiplicity when Γ is torsion-free and $\Omega = \partial_{\pm}^1 \mathcal{D}$.

(1) If for every $i \in I$, the quotient $\Gamma_{D_i} \backslash D_i$ of D_i by its stabiliser Γ_{D_i} maps injectively in M by the map induced by the inclusion of D_i in \widetilde{M} , and if for every $i, j \in I$ such that $j \notin \Gamma i$, the intersection $D_i \cap D_j$ is empty, then the nonzero multiplicities $m_\Omega(\ell)$ are all equal to 1.

(2) Here is a simple example of a multiplicity different from 0 or 1. Let c be a closed geodesic in the Riemannian manifold M , let \tilde{c} be a geodesic line in \widetilde{M} mapping to c in M , let $\mathcal{D} = (\gamma \tilde{c})_{\gamma \in \Gamma}$, let x be a double point of c , let $v \in T_x^1 M$ be orthogonal to the two tangent lines to c at x (this requires the dimension of \widetilde{M} to be at least 3, if x is a transverse self-intersection point). Then $m_{\partial_\pm^1 \mathcal{D}}(v) = 2$.



Given $t > 0$ and two unit tangent vectors $v, w \in T^1 M$, we define the number $n_t(v, w)$ of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, weighted by the potential F , with length at most t , by

$$n_t(v, w) = \sum_{\alpha} \text{Card}(\Gamma_{\alpha}) e^{\int_{\alpha} F},$$

where the sum ranges over the locally geodesic paths $\alpha : [0, s] \rightarrow M$ in the Riemannian orbifold M such that $\dot{\alpha}(0) = v$, $\dot{\alpha}(s) = w$ and $s \in]0, t]$, and Γ_{α} is the stabiliser in Γ of any geodesic path $\tilde{\alpha}$ in \widetilde{M} mapping to α by the quotient map $\widetilde{M} \rightarrow M$. If $F = 0$ and Γ is torsion free, then $n_t(v, w)$ is precisely the number of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, with length at most t .

Let $\Omega^- = (\Omega_i^-)_{i \in I^-}$ and $\Omega^+ = (\Omega_j^+)_{j \in I^+}$ be Γ -equivariant families of subsets of $T^1 \widetilde{M}$, where Ω_k^\mp is a measurable subset of $\partial_\pm^1 D_k^\mp$ for all $k \in I^\mp$. We will denote by $\mathcal{N}_{\Omega^-, \Omega^+, F} :]0, +\infty[\rightarrow \mathbb{R}$ the following *counting function*: for every $t > 0$, let $\mathcal{N}_{\Omega^-, \Omega^+, F}(t)$ be the number of common perpendiculars whose initial vectors belong to the images in $T^1 M$ of the elements of Ω^- and terminal vectors to the images in $T^1 M$ of the elements of Ω^+ , counted with multiplicities and weighted by the potential F , that is:

$$\mathcal{N}_{\Omega^-, \Omega^+, F}(t) = \sum_{v, w \in T^1 M} m_{\Omega^-}(v) m_{\Omega^+}(w) n_t(v, w).$$

When $\Omega^\pm = \partial_\pm^1 \mathcal{D}^\pm$, we denote $\mathcal{N}_{\Omega^-, \Omega^+, F}$ by $\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}$.

Remark 12.2. Let Y be a negatively curved complete connected Riemannian manifold and let $\tilde{Y} \rightarrow Y$ be its Riemannian universal cover. Let D^\pm be a locally convex¹ geodesic metric space endowed with a continuous map $f^\pm : D^\pm \rightarrow Y$ such that if $\tilde{D}^\pm \rightarrow D^\pm$ is a locally isometric universal cover and if $\tilde{f}^\pm : \tilde{D}^\pm \rightarrow \tilde{Y}$ is a lift of f^\pm , then \tilde{f}^\pm is on each connected component of \tilde{D}^\pm an isometric embedding whose image is a proper nonempty closed locally convex subset of \tilde{Y} , and the family of images under the covering group of $\tilde{Y} \rightarrow Y$ of the images by \tilde{f}^\pm of the connected components of \tilde{D}^\pm is locally finite. Then D^\pm (or the pair (D^\pm, f^\pm)) is a *proper nonempty properly immersed closed locally convex subset* of Y .

If Γ is a discrete subgroup without torsion of isometries of a CAT(-1) Riemannian manifold X , if $\mathcal{D}^\pm = (\gamma \tilde{D}^\pm)_{\gamma \in \Gamma}$ where \tilde{D}^\pm is a nonempty proper closed convex subset of X such that the family \mathcal{D}^\pm is locally finite, and if D^\pm is the image of \tilde{D}^\pm by the covering map $X \rightarrow \Gamma \backslash X$, then D^\pm is a proper nonempty properly immersed closed convex subset of $\Gamma \backslash X$. Under these assumptions, $\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}$ is the counting function $\mathcal{N}_{D^-, D^+, F}$ given in the introduction.

¹not necessarily connected

Let us continue fixing the notation used in Sections 12.2 and 12.3. For every (i, j) in $I^- \times I^+$ such that D_i^- and D_j^+ have a common perpendicular², we denote by $\alpha_{i,j}$ this common perpendicular, by $\lambda_{i,j}$ its length, by $v_{i,j}^- \in \partial_+^1 D_i^-$ its initial tangent vector and by $v_{i,j}^+ \in \partial_-^1 D_i^+$ its terminal tangent vector. Note that if $i' \sim i$, $j' \sim j$ and $\gamma \in \Gamma$, then

$$\gamma \alpha_{i',j'} = \alpha_{\gamma i, \gamma j}, \quad \lambda_{i',j'} = \lambda_{\gamma i, \gamma j} \quad \text{and} \quad \gamma v_{i',j'}^\pm = v_{\gamma i, \gamma j}^\pm. \quad (12.1)$$

When Γ has no torsion, we have, for the diagonal action of Γ on $I^- \times I^+$,

$$\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t) = \sum_{(i,j) \in \Gamma \backslash ((I^- / \sim) \times (I^+ / \sim)) : \overline{D_i^-} \cap \overline{D_j^+} = \emptyset, \lambda_{i,j} \leq t} e^{\int \alpha_{i,j} \tilde{F}}.$$

When the potential F is zero and Γ acts without torsion, $\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t)$ is the number of common perpendiculars of length at most t , and the counting function $t \mapsto \mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, 0}(t)$ has been studied in various special cases of negatively curved manifolds since the 1950's and in a number of recent works, see the Introduction. The asymptotics of $\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, 0}(t)$ as $t \rightarrow +\infty$ in the case when X is a Riemannian manifold with pinched negative curvature are described in general in [PaP16c, Thm. 1], where it is shown that if the skinning measures $\sigma_{\mathcal{D}^-}^+$ and $\sigma_{\mathcal{D}^+}^-$ are finite and nonzero, then as $s \rightarrow +\infty$,

$$\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, 0}(s) \sim \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\|m_{\text{BM}}\|} \frac{e^{\delta_\Gamma s}}{\delta_\Gamma}. \quad (12.2)$$

12.2 Common perpendiculars in Riemannian orbifolds

Corollary 12.3 below is the main result of this text on the counting with weights of common perpendiculars and on the equidistribution of their initial and terminal tangent vectors in negatively curved Riemannian manifolds endowed with a Hölder potential. We use the notation of Section 12.1.

The following observation on the behaviour of induced³ measures under quotients by properly discontinuous group actions will be used in the proof of the following result and also those of its analogues in Section 12.4. Let G be a discrete group that acts properly on a Polish space \tilde{Y} and let $Y = G \backslash \tilde{Y}$. Let $\tilde{\mu}_k$ for $k \in \mathbb{N}$ and $\tilde{\mu}$ be G -invariant locally finite measures on \tilde{Y} , with finite induced measures μ_k for $k \in \mathbb{N}$ and μ on Y . If for every Borel subset B of \tilde{Y} with $\tilde{\mu}(B)$ finite and $\tilde{\mu}(\partial B) = 0$ we have $\lim_{k \rightarrow \infty} \tilde{\mu}_k(B) = \tilde{\mu}(B)$, then the sequence $(\mu_k)_{k \in \mathbb{N}}$ narrowly converges to μ .

Corollary 12.3. *Let \tilde{M} be a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1 . Let Γ be a nonelementary discrete group of isometries of \tilde{M} . Let $\tilde{F} : T^1 \tilde{M} \rightarrow \mathbb{R}$ be a bounded Γ -invariant Hölder-continuous function with positive critical exponent δ . Let $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ and $\mathcal{D}^+ = (D_j^+)_{j \in I^+}$ be locally finite Γ -equivariant families of nonempty proper closed convex subsets of \tilde{M} . Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow on $T^1 M$. Then,*

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{v, w \in T^1 M} m_{\partial_+^1 \mathcal{D}^-}(v) m_{\partial_-^1 \mathcal{D}^+}(w) n_t(v, w) \Delta_v \otimes \Delta_w = \sigma_{\mathcal{D}^-}^+ \otimes \sigma_{\mathcal{D}^+}^- \quad (12.3)$$

²that is, whose closures $\overline{D_i^-}$ and $\overline{D_j^+}$ in $X \cup \partial_\infty X$ have empty intersection

³see for instance [PauPS, §2.6] for a definition

for the weak-star convergence of measures on the locally compact space $T^1M \times T^1M$. If $\sigma_{\mathcal{D}^+}^+$ and $\sigma_{\mathcal{D}^+}^-$ are finite, the result also holds for the narrow convergence.

Furthermore, for all Γ -equivariant families $\Omega^\pm = (\Omega_k^\pm)_{k \in I^\pm}$ of subsets of $T^1\widetilde{M}$ with Ω_k^\mp a Borel subset of $\partial_\pm^1 D_k^\mp$ for all $k \in I^\mp$, with nonzero finite skinning measure and with boundary in $\partial_\pm^1 D_k^\mp$ of zero skinning measure, we have, as $t \rightarrow +\infty$,

$$\mathcal{N}_{\Omega^-, \Omega^+, F}(t) \sim \frac{\|\sigma_{\Omega^-}^+\| \|\sigma_{\Omega^+}^-\|}{\delta \|m_F\|} e^{\delta t}.$$

Proof. Note that the sum in Equation (12.3) is locally finite, hence it defines a locally finite measure on $T^1M \times T^1M$. We are going to rewrite the sum in the statement of Theorem 11.1 in a way which makes it easier to push it down from $T^1\widetilde{M} \times T^1\widetilde{M}$ to $T^1M \times T^1M$.

For every $\tilde{v} \in T^1\widetilde{M}$, let

$$m^\mp(\tilde{v}) = \text{Card} \{k \in I^\mp / \sim : \tilde{v} \in \partial_\pm^1 D_k^\mp\},$$

so that for every $v \in T^1M$, the multiplicity of v with respect to the family $\partial_\pm^1 \mathcal{D}^\mp$ is⁴

$$m_{\partial_\pm^1 \mathcal{D}^\mp}(v) = \frac{m^\mp(\tilde{v})}{\text{Card}(\text{Stab}_\Gamma \tilde{v})},$$

for any preimage \tilde{v} of v in $T^1\widetilde{M}$.

For all $\gamma \in \Gamma$ and $\tilde{v}, \tilde{w} \in T^1\widetilde{M}$, there exists $(i, j) \in (I^- / \sim) \times (I^+ / \sim)$ such that $\tilde{v} = v_{i, \gamma j}^-$ and $\tilde{w} = v_{\gamma^{-1}i, j}^+ = \gamma^{-1}v_{i, \gamma j}^+$ if and only if $\gamma\tilde{w} \in \mathbf{g}^{\mathbb{R}} \tilde{v}$, there exists $i' \in I^- / \sim$ such that $\tilde{v} \in \partial_+^1 D_{i'}^-$ and there exists $j' \in I^+ / \sim$ such that $\gamma\tilde{w} \in \partial_-^1 D_{j'}^+$. Then the choice of such elements (i, j) , as well as i' and j' , is free. We hence have

$$\begin{aligned} & \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ 0 < \lambda_{i, \gamma j} \leq t, v_{i, \gamma j}^- = \tilde{v}, v_{\gamma^{-1}i, j}^+ = \tilde{w}}} e^{\int \alpha_{i, \gamma j} \tilde{F}} \Delta_{v_{i, \gamma j}^-} \otimes \Delta_{v_{\gamma^{-1}i, j}^+} \\ &= \sum_{\substack{\gamma \in \Gamma, 0 < s \leq t \\ \gamma\tilde{w} = \mathbf{g}^s \tilde{v}}} e^{\int \pi(\tilde{w}) \tilde{F}} \text{Card} \{(i, j) \in (I^- / \sim) \times (I^+ / \sim) : v_{i, \gamma j}^- = \tilde{v}, v_{\gamma^{-1}i, j}^+ = \tilde{w}\} \Delta_{\tilde{v}} \otimes \Delta_{\tilde{w}} \\ &= \sum_{\substack{\gamma \in \Gamma, 0 < s \leq t \\ \gamma\tilde{w} = \mathbf{g}^s \tilde{v}}} e^{\int \pi(\tilde{v}) \tilde{F}} m^-(\tilde{v}) m^+(\gamma\tilde{w}) \Delta_{\tilde{v}} \otimes \Delta_{\tilde{w}}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ 0 < \lambda_{i, \gamma j} \leq t}} e^{\int \alpha_{i, \gamma j} \tilde{F}} \Delta_{v_{i, \gamma j}^-} \otimes \Delta_{v_{\gamma^{-1}i, j}^+} \\ &= \sum_{\tilde{v}, \tilde{w} \in T^1\widetilde{M}} \left(\sum_{\substack{\gamma \in \Gamma, 0 < s \leq t \\ \gamma\tilde{w} = \mathbf{g}^s \tilde{v}}} e^{\int \pi(\tilde{w}) \tilde{F}} \right) m^-(\tilde{v}) m^+(\tilde{w}) \Delta_{\tilde{v}} \otimes \Delta_{\tilde{w}}. \end{aligned}$$

⁴See Section 12.1

By definition, $\sigma_{\mathcal{D}^\pm}^\pm$ is the measure on T^1M induced by the Γ -invariant measure $\tilde{\sigma}_{\mathcal{D}^\pm}^\pm$. Thus Corollary 12.3 follows from Theorem 11.1 and Equation (11.4) after a similar reduction⁵ as in Section 11.1, and since no compactness assumptions were made on Ω^\pm to get this equation, by [PauPS, §2.6]. \square

In particular, if the skinning measures $\sigma_{\mathcal{D}^-}^+$ and $\sigma_{\mathcal{D}^+}^-$ are positive and finite, Corollary 12.3 gives, as $t \rightarrow +\infty$,

$$\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t) \sim \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta \|m_F\|} e^{\delta t}.$$

Remark 12.4. Under the assumptions of Corollary 12.3 with the exception that δ may now be nonpositive, we have the following asymptotic result as $t \rightarrow +\infty$ for the growth of the weighted number of common perpendiculars with lengths in $]t - \tau, t]$ for every fixed $\tau > 0$:

$$\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t) - \mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t - \tau) \sim \frac{(1 - e^{-\delta\tau}) \|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta \|m_F\|} e^{\delta t}.$$

This result follows by considering a large enough constant σ such that $\delta_{\Gamma, F+\sigma} = \delta + \sigma > 0$, by applying Corollary 12.3 with the potential $F + \sigma$ (see Remark 7.1 (2)), and by an easy subdivision and geometric series argument, see [PauPS, Ch. 9].

Using the continuity of the pushforwards of measures for the weak-star and the narrow topologies, applied to the basepoint maps $\pi \times \pi$ from $T^1\tilde{M} \times T^1\tilde{M}$ to $\tilde{M} \times \tilde{M}$, and from $T^1M \times T^1M$ to $M \times M$, we have the following result of equidistribution of the ordered pairs of endpoints of common perpendiculars between two equivariant families of convex sets in \tilde{M} or two families of locally convex sets in M . When M has constant curvature and finite volume, $F = 0$ and \mathcal{D}^- is the Γ -orbit of a point and \mathcal{D}^+ is the Γ -orbit of a totally geodesic cocompact submanifold, this result is due to Herrmann [Her]. When \mathcal{D}^\pm are Γ -orbits of points and F is a Hölder potential, see [PauPS, Thm. 9.1,9.3], and we refer for instance to [BoyM] for an application of this particular case.

Corollary 12.5. *Let $\tilde{M}, \Gamma, \tilde{F}, \mathcal{D}^-, \mathcal{D}^+$ be as in Corollary 12.3. Then*

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{\substack{i \in I^-/\sim, j \in I^+/\sim, \gamma \in \Gamma \\ 0 < \lambda_{i, \gamma j} \leq t}} e^{\int_{\alpha_i, \gamma j} \tilde{F}} \Delta_{\pi(v_{i, \gamma j}^-)} \otimes \Delta_{\pi(v_{\gamma^{-1}i, j}^+)} = \pi_* \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \pi_* \tilde{\sigma}_{\mathcal{D}^+}^- ,$$

for the weak-star convergence of measures on the locally compact space $\tilde{M} \times \tilde{M}$, and

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{v, w \in T^1M} m_{\partial_+^1 \mathcal{D}^-}(v) m_{\partial_-^1 \mathcal{D}^+}(w) n_t(v, w) \Delta_{\pi(v)} \otimes \Delta_{\pi(w)} \\ &= \pi_* \sigma_{\mathcal{D}^-}^+ \otimes \pi_* \sigma_{\mathcal{D}^+}^- , \end{aligned}$$

for the weak-star convergence of measures on $M \times M$. If the measures $\sigma_{\mathcal{D}^\pm}^\pm$ are finite, then the above claim holds for the narrow convergence of measures on $M \times M$. \square

⁵See Step 1 of the proof of Theorem 11.1.

We will now prove Theorems 1.4 and 1.5 (1) in the Introduction for Riemannian manifolds. Recall from Remark 12.2 the definition of proper nonempty properly immersed closed locally convex subsets D^\pm in a pinched negatively curved complete connected Riemannian manifold Y and the associated maps $\tilde{f}^\pm : \tilde{D}^\pm \rightarrow \tilde{Y}$.

Proof of Theorems 1.4 and 1.5 (1) for Riemannian manifolds. Let Y, F, D^\pm be as in these statements and assume that Y is a Riemannian manifold. Let Γ be the covering group of the universal Riemannian cover $\tilde{Y} \rightarrow Y$. Let $I^\pm = \Gamma \times \pi_0(\tilde{D}^\pm)$ with the action of Γ defined by $\gamma \cdot (\alpha, c) = (\gamma\alpha, c)$ for all $\gamma, \alpha \in \Gamma$ and every component c of \tilde{D}^\pm . Consider the families $\mathcal{D}^\pm = (D_k^\pm)_{k \in I^\pm}$ where $D_k^\pm = \alpha \tilde{f}^\pm(c)$ if $k = (\alpha, c)$. Then \mathcal{D}^\pm are Γ -equivariant families of nonempty proper closed convex subsets of \tilde{Y} , which are locally finite since D^\pm are properly immersed in Y . The conclusions in Theorems 1.4 and 1.5 (1) when Y is a manifold then follow from Corollary 12.3, applied with $\tilde{M} = \tilde{Y}$ and with \tilde{F} the lift of F to $T^1\tilde{M}$. \square

Corollary 12.6. *Let $\tilde{M}, \Gamma, \tilde{F}, \mathcal{D}^-, \mathcal{D}^+$ be as in Corollary 12.3. Assume that $\sigma_{\mathcal{D}^\pm}^\pm$ are finite and nonzero. Then*

$$\lim_{s \rightarrow +\infty} \lim_{t \rightarrow +\infty} \frac{\delta \|m_F\|^2 e^{-\delta t}}{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|} \sum_{v \in T^1M} m_{\partial_+^1 \mathcal{D}^-}(v) n_{t, \mathcal{D}^+}(v) \Delta_{\mathbf{g}^s v} = m_F,$$

where

$$n_{t, \mathcal{D}^+}(v) = \sum_{w \in T^1M} m_{\partial_+^1 \mathcal{D}^+}(w) n_t(v, w)$$

is the number (counted with multiplicities) of locally geodesic paths in M of length at most t , with initial vector v , arriving perpendicularly to \mathcal{D}^+ .

Proof. For every $s \in \mathbb{R}$, by Corollary 12.3, using the continuity of the pushforwards of measures by the first projection $(v, w) \mapsto v$ from $T^1M \times T^1M$ to T^1M , and by the geodesic flow on T^1M at time s , since $(\mathbf{g}^s)_* \Delta_v = \Delta_{\mathbf{g}^s v}$, we have

$$\lim_{t \rightarrow +\infty} \delta \|m_F\| e^{-\delta t} \sum_{v \in T^1M} m_{\partial_+^1 \mathcal{D}^-}(v) n_{t, \mathcal{D}^+}(v) \Delta_{\mathbf{g}^s v} = (\mathbf{g}^s)_* \sigma_{\mathcal{D}^-}^+ \|\sigma_{\mathcal{D}^+}^-\|.$$

The result then follows from Theorem 10.2 with $\Omega = \partial_+^1 \mathcal{D}^-$. \square

12.3 Error terms for equidistribution and counting for Riemannian orbifolds

In Section 9.1, we discussed various results on the rate of mixing of the geodesic flow for Riemannian manifolds. In this Section, we apply these results to give error bounds to the statements of equidistribution and counting of common perpendicular arcs given in Section 12.2. We use again the notation of Section 12.1.

Theorem 12.7. *Let \tilde{M} be a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1 . Let Γ be a nonelementary discrete group of isometries of \tilde{M} . Let $\tilde{F} : T^1\tilde{M} \rightarrow \mathbb{R}$ be a bounded Γ -invariant Hölder-continuous function with positive critical exponent δ . Assume that $(\tilde{M}, \Gamma, \tilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses. Let $\mathcal{D}^- = (D_i^-)_{i \in I^-}$ and $\mathcal{D}^+ = (D_j^+)_{j \in I^+}$ be locally finite Γ -equivariant families of nonempty proper closed convex subsets of \tilde{M} , with finite nonzero skinning measure $\sigma_{\mathcal{D}^-}$ and $\sigma_{\mathcal{D}^+}$. Let $M = \Gamma \backslash \tilde{M}$ and let $F : T^1M \rightarrow \mathbb{R}$ be the potential induced by \tilde{F} .*

(1) Assume that M is compact and that the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F . Then there exist $\alpha \in]0, 1]$ and $\kappa' > 0$ such that for all nonnegative $\psi^\pm \in \mathcal{C}_c^\alpha(T^1M)$, we have, as $t \rightarrow +\infty$,

$$\begin{aligned} \frac{\delta \|m_F\|}{e^{\delta t}} \sum_{v, w \in T^1M} m_{\partial_+^1 \mathcal{D}^-}(v) m_{\partial_-^1 \mathcal{D}^+}(w) n_t(v, w) \psi^-(v) \psi^+(w) \\ = \int_{T^1M} \psi^- d\sigma_{\mathcal{D}^-}^+ \int_{T^1M} \psi^+ d\sigma_{\mathcal{D}^+}^- + O(e^{-\kappa' t} \|\psi^-\|_\alpha \|\psi^+\|_\alpha). \end{aligned}$$

(2) Assume that \widetilde{M} is a symmetric space, that D_k^\pm has smooth boundary for every $k \in I^\pm$, that m_F is finite and smooth, and that the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F . Then there exist $\ell \in \mathbb{N}$ and $\kappa' > 0$ such that for all nonnegative maps $\psi^\pm \in \mathcal{C}_c^\ell(T^1M)$, we have, as $t \rightarrow +\infty$,

$$\begin{aligned} \frac{\delta \|m_F\|}{e^{\delta t}} \sum_{v, w \in T^1M} m_{\partial_+^1 \mathcal{D}^-}(v) m_{\partial_-^1 \mathcal{D}^+}(w) n_t(v, w) \psi^-(v) \psi^+(w) \\ = \int_{T^1M} \psi^- d\sigma_{\mathcal{D}^-}^+ \int_{T^1M} \psi^+ d\sigma_{\mathcal{D}^+}^- + O(e^{-\kappa' t} \|\psi^-\|_\ell \|\psi^+\|_\ell). \end{aligned}$$

Furthermore, if \mathcal{D}^- and \mathcal{D}^+ respectively have nonzero finite outer and inner skinning measures, and if $(\widetilde{M}, \Gamma, \widetilde{F})$ satisfies the conditions of (1) or (2) above, then there exists $\kappa'' > 0$ such that, as $t \rightarrow +\infty$,

$$\mathcal{N}_{\mathcal{D}^-, \mathcal{D}^+, F}(t) = \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta \|m_F\|} e^{\delta t} (1 + O(e^{-\kappa'' t})).$$

The maps $O(\cdot)$ depend on $\widetilde{M}, \Gamma, F, \mathcal{D}$, and the speeds of mixing. The proof is a generalization to nonzero potential of [PaP16c, Thm. 15].

Proof. We will follow the proofs of Theorem 11.1 and Corollary 12.3 to prove generalizations of the assertions (1) and (2) by adding to these proofs a regularisation process of the test functions $\tilde{\phi}_\eta^\pm$ as for the deduction of [PaP14a, Theo. 20] from [PaP14a, Theo. 19]. We will then deduce the last statement of Theorem 12.7 from these generalisations, again using this regularisation process.

Let β be either $\alpha \in]0, 1]$ small enough in the Hölder regularity case or $\ell \in \mathbb{N}$ big enough in the Sobolev regularity case. We fix $i \in I^-, j \in I^+$, and we use the notation $D^\pm, \alpha_\gamma, \lambda_\gamma, v_\gamma^\pm$ and $\tilde{\sigma}^\pm$ of Equation (11.2). Let $\tilde{\psi}^\pm \in \mathcal{C}_c^\beta(\partial_\mp^1 D^\pm)$. Under the assumptions of Assertion (1) or (2), we first prove the following avatar of Equation (11.4), indicating only the required changes in its proof: there exists $\kappa_0 > 0$ (independent of $\tilde{\psi}^\pm$) such that, as $T \rightarrow +\infty$,

$$\begin{aligned} \delta \|m_F\| e^{-\delta T} \sum_{\gamma \in \Gamma, 0 < \lambda_\gamma \leq T} e^{\int \alpha_\gamma \tilde{F}} \tilde{\psi}^-(v_\gamma^-) \tilde{\psi}^+(v_\gamma^+) \\ = \int_{\partial_+^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_-^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa_0 T} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta). \end{aligned} \quad (12.4)$$

By [PaP16c, Lem. 6] and the Hölder regularity of the strong stable and unstable foliations under the assumptions of Assertion (1), or by the smoothness of the boundary of D^\pm under

the assumptions of Assertion (2), the maps $f_{D^\pm}^\pm : \mathcal{V}_{\eta,R}^\pm(\partial_\pm^1 D^\mp) \rightarrow \partial_\pm^1 D^\mp$ are respectively Hölder-continuous or smooth fibrations, whose fiber over $w \in \partial_\pm^1 D^\mp$ is exactly $V_{w,\eta,R}^\pm$. By applying leafwise the regularisation process described in the proof of [PaP14a, Theo. 20] to characteristic functions, there exist a constant $\kappa_1 > 0$ and $\chi_{\eta,R}^\pm \in \mathcal{C}^\beta(T^1\widetilde{M})$ such that

- $\|\chi_{\eta,R}^\pm\|_\beta = O(\eta^{-\kappa_1})$,
- $\mathbb{1}_{\mathcal{V}_{\eta e^{-O(\eta)}, R e^{-O(\eta)}}(\partial_\pm^1 D^\pm)} \leq \chi_{\eta,R}^\pm \leq \mathbb{1}_{\mathcal{V}_{\eta,R}^\pm(\partial_\pm^1 D^\pm)}$,
- for every $w \in \partial_\pm^1 D^\pm$, we have

$$\int_{V_{w,\eta,R}^\mp} \chi_{\eta,R}^\pm d\nu_w^\pm = \nu_w^\pm(V_{w,\eta,R}^\mp) e^{-O(\eta)} = \nu_w^\pm(V_{w,\eta e^{-O(\eta)}, R e^{-O(\eta)}}^\mp) e^{O(\eta)}.$$

We now define the new test functions (compare with Section 10.1). For every $w \in \partial_\pm^1 D^\pm$, let

$$H_{\eta,R}^\pm(w) = \frac{1}{\int_{V_{w,\eta,R}^\mp} \chi_{\eta,R}^\pm d\nu_w^\pm}.$$

Let $\Phi_\eta^\pm : T^1\widetilde{M} \rightarrow \mathbb{R}$ be the map defined by

$$\Phi_\eta^\pm = (H_{\eta,R}^\pm \tilde{\psi}^\pm) \circ f_{D^\pm}^\mp \chi_{\eta,R}^\pm.$$

The support of this map is contained in $\mathcal{V}_{\eta,R}^\mp(\partial_\pm^1 D^\pm)$. Since M is compact in Assertion (1) and by homogeneity in Assertion (2), if R is large enough, by the definitions of the measures ν_w^\pm , the denominator of $H_{\eta,R}^\pm(w)$ is at least $c\eta$ where $c > 0$. The map $H_{\eta,R}^\pm$ is hence Hölder-continuous under the assumptions of Assertion (1), and it is smooth under the assumptions of Assertion (2). Therefore $\Phi_\eta^\pm \in \mathcal{C}^\beta(T^1\widetilde{M})$ and there exists a constant $\kappa_2 > 0$ such that

$$\|\Phi_\eta^\pm\|_\beta = O(\eta^{-\kappa_2} \|\tilde{\psi}^\pm\|_\beta).$$

As in Lemma 10.1, the functions Φ_η^\mp are measurable, nonnegative and satisfy

$$\int_{T^1\widetilde{M}} \Phi_\eta^\mp d\tilde{m}_F = \int_{\partial_\pm^1 D^\pm} \tilde{\psi}^\pm d\tilde{\sigma}^\mp.$$

As in the proof of Theorem 11.1, we will estimate in two ways the quantity

$$I_\eta(T) = \int_0^T e^{\delta t} \sum_{\gamma \in \Gamma} \int_{T^1\widetilde{M}} (\Phi_\eta^- \circ \mathbf{g}^{-t/2}) (\Phi_\eta^+ \circ \mathbf{g}^{t/2} \circ \gamma^{-1}) d\tilde{m}_F dt. \quad (12.5)$$

We first apply the mixing property, now with exponential decay of correlations, as in Step 2 of the proof of Theorem 11.1. For all $t \geq 0$, let

$$A_\eta(t) = \sum_{\gamma \in \Gamma} \int_{v \in T^1\widetilde{M}} \Phi_\eta^-(\mathbf{g}^{-t/2}v) \Phi_\eta^+(\mathbf{g}^{t/2}\gamma^{-1}v) d\tilde{m}_F(v).$$

Then with $\kappa > 0$ as in the definitions of the exponential mixing for the Hölder or Sobolev regularity, we have

$$\begin{aligned} A_\eta(t) &= \frac{1}{\|m_F\|} \int_{T^1\widetilde{M}} \Phi_\eta^- d\tilde{m}_F \int_{T^1\widetilde{M}} \Phi_\eta^+ d\tilde{m}_F + O(e^{-\kappa t} \|\Phi_\eta^-\|_\beta \|\Phi_\eta^+\|_\beta) \\ &= \frac{1}{\|m_F\|} \int_{\partial_\pm^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_\pm^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa t} \eta^{-2\kappa_2} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta). \end{aligned}$$

Hence by integrating,

$$I_\eta(T) = \frac{e^{\delta T}}{\delta \|m_F\|} \left(\int_{\partial_+^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_-^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa T} \eta^{-2\kappa_2} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta) \right). \quad (12.6)$$

Now, as in Step 3 of the proof of Theorem 11.1, we exchange the integral over t and the summation over γ in the definition of $I_\eta(T)$, and we estimate the integral term independently of γ :

$$I_\eta(T) = \sum_{\gamma \in \Gamma} \int_0^T e^{\delta t} \int_{T^1 \tilde{M}} (\Phi_\eta^- \circ \mathbf{g}^{-t/2}) (\Phi_\eta^+ \circ \mathbf{g}^{t/2} \circ \gamma^{-1}) d\tilde{m}_F dt.$$

Let $\hat{\Phi}_\eta^\pm = H_{\eta,R}^\pm \circ f_{D^\pm}^\mp \chi_{\eta,R}^\pm$, so that $\Phi_\eta^\pm = \tilde{\psi}^\pm \circ f_{D^\pm}^\mp \hat{\Phi}_\eta^\pm$. By the last two properties of the regularised maps $\chi_{\eta,R}^\pm$, we have, with ϕ_η^\pm defined as in Equation (10.4),

$$\phi_\eta^\pm e^{-O(\eta)}, \partial_{\pm}^1 D^\pm e^{-O(\eta)} \leq \hat{\Phi}_\eta^\pm \leq \phi_\eta^\pm e^{O(\eta)}. \quad (12.7)$$

If $v \in T^1 \tilde{M}$ belongs to the support of $(\Phi_\eta^- \circ \mathbf{g}^{-t/2}) (\Phi_\eta^+ \circ \mathbf{g}^{t/2} \circ \gamma^{-1})$, then we have $v \in \mathbf{g}^{t/2} \mathcal{V}_{\eta,R}^+ (\partial_+^1 D^-) \cap \mathbf{g}^{-t/2} \mathcal{V}_{\eta,R}^- (\gamma \partial_-^1 D^+)$. Hence the properties (i), (ii) and (iii) of Step 3M of the proof of Theorem 11.1 still hold (with $\Omega_- = \partial_+^1 D^-$ and $\Omega_+ = \partial_-^1 (\gamma D^+)$). In particular, if $w^- = f_{D^-}^+(v)$ and $w^+ = f_{\gamma D^+}^-(v)$, we have, by Assertion (iii) in Step 3M of the proof of Theorem 11.1,⁶ that

$$d(w^\pm, v_\gamma^\pm) = O(\eta + e^{-\lambda\gamma/2}).$$

Hence, with $\kappa_3 = \alpha$ in the Hölder case and $\kappa_3 = 1$ in the Sobolev case (we may assume that $\ell \geq 1$), we have

$$|\tilde{\psi}^\pm(w^\pm) - \tilde{\psi}^\pm(v_\gamma^\pm)| = O((\eta + e^{-\lambda\gamma/2})^{\kappa_3} \|\tilde{\psi}^\pm\|_\beta).$$

Therefore there exists a constant $\kappa_4 > 0$ such that

$$I_\eta(T) = \sum_{\gamma \in \Gamma} (\tilde{\psi}^-(v_\gamma^-) \tilde{\psi}^+(v_\gamma^+) + O((\eta + e^{-\lambda\gamma/2})^{\kappa_4} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta)) \times \int_0^T e^{\delta t} \int_{v \in T^1 \tilde{M}} \hat{\Phi}_\eta^-(\mathbf{g}^{-t/2} v) \hat{\Phi}_\eta^+(\gamma^{-1} \mathbf{g}^{t/2} v) d\tilde{m}_F(v) dt.$$

Now, using the inequalities (12.7), Equation (12.4) follows as in Steps 3M and 4M of the proof of Theorem 11.1, by taking $\eta = e^{-\kappa_5 T}$ for some $\kappa_5 > 0$ and using the effective control given by Equation (11.19) in Step 4M.

In order to prove Assertions (1) and (2) of Theorem 12.7, we may assume that the supports of ψ^\pm are small enough, say contained in $B(x^\pm, \epsilon)$ for some $x^\pm \in T^1 M$ and ϵ small enough. Let \tilde{x}^\pm be lifts of x^\pm and let $\tilde{\psi}^\pm \in \mathcal{C}_c^\beta(T^1 \tilde{M})$ with support in $B(\tilde{x}^\pm, \epsilon)$ be such that $\tilde{\psi}^\pm = \psi^\pm \circ T p$ where $p : \tilde{M} \rightarrow M$ is the universal cover. By a finite summation argument and Equation (12.4), we have

$$\begin{aligned} & \delta \|m_F\| e^{-\delta T} \sum_{\substack{i \in I^- / \sim, j \in I^+ / \sim, \gamma \in \Gamma \\ 0 < \lambda_i, \gamma_j \leq T}} e^{\int \alpha_\gamma \tilde{F}} \tilde{\psi}^-(v_\gamma^-) \tilde{\psi}^+(v_\gamma^+) \\ &= \int_{\partial_+^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_-^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa_0 T} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta). \end{aligned} \quad (12.8)$$

⁶See also the picture at the beginning of the proof of Lemma 11.7.

Assertions (1) and (2) are deduced from this equation in the same way that Corollary 12.3 is deduced from Theorem 11.1. Taking the functions ψ_k^\pm to be the constant functions 1 in Assertion (1) gives the last statement of Theorem 12.7 under the assumptions of Assertion (1). An approximation argument gives the result under the assumptions of Assertion (2). \square

12.4 Equidistribution and counting for quotient simplicial and metric trees

In this Section, we assume that X is the geometric realisation of a locally finite metric tree without terminal vertices (\mathbb{X}, λ) , and that Γ is a (nonelementary discrete) subgroup of $\text{Aut}(\mathbb{X}, \lambda)$. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a system of conductances for Γ , and let $c : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ be its quotient function. We assume in this Section that the potential \tilde{F} is the potential \tilde{F}_c associated⁷ with c . Let $\delta_c = \delta_{\Gamma, F_c}$ be the critical exponent of (Γ, F_c) and let $\tilde{m}_c = \tilde{m}_{F_c}$ and $m_c = m_{F_c}$ be the Gibbs measures of F_c for the continuous time geodesic flow on respectively $\mathcal{G}X$ and $\Gamma \backslash \mathcal{G}X$, as well as for the discrete time geodesic flow on respectively $\mathcal{G}\mathbb{X}$ and $\Gamma \backslash \mathcal{G}\mathbb{X}$ when (\mathbb{X}, λ) is simplicial, that is, if λ is constant with value 1.

Let \mathbb{D}^\pm be simplicial subtrees of \mathbb{X} , with the edge length map induced by λ ,⁸ such that the Γ -equivariant families $\mathcal{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite in X .⁹

For all γ, γ' in Γ such that $\gamma\mathbb{D}^-$ and $\gamma'\mathbb{D}^+$ are disjoint, we denote by $\alpha_{\gamma, \gamma'}$ the common perpendicular from $\gamma\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ (which is an edge path in \mathbb{X}), with length $\lambda_{\gamma, \gamma'} = d(\gamma\mathbb{D}^-, \gamma'\mathbb{D}^+) \in \mathbb{N}$, and by $\alpha_{\gamma, \gamma'}^\pm \in \check{\mathcal{G}}\mathbb{X}$ its parametrisations as in the beginning of Chapter 11: it is the unique map from \mathbb{R} to X such that $\alpha_{\gamma, \gamma'}^-(t) \in \gamma V\mathbb{D}^-$ is the origin $o(\alpha_{\gamma, \gamma'})$ of the edge path $\alpha_{\gamma, \gamma'}$ if $t \leq 0$, $\alpha_{\gamma, \gamma'}^-(t) \in \gamma' V\mathbb{D}^+$ is the endpoint $t(\alpha_{\gamma, \gamma'})$ of the edge path $\alpha_{\gamma, \gamma'}$ if $t \geq \lambda_{\gamma, \gamma'}$, and $\alpha_{\gamma, \gamma'}^-|_{[0, \lambda_{\gamma, \gamma}]}$ is the shortest geodesic arc starting from a point of $\gamma\mathbb{D}^-$ and ending at a point of $\gamma'\mathbb{D}^+$.

For all γ, γ' in Γ such that $\gamma\mathbb{D}^-$ and $\gamma'\mathbb{D}^+$ are disjoint, we define the *multiplicity* of the common perpendicular $\alpha_{\gamma, \gamma'}$ from $\gamma\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ as

$$m_{\gamma, \gamma'} = \frac{1}{\text{Card}(\gamma\Gamma_{\mathbb{D}^-} \gamma^{-1} \cap \gamma'\Gamma_{\mathbb{D}^+} \gamma'^{-1})}. \quad (12.9)$$

Note that $m_{\gamma, \gamma'} = 1$ for all $\gamma, \gamma' \in \Gamma$ when Γ acts freely on $E\mathbb{X}$ (for instance when Γ is torsion-free). Generalising the definition for simplicial trees in Section 11.4, we set

$$c(\alpha) = \sum_{i=1}^k c(e_i) \lambda(e_i),$$

for any edge path $\alpha = (e_1, \dots, e_k)$.

For $n \in \mathbb{N} - \{0\}$, let

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) = \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})},$$

⁷See Section 3.5.

⁸By abuse, we will still denote by \mathbb{D}^\pm the geometric realisation $|\mathbb{D}^\pm|_\lambda$.

⁹We leave to the reader the extension to more general locally finite families of subtrees, as for instance finite unions of those above.

where Γ acts diagonally on $(\Gamma/\Gamma_{\mathbb{D}^-}) \times (\Gamma/\Gamma_{\mathbb{D}^+})$ and d is the distance on $X = |\mathbb{X}|_\lambda$. When Γ is torsion-free, $\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n)$ is the number of edge paths in the graph $\Gamma \backslash \mathbb{X}$ of length at most n , starting by an outgoing edge from the image of \mathbb{D}^- and ending by the opposite of an outgoing edge from the image of \mathbb{D}^+ , with multiplicities coming from the fact that $\Gamma_{\mathbb{D}^\pm} \backslash \mathbb{D}^\pm$ is not assumed to be embedded in $\Gamma \backslash \mathbb{X}$, and with weights coming from the conductances.

In the next results, we distinguish the continuous time case (Theorem 12.8) from the discrete time case (Theorem 12.9). We leave to the reader the versions without the assumption $\delta_c > 0$, giving for every $\tau \in \mathbb{N} - \{0\}$ an asymptotic on

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+, \tau}(n) = \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ n - \tau < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})}.$$

When $\Gamma \backslash X$ is compact, $c = 0$ and \mathbb{D}^\pm are reduced to points, the counting results in Theorems 12.8 and 12.9 are proved in [Gui]. When \mathbb{D}^\pm are singletons, Theorem 12.8 is due to [Rob2] if $c = 0$. Otherwise, the result seems to be new.

Theorem 12.8. *Let (\mathbb{X}, λ) , Γ , \mathbb{D}^\pm and c be as in the beginning of this Section. Assume that the critical exponent δ_c is finite and positive, that the skinning measures $\sigma_{\mathcal{D}^\pm}^\pm$ are finite and nonzero, and that the Gibbs measure m_c is finite and mixing for the continuous time geodesic flow. Then as $t \rightarrow +\infty$, the measures*

$$\delta_c \|m_c\| e^{-\delta_c t} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq t}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})} \Delta_{\Gamma \alpha_{e, \gamma}^-} \otimes \Delta_{\Gamma \alpha_{\gamma^{-1}, e}^+}$$

narrow converge to $\sigma_{\mathcal{D}^-}^+ \otimes \sigma_{\mathcal{D}^+}^-$ in $\Gamma \backslash \check{\mathcal{G}}X \times \Gamma \backslash \check{\mathcal{G}}X$, and

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(t) \sim \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta_c \|m_c\|} e^{\delta_c t}.$$

Proof. By Theorem 11.1, we have

$$\lim_{t \rightarrow +\infty} \delta_c \|m_c\| e^{-\delta_c t} \sum_{\substack{(a, b, \gamma) \in \Gamma / \Gamma_{\mathbb{D}^-} \times \Gamma / \Gamma_{\mathbb{D}^+} \times \Gamma \\ 0 < d(a \mathbb{D}^-, \gamma b \mathbb{D}^+) \leq t}} e^{\int \alpha_{a, \gamma b} \tilde{F}_c} \Delta_{\alpha_{a, \gamma b}^-} \otimes \Delta_{\alpha_{\gamma^{-1} a, b}^+} = \tilde{\sigma}_{\mathcal{D}^-}^+ \otimes \tilde{\sigma}_{\mathcal{D}^+}^- ,$$

not only for the weak-star convergence on $\check{\mathcal{G}}X \times \check{\mathcal{G}}X$, but also by Step 1 of the proof of Theorem 11.1, for the narrow convergence, as $\sigma_{\mathcal{D}^-}^+$ and $\sigma_{\mathcal{D}^+}^-$ are finite. Recall that given a discrete group G acting properly (but not necessarily freely) on a locally compact space Z , the induced measure on $G \backslash Z$ of a (positive, Radon) measure μ on Z is a measure $\bar{\mu}$ which depends linearly and continuously (both for the weak-star and narrow topologies) on μ , and satisfies $\overline{\Delta_z} = \frac{1}{|Gz|} \Delta_{Gz}$ for every $z \in Z$. See for instance Section [PauPS, §2.4] for more details.

The group $\Gamma \times \Gamma$ acts on $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ by

$$(a', b') \cdot (a, b, \gamma) = (a'a, b'b, a'\gamma(b')^{-1}).$$

and the map from the discrete set $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ to $\check{\mathcal{G}}X \times \check{\mathcal{G}}X$ which sends (a, b, γ) to $(\alpha_{a, \gamma b}^-, \alpha_{\gamma^{-1} a, b}^+)$ is $(\Gamma \times \Gamma)$ -equivariant. In particular, the pushforward of measures by this map sends the unit Dirac mass at (a, b, γ) to $\Delta_{\alpha_{a, \gamma b}^-} \otimes \Delta_{\alpha_{\gamma^{-1} a, b}^+}$.

Every orbit of $\Gamma \times \Gamma$ on $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ has a representative of the form $(\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma)$ for some $\gamma \in \Gamma$, since $(a, b) \cdot (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, a^{-1}\gamma b) = (a\Gamma_{\mathbb{D}^-}, b\Gamma_{\mathbb{D}^+}, \gamma)$. Furthermore the double class in $\Gamma_{\mathbb{D}^-} \backslash \Gamma/\Gamma_{\mathbb{D}^+}$ of such a γ is uniquely defined, and the stabiliser of $(\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma)$ has cardinality $|\Gamma_{\mathbb{D}^-} \cap \gamma\Gamma_{\mathbb{D}^+}\gamma^{-1}|$, since $(a, b) \cdot (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma) = (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma')$ if and only if $a \in \Gamma_{\mathbb{D}^-}$, $b \in \Gamma_{\mathbb{D}^+}$ and $a\gamma b^{-1} = \gamma'$. When $\gamma' = \gamma$, this happens if and only if $b = \gamma^{-1}a\gamma$ and $a \in \Gamma_{\mathbb{D}^-} \cap \gamma\Gamma_{\mathbb{D}^+}\gamma^{-1}$.

Hence the measures

$$\delta_c \|m_c\| e^{-\delta_c t} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq t}} \frac{1}{|\Gamma_{\mathbb{D}^-} \cap \gamma\Gamma_{\mathbb{D}^+}\gamma^{-1}|} e^{\int_{\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}} F_c} \Delta_{\Gamma_{\mathbb{D}^-}} \otimes \Delta_{\Gamma_{\mathbb{D}^+}}$$

narrow converge as $t \rightarrow +\infty$ to $\sigma_{\mathcal{D}^-}^+ \otimes \sigma_{\mathcal{D}^+}^-$ in $\Gamma \backslash \check{\mathcal{G}} \mathbb{X} \times \Gamma \backslash \check{\mathcal{G}} \mathbb{X}$. By applying this convergence to the constant function 1, and by the finiteness and nonvanishing of $\sigma_{\mathcal{D}^-}^+$ and $\sigma_{\mathcal{D}^+}^-$, the result follows using the defining property of the potential F_c , see Proposition 3.11. \square

In the remainder of this Section, we consider simplicial trees with the discrete time geodesic flow.

Theorem 12.9. *Let (\mathbb{X}, λ) , Γ , \tilde{c} and \mathbb{D}^\pm be as in the beginning of this Section, with λ constant with value 1. Assume that the critical exponent δ_c is finite and positive. If the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow and the skinning measures $\sigma_{\mathcal{D}^\mp}^\pm$ are finite and nonzero, then as $n \rightarrow +\infty$, the measures*

$$\frac{e^{\delta_c} - 1}{e^{\delta_c}} \|m_c\| e^{-\delta_c n} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} m_{e, \gamma} e^{c(\alpha_e, \gamma)} \Delta_{\Gamma_{\mathbb{D}^-}} \otimes \Delta_{\Gamma_{\mathbb{D}^+}}$$

narrow converge to $\sigma_{\mathcal{D}^-}^+ \otimes \sigma_{\mathcal{D}^+}^-$ in $\Gamma \backslash \check{\mathcal{G}} \mathbb{X} \times \Gamma \backslash \check{\mathcal{G}} \mathbb{X}$ and

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) \sim \frac{e^{\delta_c} \|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{(e^{\delta_c} - 1) \|m_c\|} e^{\delta_c n}.$$

Proof. The claims follow as in Theorem 12.8, replacing Theorem 11.1 by Theorem 11.8. \square

Examples 12.10. (1) Let \mathbb{X}, Γ, c be as in Theorem 12.9, and let $\mathbb{D}^- = \{x\}$ and $\mathbb{D}^+ = \{y\}$ for some $x, y \in V\mathbb{X}$. If the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow, then we have a version of Roblin's simultaneous equidistribution theorem with potential, see Corollary 11.2, and the number $\mathcal{N}_{x, y}(n)$ of nonbacktracking edge paths of length at most n from x to y (counted with weights and multiplicities) satisfies

$$\mathcal{N}_{x, y}(n) \sim \frac{e^{\delta_c} \|\mu_x^+\| \|\mu_y^-\|}{(e^{\delta_c} - 1) \|m_c\| |\Gamma_x| |\Gamma_y|} e^{\delta_c n}.$$

(2) If \mathbb{Y} is a finite connected nonbipartite $(q+1)$ -regular graph (with $q \geq 2$) and \mathbb{Y}^\pm are points, then the number of nonbacktracking edge paths from \mathbb{Y}^- to \mathbb{Y}^+ of length at most n is equivalent as $n \rightarrow +\infty$ to

$$\frac{q+1}{q-1} \frac{q^n}{|V\mathbb{Y}|} + O(r^n) \tag{12.10}$$

for some $r < q$. Indeed, by Theorem 12.9 with \mathbb{X} the universal cover of \mathbb{Y} , Γ its covering group and $c = 0$, we have $\delta_c = \ln q$ and m_c is the Bowen-Margulis measure, so that normalizing the

Patterson measures to be probability measures, we have $\|m_c\| = \frac{q}{q+1} |V\mathbb{Y}|$ by Equation (8.3). We refer to Section 12.6 (see Remark (i) following Theorem 12.17) for the error term.

Let \mathbb{Y} the figure 8-graph with a single vertex and four directed edges, and let \mathbb{Y}^\pm be the the singleton consisting of its vertex. In this simple example, it is easy to count by hand that the number of loops of length exactly n without backtracking in \mathbb{Y} is $4 \cdot 3^{n-1}$. Thus the number $\mathcal{N}(n)$ of common perpendiculars of the vertex to itself of length at most n is by a simple geometric sum $2(3^n - 1)$. This agrees with Equation (12.10) that gives $\mathcal{N}(n) \sim 2 \cdot 3^n$ as $n \rightarrow +\infty$.

(3) Let \mathbb{Y} be a finite connected nonbipartite $(q+1)$ -regular graph (with $q \geq 2$). Let \mathbb{Y}^\pm be regular connected subgraphs of degrees $q^\pm \geq 0$. Then the number $\mathcal{N}(n)$ of edge paths of length at most n starting transversally to \mathbb{Y}^- and ending transversally to \mathbb{Y}^+ satisfies

$$\mathcal{N}(n) = \frac{(q+1-q^-)(q+1-q^+) |V\mathbb{Y}^-| |V\mathbb{Y}^+|}{(q^2-1) |V\mathbb{Y}|} q^n + O(r^n)$$

for some $r < q$. This is a direct consequence of Theorem 12.9, using Proposition 8.1 (3) and Proposition 8.4 (3), and referring to Section 12.6 (see Remark (i) following Theorem 12.17) for the error term.

We refer for instance to Section 15.2 for examples of counting results in graphs of groups where the underlying graph is infinite.

Remark 12.11. A common perpendicular in a simplicial tree is, in the language of graph theory, a non-backtracking walk. Among other applications (when restricting to groups Γ acting freely, which is never the case if Γ is a nonuniform lattice in the tree \mathbb{X} , that is, when the quotient graph of groups $\Gamma \backslash \mathbb{X}$ is infinite but has finite volume), Theorem 12.9 gives a complete asymptotic solution to the problem of counting non-backtracking walks from a given vertex to a given vertex of a (finite) nonbipartite graph. See Theorem 12.12 for the corresponding result in bipartite graphs and for example [ABLS, Th. 1.1], [AnFH, p. 4290,4302], [Fri2, L. 2.3], [Sod, Prop. 6.4] for related results. Anticipating on the error terms that we will give in Section 12.6, note that the paper [ABLS, Th. 1.1] for instance gives a precise speed using spectral properties, more precise than the ones we obtain.

In some applications (see the examples at the end of this Section), we encounter bipartite simplicial graphs and, consequently, their discrete time geodesic flow is not mixing. The following result applies in this context.

Until the end of this section, we assume that the simplicial tree \mathbb{X} has a Γ -invariant structure of a bipartite graph, and we denote by $V\mathbb{X} = V_1\mathbb{X} \sqcup V_2\mathbb{X}$ the corresponding partition of its set of vertices. For every $i \in \{1, 2\}$, we denote by $\check{\mathcal{G}}_i \mathbb{X}$ the space of generalised discrete geodesic lines $\ell \in \check{\mathcal{G}} \mathbb{X}$ such that $\ell(0) \in V_i\mathbb{X}$, so that we have a partition $\check{\mathcal{G}} \mathbb{X} = \check{\mathcal{G}}_1 \mathbb{X} \sqcup \check{\mathcal{G}}_2 \mathbb{X}$. Note that if the basepoint x_0 lies in $V_i\mathbb{X}$, then $\mathcal{G}_{\text{even}} \mathbb{X}$ is equal to $\check{\mathcal{G}}_i \mathbb{X} \cap \mathcal{G} \mathbb{X}$. For all $i, j \in \{1, 2\}$, we define

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+, i, j}(n) = \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n \\ o(\alpha_{e, \gamma}) \in V_i \mathbb{X}, t(\alpha_{e, \gamma}) \in V_j \mathbb{X}}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})}.$$

Theorem 12.12. *Let (\mathbb{X}, λ) , Γ and c be as in the beginning of this Section, with λ constant with value 1. Assume that the critical exponent δ_c is finite and positive. If \mathbb{X} has a Γ -invariant*

structure of a bipartite graph as above, if the restriction to $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$ of the Gibbs measure m_c is finite and mixing for the square of the discrete time geodesic flow, then for all $i, j \in \{1, 2\}$ such that the measures $\sigma_{\mathcal{D}^-, i}^-$ and $\sigma_{\mathcal{D}^+, j}^+$ are finite and nonzero, as n tends to $+\infty$ with $n \equiv i - j \pmod{2}$, the measures

$$\frac{e^{2\delta_c} - 1}{2e^{2\delta_c}} \|m_c\| e^{-\delta_c n} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n \\ o(\alpha_{e, \gamma}) \in V_i \mathbb{X}, t(\alpha_{e, \gamma}) \in V_j \mathbb{X}}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})} \Delta_{\Gamma \alpha_{e, \gamma}^-} \otimes \Delta_{\Gamma \alpha_{\gamma^{-1}, e}^+}$$

narrow converge to $\sigma_{\mathcal{D}^-, i}^+ \otimes \sigma_{\mathcal{D}^+, j}^-$ in $\Gamma \backslash \check{\mathcal{G}} \mathbb{X} \times \Gamma \backslash \check{\mathcal{G}} \mathbb{X}$ and

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+, i, j}(n) \sim \frac{2e^{2\delta_c} \|\sigma_{\mathcal{D}^-, i}^+\| \|\sigma_{\mathcal{D}^+, j}^-\|}{(e^{2\delta_c} - 1) \|m_c\|} e^{\delta_c n}.$$

Proof. This Theorem is proved in the same way as the above Theorem 12.9 using Theorem 11.10. Note that we have a $(\Gamma \times \Gamma)$ -invariant partition

$$\check{\mathcal{G}} \mathbb{X} \times \check{\mathcal{G}} \mathbb{X} = \bigsqcup_{(i, j) \in \{1, 2\}^2} \check{\mathcal{G}}_i \mathbb{X} \times \check{\mathcal{G}}_j \mathbb{X},$$

that $\alpha_{e, \gamma}^- \in \check{\mathcal{G}}_i \mathbb{X}$ if and only if $o(\alpha_{e, \gamma}) \in V_i \mathbb{X}$, and that $\alpha_{\gamma^{-1}, e}^+ \in \check{\mathcal{G}}_j \mathbb{X}$ if and only if $t(\alpha_{e, \gamma}) \in V_j \mathbb{X}$, since $\alpha_{\gamma^{-1}, e}^+(0) = \gamma^{-1} \alpha_{e, \gamma}^+(0) = \gamma^{-1} t(\alpha_{e, \gamma})$. \square

Examples 12.13. (1) Let \mathbb{X}, Γ, c be as in Theorem 12.12, and let $\mathbb{D}^- = \{x\}$ and $\mathbb{D}^+ = \{y\}$ for some vertices x, y in the same $V_i \mathbb{X}$ for $i \in \{1, 2\}$. If the restriction to $\Gamma \backslash \mathcal{G}_{\text{even}} \mathbb{X}$ of the Gibbs measure m_c is finite and mixing for the square of the discrete time geodesic flow, then as $n \rightarrow +\infty$ is even,

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) \sim \frac{2e^{2\delta_c}}{e^{2\delta_c} - 1} \frac{\|\mu_x^+\| \|\mu_y^-\|}{\|m_c\| |\Gamma_x| |\Gamma_y|} e^{\delta_c n}.$$

Indeed, we have $\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) = \mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+, i, i}(n)$ and $\sigma_{\mathcal{D}^\pm, i}^\pm = \sigma_{\mathcal{D}^\pm}^\pm$.

(2) Let \mathbb{Y} be the complete biregular graph with $q + 1$ vertices of degree $p + 1$ and $p + 1$ vertices of degree $q + 1$. Let $\mathbb{Y}^\pm = \{y\}$ be a fixed vertex of degree $p + 1$. Note that \mathbb{Y} being bipartite, all common perpendiculars from y to y have even length, (the shortest one having length 4). Then as n is even and tends to $+\infty$, we have

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n) \sim \frac{q(p + 1)}{(q + 1)(pq - 1)} (pq)^{n/2}.$$

Indeed, the biregular tree $\mathbb{X}_{p, q}$ of degrees $(p + 1, q + 1)$ is a universal cover of \mathbb{Y} with covering group Γ acting freely and cocompactly, so that with $c = 0$ we have $\delta_c = \ln \sqrt{pq}$ and the Gibbs measure m_c is the Bowen-Margulis measure m_{BM} . If we normalise the Patterson density such that $\|\mu_y\| = \frac{p+1}{\sqrt{p}}$, then by Proposition 8.1 (2), we have $\|m_{\text{BM}}\| = 2(p + 1)(q + 1)$. Thus the result follows from Example (1). Note that if $p = q$, then

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n) \sim \frac{q}{q^2 - 1} q^n,$$

and the constant in front of q^n is indeed different from that in the nonbipartite case.

(3) Let \mathbb{Y} be a finite biregular graph with vertices of degrees $p + 1$ and $q + 1$, where $p, q \geq 2$, and let $V\mathbb{Y} = V_p\mathbb{Y} \sqcup V_q\mathbb{Y}$ be the corresponding partition. If $\mathbb{Y}^- = \{v\}$ where $v \in V_p\mathbb{Y}$ and \mathbb{Y}^+ is a cycle of length $L \geq 2$, then as $N \rightarrow +\infty$, the number of common perpendiculars of even length at most $2N$ from \mathbb{Y}^- to \mathbb{Y}^+ is equivalent to

$$\frac{L q (p - 1)}{2 (pq - 1) |V_p\mathbb{Y}|} (pq)^N,$$

and the number of common perpendiculars of odd length at most $2N - 1$ from v to \mathbb{Y}^+ is equivalent to

$$\frac{L (q - 1)}{2 (pq - 1) |V_p\mathbb{Y}|} (pq)^N.$$

Proof. The cycle \mathbb{Y}^+ has even length L and has $\frac{L}{2}$ vertices in both $V_p\mathbb{Y}$ and $V_q\mathbb{Y}$. A common perpendicular from \mathbb{Y}^- to \mathbb{Y}^+ has even length if and only if it ends at a vertex in $V_p\mathbb{Y}$.

Let $\mathbb{X} \rightarrow \mathbb{Y}$ be a universal cover of \mathbb{Y} , whose covering group Γ acts freely and cocompactly on \mathbb{X} . Let $\mathbb{D}^- = \{\tilde{v}\}$ where $\tilde{v} \in V_p\mathbb{X}$ is a lift of v , and let \mathbb{D}^- be a geodesic line in \mathbb{X} mapping to \mathbb{Y}^+ . We use Theorem 12.12 with $V_1\mathbb{X}$ the (full) preimage of $V_p\mathbb{Y}$ in \mathbb{X} , with $V_2\mathbb{X}$ the preimage of $V_q\mathbb{Y}$ in \mathbb{X} and with $c = 0$, so that $\delta_c = \ln \sqrt{pq}$ and $m_c = m_{\text{BM}}$. Let us normalise the Patterson density of Γ as in Proposition 8.1 (2), so that

$$\|\sigma_{\mathcal{D}^-, 1}^+\| = \|\mu_{\tilde{v}}\| = \frac{p + 1}{\sqrt{p}}.$$

The mass for the skinning measure of the part of the inner unit normal bundle of \mathbb{Y}^+ with basepoint in $V_p\mathbb{Y}$ is (see Corollary 8.5)

$$\frac{L}{2} \frac{p + 1}{\sqrt{p}} \frac{p - 1}{p + 1} = \frac{L(p - 1)}{2\sqrt{p}}$$

and its complement has mass $\frac{L(q-1)}{2\sqrt{q}}$. Recall also that, by Proposition 8.1 (2), considering the graph \mathbb{Y} as a graph of groups with trivial groups,

$$\|m_{\text{BM}}\| = \text{TVol}(\mathbb{Y}) = |E\mathbb{Y}| = 2(p + 1)|V_p\mathbb{Y}| = 2(q + 1)|V_q\mathbb{Y}|.$$

The claim about the common perpendiculars of even length at most $2N$ follows from Theorem 12.12 with $i = j = 1$, since

$$\frac{2 e^{2\delta_c} \|\sigma_{\mathcal{D}^-, i}^+\| \|\sigma_{\mathcal{D}^+, j}^-\|}{(e^{2\delta_c} - 1) \|m_c\|} = \frac{2 pq \frac{p+1}{\sqrt{p}} \frac{L(p-1)}{2\sqrt{p}}}{(pq - 1) 2 (p + 1) |V_p\mathbb{Y}|} = \frac{L q (p - 1)}{2 (pq - 1) |V_p\mathbb{Y}|}.$$

The claim about the common perpendiculars of odd length at most $2N - 1$ follows similarly from Theorem 12.12 with $i = 1$ and $j = 2$. \square

(4) Let \mathbb{Y} be a finite biregular graph with vertices of degrees $p + 1$ and $q + 1$, where $p, q \geq 2$, and let $V\mathbb{Y} = V_p\mathbb{Y} \sqcup V_q\mathbb{Y}$ be the corresponding partition. If \mathbb{Y}^- and \mathbb{Y}^+ are cycles of length $L^- \geq 2$ and $L^+ \geq 2$ respectively, then as $N \rightarrow +\infty$, the number of common perpendiculars of length at most N from \mathbb{Y}^- to \mathbb{Y}^+ is equal to

$$\frac{(\sqrt{q} + \sqrt{p})^2 L^- L^+}{2(pq - 1) \text{Card}(E\mathbb{Y})} (\sqrt{pq})^{N+2} + O(r^N) \tag{12.11}$$

for some $r < \sqrt{pq}$.

Proof. As in the above proof of Example (3), let $\mathbb{X} \rightarrow \mathbb{Y}$ be a universal cover of \mathbb{Y} , with covering group Γ and let \mathbb{D}^\pm be a geodesic line in \mathbb{X} mapping to \mathbb{Y}^\pm . We normalise the Patterson density $(\mu_x)_{x \in V\mathbb{X}}$ of Γ so that $\|\mu_x\| = \frac{\deg_{\mathbb{X}}(x)}{\sqrt{\deg_{\mathbb{X}}(x)-1}}$. By Proposition 8.4 (3) with $k = 1$ and trivial vertex stabilisers, and since a simple cycle of length λ in a biregular graph of different degrees $p + 1$ and $q + 1$ has exactly $\frac{\lambda}{2}$ vertices of degree either $p + 1$ or $q + 1$, we have

$$\|\sigma_{\mathcal{D}^\pm}^\pm\| = \sum_{\Gamma x \in \mathbb{Y}^\mp} \frac{\|\mu_x\| (\deg_{\mathbb{X}}(x) - k)}{\deg_{\mathbb{X}}(x)} = \sum_{y \in V_p \mathbb{Y}^\mp} \sqrt{p} + \sum_{y \in V_q \mathbb{Y}^\mp} \sqrt{q} = L^\mp \frac{\sqrt{p} + \sqrt{q}}{2}.$$

The result without the error term then follows from Theorem 12.12, using Proposition 8.1 (2) and Remark 2.11.

We refer to Section 13.2 (see Remark (ii) following Theorem 12.17) for the error term. \square

Remark 12.14. If \mathbb{Y} is a finite bipartite $(p + 1)$ -regular graph, \mathbb{Y}^- consists in a vertex and \mathbb{Y}^+ is a cycle of length L , then Example (3) above gives

$$\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n) \sim \frac{L p (p - 1)}{2 (p^2 - 1) |V\mathbb{Y}|} p^n + \frac{L (p - 1)}{2 (p^2 - 1) |V\mathbb{Y}|} p^n = \frac{L}{|V\mathbb{Y}|} p^n$$

for the number $\mathcal{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n)$ of common perpendiculars from \mathbb{Y}^- to \mathbb{Y}^+ with length at most n . This is the same result as for nonbipartite trees.

12.5 Counting for simplicial graphs of groups

In this Section, we give an intrinsic translation “a la Bass-Serre” of the counting result in Theorem 12.9 using graphs of groups (see [Ser3] and Section 2.7 for background information).

Let (\mathbb{Y}, G_*) be a locally finite, connected graph of finite groups, and let $(\mathbb{Y}^\pm, G_*^\pm)$ be connected subgraphs of subgroups.¹⁰ Let $c : E\mathbb{Y} \rightarrow \mathbb{R}$ be a system of conductances on \mathbb{Y} .

Let \mathbb{X} be the Bass-Serre tree of the graph of groups (\mathbb{Y}, G_*) (with geometric realisation $X = |\mathbb{X}|_1$) and Γ its fundamental group (for an indifferent choice of basepoint). Assume that Γ is nonelementary. We denote by $\mathcal{G}(\mathbb{Y}, G_*) = \Gamma \backslash \mathcal{G}\mathbb{X}$ and $(\mathbf{g}^t : \mathcal{G}(\mathbb{Y}, G_*) \rightarrow \mathcal{G}(\mathbb{Y}, G_*))_{t \in \mathbb{Z}}$ the quotient of the (discrete time) geodesic flow on $\mathcal{G}\mathbb{X}$, by $\tilde{c} : \mathbb{X} \rightarrow \mathbb{R}$ the $(\Gamma$ -invariant) lift of c , with δ_c its critical exponent and $\tilde{F}_c : T^1 X \rightarrow \mathbb{R}$ its associated potential, by m_c the Gibbs measure on $\mathcal{G}(\mathbb{Y}, G_*)$ associated with a choice of Patterson densities $(\mu_x^\pm)_{x \in X}$ for the pairs (Γ, F_c^\pm) , by \mathbb{D}^\pm two subtrees in \mathbb{X} such that the quotient graphs of groups $\Gamma_{\mathbb{D}^\pm} \backslash \mathbb{D}^\pm$ identify with $(\mathbb{Y}^\pm, G_*^\pm)$ (see below for precisions), and by $\sigma_{(\mathbb{Y}^\mp, G_*^\mp)}^\pm$ the associated skinning measures.

The *fundamental groupoid* $\pi(\mathbb{Y}, G_*)$ of (\mathbb{Y}, G_*) ¹¹ is the quotient of the free product of the groups G_v for $v \in V\mathbb{Y}$ and of the free group on $E\mathbb{Y}$ by the normal subgroup generated by the elements $e \bar{e}$ and $e \rho_e(g) \bar{e} \rho_{\bar{e}}(g)^{-1}$ for all $e \in E\mathbb{Y}$ and $g \in G_e$. We identify each G_x for $x \in V\mathbb{Y}$ with its image in $\pi(\mathbb{Y}, G_*)$.

Let $n \in \mathbb{N} - \{0\}$. A (locally) geodesic path of length n in the graph of groups (\mathbb{Y}, G_*) is the image α in $\pi(\mathbb{Y}, G_*)$ of a word, called *reduced* in [Bass, 1.7],

$$h_0 e_1 h_1 e_2 \dots h_{n-1} e_n h_n$$

¹⁰See Section 2.7 for definitions and background.

¹¹denoted by $F(\mathbb{Y}, G_*)$ in [Ser3, §5.1], called the *path group* in [Bass, 1.5], see also [Hig]

with

- $e_i \in E\mathbb{Y}$ and $t(e_i) = o(e_{i+1})$ for $1 \leq i \leq n-1$ (so that (e_1, \dots, e_n) is an edge path in the graph \mathbb{Y} ;
- $h_0 \in G_{o(e_1)}$ and $h_i \in G_{t(e_i)}$ for $1 \leq i \leq n$;
- if $e_{i+1} = \bar{e}_i$ then h_i does not belong to $\rho_{e_i}(G_{e_i})$, for $1 \leq i \leq n-1$.

Its *origin* is $o(\alpha) = o(e_1)$ and its *endpoint* is $t(\alpha) = t(e_n)$. They do not depend on the chosen words with image α in $\pi(\mathbb{Y}, G_*)$.

A *common perpendicular* of length n from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) in the graph of groups (\mathbb{Y}, G_*) is the double coset

$$[\alpha] = G_{o(\alpha)}^- \alpha G_{t(\alpha)}^+$$

of a geodesic path α of length n as above, such that:

- α starts transversally from (\mathbb{Y}^-, G_*^-) , that is, its origin $o(\alpha) = o(e_1)$ belongs to $V\mathbb{Y}^-$ and $h_0 \notin G_{o(e_1)}^- \rho_{\bar{e}_1}(G_{e_1})$ if $e_1 \in E\mathbb{Y}^-$,
- α ends transversally in (\mathbb{Y}^+, G_*^+) , that is, its endpoint $t(\alpha) = t(e_n)$ belongs to \mathbb{Y}^+ and $h_n \notin \rho_{e_n}(G_{e_n}) G_{t(e_n)}^+$ if $e_n \in E\mathbb{Y}^+$.

Note that these two notions do not depend on the representative of the double coset $G_{o(\alpha)}^- \alpha G_{t(\alpha)}^+$, and we also say that the double coset $[\alpha]$ starts transversally from (\mathbb{Y}^-, G_*^-) or ends transversally in (\mathbb{Y}^+, G_*^+) .

We denote by $\text{Perp}((\mathbb{Y}^\pm, G_*^\pm), n)$ the set of common perpendiculars in (\mathbb{Y}, G_*) of length n from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) . We denote by

$$c(\alpha) = \sum_{i=1}^n c(e_i)$$

the *conductance* of a geodesic path α as above, which depends only on the double class $[\alpha]$. We define the *multiplicity* m_α of a geodesic path α as above by

$$m_\alpha = \frac{1}{\text{Card}(G_{o(\alpha)}^- \cap \alpha G_{t(\alpha)}^+ \alpha^{-1})}.$$

It depends only on the double class $[\alpha]$ of α . We define the *counting function* of the common perpendiculars in (\mathbb{Y}, G_*) of length at most n from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) (counted with multiplicities and with weights given by the system of conductances c) as

$$\mathcal{N}_{(\mathbb{Y}^-, G_*^-), (\mathbb{Y}^+, G_*^+)}(n) = \sum_{[\alpha] \in \text{Perp}((\mathbb{Y}^\pm, G_*^\pm), n)} m_\alpha e^{c(\alpha)}.$$

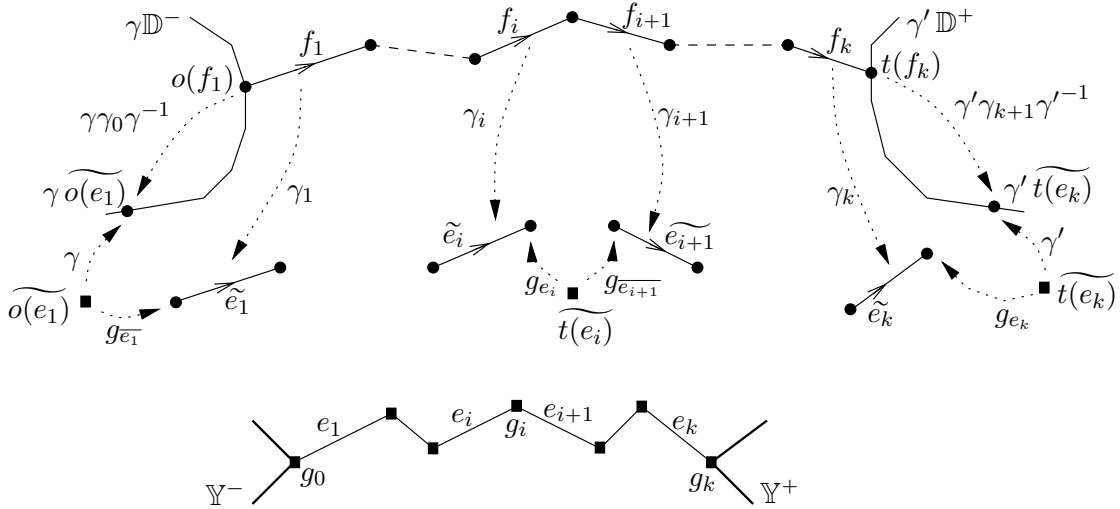
Theorem 12.15. *Let (\mathbb{Y}, G_*) , $(\mathbb{Y}^\pm, G_*^\pm)$ and c be as in the beginning of this Section. Assume that the critical exponent δ_c of c is positive and that the Gibbs measure m_c on $\mathcal{G}(\mathbb{Y}, G_*)$ is finite and mixing for the discrete time geodesic flow. Assume that the skinning measures $\sigma_{(\mathbb{Y}^\mp, G_*^\mp)}^\pm$ are finite and nonzero. Then as $n \in \mathbb{N}$ tends to ∞*

$$\mathcal{N}_{(\mathbb{Y}^-, G_*^-), (\mathbb{Y}^+, G_*^+)}(n) \sim \frac{e^{\delta_c} \|\sigma_{(\mathbb{Y}^-, G_*^-)}^+\| \|\sigma_{(\mathbb{Y}^+, G_*^+)}^-\|}{(e^{\delta_c} - 1) \|m_c\|} e^{\delta_c n}.$$

Proof. Let \mathbb{X} be the Bass-Serre tree of (\mathbb{Y}, G_*) and Γ its fundamental group (for an indifferent choice of basepoint). As seen in Section 2.7, the Bass-Serre trees \mathbb{D}^\pm of $(\mathbb{Y}^\pm, G_*^\pm)$, with fundamental groups Γ^\pm , identify with simplicial subtrees \mathbb{D}^\pm of \mathbb{X} , such that Γ^\pm are the stabilisers $\Gamma_{\mathbb{D}^\pm}$ of \mathbb{D}^\pm in Γ . In particular, the maps $(\Gamma_{\mathbb{D}^\pm} \backslash \mathbb{D}^\pm) \rightarrow (\Gamma \backslash \mathbb{X})$ induced by the inclusion maps $\mathbb{D}^\pm \rightarrow \mathbb{X}$ by taking quotient, are injective:

$$\forall \gamma \in \Gamma, \forall z \in V\mathbb{D}^\pm \cup E\mathbb{D}^\pm, \text{ if } \gamma z \in V\mathbb{D}^\pm \cup E\mathbb{D}^\pm, \text{ then } \exists \gamma_\pm \in \Gamma_{\mathbb{D}^\pm}, \gamma_\pm z = \gamma z. \quad (12.12)$$

As in Definition 2.10, for all $z \in V\mathbb{Y} \cup E\mathbb{Y}$ and $e \in E\mathbb{Y}$, we fix a lift $\tilde{z} \in V\mathbb{X} \cup E\mathbb{X}$ of z and $g_e \in \Gamma$, such that $\tilde{e} = \tilde{z}$, $g_e \tilde{t}(e) = t(\tilde{e})$, $G_z = \Gamma_{\tilde{z}}$, and the monomorphism $\rho_e : G_e \rightarrow G_{t(e)}$ is $\gamma \mapsto g_e^{-1} \gamma g_e$. We assume, as we may, that $\tilde{z} \in V\mathbb{D}^\pm \cup E\mathbb{D}^\pm$ if $z \in V\mathbb{Y}^\pm \cup E\mathbb{Y}^\pm$. We assume, as we may using Equation (12.12), that if $e \in E\mathbb{Y}^\pm$, then $g_e \in \Gamma_{\mathbb{D}^\pm}$. We denote by $p : \mathbb{X} \rightarrow \mathbb{Y} = \Gamma \backslash \mathbb{X}$ the canonical projection.



For all $\gamma, \gamma' \in \Gamma$ such that $\gamma\mathbb{D}^-$ and $\gamma'\mathbb{D}^+$ are disjoint, the common perpendicular $\alpha_{\gamma\mathbb{D}^-, \gamma'\mathbb{D}^+}$ from $\gamma\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ is an edge path (f_1, f_2, \dots, f_k) with $o(f_1) \in \gamma\mathbb{D}^-$ and $t(f_k) \in \gamma'\mathbb{D}^+$. Note that $\gamma^{-1}o(f_1)$ and $p(o(f_1))$ are two vertices of \mathbb{D}^- in the same Γ -orbit, and that $\gamma'^{-1}t(f_k)$ and $p(t(f_k))$ are two vertices of \mathbb{D}^+ in the same Γ -orbit. Hence by Equation (12.12), we may choose $\gamma_0 \in \Gamma_{\mathbb{D}^-}$ such that $\gamma_0\gamma^{-1}o(f_1) = p(o(f_1))$ and $\gamma_{k+1} \in \Gamma_{\mathbb{D}^-}$ such that $\gamma_{k+1}\gamma'^{-1}t(f_k) = p(t(f_k))$. For $1 \leq i \leq k$, choose $\gamma_i \in \Gamma$ such that $\gamma_i f_i = p(f_i)$. We define

- $e_i = p(f_i)$ for $1 \leq i \leq k$,
- $h_i = g_{e_i}^{-1} \gamma_i \gamma_{i+1}^{-1} g_{e_{i+1}}$, which belongs to $\Gamma_{\tilde{t}(e_i)} = G_{t(e_i)}$ for $1 \leq i \leq k-1$,
- $h_0 = \gamma_0 \gamma^{-1} \gamma_1^{-1} g_{e_1} = \gamma^{-1} (\gamma_0 \gamma_1^{-1}) \gamma_1^{-1} g_{e_1}$, which belongs to $\Gamma_{\tilde{o}(e_1)} = G_{o(e_1)}$,
- $h_k = g_{e_k}^{-1} \gamma_k \gamma' \gamma_{k+1}^{-1} = g_{e_k}^{-1} \gamma_k (\gamma' \gamma_{k+1} \gamma'^{-1})^{-1} \gamma'$, which belongs to $\Gamma_{\tilde{t}(e_k)} = G_{t(e_k)}$.

Lemma 12.16. (1) *The word $h_0 e_1 h_1 \dots h_{k-1} e_k h_k$ is reduced. Its image α in the fundamental groupoid $\pi(\mathbb{Y}, G_*)$ does not depend on the choices of $\gamma_1, \dots, \gamma_k$, and starts transversally from (\mathbb{Y}^-, G_*^-) and ends transversally in (\mathbb{Y}^+, G_*^+) . The double class $[\alpha]$ of α is independent of the choices of γ_0 and γ_{k+1} .*

- (2) The map $\tilde{\Theta}$ from the set of common perpendiculars in \mathbb{X} between disjoint images of \mathbb{D}^- and \mathbb{D}^+ under elements of Γ , into the set of common perpendiculars in (\mathbb{Y}, G_*) from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) , sending $\alpha_{\gamma\mathbb{D}^-, \gamma\mathbb{D}^+}$ to $[\alpha]$, is constant under the action of Γ at the source, and preserves the lengths and the multiplicities.
- (3) The map Θ induced by $\tilde{\Theta}$ from the set of Γ -orbits of common perpendiculars in \mathbb{X} between disjoint images of \mathbb{D}^- and \mathbb{D}^+ under elements of Γ into the set of common perpendiculars in (\mathbb{Y}, G_*) from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) is a bijection, preserving the lengths and the multiplicities.

Proof. (1) If $e_{i+1} = \bar{e}_i$, then by the definition of h_i , we have

$$\begin{aligned}
h_i \in \rho_{e_i}(G_{e_i}) = g_{e_i}^{-1} \Gamma_{\tilde{e}_i} g_{e_i} &\iff g_{e_i} h_i g_{e_i}^{-1} \tilde{e}_i = \tilde{e}_i \\
&\iff g_{e_i} g_{e_i}^{-1} \gamma_i \gamma_{i+1}^{-1} g_{\bar{e}_{i+1}} g_{e_i}^{-1} \tilde{e}_i = \tilde{e}_i \\
&\iff \gamma_i \gamma_{i+1}^{-1} \widetilde{\bar{e}_{i+1}} = \tilde{e}_i \\
&\iff \overline{\gamma_{i+1}^{-1} \widetilde{\bar{e}_{i+1}}} = \gamma_i^{-1} \tilde{e}_i \iff \overline{f_{i+1}} = f_i.
\end{aligned}$$

Hence the word $h_0 e_1 h_1 \dots h_{k-1} e_k h_k$ is reduced.

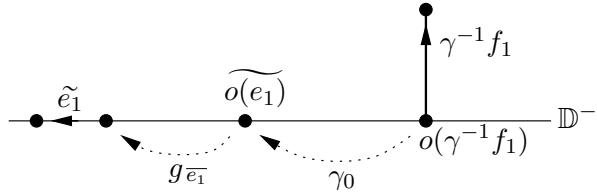
The element γ_i for $i \in \{1, \dots, k\}$ is uniquely determined up to multiplication on the left by an element of $\Gamma_{\tilde{e}_i} = G_{e_i}$. If we fix¹² $i \in \{1, \dots, k\}$ and if we replace γ_i by $\gamma'_i = \alpha \gamma_i$ for some $\alpha \in G_{e_i}$, then only the elements h_{i-1} and h_i change, replaced by elements that we denote by h'_{i-1} and h'_i respectively. We have (if $2 \leq i \leq k-1$, but otherwise the argument is similar by the definitions of h_0 and h_k)

$$\begin{aligned}
h'_{i-1} e_i h'_i &= g_{e_{i-1}}^{-1} \gamma_{i-1} \gamma_i^{-1} \alpha^{-1} g_{\bar{e}_i} e_i g_{e_i}^{-1} \alpha \gamma_i \gamma_{i+1}^{-1} g_{\bar{e}_{i+1}} \\
&= g_{e_{i-1}}^{-1} \gamma_{i-1} \gamma_i^{-1} g_{\bar{e}_i} \rho_{\bar{e}_i}(\alpha)^{-1} e_i \rho_{e_i}(\alpha) g_{e_i}^{-1} \gamma_i \gamma_{i+1}^{-1} g_{\bar{e}_{i+1}}.
\end{aligned}$$

Since $\rho_{\bar{e}_i}(\alpha)^{-1} e_i \rho_{e_i}(\alpha)$ is equal to $\bar{e}_i^{-1} = e_i$ in the fundamental groupoid, the words $h'_{i-1} e_i h'_i$ and $h_{i-1} e_i h_i$ have the same image in $\pi(\mathbb{Y}, G_*)$. Therefore α is independent on the choices of $\gamma_1, \dots, \gamma_k$.

We have $o(\alpha) = o(e_1) \in V\mathbb{Y}^-$ and $t(\alpha) = t(e_k) \in V\mathbb{Y}^+$, hence α starts from \mathbb{Y}^- and ends in \mathbb{Y}^+ .

Assume that $e_1 \in E\mathbb{Y}^-$. Let us prove that $h_0 \in G_{o(e_1)}^- \rho_{\bar{e}_1}(G_{e_1})$ if and only if $f_1 \in \gamma E\mathbb{D}^-$.



By the definition of $\rho_{\bar{e}_1}$, we have $h_0 \in G_{o(e_1)}^- \rho_{\bar{e}_1}(G_{e_1})$ if and only if there exists $\alpha \in \Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-}$ such that $\alpha^{-1} h_0 \in g_{\bar{e}_1}^{-1} \Gamma_{\tilde{e}_1} g_{\bar{e}_1}$. By the definition of h_0 and since γ_1 maps f_1 to \tilde{e}_1 , we have

$$\begin{aligned}
\alpha^{-1} h_0 \in g_{\bar{e}_1}^{-1} \Gamma_{\tilde{e}_1} g_{\bar{e}_1} &\iff g_{\bar{e}_1} \alpha^{-1} (\gamma_0 \gamma_1^{-1} \gamma_1^{-1} g_{\bar{e}_1}) g_{\bar{e}_1}^{-1} \tilde{e}_1 = \tilde{e}_1 \\
&\iff f_1 = \gamma \gamma_0^{-1} \alpha g_{\bar{e}_1}^{-1} \tilde{e}_1.
\end{aligned}$$

¹²We leave to the reader the verification that the changes induced by various i 's do not overlap.

Since $\tilde{e}_1 \in E\mathbb{D}^-$ and $\gamma_0, \alpha, g_{\tilde{e}_1}$ all belong to $\Gamma_{\mathbb{D}^-}$, this last condition implies that $f_1 \in \gamma E\mathbb{D}^-$. Conversely (for future use), if $f_1 \in \gamma E\mathbb{D}^-$, then (see the above picture) $\gamma_0 \gamma^{-1} f_1$ is an edge of \mathbb{D}^- with origin $\widetilde{o(e_1)}$, in the same orbit that the edge $g_{\tilde{e}_1}^{-1} \tilde{e}_1$ of \mathbb{D}^- , which also has origin $\widetilde{o(e_1)}$. By Equation (12.12), this implies that there exists $\alpha \in \Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-}$ such that $f_1 = \gamma \gamma_0^{-1} \alpha g_{\tilde{e}_1}^{-1} \tilde{e}_1$. By the above equivalences, we hence have that $h_0 \in G_{o(e_1)}^- \rho_{\tilde{e}_1}(G_{e_1})$.

Similarly, one proves that if $e_k \in E\mathbb{Y}^+$, then $h_k \in \rho_{e_k}(G_{e_k}) G_{t(e_k)}^+$ if and only if $f_k \in \gamma' E\mathbb{D}^+$. Since (f_1, \dots, f_n) is the common perpendicular edge path from $\gamma \mathbb{D}^-$ to $\gamma' \mathbb{D}^+$, this proves that α starts transversally from \mathbb{Y}^- and ends transversally in \mathbb{Y}^+ .

Note that the element $\gamma_0 \in \Gamma_{\mathbb{D}^-}$ is uniquely defined up to multiplication on the left by an element of $\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-} = G_{o(e_1)}^-$, and appears only as the first letter in the expression of h_0 . Note that the element $\gamma_{k+1} \in \Gamma_{\mathbb{D}^+}$ is uniquely defined up to multiplication on the left by an element of $\Gamma_{\widetilde{t(e_k)}} \cap \Gamma_{\mathbb{D}^+} = G_{t(e_k)}^+$, hence γ_{k+1}^{-1} is uniquely defined up to multiplication on the right by an element of $G_{t(e_k)}^+$, and appears only as the last letter in the expression of h_k . Therefore α is uniquely defined in the fundamental groupoid $\pi(\mathbb{Y}, G_*)$ up to multiplication on the left by an element of $G_{o(e_1)}^-$ and multiplication on the right by an element of $G_{t(e_k)}^+$, that is, the double class $[\alpha] \subset \pi(\mathbb{Y}, G_*)$ is uniquely defined.

(2) Let β be an element in Γ and let $x = \alpha_{\gamma \mathbb{D}^-, \gamma' \mathbb{D}^+}$ be a common perpendicular in \mathbb{X} between disjoint images of \mathbb{D}^- and \mathbb{D}^+ under elements of Γ . Let us prove that $\tilde{\Theta}(\beta x) = \tilde{\Theta}(x)$.

Since $\beta x = \alpha_{\beta \gamma \mathbb{D}^-, \beta \gamma' \mathbb{D}^+}$, in the construction of $\tilde{\Theta}(\beta x)$, we may take, instead of the elements $\gamma_0, \gamma_1, \dots, \gamma_k, \gamma_{k+1}$ used to construct $\tilde{\Theta}(x)$, the elements

$$\gamma_0^\sharp = \gamma_0, \quad \gamma_1^\sharp = \gamma_1 \beta^{-1}, \dots, \quad \gamma_k^\sharp = \gamma_k \beta^{-1}, \quad \gamma_{k+1}^\sharp = \gamma_{k+1}.$$

And instead of γ and γ' , we now may use $\gamma^\sharp = \beta \gamma$ and $\gamma'^\sharp = \beta \gamma'$.

The only terms involving $\gamma, \gamma', \gamma_1, \dots, \gamma_k$ in the construction of $\tilde{\Theta}(x)$ come under the form $\gamma^{-1} \gamma_1^{-1}$ in h_0 , $\gamma_i \gamma_{i+1}^{-1}$ in h_i for $1 \leq i \leq k-1$, and $\gamma_k \gamma'$ in h_k . Since $(\gamma^\sharp)^{-1} (\gamma_1^\sharp)^{-1} = \gamma^{-1} \gamma_1^{-1}$, $(\gamma_i^\sharp) (\gamma_{i+1}^\sharp)^{-1} = \gamma_i \gamma_{i+1}^{-1}$ for $1 \leq i \leq k-1$, and $(\gamma_k^\sharp) (\gamma'^\sharp) = \gamma_k \gamma'$, this proves that $\tilde{\Theta}(\beta x) = \tilde{\Theta}(x)$, as wanted.

It is immediate that if the length of $\alpha_{\gamma \mathbb{D}^-, \gamma' \mathbb{D}^+}$ is k , then the length of $[\alpha]$ is k .

Let us prove that the multiplicity, given in Equation (12.9),

$$m_{\gamma, \gamma'} = \frac{1}{\text{Card}(\gamma \Gamma_{\mathbb{D}^-} \gamma^{-1} \cap \gamma' \Gamma_{\mathbb{D}^+} \gamma'^{-1})}$$

of the common perpendicular $\alpha_{\gamma \mathbb{D}^-, \gamma' \mathbb{D}^+}$ in \mathbb{X} between $\gamma \mathbb{D}^-$ and $\gamma' \mathbb{D}^+$ is equal to the multiplicity

$$m_\alpha = \frac{1}{\text{Card}(G_{o(\alpha)}^- \cap \alpha G_{t(\alpha)}^+ \alpha^{-1})}$$

of the common perpendicular α in (\mathbb{Y}, G_*) from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) .

Since the multiplicity $m_{\gamma, \gamma'}$ is invariant under the diagonal action by left translations of $\gamma_0^{-1} \gamma^{-1} \in \Gamma$ on (γ, γ') , we may assume that $\gamma = \gamma_0 = \text{id}$. Since the multiplicity $m_{\gamma, \gamma'}$ is invariant under right translation by γ_{k+1}^{-1} , which stabilises \mathbb{D}^+ , on the element γ' , we may assume that $\gamma_{k+1} = \text{id}$. In particular, we have

$$o(f_1) = \widetilde{o(e_1)} \quad \text{and} \quad t(f_k) = \gamma' \widetilde{t(e_k)}.$$

We use the basepoint $x_0 = o(e_1)$ in the construction of the fundamental group and the Bass-Serre tree of (\mathbb{Y}, G_*) , so that (see in particular [Bass, Eq. (1.3)])

$$V\mathbb{X} = \coprod_{\beta \in \pi(\mathbb{Y}, G_*) : o(\beta) = x_0} \beta G_{t(\beta)}$$

and

$$\Gamma = \pi_1(\mathbb{Y}, G_*) = \{\beta \in \pi(\mathbb{Y}, G_*) : o(\beta) = t(\beta) = x_0\}.$$

Since an element in Γ which preserves \mathbb{D}^- and $\gamma' \mathbb{D}^+$ fixes pointwise its (unique) common perpendicular in \mathbb{X} , we have

$$\begin{aligned} \Gamma_{\mathbb{D}^-} \cap \gamma' \Gamma_{\mathbb{D}^+} \gamma'^{-1} &= \Gamma_{\mathbb{D}^-} \cap \Gamma_{\gamma' \mathbb{D}^+} = (\Gamma_{o(f_1)} \cap \Gamma_{\mathbb{D}^-}) \cap (\Gamma_{t(f_k)} \cap \Gamma_{\gamma' \mathbb{D}^+}) \\ &= (\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-}) \cap (\Gamma_{\gamma' t(e_k)} \cap \Gamma_{\gamma' \mathbb{D}^+}). \end{aligned}$$

Note that $\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-} = G_{o(e_1)}^-$. By the construction of the edges in the Bass-Serre tree of a graph of groups (see [Bass, page 11]), the vertex $\alpha G_{t(e_k)}$ is exactly the vertex $t(f_k) = \gamma' t(e_k)$. By [Bass, Eq. (1.4)], we hence have

$$\alpha G_{t(e_k)} \alpha^{-1} = \text{Stab}_{\pi_1(\mathbb{Y}, G_*)}(\alpha G_{t(e_k)}) = \Gamma_{\gamma' t(e_k)}.$$

Therefore $m_{\gamma, \gamma'} = m_\alpha$.

(3) Let $[\alpha] = G_{o(\alpha)} \alpha G_{t(\alpha)}$ be a common perpendicular in (\mathbb{Y}, G_*) from (\mathbb{Y}^-, G_*^-) to (\mathbb{Y}^+, G_*^+) , with representative $\alpha \in \pi(\mathbb{Y}, G_*)$, and let $h_0 e_1 h_1 \dots e_k h_k$ be a reduced word whose image in $\pi(\mathbb{Y}, G_*)$ is α .

We define

- $\gamma_1 = g_{\bar{e}_1} h_0^{-1}$,
- $f_1 = \gamma_1^{-1} \tilde{e}_1$,
- assuming that γ_i and f_i for some $1 \leq i \leq k-1$ are constructed, let

$$\gamma_{i+1} = g_{\bar{e}_{i+1}} h_i^{-1} g_{e_i}^{-1} \gamma_i \quad \text{and} \quad f_{i+1} = \gamma_{i+1}^{-1} \widetilde{e_{i+1}},$$

- with γ_k and f_k constructed by induction, finally let $\gamma' = \gamma_k^{-1} g_{e_k} h_k$.

It is easy to check, using the equivalences in the proof of Lemma 12.16 (1) with $\gamma = \gamma_0 = \gamma_{k+1} = \text{id}$, that the sequence (f_1, \dots, f_k) is the edge path of a common perpendicular in \mathbb{X} from \mathbb{D}^- to $\gamma' \mathbb{D}^+$ with origin $o(e_1)$ and endpoint $\gamma' t(e_k)$.

If h_0 is replaced by αh_0 with $\alpha \in G_{o(e_1)}^-$, then by induction, f_1, f_2, \dots, f_k are replaced by $\alpha f_1, \alpha f_2, \dots, \alpha f_k$ and γ' is replaced by $\alpha \gamma'$. Note that $(\alpha f_1, \alpha f_2, \dots, \alpha f_k)$ is then the common perpendicular edge path from $\mathbb{D}^- = \alpha \mathbb{D}^-$ to $\alpha \gamma' \mathbb{D}^+$. If h_k is replaced by $h_k \alpha$ with $\alpha \in G_{t(e_k)}^+$, then f_1, f_2, \dots, f_k are unchanged, and γ' is replaced by $\gamma' \alpha$. Note that $\gamma' \alpha \mathbb{D}^+ = \gamma' \mathbb{D}^+$.

Hence the map which associates to $[\alpha]$ the Γ -orbit of the common perpendicular in \mathbb{X} from \mathbb{D}^- to $\gamma' \mathbb{D}^+$ with edge path (f_1, \dots, f_k) is well defined. It is easy to see by construction that this map is the inverse of Θ . □

Theorem 12.15 now follows from Theorem 12.9. □

12.6 Error terms for equidistribution and counting for metric and simplicial graphs of groups

In this Section, we give error terms to the equidistribution and counting results of Section 12.4, given by Theorem 12.8 for metric trees (and their continuous time geodesic flows) and by Theorem 12.9 for simplicial trees (and their discrete time geodesic flows), under appropriate bounded geometry and rate of mixing properties.

Let (\mathbb{X}, λ) , X , Γ , \tilde{c} , c , \tilde{F}_c , F_c , δ_c , \mathbb{D}^\pm , \mathcal{D}^\pm , $\lambda_{\gamma, \gamma'}$, $\alpha_{\gamma, \gamma'}$, $\alpha_{\gamma, \gamma'}^\pm$, $m_{\gamma, \gamma'}$ be as in Section 12.4. We first consider the simplicial case (when $\lambda = 1$), for the discrete time geodesic flow.

Theorem 12.17. *Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$, let \tilde{c} be a system of conductances on \mathbb{X} for Γ and let \mathbb{D}^\pm be nonempty proper simplicial subtrees of \mathbb{X} . Assume that the critical exponent δ_c is finite and positive, that the Gibbs measure m_c (for the discrete time geodesic flow) is finite and that the skinning measures $\sigma_{\mathcal{D}^\pm}^\pm$ are finite and nonzero. Assume furthermore that*

(1) *at least one of the following holds :*

- $\Gamma_{\mathbb{D}^\pm} \setminus \partial \mathbb{D}^\pm$ is compact
- $\mathcal{C}\Lambda\Gamma$ is uniform and Γ is a lattice of $\mathcal{C}\Lambda\Gamma$,

(2) *there exists $\beta \in]0, 1]$ such that the discrete time geodesic flow on $(\Gamma \backslash \mathcal{G}\mathbb{X}, m_c)$ is exponentially mixing for the β -Hölder regularity.*

Then there exists $\kappa' > 0$ such that for all $\psi^\pm \in \mathcal{C}_c^\beta(\Gamma \backslash \check{\mathcal{G}}\mathbb{X})$, we have, as $n \rightarrow +\infty$,

$$\begin{aligned} & \frac{e^{\delta_c} - 1}{e^{\delta_c}} \|m_c\| e^{-\delta_c n} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})} \psi^-(\Gamma \alpha_{e, \gamma}^-) \psi^+(\Gamma \alpha_{\gamma^{-1}, e}^+) \\ &= \int \psi^- d\sigma_{\mathcal{D}^-}^+ \int \psi^+ d\sigma_{\mathcal{D}^+}^- + O(e^{-\kappa' n} \|\psi^-\|_\beta \|\psi^+\|_\beta) \end{aligned}$$

and if $\Gamma_{\mathbb{D}^\pm} \setminus \partial \mathbb{D}^\pm$ is compact, then

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) = \frac{e^{\delta_c} \|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{(e^{\delta_c} - 1) \|m_c\|} e^{\delta_c n} + O(e^{(\delta_c - \kappa')n}).$$

Proof. We follow the scheme of proof of Theorem 12.7, replacing aspects of Riemannian manifolds by aspects of simplicial trees as in the proof of Theorem 11.8. Let $\tilde{\psi}^\pm \in \mathcal{C}_c^\beta(\check{\mathcal{G}}\mathbb{X})$. In order to simplify the notation, let $\lambda_\gamma = \lambda_{e, \gamma}$, $\alpha_\gamma = \alpha_{e, \gamma}$, $\alpha_{\bar{\gamma}} = \alpha_{e, \gamma}^-$, $\alpha_\gamma^+ = \alpha_{\gamma^{-1}, e}^+$ and $\tilde{\sigma}^\pm = \tilde{\sigma}_{\mathbb{D}^\pm}^\pm$.

Let us first prove the following avatar of Equation (12.4), indicating only the required changes in its proof: there exists $\kappa_0 > 0$ (independent of $\tilde{\psi}^\pm$) such that, as $n \rightarrow +\infty$,

$$\begin{aligned} & \frac{e^{\delta_c} - 1}{e^{\delta_c}} \|m_c\| e^{-\delta_c n} \sum_{\gamma \in \Gamma, 0 < \lambda_\gamma \leq n} e^{c(\alpha_\gamma)} \tilde{\psi}^-(\alpha_{\bar{\gamma}}) \tilde{\psi}^+(\alpha_\gamma^+) \\ &= \int_{\partial_{\mathbb{D}^+} \mathbb{D}^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_{\mathbb{D}^-} \mathbb{D}^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa_0 n} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta). \end{aligned} \quad (12.13)$$

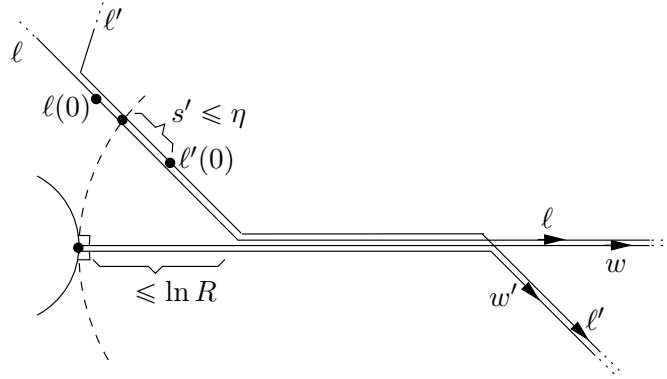
Most of the new work to be done in order to prove this formula concerns regularity properties of the test functions that will be introduced later on. We start with the regularity of the fibration of the dynamical neighbourhoods.¹³

Let $\eta, R > 0$ be such that $\eta < 1 < \ln R$.

Lemma 12.18. *Let Y be an \mathbb{R} -tree and let D be a nonempty closed convex subset of Y . Then the restriction to $\mathcal{V}_{\eta, R}^{\pm}(\partial_{\pm}^1 D)$ of the fibration f_D^{\pm} is (uniformly locally) Lipschitz, with constants independent of η .*

Proof. We assume for instance that $\pm = +$. Let $\ell, \ell' \in \mathcal{V}_{\eta, R}^+(\partial_+^1 D)$ and let $w = f_D^+(\ell), w' = f_D^+(\ell')$.

Since the fiber over $\rho \in \partial_+^1 D$ of the restriction to $\mathcal{V}_{\eta, R}^+(\partial_+^1 D)$ of f_D^+ is $V_{\rho, \eta, R}^+$ (see the end of Section 2.5), there exist $s, s' \in]-\eta, \eta[$ such that $\mathbf{g}^s \ell \in B^+(w, R)$ and $\mathbf{g}^{s'} \ell' \in B^+(w', R)$, so that $\mathbf{g}^s \ell(t) = w(t)$ and $\mathbf{g}^{s'} \ell'(t) = w'(t)$ for all $t \geq \ln R$ by the definition of the Hamenstädt balls. Up to permuting ℓ and ℓ' , we assume that $s' \geq s$.



By (the proof of) Lemma 10.11 (1), there exists a constant $c_R > 0$ depending only on R such that if $d(\ell, \ell') \leq c_R$ and $s'' = d(\ell(0), \ell'(0))$, then $s'' = s' - s$ and the geodesic lines $\mathbf{g}^s \ell$ and $\mathbf{g}^{s'} \ell'$ coincide at least on $[-\ln R - 1, \ln R + 1]$. In particular, we have

$$w(\ln R) = \ell(s + \ln R) = \ell'(s' + \ln R) = w'(\ln R) .$$

Since the origin of w is the closest point on D to any point of $w([0, +\infty[)$, we hence have that $w(t) = w'(t)$ for all $t \in [0, \ln R]$. Therefore (using Equation (2.5) for the last inequality),

$$\begin{aligned} d(w, w') &= \int_{\ln R}^{+\infty} d(w(t), w'(t)) e^{-2t} dt = \int_{\ln R}^{+\infty} d(\mathbf{g}^s \ell(t), \mathbf{g}^{s'} \ell'(t)) e^{-2t} dt \\ &= e^{2s} \int_{\ln R + s}^{+\infty} d(\ell(u), \mathbf{g}^{s''} \ell'(u)) e^{-2u} du \leq e^{2s} d(\ell, \mathbf{g}^{s''} \ell') \\ &\leq e^{2s} (d(\ell, \ell') + d(\ell', \mathbf{g}^{s''} \ell')) \leq e^{2s} (d(\ell, \ell') + s'') \\ &= e^{2s} (d(\ell, \ell') + d(\ell(0), \ell'(0))) , \end{aligned}$$

so that the result follows from Lemma 10.11 (2).

Note that when Y is (the geometric realisation of) a simplicial tree, then we have $s = s' = s'' = 0$ and the above computations simplify to give $d(w, w') \leq d(\ell, \ell')$. \square

¹³See Section 2.5 for notations.

We fix $R > 0$ big enough. With $D^\pm = |\mathbb{D}^\pm|_1$, we introduce the following modification of the test functions ϕ_η^\pm .¹⁴

$$\Phi_\eta^\pm = (h_{\eta,R}^\pm \tilde{\psi}^\pm) \circ f_{D^\pm}^\mp \mathbb{1}_{\mathcal{V}_{\eta,R}^\mp(\partial_\pm^1 \mathbb{D}^\pm)}.$$

As in Lemma 10.1, the functions Φ_η^\pm are measurable and satisfy

$$\int_{\mathcal{G}\mathbb{X}} \Phi_\eta^\pm d\tilde{m}_c = \int_{\partial_\pm^1 \mathbb{D}^\pm} \tilde{\psi}^\pm d\tilde{\sigma}^\mp. \quad (12.14)$$

Lemma 12.19. *The maps Φ_η^\pm are β -Hölder-continuous with*

$$\|\Phi_\eta^\pm\|_\beta = O(\|\tilde{\psi}^\pm\|_\beta). \quad (12.15)$$

Proof. Since \mathbb{X} is a simplicial tree and $\eta < 1$, we have $V_{w,\eta,R}^\pm = B^\pm(w, R)$ for every $w \in \partial_\pm^1 \mathbb{D}^\mp$. As seen above, there exists $c_R > 0$ depending only on R such that if $\ell \in B^\pm(w, R)$ and $\ell' \in \mathcal{G}\mathbb{X}$ satisfy $d(\ell, \ell') \leq c_R$, then $\ell' \in B^\pm(w, R)$. Hence (see Section 3.1) the characteristic function $\mathbb{1}_{\mathcal{V}_{\eta,R}^\pm(\partial_\pm^1 \mathbb{D}^\mp)}$ is c_R -locally constant, thus β -Hölder-continuous by Remark 3.2.

By Assumption (1) in the statement of Theorem 12.17, the denominator of

$$h_{\eta,R}^\mp(w) = \frac{1}{\mu_{W^\pm(w)}(B^\pm(w, R))}$$

is at least a positive constant depending only on R , hence $h_{\eta,R}^\mp$ is bounded by a constant depending only on R . Since the map $\mathbb{1}_{B^\pm(w,R)}$ is c_R -locally constant, so is the map $h_{\eta,R}^\mp$. The result then follows from Lemma 12.18 and Equation (3.1). \square

In order to prove Equation (12.13), as in the proofs of Theorems 12.7 and 11.8, for all $N \in \mathbb{N}$, we estimate in two ways the quantity

$$I_\eta(N) = \sum_{n=0}^N e^{\delta_c n} \sum_{\gamma \in \Gamma} \int_{\ell \in \mathcal{G}\mathbb{X}} \Phi_\eta^-(\mathbf{g}^{-\lfloor n/2 \rfloor} \ell) \Phi_\eta^+(\mathbf{g}^{\lfloor n/2 \rfloor} \gamma^{-1} \ell) d\tilde{m}_c(\ell). \quad (12.16)$$

On the one hand, as in order to obtain Equation (12.6), using now Assumption (2) in the statement of Theorem 12.17 on the exponential mixing for the discrete time geodesic flow, a geometric sum argument and Equations (12.14) and (12.15), we have

$$I_\eta(N) = \frac{e^{\delta_c(N+1)}}{(e^{\delta_c} - 1) \|m_c\|} \left(\int_{\partial_+^1 \mathbb{D}^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_-^1 \mathbb{D}^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(e^{-\kappa N} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta) \right). \quad (12.17)$$

On the other hand, exchanging the summations over γ and n in the definition of $I_\eta(N)$, we have

$$I_\eta(N) = \sum_{\gamma \in \Gamma} \sum_{n=0}^N e^{\delta_c n} \int_{\mathcal{G}\mathbb{X}} \Phi_\eta^-(\mathbf{g}^{-\lfloor n/2 \rfloor} \ell) \Phi_\eta^+(\mathbf{g}^{\lfloor n/2 \rfloor} \gamma^{-1} \ell) d\tilde{m}_c(\ell).$$

¹⁴See Equation (10.1) for the definition of $h_{\eta,R}^\pm$, that simplifies as $h_{\eta,R}^\mp(w) = (\mu_{W^\pm(w)}(B^\pm(w, R)))^{-1}$ since \mathbb{X} is simplicial, as seen in Equation (11.24).

With the simplifications in Step 3T of the proof of Theorem 11.1 given by the proof of Theorem 11.8, if $\eta < \frac{1}{2}$, if $\ell \in \mathcal{G}\mathbb{X}$ belongs to the support of $\Phi_\eta^- \circ \mathbf{g}^{-\lfloor n/2 \rfloor} \Phi_\eta^+ \circ \mathbf{g}^{\lfloor n/2 \rfloor} \circ \gamma^{-1}$, setting $w^- = f_{D^-}^+(\ell)$ and $w^+ = f_{\gamma D^+}^-(\ell)$, we then have $\lambda_\gamma = n$, $w^\pm(0) = \alpha_\gamma^\pm(0)$ and

$$w^-([n/2]) = w^+(-[n/2]) = \ell(0) = \alpha_\gamma^-([n/2]) = \alpha_\gamma^+(-[n/2]),$$

hence

$$d(w^\pm, \alpha_\gamma^\pm) = O(e^{-\lambda_\gamma}).$$

Therefore, since $\tilde{\psi}^\pm$ is β -Hölder-continuous,

$$|\tilde{\psi}^\pm(w^\pm) - \tilde{\psi}^\pm(v_\gamma^\pm)| = O(e^{-\beta\lambda_\gamma} \|\tilde{\psi}^\pm\|_\beta).$$

Note that now $\Phi_\eta^\pm = \tilde{\psi}^\pm \circ f_{D^\pm}^\mp \phi_\eta^\pm$, so that

$$\begin{aligned} I_\eta(N) &= \sum_{\gamma \in \Gamma} (\tilde{\psi}^-(v_\gamma^-) \tilde{\psi}^+(v_\gamma^+) + O(e^{-2\beta\lambda_\gamma} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta)) \times \\ &\quad \sum_{n=0}^N e^{\delta_c n} \int_{\mathcal{G}\mathbb{X}} \phi_\eta^-(\mathbf{g}^{-\lfloor n/2 \rfloor} \ell) \phi_\eta^+(\mathbf{g}^{\lfloor n/2 \rfloor} \gamma^{-1} \ell) d\tilde{m}_c(\ell). \end{aligned}$$

Now if $\eta < \frac{1}{2}$, Equation (12.13) with $\kappa_0 = \min\{2\beta, \kappa\}$ follows as in Steps 3T and 4T of the proof of Theorem 11.1 with the simplifications given by the proof of Theorem 11.8.

The end of the proof of the equidistribution claim of Theorem 12.17 follows from Equation (12.13) as the one of Theorem 12.7 from Equation (12.4).

The counting claim follows from the equidistribution one by taking $\psi^\pm = \mathbb{1}_{\Gamma \mathcal{V}_{\eta, R}(\partial_\mp^\perp \mathbb{D}^\pm)}$, which has compact support since $\Gamma_{\mathbb{D}^\pm} \setminus \partial \mathbb{D}^\pm$ is assumed to be compact, and is β -Hölder-continuous by previous arguments. \square

Remarks. (i) Assume that $\tilde{c} = 0$, that the simplicial tree \mathbb{X}' with $|\mathbb{X}'|_1 = \mathcal{C}\Lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma = \mathbb{Z}$ and that Γ is a geometrically finite lattice of \mathbb{X}' . Then all assumptions of Theorem 12.17 are satisfied by the results of Section 4.4 and by Corollary 9.6. Therefore we have an exponentially small error term in the (joint) equidistribution of the common perpendiculars, and in their counting if $\Gamma_{\mathbb{D}^\pm} \setminus \partial \mathbb{D}^\pm$ is compact, see Example 12.10 (2).

(ii) Assume in this remark that Assumption (2) of the above theorem is replaced by the assumptions that $\mathcal{C}\Lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma = 2\mathbb{Z}$, and that there exists $\beta \in]0, 1]$ such that the square of the discrete time geodesic flow on $(\Gamma \setminus \mathcal{G}_{\text{even}}\mathbb{X}, m_c)$ is exponentially mixing for the β -Hölder regularity, for instance if Γ is geometrically finite by Corollary 9.6 (2). Then a similar proof (replacing the references to Theorem 11.8 by references to Theorem 11.10) shows that there exists $\kappa' > 0$ such that for all $\psi^\pm \in \mathcal{C}_c^\beta(\Gamma \setminus \check{\mathcal{G}}\mathbb{X})$, we have, as $n \rightarrow +\infty$,

$$\begin{aligned} & \frac{e^{2\delta_c} - 1}{2e^{2\delta_c}} \|m_c\| e^{-\delta_c n} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \setminus \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} m_{e, \gamma} e^{c(\alpha_e, \gamma)} \psi^-(\Gamma \alpha_{e, \gamma}^-) \psi^+(\Gamma \alpha_{\gamma^{-1}, e}^+) \\ &= \int \psi^- d\sigma_{\mathcal{D}^-}^+ \int \psi^+ d\sigma_{\mathcal{D}^+}^- + O(e^{-\kappa' n} \|\psi^-\|_\beta \|\psi^+\|_\beta) \end{aligned}$$

and if $\Gamma_{\mathbb{D}^\pm} \setminus \partial \mathbb{D}^\pm$ is compact, then

$$\mathcal{N}_{\mathbb{D}^-, \mathbb{D}^+}(n) = \frac{2 e^{2\delta_c} \|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{(e^{2\delta_c} - 1) \|m_c\|} e^{\delta_c n} + O(e^{(\delta_c - \kappa')n}).$$

Let us now consider the metric tree case, for the continuous time geodesic flow, where the main change is to assume a superpolynomial decay of correlations and hence get a superpolynomial error term.

Theorem 12.20. *Let (\mathbb{X}, λ) , Γ , \tilde{c} and \mathbb{D}^\pm be as in the beginning of this Section, and let $D^\pm = |\mathbb{D}^\pm|_\lambda$. Assume that the critical exponent δ_c is finite and positive, that the Gibbs measure m_c (for the continuous time geodesic flow) is finite and that the skinning measures $\sigma_{\mathcal{D}^\mp}^\pm$ are finite and nonzero. Assume furthermore that*

(1) *at least one of the following holds :*

- $\Gamma_{D^\pm} \setminus \partial D^\pm$ *is compact*
- *the metric subtree $\mathcal{C}\Lambda\Gamma$ is uniform and Γ is a lattice of $\mathcal{C}\Lambda\Gamma$,*

(2) *there exists $\beta \in]0, 1]$ such that the continuous time geodesic flow on $(\Gamma \setminus \mathcal{G}X, m_c)$ has superpolynomial decay of β -Hölder correlations.*

Then for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $\psi^\pm \in \mathcal{C}_c^{k, \beta}(\Gamma \setminus \check{\mathcal{G}}X)$, we have, as $T \rightarrow +\infty$,

$$\begin{aligned} & \delta_c \|m_c\| e^{-\delta_c T} \sum_{\substack{[\gamma] \in \Gamma_{D^-} \setminus \Gamma / \Gamma_{D^+} \\ 0 < d(D^-, \gamma D^+) \leq T}} m_{e, \gamma} e^{c(\alpha_{e, \gamma})} \psi^-(\Gamma \alpha_{e, \gamma}^-) \psi^+(\Gamma \alpha_{\gamma^{-1}, e}^+) \\ &= \int_{\Gamma \setminus \check{\mathcal{G}}X} \psi^- d\sigma_{\mathcal{D}^-}^+ - \int_{\Gamma \setminus \check{\mathcal{G}}X} \psi^+ d\sigma_{\mathcal{D}^+}^- + O(T^{-n} \|\psi^-\|_{k, \beta} \|\psi^-\|_{k, \beta}) \end{aligned}$$

and if $\Gamma_{D^\pm} \setminus \partial D^\pm$ is compact, then for every $n \in \mathbb{N}$

$$\mathcal{N}_{D^-, D^+}(T) = \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta_c \|m_c\|} e^{\delta_c T} + O(e^{\delta_c T} T^{-n}).$$

Remark. Assume that $\tilde{c} = 0$, that the metric tree $\mathcal{C}\Lambda\Gamma$ is uniform, either that $\Gamma \setminus \mathbb{X}$ is finite and the length spectrum L_Γ of Γ is 2-Diophantine or that Γ is a geometrically finite lattice of $\mathcal{C}\Lambda\Gamma$ and that L_Γ is 4-Diophantine. Then all assumptions of Theorem 12.20 are satisfied by the results of Section 4.4 and by Corollary 9.10. Therefore we have a superpolynomially small error term in the (joint) equidistribution of the common perpendiculars (and in their counting if $\Gamma_{D^\pm} \setminus \partial D^\pm$ is compact).

Proof. The proof is similar to the one of Theorem 12.17, except that since the time is now continuous, we need to regularise our test functions in the time direction in order to obtain the regularity required for the application of the assumption on the mixing rate. We again use the simplifying notation $\lambda_\gamma = \lambda_{e, \gamma}$, $\alpha_\gamma = \alpha_{e, \gamma}$, $\alpha_\gamma^- = \alpha_{e, \gamma}^-$, $\alpha_\gamma^+ = \alpha_{\gamma^{-1}, e}^+$ and $\tilde{\sigma}^\pm = \tilde{\sigma}_{D^\mp}^\pm$.

We fix $n \in \mathbb{N} - \{0\}$. Using the rapid mixing property, there exists a regularity k such that for all $\psi, \psi' \in \mathcal{C}_b^{k, \beta}(\Gamma \setminus \mathcal{G}X)$ we have as $t \rightarrow +\infty$

$$\text{cov}_{\overline{m}_c, t}(\psi, \psi') = O(t^{-Nn} \|\psi\|_{k, \beta} \|\psi'\|_{k, \beta}), \quad (12.18)$$

where $N \in \mathbb{N} - \{0\}$ is a constant which will be made precise later on.

Let us first prove that for all $\tilde{\psi}^\pm \in \mathcal{C}_c^{k,\beta}(\check{\mathcal{G}}X)$, we have, as $T \rightarrow +\infty$,

$$\begin{aligned} & \delta_c \|m_c\| e^{-\delta_c T} \sum_{\gamma \in \Gamma, 0 < \lambda_\gamma \leq T} e^{c(\alpha_\gamma)} \tilde{\psi}^-(\alpha_\gamma^-) \tilde{\psi}^+(\alpha_\gamma^+) \\ &= \int_{\partial_+^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ \int_{\partial_-^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(T^{-n} \|\tilde{\psi}^-\|_{k,\beta} \|\tilde{\psi}^+\|_{k,\beta}). \end{aligned} \quad (12.19)$$

In order to prove this formula, we introduce modified test functions, making them with bounded Hölder-continuous derivatives up to order k (by a standard construction) in the time direction (the stable leaf and unstable leaf directions remain discrete). We fix $R > 0$ big enough.

For every $\eta \in]0, 1[$, there exists a map $\widehat{\mathbb{1}}_\eta : \mathbb{R} \rightarrow [0, 1]$ which has bounded β -Hölder-continuous derivatives up to order k , which is equal to 0 if $t \notin [-\eta, \eta]$ and to 1 if $t \in [-\eta e^{-\eta}, \eta e^{-\eta}]$ (when $k = 0$, just take $\widehat{\mathbb{1}}_\eta$ to be continuous and linear on each remaining segment $[-\eta, -\eta e^{-\eta}]$ and $[\eta e^{-\eta}, \eta]$), such that, for some constant $\kappa_1 > 0$,

$$\|\widehat{\mathbb{1}}_\eta\|_{k,\beta} = O(\eta^{-\kappa_1}).$$

Using leafwise this regularisation process, there exists $\chi_{\eta,R}^\pm \in \mathcal{C}_b^{k,\beta}(\mathcal{G}X)$ such that

- $\|\chi_{\eta,R}^\pm\|_{k,\beta} = O(\eta^{-\kappa_1})$,
- $\mathbb{1}_{\mathcal{V}_{\eta e^{-\eta}, R}^\mp(\partial_\mp^1 D^\pm)} \leq \chi_{\eta,R}^\pm \leq \mathbb{1}_{\mathcal{V}_{\eta,R}^\mp(\partial_\mp^1 D^\pm)}$,
- for every $w \in \partial_\mp^1 D^\pm$, we have

$$\int_{V_{w,\eta,R}^\mp} \chi_{\eta,R}^\pm d\nu_w^\pm = \nu_w^\pm(V_{w,\eta,R}^\mp) e^{-O(\eta)} = \nu_w^\pm(V_{w,\eta e^{-\eta}, R}^\mp) e^{O(\eta)}.$$

As in the proof of Theorem 12.7 in the manifold case, the new test functions are defined, with

$$H_{\eta,R}^\pm : w \in \partial_\mp^1 D^\pm \mapsto \frac{1}{\int_{V_{w,\eta,R}^\mp} \chi_{\eta,R}^\pm d\nu_w^\pm},$$

by

$$\Phi_\eta^\pm = (H_{\eta,R}^\pm \tilde{\psi}^\pm) \circ f_{D^\pm}^\mp \chi_{\eta,R}^\pm : \mathcal{G}X \rightarrow \mathbb{R}.$$

Let $\widehat{\Phi}_\eta^\pm = H_{\eta,R}^\pm \circ f_{D^\pm}^\mp \chi_{\eta,R}^\pm$, so that $\Phi_\eta^\pm = \tilde{\psi}^\pm \circ f_{D^\pm}^\mp \widehat{\Phi}_\eta^\pm$. By the last two properties of the regularised maps $\chi_{\eta,R}^\pm$, we have, with ϕ_η^\pm defined as in Equation (10.4),

$$\phi_{\eta e^{-\eta}}^\pm e^{-O(\eta)} \leq \widehat{\Phi}_\eta^\pm \leq \phi_\eta^\pm e^{O(\eta)}. \quad (12.20)$$

By Assumption (1), if R is large enough, by the definitions of the measures ν_w^\pm , the denominator of $H_{\eta,R}^\pm(w)$ is at least $c\eta$ where $c > 0$. As in the proof of Theorem 12.7, there exists $\kappa'' > 0$ such that

$$\int_{\mathcal{G}X} \Phi_\eta^\mp d\tilde{m}_c = \int_{\partial_\mp^1 D^\pm} \tilde{\psi}^\pm d\tilde{\sigma}^\mp$$

and

$$\|\Phi_\eta^\pm\|_{k,\beta} = O(\eta^{-\kappa''} \|\tilde{\psi}^\pm\|_{k,\beta}).$$

We again estimate in two ways as $T \rightarrow +\infty$ the quantity

$$I_\eta(T) = \int_0^T e^{\delta_c t} \sum_{\gamma \in \Gamma} \int_{\ell \in \mathcal{G}X} \Phi_\eta^-(\mathbf{g}^{-t/2} \ell) \Phi_\eta^+(\mathbf{g}^{t/2} \gamma^{-1} \ell) d\tilde{m}_c(\ell) dt. \quad (12.21)$$

Note that as $T \rightarrow +\infty$,

$$\begin{aligned} e^{-\delta_c T} \int_1^T e^{\delta_c t} t^{-Nn} dt &= e^{-\delta_c T} \int_1^{T/2} e^{\delta_c t} t^{-Nn} dt + e^{-\delta_c T} \int_{T/2}^T e^{\delta_c t} t^{-Nn} dt \\ &= O(e^{-\delta_c T/2}) + O(T^{-Nn+1}) = O(T^{-(N-1)n}). \end{aligned}$$

Using Equation (12.18), an integration argument and the above two properties of the test functions, we hence have

$$I_\eta(T) = \frac{e^{\delta_c T}}{\delta_c \|m_c\|} \left(\int_{\partial_{\pm}^1 D^-} \tilde{\psi}^- d\tilde{\sigma}^+ + \int_{\partial_{\pm}^1 D^+} \tilde{\psi}^+ d\tilde{\sigma}^- + O(T^{-(N-1)n} \eta^{-2\kappa''} \|\tilde{\psi}^-\|_{k,\beta} \|\tilde{\psi}^+\|_{k,\beta}) \right). \quad (12.22)$$

As in Step 3T of the proof of Theorem 11.1, for all $\gamma \in \Gamma$ and $t > 0$ big enough, if $\ell \in \mathcal{G}X$ belongs to the support of $\Phi_\eta^- \circ \mathbf{g}^{-t/2} \Phi_\eta^+ \circ \mathbf{g}^{t/2} \circ \gamma^{-1}$ (which is contained in the support of $\phi_\eta^- \circ \mathbf{g}^{-t/2} \phi_\eta^+ \circ \mathbf{g}^{t/2} \circ \gamma^{-1}$), then we may define $w^- = f_{D^-}^-(\ell)$ and $w^+ = f_{D^+}^-(\ell)$.

By the property (iii) in Step 3T of the proof of Theorem 11.1, the generalised geodesic lines w^- and α_γ^- coincide, besides on $] -\infty, 0]$, at least on $[0, \frac{t}{2} - \eta]$, and similarly, w^+ and α_γ^+ coincide, besides on $[0, +\infty[$, at least on $[-\frac{t}{2} + \eta, 0]$. Therefore, by an easy change of variable and since $|\frac{t}{2} - \frac{\lambda_\gamma}{2}| \leq \eta$,

$$\begin{aligned} d(w^-, \alpha_\gamma^-) &\leq \int_{\frac{t}{2} - \eta}^{+\infty} d(w^-(s), \alpha_\gamma^-(s)) e^{-2s} ds \leq e^{-2(\frac{t}{2} - \eta)} \int_0^{+\infty} 2s e^{-2s} ds \\ &= O(e^{-t}) = O(e^{-\lambda_\gamma}). \end{aligned}$$

Similarly, $d(w^+, \alpha_\gamma^+) = O(e^{-\lambda_\gamma})$. Hence since $\tilde{\psi}^\pm$ is β -Hölder-continuous, we have

$$|\tilde{\psi}^\pm(w^\pm) - \tilde{\psi}^\pm(\alpha_\gamma^\pm)| = O(e^{-\beta\lambda_\gamma} \|\tilde{\psi}^\pm\|_\beta).$$

Therefore, as in the proof of Theorem 12.17, we have

$$\begin{aligned} I_\eta(T) &= \sum_{\gamma \in \Gamma} (\tilde{\psi}^-(\alpha_\gamma^-) \tilde{\psi}^+(\alpha_\gamma^+) + O(e^{-2\beta\lambda_\gamma} \|\tilde{\psi}^-\|_\beta \|\tilde{\psi}^+\|_\beta)) \times \\ &\quad \int_0^T e^{\delta t} \int_{\ell \in \mathcal{G}X} \hat{\Phi}_\eta^-(\mathbf{g}^{-t/2} \ell) \hat{\Phi}_\eta^+(\gamma^{-1} \mathbf{g}^{t/2} \ell) d\tilde{m}_c(\ell) dt. \end{aligned}$$

Finally, Equation (12.19) follows as in the end of the proof of Equation (12.4), using Equations (12.20) and (11.16) instead of Equations (12.7) and (11.19), by taking $\eta = T^{-n}$ and $N = 2(\lceil \kappa'' \rceil + 1)$.

The end of the proof of the equidistribution claim of Theorem 12.20 follows from Equation (12.19) as the one of Theorem 12.7 from Equation (12.4).

The counting claim follows from the equidistribution one by taking ψ^\pm to be β -Hölder-continuous plateau functions around $\Gamma \mathcal{V}_{\eta,R}(\partial_{\mp}^1 \mathbb{D}^\pm)$. \square

We are now in a position to prove one of the counting results in the introduction.

Proof of Theorem 1.9. Let \mathbb{X} be the universal cover of \mathbb{Y} , with fundamental group Γ for an indifferent choice of basepoint, and let \mathbb{D}^\pm be connected components of the preimages of \mathbb{Y}^\pm in \mathbb{X} . Assertion (1) of Theorem 1.9 follows from Theorem 12.20 and its subsequent Remark. Assertion (2) of Theorem 1.9 follows from Theorem 12.17 and its subsequent Remarks (ii) and (i), respectively if \mathbb{Y} is bipartite or not. \square

Chapter 13

Geometric applications

In this final Chapter of Part II, we apply the equidistribution and counting results obtained in the previous Chapters in order to study geometric equidistribution and counting problems for metric and simplicial trees concerning conjugacy classes in discrete isometry groups and closed orbits of the geodesic flows.

13.1 Orbit counting in conjugacy classes for groups acting on trees

In this Section, we study the orbital counting problem for groups acting on metric or simplicial trees when we consider only the orbit points by elements in a given conjugacy class. We refer to the Introduction for motivations and previously known results for manifolds (see [Hub1] and [PaP15]) and graphs (see [Dou] and [KeS]). The main tools we use are Theorem 12.8 for the metric tree case and Theorem 12.17 for the simplicial tree case, as well as their error terms. We in particular obtain a much more general version of Theorem 1.12 in the Introduction.

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices, let $X = |\mathbb{X}|_\lambda$ be its geometric realisation, let $x_0 \in V\mathbb{X}$ and let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$.¹ Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant system of conductances, let \tilde{F}_c and F_c be its associated potentials on T^1X and $\Gamma \backslash T^1X$ respectively, and let $\delta_c = \delta_{\Gamma, F_c^\pm}$ be its critical exponent.² Let $(\mu_x^\pm)_{x \in X}$ (respectively $(\mu_x^\pm)_{x \in V\mathbb{X}}$) be Patterson densities for the pairs (Γ, F_c^\pm) , and let $\tilde{m}_c = \tilde{m}_{F_c}$ and $m_c = m_{F_c}$ be the associated Gibbs measures on $\mathcal{G}X$ and $\Gamma \backslash \mathcal{G}X$ (respectively $\mathcal{G}\mathbb{X}$ and $\Gamma \backslash \mathcal{G}\mathbb{X}$) for the continuous time geodesic flow (respectively the discrete time geodesic flow, when $\lambda \equiv 1$).³

Recall that the *virtual centre* $Z^{\text{virt}}(\Gamma)$ of Γ is the finite (normal) subgroup of Γ consisting of the elements $\gamma \in \Gamma$ acting by the identity on the limit set $\Lambda\Gamma$ of Γ in $\partial_\infty X$, see for instance [Cha, §5.1]. If $\Lambda\Gamma = \partial_\infty X$ (for instance if Γ is a lattice), then $Z^{\text{virt}}(\Gamma) = \{e\}$.

For any nontrivial element γ in Γ with translation length $\lambda(\gamma)$ in X , let C_γ be

- the translation axis of γ if γ is loxodromic on X ,
- the fixed point set of γ if γ is elliptic on X ,

¹See Section 2.7 for definitions and notation.

²See Section 3.5 for definitions and notation.

³See Sections 4.3 and 4.4 for definitions and notation.

and let Γ_{C_γ} be the stabiliser of C_γ in Γ . In the simplicial case, C_γ is a simplicial subtree of \mathbb{X} . Note that $\lambda(\gamma) = \lambda(\gamma'\gamma(\gamma')^{-1})$ and $\gamma'C_\gamma = C_{\gamma'\gamma(\gamma')^{-1}}$ for all $\gamma' \in \Gamma$, and that for any $x_0 \in X$

$$d(x_0, C_\gamma) = \frac{d(x_0, \gamma x_0) - \lambda(\gamma)}{2}. \quad (13.1)$$

By the equivariance properties of the skinning measures, the total mass of the skinning measure⁴ $\sigma_{\mathcal{D}}^-$ where $\mathcal{D} = (\gamma'C_\gamma)_{\gamma' \in \Gamma/\Gamma_{C_\gamma}}$ depends only on the conjugacy class \mathfrak{K} of γ in Γ , and will be denoted by $\|\sigma_{\mathfrak{K}}^-\|$. This quantity, called the *skinning measure* of \mathfrak{K} , is positive unless $\partial_\infty C_\gamma = \Lambda\Gamma$, which is equivalent to $\gamma \in Z^{\text{virt}}(\Gamma)$ (and implies in particular that γ is elliptic). Furthermore, $\|\sigma_{\mathfrak{K}}^-\|$ is finite if γ is loxodromic, and it is finite if γ is elliptic and $\Gamma_{C_\gamma} \setminus (C_\gamma \cap \mathcal{C}\Lambda\Gamma)$ is compact. This last condition is in particular satisfied if $C_\gamma \cap \mathcal{C}\Lambda\Gamma$ itself is compact, and this is the case for instance if, for some $k \geq 0$, the action of Γ on X is *k-acylindrical* (see for instance [Sel, GuL]), that is, if any element of Γ fixing a segment of length k in $\mathcal{C}\Lambda\Gamma$ is the identity.

For every $\gamma \in \Gamma - \{e\}$, we define

$$m_\gamma = \frac{1}{\text{Card}(\Gamma_{x_0} \cap \Gamma_{C_\gamma})},$$

which is a natural multiplicity of γ , and equals 1 if the stabiliser of x_0 in Γ is trivial (for instance if Γ is torsion-free). Note that for every $\alpha \in \Gamma$, the real number $m_{\alpha\gamma\alpha^{-1}}$ depends only on the double coset of α in $\Gamma_{x_0} \setminus \Gamma/\Gamma_{C_\gamma}$.

The centraliser $Z_\Gamma(\gamma)$ of γ in Γ is contained in the stabiliser of C_γ in Γ . The index

$$i_{\mathfrak{K}} = [\Gamma_{C_\gamma} : Z_\Gamma(\gamma)]$$

depends only on the conjugacy class \mathfrak{K} of γ ; it will be called the *index* of \mathfrak{K} . The index $i_{\mathfrak{K}}$ is finite if γ is loxodromic (the stabiliser of its translation axis C_γ is then virtually cyclic), and also finite if C_γ is compact (as for instance if the action of Γ on X is *k-acylindrical* for some $k \geq 0$).

We define

$$c_\gamma = \sum_{i=1}^k c(e_i) \lambda(e_i),$$

where (e_1, \dots, e_k) is the shortest edge path from x_0 to C_γ .

We finally define the orbital counting function in conjugacy classes, counting with multiplicities and weights coming from the system of conductances, as

$$N_{\mathfrak{K}, x_0}(t) = \sum_{\alpha \in \mathfrak{K}, d(x_0, \alpha x_0) \leq t} m_\alpha e^{c_\alpha}.$$

for $t \in [0, +\infty[$ (simply $t \in \mathbb{N}$ in the simplicial case). When the stabiliser of x_0 in Γ is trivial and when the system of conductances c vanishes, we recover the definition of the Introduction.

Theorem 13.1. *Let \mathfrak{K} be the conjugacy class of a nontrivial element γ_0 of Γ , with finite index $i_{\mathfrak{K}}$, and with positive and finite skinning measure $\|\sigma_{\mathfrak{K}}^-\|$. Assume that δ_c is finite and positive.*

⁴See Chapter 7 for definitions and notation.

(1) Assume that m_c is finite and mixing for the continuous time geodesic flow on $\Gamma \backslash X$. Then, as $t \rightarrow +\infty$,

$$N_{\mathfrak{K}, x_0}(t) \sim \frac{i_{\mathfrak{K}} \|\mu_{x_0}^+\| \|\sigma_{\mathfrak{K}}^-\| e^{-\frac{\lambda(\gamma_0)}{2} t}}{\delta_c \|m_c\|} e^{\frac{\delta_c}{2} t}.$$

If $\Gamma_{C_\gamma} \backslash (C_\gamma \cap \mathcal{C}\Lambda\Gamma)$ is compact when $\gamma \in \mathfrak{K}$ is elliptic and if there exists $\beta \in]0, 1]$ such that the continuous time geodesic flow on $(\Gamma \backslash \mathcal{G}X, m_c)$ has superpolynomial decay of β -Hölder correlations, then the error term is $O(t^{-n} e^{\frac{\delta_c}{2} t})$ for every $n \in \mathbb{N}$.

(2) Assume that $\lambda \equiv 1$ and that m_c is finite and mixing for the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$. Then, as $n \rightarrow +\infty$,

$$N_{\mathfrak{K}, x_0}(n) \sim \frac{e^{\delta_c} i_{\mathfrak{K}} \|\mu_{x_0}^+\| \|\sigma_{\mathfrak{K}}^-\|}{(e^{\delta_c} - 1) \|m_c\|} e^{\delta_c \lfloor \frac{n - \lambda(\gamma_0)}{2} \rfloor}.$$

If $\Gamma_{C_\gamma} \backslash (C_\gamma \cap \mathcal{C}\Lambda\Gamma)$ is compact when $\gamma \in \mathfrak{K}$ is elliptic and if there exists $\beta \in]0, 1]$ such that the discrete time geodesic flow on $(\Gamma \backslash \mathcal{G}\mathbb{X}, m_c)$ is exponentially mixing for the β -Hölder regularity, then the error term is $O(e^{(\delta_c - \kappa)n/2})$ for some $\kappa > 0$.

One can also formulate a version of the above result for groups acting on bipartite simplicial trees based on Theorem 12.12 and Remark (ii) following the proof of Theorem 12.17.

The error term in Assertion (1) holds for instance if $\tilde{c} = 0$, X is uniform, and either $\Gamma \backslash X$ is compact and the length spectrum L_Γ is 2-Diophantine or Γ is a geometrically finite lattice of X whose length spectrum L_Γ is 4-Diophantine, by the Remark following Theorem 12.20. When $\Gamma \backslash X$ is compact and Γ has no torsion (in particular, Γ has then a very restricted group structure, as it is then a free group), we thus recover a result of [KeS].

The error term in Assertion (2) holds for instance if $\tilde{c} = 0$, \mathbb{X} is uniform with vertices of degrees at least 3, Γ is a geometrically finite lattice of \mathbb{X} with length spectrum equal to \mathbb{Z} , by Remark (i) following the proof of Theorem 12.17.

Theorem 1.12 in the introduction follows from this theorem, using Proposition 4.14 (3) and Proposition 4.15.

Proof. We only give a full proof of Assertion (1) of this theorem, Assertion (2) follows similarly using Theorems 12.9 and 12.17 instead of Theorems 12.8 and 12.20.

The proof is similar to the proof of [PaP15, Theo. 8]. Let $D^- = \{x_0\}$ and $D^+ = C_{\gamma_0}$. Let $\mathcal{D}^- = (\gamma D^-)_{\gamma \in \Gamma / \Gamma_{D^-}}$ and $\mathcal{D}^+ = (\gamma D^+)_{\gamma \in \Gamma / \Gamma_{D^+}}$. By Equation (7.14), we have

$$\|\sigma_{\mathcal{D}^-}^+\| = \frac{\|\mu_{x_0}^+\|}{|\Gamma_{x_0}|}.$$

By Equation (13.1), by the definition⁵ of the counting function \mathcal{N}_{D^-, D^+} and by the last claim

⁵See Section 12.4.

of Theorem 12.8, we have, as $t \rightarrow +\infty$,

$$\begin{aligned}
\sum_{\alpha \in \mathfrak{K}, 0 < d(x_0, \alpha x_0) \leq t} m_\alpha e^{c\alpha} &= \sum_{\alpha \in \mathfrak{K}, 0 < d(x_0, C_\alpha) \leq \frac{t - \lambda(\gamma_0)}{2}} m_\alpha e^{c\alpha} \\
&= \sum_{\gamma \in \Gamma / Z_\Gamma(\gamma_0), 0 < d(x_0, \gamma C_{\gamma_0}) \leq \frac{t - \lambda(\gamma_0)}{2}} m_{\gamma \gamma_0 \gamma^{-1}} e^{c_{\gamma \gamma_0 \gamma^{-1}}} \\
&= |\Gamma_{x_0}| i_{\mathfrak{K}} \sum_{\gamma \in \Gamma_{x_0} \backslash \Gamma / \Gamma_{C_{\gamma_0}}, 0 < d(x_0, \gamma C_{\gamma_0}) \leq \frac{t - \lambda(\gamma_0)}{2}} m_{\gamma \gamma_0 \gamma^{-1}} e^{c_{\gamma \gamma_0 \gamma^{-1}}} \\
&= |\Gamma_{x_0}| i_{\mathfrak{K}} \mathcal{N}_{D^-, D^+} \left(\frac{t - \lambda(\gamma_0)}{2} \right) \\
&\sim |\Gamma_{x_0}| i_{\mathfrak{K}} \frac{\|\sigma_{\mathcal{D}^-}^+\| \|\sigma_{\mathcal{D}^+}^-\|}{\delta_c \|m_c\|} e^{\delta_c \frac{t - \lambda(\gamma_0)}{2}}.
\end{aligned}$$

Assertion (1) without the error term follows, and the error term statement follows similarly from Theorem 12.20. \square

Theorem 13.1 (1) without an explicit form of the multiplicative constant in the asymptotic is due to [KeS] under the strong restriction that Γ is a free group acting freely on X and $\Gamma \backslash X$ is a finite graph. The following result is due to [Dou, Thm. 1] in the very special case when \mathbb{X} is a regular tree and the group Γ has no torsion and finite quotient $\Gamma \backslash \mathbb{X}$.

Corollary 13.2. *Let \mathbb{X} be a regular simplicial tree with vertices of degree $q + 1 \geq 3$, let $x_0 \in V\mathbb{X}$, let Γ be a lattice of \mathbb{X} such that $\Gamma \backslash \mathbb{X}$ is nonbipartite, and let \mathfrak{K} be the conjugacy class of a loxodromic element $\gamma_0 \in \Gamma$. Then, as $n \rightarrow +\infty$,*

$$\sum_{\alpha \in \mathfrak{K}, d(x_0, \alpha x_0) \leq n} m_\alpha \sim \frac{\lambda(\gamma_0)}{[Z_\Gamma(\gamma_0) : \gamma_0^{\mathbb{Z}}] \text{Vol}(\Gamma \backslash \mathbb{X})} q^{\lfloor \frac{n - \lambda(\gamma_0)}{2} \rfloor}.$$

If we assume furthermore that Γ has no torsion, then the result holds also when $\Gamma \backslash \mathbb{X}$ is bipartite. In this case, we have as $n \rightarrow +\infty$,

$$\text{Card}\{\alpha \in \mathfrak{K} : d(x_0, \alpha x_0) \leq n\} \sim \frac{\lambda(\gamma_0)}{[\Gamma \backslash V\mathbb{X}]} q^{\lfloor \frac{n - \lambda(\gamma_0)}{2} \rfloor}.$$

Proof. Under these assumptions, taking $c \equiv 0$ in Theorem 13.1 so that the Gibbs measure is the Bowen-Margulis measure, the discrete time geodesic flow on $\Gamma \backslash \mathcal{G}\mathbb{X}$ is finite and mixing by Proposition 4.14 (3) and Proposition 4.15. We also have $\delta_c = \ln q$. Using the normalisation of the Patterson density $(\mu_x^\pm)_{x \in V\mathbb{X}}$ to probability measures, Proposition 8.1 (3) and Equation (8.11), the result follows, since when γ is loxodromic,

$$\text{Vol}(\Gamma_{C_\gamma} \backslash C_\gamma) = \frac{\text{Vol}(\gamma^{\mathbb{Z}} \backslash C_\gamma)}{[\Gamma_{C_\gamma} : \gamma^{\mathbb{Z}}]} = \frac{\lambda(\gamma)}{[\Gamma_{C_\gamma} : Z_\Gamma(\gamma)] [Z_\Gamma(\gamma) : \gamma^{\mathbb{Z}}]}.$$

The claim for the bipartite $(p + 1)$ -regular case follows from Remark 12.14. \square

The value of C' given below Theorem 1.12 in the Introduction follows from this corollary.

We leave to the reader an extension with nonzero potential F of the results for manifolds in [PaP15], along the lines of the above proofs.

13.2 Equidistribution and counting of closed orbits on metric and simplicial graphs (of groups)

Classically, an important characterization of the Bowen-Margulis measure on compact negatively curved Riemannian manifolds is that it coincides with the weak-star limit of properly normalised sums of Lebesgue measures supported on periodic orbits, see [Bowe1]. Under much weaker assumptions than compactness, this result was extended to CAT(−1) spaces with zero potential in [Rob2] and to Gibbs measures in the manifold case in [PauPS, Thm. 9.11]. As a corollary of the simultaneous equidistribution results from Chapter 11, we prove in this Section the equidistribution towards the Gibbs measure of weighted closed orbits in quotients of metric and simplicial graphs of groups and as a corollary, in the standard manner, we obtain asymptotic counting results for weighted (primitive) closed orbits.

Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices, and $X = |\mathbb{X}|_\lambda$ its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$. Let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant system of conductances, and $c : \Gamma \backslash E\mathbb{X} \rightarrow \mathbb{R}$ its induced function.

Given a periodic orbit g of the geodesic flow on $\Gamma \backslash \mathcal{G}X$, if (e_1, \dots, e_k) is the sequence of edges followed by g , we denote by \mathcal{L}_g the Lebesgue measure along g , by $\lambda(g)$ the length of g and by $c(g)$ its *period* for the system of conductances c :

$$\lambda(g) = \sum_{i=1}^k \lambda(e_i) \quad \text{and} \quad c(g) = \sum_{i=1}^k \lambda(e_i) c(e_i) .$$

Let $\mathfrak{Per}(t)$ be the set of periodic orbits of the geodesic flow on $\Gamma \backslash \mathcal{G}X$ and let $\mathfrak{Per}'(t)$ be the subset of prime periodic orbits.

Theorem 13.3. *Assume that the critical exponent δ_c of c is finite and positive and that the Gibbs measure m_c of c is finite and mixing for the continuous time geodesic flow. As $t \rightarrow +\infty$, the measures*

$$\delta_c e^{-\delta_c t} \sum_{g \in \mathfrak{Per}'(t)} e^{c(g)} \mathcal{L}_g$$

and

$$\delta_c t e^{-\delta_c t} \sum_{g \in \mathfrak{Per}'(t)} e^{c(g)} \frac{\mathcal{L}_g}{\ell(g)}$$

converge to $\frac{m_c}{|m_c|}$ for the weak-star convergence of measures. If Γ is geometrically finite, the convergence holds for the narrow convergence.

We conjecture that if Γ is geometrically finite and if its length spectrum is 4-Diophantine⁶, then for all $n \in \mathbb{N}$ and $\beta \in]0, 1]$, there exists $k \in \mathbb{N}$ and an error term of the form $O(t^n \|\psi\|_{k, \beta})$ for these equidistribution claims evaluated on any $\psi \in \mathcal{C}_c^{b, \beta}(\Gamma \backslash \mathcal{G}X)$. But since we will not need this result and since the proof is likely to be very long, we do not address the problem here.

Proof. Let \tilde{F}_c and F_c be the potentials on T^1X and $\Gamma \backslash T^1X$ respectively associated with⁷ c , and note that the period⁸ of a periodic orbit g for the geodesic flow on $\Gamma \backslash \mathcal{G}X$ satisfies

$$c(g) = \mathcal{L}_g(F_c^\#) = \text{Per}_{F_c}(\gamma) ,$$

⁶See definition in Section 9.3.

⁷See Section 3.5.

⁸See Section 3.2.

where F_c^\sharp is the composition of the canonical map $\Gamma \backslash \mathcal{G}X \rightarrow \Gamma \backslash T^1X$ with $F_c : \Gamma \backslash T^1X \rightarrow \mathbb{R}$, and $\gamma \in \Gamma$ is the loxodromic element of Γ whose conjugacy class corresponds to g .

Let $\mathcal{H}_{\Gamma,t}$ be the subset of Γ that consists of loxodromic elements whose translation length is at most t , and let $\mathcal{H}'_{\Gamma,t}$ be the subset of $\mathcal{H}_{\Gamma,t}$ that consists of the primitive elements. The first claim is equivalent to the following assertion: we have

$$\delta_c e^{-\delta_c t} \sum_{\gamma \in \mathcal{H}'_{\Gamma,t}} e^{\text{Per}_{F_c}(\gamma)} \mathcal{L}_g \stackrel{*}{\rightarrow} \frac{m_c}{\|m_c\|} \quad (13.2)$$

as $t \rightarrow +\infty$. We proceed with the proof of the convergence claimed in Equation (13.2) as in [PauPS, Thm. 9.11]. We first prove that

$$\nu_t'' = \delta_c e^{-\delta_c t} \sum_{\gamma \in \mathcal{H}'_{\Gamma,t}} e^{\text{Per}_{F_c}(\gamma)} \mathcal{L}_g \stackrel{*}{\rightarrow} \frac{m_c}{\|m_c\|}. \quad (13.3)$$

We then refer to Step 2 of the proof of [PauPS, Thm. 9.11] for the fact that the contribution of the periods that are not primitive is negligible. Although the proof in loc. cit. is written for manifolds, the arguments are directly applicable for any $\text{CAT}(-1)$ space X and potential F satisfying the HC-property.⁹ In particular, the use of Proposition 5.13 (i) and (ii) of loc. cit. in the proof of Step 2 in loc. cit. is replaced now by the use of Theorem 4.5 (1) and (4) respectively.

Let us fix $x \in X$. Let

$$V(x) = \{(\xi, \eta) \in (X \cup \partial_\infty X)^2 : \xi \neq \eta, x \in]\xi, \eta[\},$$

which is an open subset of $X \cup \partial_\infty X$. Note that the family $(V(y))_{y \in X}$ covers the set of pairs of distinct points of $\partial_\infty X$. For every $t > 0$, let ν_t be the measure on $(X \cup \partial_\infty X)^2$ defined by

$$\nu_t = \delta_c \|m_c\| e^{-\delta_c t} \sum_{\gamma \in \Gamma : d(x, \gamma x) \leq t} e^{\int_x^{\gamma x} \tilde{F}_c} \Delta_{\gamma^{-1}x} \otimes \Delta_{\gamma x}.$$

The measures ν_t weak-star converge to $\mu_x^- \otimes \mu_x^+$ as $t \rightarrow +\infty$ by Corollary 11.2 (taking in its statement $y = x$).

Let γ_\pm be the attracting and repelling fixed points of any loxodromic element $\gamma \in \Gamma$. Let

$$\nu_t''' = \delta_c \|m_c\| e^{-\delta_c t} \sum_{\gamma \in \mathcal{H}_{\Gamma,t}} e^{\text{Per}_{F_c}(\gamma)} \Delta_{\gamma_-} \otimes \Delta_{\gamma_+}.$$

Since \mathbb{X} is an \mathbb{R} -tree, every element $\gamma \in \Gamma$ such that $x \in]\gamma^{-1}x, \gamma x[$ is loxodromic, and we have

$$d(x, \gamma x) = \lambda(\gamma) \quad \text{and} \quad \int_x^{\gamma x} \tilde{F}_c = \text{Per}_{F_c}(\gamma).$$

If furthermore $d(x, \gamma x)$ is big, then $\gamma^{-1}x$ and γx are respectively close to γ_- and γ_+ in $X \cup \partial_\infty X$.

Hence, for every continuous map $\psi : (X \cup \partial_\infty X)^2 \rightarrow [0, +\infty[$ with support contained in $V(x)$, and for every $\epsilon > 0$, if t is big enough, we have

$$e^{-\epsilon} \nu_t(\psi) \leq \nu_t'''(\psi) \leq e^\epsilon \nu_t(\psi).$$

⁹See Definition 3.4.

Using Hopf's parametrisation with basepoint x , we have $\nu_t''' \otimes ds = \|m_c\| \nu_t''$, and the support of any continuous function with compact support on $\mathcal{G}X$ may be covered by finitely many open sets $V(x) \times \mathbb{R}$ where $x \in X$. The proof of the claim (13.3) now follows as in Step 1 of the proof of Theorem [PauPS, Thm. 9.11].

The second claim of Theorem 13.3 follows from the first one in the same way as in [PauPS, Thm. 9.11] to which we refer for the proof. \square

In a similar way, replacing in the above proof Corollary 11.2 of Theorem 11.1 by the similar corollary of Theorem 11.8 with $\mathcal{D}^- = (\gamma x)_{\gamma \in \Gamma}$ and $\mathcal{D}^+ = (\gamma y)_{\gamma \in \Gamma}$ for any $x, y \in V\mathbb{X}$, we get the following analogous result for simplicial trees.

Theorem 13.4. *Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, let Γ be a nonelementary discrete subgroup of $\text{Aut}(\mathbb{X})$ and let $\tilde{c} : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant system of conductances. Assume that the critical exponent δ_c of c is finite and positive and that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow. As $n \rightarrow +\infty$, the measures*

$$\frac{e^{\delta_c} - 1}{e^{\delta_c}} e^{-\delta_c n} \sum_{g \in \mathfrak{P}\text{er}'(n)} e^{c(g)} \mathcal{L}_g$$

and

$$\frac{e^{\delta_c} - 1}{e^{\delta_c}} n e^{-\delta_c n} \sum_{g \in \mathfrak{P}\text{er}'(n)} e^{c(g)} \frac{\mathcal{L}_g}{\ell(g)}$$

converge to $\frac{m_c}{|m_c|}$ for the weak-star convergence of measures. If Γ is geometrically finite, the convergence holds for narrow convergence. \square

In the special case when $\Gamma \backslash X$ is a compact graph and $F = 0$, the following immediate corollary of Theorem 13.3 is proved in [Gui], and it follows from the results of [ParP].¹⁰ There are also some works on non-backtracking random walks with related results. For example, for regular finite graphs, [LuPS1] and [Fri1] (see [Fri2, Lem. 2.3]) give an expression of the irreducible trace which is the number of closed walks of a given length.

Corollary 13.5. *Let (\mathbb{X}, λ) be a locally finite metric tree without terminal vertices. Let Γ be a geometrically finite discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$. Let $c : E\mathbb{X} \rightarrow \mathbb{R}$ be a Γ -invariant system of conductances, with finite and positive critical exponent δ_c .*

(1) *If the Gibbs measure m_c is finite and mixing for the continuous time geodesic flow, then*

$$\sum_{g \in \mathfrak{P}\text{er}'(t)} e^{c(g)} \sim \frac{e^{\delta_c t}}{\delta_c t}$$

as $t \rightarrow +\infty$.

(2) *If $\lambda = 1$ and if the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow, then*

$$\sum_{g \in \mathfrak{P}\text{er}'(n)} e^{c(g)} \sim \frac{e^{\delta_c}}{e^{\delta_c} - 1} \frac{e^{\delta_c n}}{n}$$

as $n \rightarrow +\infty$. \square

¹⁰See the introduction of [Sha] for comments.

Part III

Arithmetic applications

Chapter 14

Fields with discrete valuations

Let \hat{K} be a non-Archimedean local field. Basic examples of such fields are the field of formal Laurent series over a finite field, and the field of p -adic numbers (see Examples 14.1 and 14.2). In Part III of this book, we apply the geometric equidistribution and counting results for simplicial trees given in Part II, in order to prove arithmetic equidistribution and counting results in such fields \hat{K} . The link between the geometry and the algebra is provided by the Bruhat-Tits tree of $(\mathrm{PGL}_2, \hat{K})$, see Chapter 15.1. We will only use the system of conductances equal to 0 in this Part III.

In the present Chapter, before embarking on our arithmetic applications, we recall basic facts on local fields for the convenience of the geometer reader. For more details, we refer for instance to [Ser2, Gos]. We refer to [BrPP] for an announcement of the results of Part III, with a presentation different from the one in the Introduction.

We will only give results for the algebraic group $\underline{G} = \mathrm{PGL}_2$ over \hat{K} and special discrete subgroups Γ of $\mathrm{PGL}_2(\hat{K})$, even though the same methods give equidistribution and counting results when \underline{G} is any semisimple connected linear algebraic group over \hat{K} of \hat{K} -rank 1 and Γ any lattice in $G = \underline{G}(\hat{K})$.

14.1 Local fields and valuations

Let F be a field and let $F^\times = (F - \{0\}, \times)$ be its multiplicative group. A surjective group morphism $v : F^\times \rightarrow \mathbb{Z}$ to the additive group \mathbb{Z} , that satisfies

$$v(a + b) \geq \min\{v(a), v(b)\}$$

for all $a, b \in F^\times$, is a (normalised discrete) *valuation* v on F . We make the usual convention and extend the definition of v to F by setting $v(0) = +\infty$. Note that $v(a+b) = \min\{v(a), v(b)\}$ if $v(a) \neq v(b)$. When F is an extension of a finite field k , the valuation v vanishes on k^\times .

The subring

$$\mathcal{O}_v = \{x \in F : v(x) \geq 0\}$$

is the *valuation ring* (or local ring) of F or of v .

The maximal ideal

$$\mathfrak{m}_v = \{x \in F : v(x) > 0\}$$

of \mathcal{O}_v is principal and it is generated as an ideal of \mathcal{O}_v by any element $\pi_v \in F$ with

$$v(\pi_v) = 1$$

which is called a *uniformiser* of F .

The *residual field* of the valuation v is

$$k_v = \mathcal{O}_v/\mathfrak{m}_v .$$

When k_v is finite, the valuation v defines a (normalised, non-Archimedean) *absolute value* $|\cdot|_v$ on F by

$$|x|_v = |k_v|^{-v(x)} ,$$

with the convention that $|k_v|^{-\infty} = 0$. This absolute value induces an ultrametric distance on F by

$$(x, y) \mapsto |x - y|_v .$$

Let F_v be the completion of F with respect to this distance. The valuation v of F uniquely extends to a (normalised discrete) valuation on F_v , again denoted by v .

Example 14.1. Let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable Y with coefficients in a finite field \mathbb{F}_q of order a positive power q of a positive prime p in \mathbb{Z} , let $\mathbb{F}_q[Y]$ be the ring of polynomials in one variable Y with coefficients in \mathbb{F}_q , and let $v_\infty : K^\times \rightarrow \mathbb{Z}$ be the *valuation at infinity* of K , defined on every $P/Q \in K$ with $P \in \mathbb{F}_q[Y]$ and $Q \in \mathbb{F}_q[Y] - \{0\}$ by

$$v_\infty(P/Q) = \deg Q - \deg P .$$

The absolute value associated with v_∞ is

$$|P/Q|_\infty = q^{\deg P - \deg Q} .$$

The completion of K for v_∞ is the field $K_{v_\infty} = \mathbb{F}_q((Y^{-1}))$ of formal Laurent series in one variable Y^{-1} with coefficients in \mathbb{F}_q . The elements x in $\mathbb{F}_q((Y^{-1}))$ are of the form

$$x = \sum_{i \in \mathbb{Z}} x_i Y^{-i}$$

where $x_i = 0 \in \mathbb{F}_q$ for $i \in \mathbb{Z}$ small enough. The valuation at infinity of $\mathbb{F}_q((Y^{-1}))$ extending the valuation at infinity of $\mathbb{F}_q(Y)$ is

$$v_\infty(x) = \sup\{i \in \mathbb{Z} : \forall j < i, \quad x_j = 0\} ,$$

that is,

$$v_\infty\left(\sum_{i=i_0}^{\infty} x_i Y^{-i}\right) = i_0$$

if $x_{i_0} \neq 0$. The valuation ring of v_∞ is the ring $\mathcal{O}_{v_\infty} = \mathbb{F}_q[[Y^{-1}]]$ of formal power series in one variable Y^{-1} with coefficients in \mathbb{F}_q . The element $\pi_{v_\infty} = Y^{-1}$ is a uniformizer of v_∞ , the residual field $\mathcal{O}_{v_\infty}/\pi_{v_\infty}\mathcal{O}_{v_\infty}$ of v_∞ is $k_{v_\infty} = \mathbb{F}_q$.

Example 14.2. Given a positive prime $p \in \mathbb{Z}$, the field of p -adic numbers \mathbb{Q}_p is the completion of \mathbb{Q} with respect to the absolute value $|\cdot|_p$ of the p -adic valuation v_p defined by setting

$$v_p\left(p^n \frac{a}{b}\right) = n$$

when $n \in \mathbb{Z}$, $a, b \in \mathbb{Z} - \{0\}$ are not divisible by p . Then the valuation ring $\mathcal{O}_{v_p} = \mathbb{Z}_p$ of \mathbb{Q}_p is the closure of \mathbb{Z} for the absolute value $|\cdot|_p$, $\pi_{v_p} = p$ is a uniformiser, and the residual field is $k_{v_p} = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, a finite field of order p .

A field endowed with a valuation is a *non-Archimedean local field* if it is complete with respect to its absolute value and if its residual field is finite.¹ Its valuation ring is then a compact open additive subgroup. Any non-Archimedean local field is isomorphic to a finite extension of the p -adic field \mathbb{Q}_p for some prime p , or to an extension of \mathbb{F}_p with transcendence degree 1 for some prime p .

The basic case of extensions of \mathbb{F}_p with transcendence degree 1 is described in Example 14.1 above, and the general case of the discussed in Section 14.2 below. The geometer reader may skip Section 14.2 and use only Example 14.1 in the remainder of Part III (using $g = 0$ when the constant g occurs).

14.2 Global function fields

In this Section, we fix a finite field \mathbb{F}_q with q elements, where q is a positive power of a positive prime $p \in \mathbb{Z}$, and we recall the definitions and basic properties of a function field K over \mathbb{F}_q , its genus g , its valuations v , its completion K_v for the associated absolute value $|\cdot|_v$ and the associated affine function ring R_v . See for instance [Gos, Ros] for the content of this Section.

Let K be a (global) *function field* over \mathbb{F}_q , which can be defined in two equivalent ways as

- (1) the field of rational functions on a geometrically irreducible smooth projective curve \mathbf{C} over \mathbb{F}_q , or
- (2) an extension of \mathbb{F}_q of transcendence degree 1, in which \mathbb{F}_q is algebraically closed.

There is a bijection between the set of closed points of \mathbf{C} and the set of (normalised discrete) valuations of its function field K , the valuation of a given element $f \in K$ being the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such an element v from now on. We denote by g the genus of the curve \mathbf{C} .

In the basic Example 14.1, \mathbf{C} is the projective line \mathbb{P}^1 over \mathbb{F}_q which is a curve of genus $g = 0$, and the closed point associated with the valuation at infinity is the point at infinity $[1 : 0]$.

We denote by K_v the completion of K for v , and by

$$\mathcal{O}_v = \{x \in K_v : v(x) \geq 0\}$$

the valuation ring of (the unique extension to K_v) of v . We choose a uniformizer π_v of v . We denote by $k_v = \mathcal{O}_v / \pi_v \mathcal{O}_v$ the residual field of v , which is a finite field of order

$$q_v = |k_v|.$$

The field k_v is from now on identified with a fixed lift in \mathcal{O}_v (see for instance [Col, Théo. 1.3]), and is an extension of the field of constants \mathbb{F}_q . The degree of this extension is denoted by $\deg v$, so that

$$q_v = q^{\deg v}.$$

We denote by $|\cdot|_v$ the (normalised) absolute value associated with v : for every $x \in K_v$, we have

$$|x|_v = (q_v)^{-v(x)} = q^{-v(x) \deg v}.$$

¹There are also two Archimedean local fields \mathbb{C} and \mathbb{R} , see for example [Cas].

Every element $x \in K_v$ is² a (converging) Laurent series $x = \sum_{i \in \mathbb{Z}} x_i (\pi_v)^i$ in the variable π_v over k_v , where $x_i \in k_v$ is zero for $i \in \mathbb{Z}$ small enough. We then have

$$|x|_v = (q_v)^{-\sup\{j \in \mathbb{Z} : \forall i < j, x_i = 0\}}, \quad (14.1)$$

and \mathcal{O}_v consists of the (converging) power series $x = \sum_{i \in \mathbb{N}} x_i (\pi_v)^i$ (where $x_i \in k_v$) in the variable π_v over k_v .

We denote by R_v the affine algebra of the affine curve $\mathbf{C} - \{v\}$, consisting of the elements of K whose only poles are at the closed point v of \mathbf{C} . Its field of fractions is equal to K , hence we will often write elements of K as x/y with $x, y \in R_v$ and $y \neq 0$. In the basic Example 14.1, we have $R_{v_\infty} = \mathbb{F}_q[Y]$. Note that

$$R_v \cap \mathcal{O}_v = \mathbb{F}_q, \quad (14.2)$$

since the only rational functions on \mathbf{C} whose only poles are at v and whose valuation at v is nonnegative are the constant ones. We have (see for instance [Ser3, II.2 Notation], [Gos, page 63])

$$(R_v)^\times = (\mathbb{F}_q)^\times. \quad (14.3)$$

The following result is immediate when $\mathbf{C} = \mathbb{P}^1$, since then $R_v + \mathcal{O}_v = K_v$.

Lemma 14.3. *The dimension of the quotient vector space $K_v/(R_v + \mathcal{O}_v)$ over \mathbb{F}_q is equal to the genus g of \mathbf{C} .*

Proof. (J.-B. Bost) We refer for instance to [Ser1] for background on sheaf cohomology. We denote in the same way the valuation v and the corresponding closed point on \mathbf{C} .

Let $\mathcal{O} = K \cap \mathcal{O}_v$ be the discrete valuation ring of v restricted to K . Since K is dense in K_v and \mathcal{O}_v is open and contains 0, we have $K_v = K + \mathcal{O}_v$. Therefore the canonical map

$$K/(R_v + \mathcal{O}) \rightarrow K_v/(R_v + \mathcal{O}_v)$$

is a linear isomorphism over \mathbb{F}_q . Let us hence prove that $\dim_{\mathbb{F}_q} K/(R_v + \mathcal{O}) = g$.

In what follows, \mathcal{V} ranges over the affine Zariski-open neighbourhoods of v in \mathbf{C} , ordered by inclusion. Let $\mathcal{O}_{\mathbf{C}}$ be the structural sheaf of \mathbf{C} . Note that by the definition of R_v , since the zeros of elements of K^\times are isolated and by the relation between valuations of K and closed points of \mathbf{C} ,

$$R_v = H^0(\mathbf{C} - \{v\}, \mathcal{O}_{\mathbf{C}}), \quad K = \varinjlim_{\mathcal{V}} H^0(\mathcal{V} - \{v\}, \mathcal{O}_{\mathbf{C}}) \quad \text{and} \quad \mathcal{O} = \varinjlim_{\mathcal{V}} H^0(\mathcal{V}, \mathcal{O}_{\mathbf{C}}).$$

Since \mathcal{V} and $\mathbf{C} - \{v\}$ are affine curves, we have $H^1(\mathbf{C} - \{v\}, \mathcal{O}_{\mathbf{C}}) = H^1(\mathcal{V}, \mathcal{O}_{\mathbf{C}}) = 0$. By the Mayer-Vietoris exact sequence since $\{\mathbf{C} - \{v\}, \mathcal{V}\}$ covers \mathbf{C} , we hence have an exact sequence

$$H^0(\mathbf{C}, \mathcal{O}_{\mathbf{C}}) \rightarrow H^0(\mathbf{C} - \{v\}, \mathcal{O}_{\mathbf{C}}) \times H^0(\mathcal{V}, \mathcal{O}_{\mathbf{C}}) \rightarrow H^0(\mathcal{V} - \{v\}, \mathcal{O}_{\mathbf{C}}) \rightarrow H^1(\mathbf{C}, \mathcal{O}_{\mathbf{C}}).$$

Therefore

$$\begin{aligned} K/(R_v + \mathcal{O}) &= \varinjlim_{\mathcal{V}} H^0(\mathcal{V} - \{v\}, \mathcal{O}_{\mathbf{C}})/(H^0(\mathbf{C} - \{v\}, \mathcal{O}_{\mathbf{C}}) + H^0(\mathcal{V}, \mathcal{O}_{\mathbf{C}})) \\ &\simeq H^1(\mathbf{C}, \mathcal{O}_{\mathbf{C}}). \end{aligned}$$

²See for instance [Col, Coro. 1.6]

Since $\dim_{\mathbb{F}_q} H^1(\mathbf{C}, \mathcal{O}_{\mathbf{C}}) = g$ by one definition of the genus of \mathbf{C} , the result follows. \square

Recall that R_v is a Dedekind ring.³ In particular, every nonzero ideal (respectively fractional ideal) I of R_v may be written uniquely as $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ where \mathfrak{p} ranges over the prime ideals in R_v and $v_{\mathfrak{p}}(I) \in \mathbb{N}$ (respectively $v_{\mathfrak{p}}(I) \in \mathbb{Z}$), with only finitely many of them nonzero. By convention $I = R_v$ if $v_{\mathfrak{p}}(I) = 0$ for all \mathfrak{p} . For every $x, y \in R_v$ (respectively $x, y \in K$), we denote by

$$\langle x, y \rangle = x R_v + y R_v$$

the ideal (respectively fractional ideal) of R_v generated by x, y . If I, J are nonzero fractional ideals of R_v , we have

$$I \cap J = \prod_{\mathfrak{p}} \mathfrak{p}^{\max\{v_{\mathfrak{p}}(I), v_{\mathfrak{p}}(J)\}} \quad \text{and} \quad I + J = \prod_{\mathfrak{p}} \mathfrak{p}^{\min\{v_{\mathfrak{p}}(I), v_{\mathfrak{p}}(J)\}} .$$

We define the (absolute) *norm* of a nonzero ideal $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ of R_v by

$$N(I) = [R_v : I] = |R_v/I| = \prod_{\mathfrak{p}} q^{v_{\mathfrak{p}}(I) \deg \mathfrak{p}} ,$$

where $\deg \mathfrak{p}$ is the degree of the field R_v/\mathfrak{p} over \mathbb{F}_q , so that $N(R_v) = 1$. By convention $N(0) = 0$. This norm is multiplicative:

$$N(IJ) = N(I)N(J) ,$$

and the norm of a nonzero fractional ideal $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ of R_v is defined by the same formula. Note that if (a) is the principal ideal in R_v generated by a , we define $N(a) = N((a))$. We have (see for instance [Gos, page 63])

$$N(a) = |a|_v . \tag{14.4}$$

Dedekind's zeta function of K is (see for instance [Gos, §7.8] or [Ros, §5])

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$$

if $\operatorname{Re} s > 1$, where the summation is over the nonzero ideals I of R_v . By for instance [Ros, §5], it has an analytic continuation on $\mathbb{C} - \{0, 1\}$ with simple poles at $s = 0, s = 1$. It is actually a rational function of q^{-s} . In particular, if $K = \mathbb{F}_q(Y)$, then (see [Ros, Theo. 5.9])

$$\zeta_{\mathbb{F}_q(Y)}(-1) = \frac{1}{(q-1)(q^2-1)} . \tag{14.5}$$

We denote by $\operatorname{Haar}_{K_v}$ the Haar measure of the (abelian) locally compact topological group $(K_v, +)$, normalised so that $\operatorname{Haar}_{K_v}(\mathcal{O}_v) = 1$.⁴ The Haar measure scales as follows under multiplication: for all $\lambda, x \in K_v$, we have

$$d \operatorname{Haar}_{K_v}(\lambda x) = |\lambda|_v d \operatorname{Haar}_{K_v}(x) . \tag{14.6}$$

Note that any fractional ideal I of R_v is a discrete subgroup of $(K_v, +)$, and we will again denote by $\operatorname{Haar}_{K_v}$ the Haar measure on the compact group K_v/I which is induced by the above normalised Haar measure of K_v .

³See for instance [Ser3, II.2 Notation]. We refer for instance to [Nar, §1.1] for background on Dedekind rings.

⁴Other normalisations are useful when considering Fourier transforms, see for instance Tate's thesis [Tat].

Lemma 14.4. *For every fractional ideal I of R_v , we have*

$$\text{Haar}_{K_v}(K_v/I) = q^{g-1} N(I) .$$

Proof. By the scaling properties of the Haar measure, we may assume that I is an ideal in R_v . By Lemma 14.3, we have $\text{Card } K_v/(R_v + \mathcal{O}_v) = q^g$. By Equation (14.2) and by the normalisation of the Haar measure, we have

$$\text{Haar}_{K_v}(R_v + \mathcal{O}_v)/R_v = \text{Haar}_{K_v} \mathcal{O}_v/(R_v \cap \mathcal{O}_v) = \text{Haar}_{K_v} \mathcal{O}_v/\mathbb{F}_q = \frac{1}{q} .$$

Hence

$$\text{Haar}_{K_v}(K_v/R_v) = \text{Card}(K_v/(R_v + \mathcal{O}_v)) \text{Haar}_{K_v}(R_v + \mathcal{O}_v)/R_v = q^{g-1} .$$

Since $\text{Haar}_{K_v}(K_v/I) = N(I) \text{Haar}_{K_v}(K_v/R_v)$, the result follows. □

Chapter 15

Bruhat-Tits trees and modular groups

In this Chapter, we give the background information and preliminary results on the main link between the geometry and the algebra used for our arithmetic applications: the (discrete time) geodesic flow on quotients of Bruhat-Tits trees by arithmetic lattices.

We denote the image in PGL_2 of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2$.

15.1 Bruhat-Tits trees

Let K_v be a non-Archimedean local field, with valuation v , valuation ring \mathcal{O}_v , choice of uniformiser π_v , and residual field k_v of order q_v (see Section 14.1 for definitions).

In this Section, we recall the construction and basic properties of the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) , see for instance [Tit]. We use its description given in [Ser3], to which we refer for proofs and further information.

An \mathcal{O}_v -lattice Λ in the K_v -vector space $K_v \times K_v$ is a rank 2 free \mathcal{O}_v -submodule of $K_v \times K_v$, generating $K_v \times K_v$ as a vector space. The Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) is the graph whose set of vertices $V\mathbb{X}_v$ is the set of homothety classes (under $(K_v)^\times$) $[\Lambda]$ of \mathcal{O}_v -lattices Λ in $K_v \times K_v$, and whose non-oriented edges are the pairs $\{x, x'\}$ of vertices such that there exist representatives Λ of x and Λ' of x' for which $\Lambda \subset \Lambda'$ and Λ'/Λ is isomorphic to $\mathcal{O}_v/\pi_v\mathcal{O}_v$. If K is any field endowed with a valuation v whose completion is K_v , then the similarly defined Bruhat-Tits tree of (PGL_2, K) coincides with \mathbb{X}_v , see [Ser3, p. 71].

The graph \mathbb{X}_v is a regular tree of degree $|\mathbb{P}_1(k_v)| = q_v + 1$. In particular, the Bruhat-Tits tree of $(\mathrm{PGL}_2, \mathbb{Q}_p)$ is regular of degree $p + 1$, and if $K_v = \mathbb{F}_q((Y^{-1}))$ and $v = v_\infty$, then the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) is regular of degree $q + 1$. More generally, if K_v is the completion of a function field over \mathbb{F}_q endowed with a valuation v as in Section 14.2, then the Bruhat-Tits tree of (PGL_2, K_v) is regular of degree $q_v + 1$.

The *standard base point* $*_v$ of \mathbb{X} is the homothety class $[\mathcal{O}_v \times \mathcal{O}_v]$ of the \mathcal{O}_v -lattice $\mathcal{O}_v \times \mathcal{O}_v$, generated by the canonical basis of $K_v \times K_v$. In particular, we have

$$d(*_v, [\mathcal{O}_v \times x\mathcal{O}_v]) = |v(x)| \tag{15.1}$$

for every $x \in (K_v)^\times$. The *link*

$$\mathrm{lk}(*_v) = \{y \in V\mathbb{X}_v : d(y, *_v) = 1\}$$

of $*_v$ in \mathbb{X}_v identifies with the projective line $\mathbb{P}_1(k_v)$.

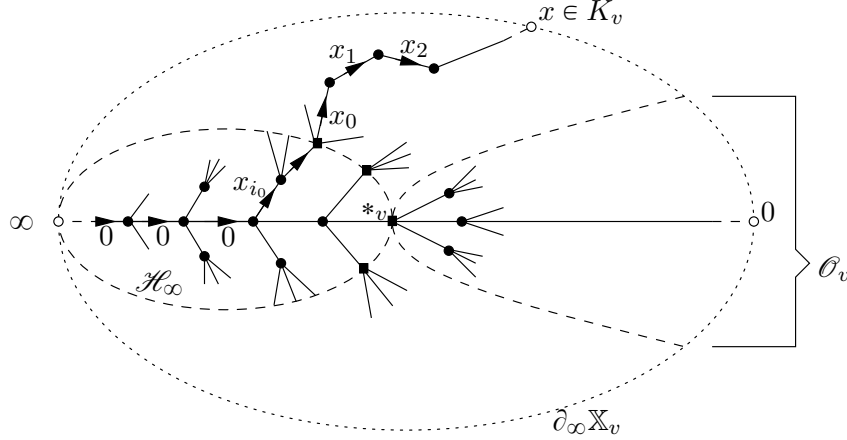
The left linear action of $\mathrm{GL}_2(K_v)$ on $K_v \times K_v$ induces a faithful, vertex-transitive left action by automorphisms of $\mathrm{PGL}_2(K_v)$ on \mathbb{X}_v . The stabiliser in $\mathrm{PGL}_2(K_v)$ of $*_v$ is $\mathrm{PGL}_2(\mathcal{O}_v)$. We will hence identify $\mathrm{PGL}_2(K_v)/\mathrm{PGL}_2(\mathcal{O}_v)$ with $V\mathbb{X}_v$ by the map $g \mathrm{PGL}_2(\mathcal{O}_v) \mapsto g*_v$.

We identify the projective line $\mathbb{P}_1(K_v)$ with $K_v \cup \{\infty\}$ using the map $K_v(x, y) \mapsto \frac{x}{y}$, so that

$$\infty = [1 : 0].$$

The projective action of $\mathrm{GL}_2(K_v)$ or $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}^1(K_v)$ is the action by *homographies*¹ on $K_v \cup \{\infty\}$, given by $(g, z) \mapsto g \cdot z = \frac{az+b}{cz+d}$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$, or $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(K_v)$. As usual we define $\infty \mapsto \frac{a}{c}$ and $-\frac{d}{c} \mapsto \infty$.

There exists a unique homeomorphism between the boundary at infinity $\partial_\infty \mathbb{X}_v$ of \mathbb{X}_v and $\mathbb{P}_1(K_v)$ such that the (continuous) extension to $\partial_\infty \mathbb{X}_v$ of the isometric action of $\mathrm{PGL}_2(K_v)$ on \mathbb{X}_v corresponds to the projective action of $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}_1(K_v)$. From now on, we identify $\partial_\infty \mathbb{X}_v$ and $\mathbb{P}_1(K_v)$ by this homeomorphism. Under this identification, \mathcal{O}_v consists of the positive endpoints ℓ_+ of the geodesic lines ℓ of \mathbb{X}_v with negative endpoint $\ell_- = \infty$ that pass through the vertex $*_v$ (see the picture below).



Let \mathcal{H}_∞ be the horoball centred at $\infty \in \partial_\infty \mathbb{X}_v$ whose associated horosphere passes through $*_v$. There is a unique labeling of the edges of \mathbb{X}_v by elements of $\mathbb{P}_1(k_v) = k_v \cup \{\infty\}$ such that

- the label of any edge of \mathbb{X}_v pointing towards $\infty \in \partial_\infty \mathbb{X}_v$ is ∞ ,
- for any $x = \sum_{i \in \mathbb{Z}} x_i (\pi_v)^i \in K_v$, the sequence $(x_i)_{i \in \mathbb{Z}}$ is the sequence of the labels of the (directed) edges that make up the geodesic line $] \infty, x[$ oriented from ∞ towards x
- x_0 is the label of the edge of $] \infty, x[$ exiting the horoball \mathcal{H}_∞ .

We refer to [Paul, Sect. 5] when $K_v = \mathbb{F}_q((Y^{-1}))$ and $v = v_\infty$.

For all $\eta, \eta' \in K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$, we have

$$|\eta - \eta'|_v = d_{\mathcal{H}_\infty}(\eta, \eta')^{\ln q_v} \quad (15.2)$$

by the definitions of the absolute value $|\cdot|_v$ and of Hamenstädt's distance, see Equation (14.1) and the above geometric interpretation, and Equation (2.8). Note that in [Paul], Hamenstädt's distance in a regular tree is defined in a different way: In that reference, the distance $|\eta - \eta'|_v$ equals Hamenstädt's distance between η and η' .

¹or linear fractional transformations

In particular, the Hölder norms $\|\psi\|_{\beta, |\cdot - \cdot|_v}$ and $\|\psi\|_{\beta', d_{\mathcal{H}_\infty}}$ of a function $\psi : K_v \rightarrow \mathbb{R}$, respectively for the distance $(x, y) \mapsto |x - y|_v$ and $d_{\mathcal{H}_\infty}$ on K_v , are related by the following formula:

$$\forall \beta \in]0, \frac{1}{\ln q_v}], \quad \|\psi\|_{\beta, |\cdot - \cdot|_v} = \|\psi\|_{\beta \ln q_v, d_{\mathcal{H}_\infty}}. \quad (15.3)$$

The group $\mathrm{PGL}_2(K_v)$ acts simply transitively on the set of ordered triples of distinct points in $\partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$. In particular, it acts transitively on the space $\mathcal{G}\mathbb{X}_v$ of discrete geodesic lines in \mathbb{X}_v . The stabiliser under this action of the geodesic line (from $\infty = [1 : 0]$ to $0 = [0 : 1]$)

$$\ell^* : n \mapsto [\mathcal{O}_v \times (\pi_v)^{-n} \mathcal{O}_v]$$

is the maximal compact-open subgroup

$$\underline{A}(\mathcal{O}_v) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in (\mathcal{O}_v)^\times \right\}$$

of the diagonal group

$$\underline{A}(K_v) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in (K_v)^\times \right\}.$$

We will hence identify $\mathrm{PGL}_2(K_v)/\underline{A}(\mathcal{O}_v)$ with $\mathcal{G}\mathbb{X}_v$ by the mapping $\tilde{\Xi} : g\underline{A}(\mathcal{O}_v) \mapsto g\ell^*$. Define

$$a_v = \begin{bmatrix} 1 & 0 \\ 0 & \pi_v^{-1} \end{bmatrix},$$

which belongs to $\underline{A}(K_v)$ and centralises $\underline{A}(\mathcal{O}_v)$. The homeomorphism $\tilde{\Xi}$ is equivariant for the actions on the left of $\mathrm{PGL}_2(K_v)$ on $\mathrm{PGL}_2(K_v)/\underline{A}(\mathcal{O}_v)$ and $\mathcal{G}\mathbb{X}_v$. It is also equivariant for the actions on $\mathrm{PGL}_2(K_v)/\underline{A}(\mathcal{O}_v)$ under translations on the right by $(a_v)^\mathbb{Z}$ and on $\mathcal{G}\mathbb{X}_v$ under the discrete geodesic flow $(\mathbf{g}^n)_{n \in \mathbb{Z}}$: for all $n \in \mathbb{Z}$ and $x \in \mathrm{PGL}_2(K_v)/\underline{A}(\mathcal{O}_v)$, we have

$$\tilde{\Xi}(x a_v^n) = \mathbf{g}^n \tilde{\Xi}(x). \quad (15.4)$$

Furthermore, the stabiliser in $\mathrm{PGL}_2(K_v)$ of the ordered pair of endpoints $(\ell_-^* = \infty, \ell_+^* = 0)$ of ℓ^* in $\partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$ is $\underline{A}(K_v)$. Therefore any element $\gamma \in \mathrm{PGL}_2(K_v)$ which is loxodromic on \mathbb{X}_v is diagonalisable over K_v . By [Ser3, page 108], the translation length on \mathbb{X}_v of $\gamma_0 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is

$$\lambda(\gamma_0) = |v(a) - v(d)|. \quad (15.5)$$

Note that if $\tilde{\gamma}_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$ is a representative of γ_0 such that $\det \tilde{\gamma}_0 \in (\mathcal{O}_v)^\times$, then as $0 = v(\det \tilde{\gamma}_0) = v(ad) = v(a) + v(d)$ and since $v(a) \neq v(d)$ if $\lambda(\gamma_0) \neq 0$, we have $v(\mathrm{tr} \tilde{\gamma}_0) = v(a + d) = \max\{v(a), v(d)\}$ and $v(a) - v(d) = 2v(a) = -2v(d)$. Thus,

$$\lambda(\gamma_0) = 2|v(\mathrm{tr} \tilde{\gamma}_0)|. \quad (15.6)$$

By conjugation, this formula is valid if $\gamma_0 \in \mathrm{PGL}_2(K_v)$ is loxodromic on \mathbb{X}_v and represented by $\tilde{\gamma}_0 \in \mathrm{GL}_2(K_v)$ such that $\det \tilde{\gamma}_0 \in (\mathcal{O}_v)^\times$.

Let \mathcal{H} be a horoball in \mathbb{X}_v whose boundary is contained in $V\mathbb{X}_v$ and whose point at infinity ξ is different from ∞ . The *height* of \mathcal{H} is

$$\text{ht}_\infty(\mathcal{H}) = \max\{\beta_\infty(x, *_v) : x \in \partial\mathcal{H}\} \in \mathbb{Z},$$

which is the signed distance between \mathcal{H}_∞ and \mathcal{H} .² It is attained at the intersection point with $\partial\mathcal{H}$ of the geodesic line from ∞ to ξ , which is then called the *highest point* of \mathcal{H} . Note that the height of \mathcal{H} is invariant under the action of the stabiliser of \mathcal{H}_∞ in $\text{PGL}_2(K_v)$ on the set of such horoballs \mathcal{H} .

The following lemma is a generalisation of [Pau1, Prop. 6.1] that covers the particular case of $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.

Lemma 15.1. *Assume that K_v is the completion of a function field K over \mathbb{F}_q endowed with a valuation v , with associated affine function ring R_v . For every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PGL}_2(K)$ with $a, b, c, d \in K$ such that $ad - bc \in (\mathcal{O}_v)^\times$ and $c \neq 0$, the image of \mathcal{H}_∞ by γ is the horoball centred at $\frac{a}{c} \in K \subset K_v = \partial_\infty\mathbb{X}_v - \{\infty\}$ with height*

$$\text{ht}_\infty(\gamma\mathcal{H}_\infty) = -2v(c).$$

Proof. It is immediate that $\gamma\infty = \frac{a}{c}$ under the projective action. Up to multiplying γ on the left by $\begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} \in \text{PGL}_2(K)$, which does not change c nor the height of $\gamma\mathcal{H}_\infty$, we may assume that $a = 0$ and that b has the form $c^{-1}u$ with $u = bc - ad \in (\mathcal{O}_v)^\times$. Multiplying γ on the right by $\begin{bmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{bmatrix} \in \text{PGL}_2(K)$ preserves $\gamma\mathcal{H}_\infty$ and does not change $a = 0$, $b = c^{-1}u$ or c . Hence we may assume that $d = 0$. Since γ then exchanges the points ∞ and 0 in $\partial_\infty\mathbb{X}_v$, the highest point of $\gamma\mathcal{H}_\infty$ is $\gamma*_v$. Assuming first that $0, \gamma*_v, *_v, \infty$ are in this order on the geodesic line from 0 to ∞ , we have by Equation (15.1)

$$\begin{aligned} \text{ht}_\infty(\gamma\mathcal{H}_\infty) &= d(*_v, \gamma*_v) = d([\mathcal{O}_v \times \mathcal{O}_v], [c^{-1}u\mathcal{O}_v \times c\mathcal{O}_v]) \\ &= d([\mathcal{O}_v \times \mathcal{O}_v], [\mathcal{O}_v \times c^2\mathcal{O}_v]) = -v(c^2) = -2v(c). \end{aligned}$$

If $0, \gamma*_v, *_v, \infty$ are in the opposite order, then the same computation holds, up to replacing the distance d by its opposite $-d$. \square

15.2 Modular graphs of groups

Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , let K_v be the completion of K associated with v , and let R_v be the affine function ring associated with v (see Section 14.2 for definitions).

The group $\Gamma_v = \text{PGL}_2(R_v)$ is a lattice in the locally compact group $\text{PGL}_2(K_v)$, and a lattice³ of the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) , called the *modular group* at v of K . The quotient graph $\Gamma_v \backslash \mathbb{X}_v$ is called the *modular graph* at v of K , and the quotient graph of groups⁴ $\Gamma_v \backslash \mathbb{X}_v$ is called the *modular graph of groups* at v of K . We refer to [Ser3] for background

²See the definition of signed distance just above Lemma 11.12.

³See Section 2.7 for a definition.

⁴See Section 2.7 for a definition.

information on these objects, and for instance to [Paul1] for a geometric treatment when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.

By for instance [Ser3], the set of cusps $\Gamma_v \backslash \mathbb{P}_1(K)$ is finite, and $\Gamma_v \backslash \mathbb{X}_v$ is the disjoint union of a finite connected subgraph containing $\Gamma_v * v$ and of maximal open geodesic rays $h_z(]0, +\infty[)$, for $z = \Gamma_v \tilde{z} \in \Gamma_v \backslash \mathbb{P}_1(K)$, where h_z (called a *cuspidal ray*) is the image by the canonical projection $\mathbb{X}_v \rightarrow \Gamma_v \backslash \mathbb{X}_v$ of a geodesic ray whose point at infinity in $\mathbb{P}_1(K) \subset \partial_\infty \mathbb{X}_v$ is equal to \tilde{z} . Conversely, any geodesic ray whose point at infinity lies in $\mathbb{P}_1(K) \subset \partial_\infty \mathbb{X}_v$ contains a subray that maps injectively by the canonical projection $\mathbb{X}_v \rightarrow \Gamma_v \backslash \mathbb{X}_v$.

The group $\Gamma_v = \mathrm{PGL}_2(R_v)$ is a geometrically finite lattice by for instance [Pau2].⁵ The set of bounded parabolic fixed points of Γ_v is exactly $\mathbb{P}_1(K) \subset \partial_\infty \mathbb{X}_v$, and the set of conical limit points of Γ_v is $\widehat{\Gamma_v \backslash \mathbb{X}_v} - \mathbb{P}_1(K)$.

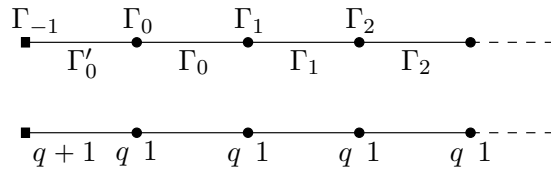
Let us denote by $\widehat{\Gamma_v \backslash \mathbb{X}_v} = (\Gamma_v \backslash \mathbb{X}_v) \sqcup \mathcal{E}_v$ Freudenthal's compactification of $\Gamma_v \backslash \mathbb{X}_v$ by its finite set of ends \mathcal{E}_v , see [Fre]. This set of ends is indeed finite, in bijection with $\Gamma_v \backslash \mathbb{P}_1(K)$ by the map which associates to $z \in \Gamma_v \backslash \mathbb{P}_1(K)$ the end towards which the cuspidal ray h_z converges. See for instance [Ser3] for a geometric interpretation of \mathcal{E}_v in terms of the curve \mathbf{C} .

Let \mathcal{I}_v be the set of classes of fractional ideals of R_v . The map which associates to an element $[x : y] \in \mathbb{P}_1(K)$ the class of the fractional ideal $xR_v + yR_v$ generated by x, y induces a bijection from the set of cusps $\Gamma \backslash \mathbb{P}_1(K)$ to \mathcal{I}_v .

The volume⁶ of the modular graph of groups can be computed using Equation (14.3) and Exercice 2 b) in [Ser3, II.2.3]:

$$\mathrm{Vol}(\mathrm{PGL}_2(R_v) \backslash \mathbb{X}_v) = (q - 1) \mathrm{Vol}(\mathrm{GL}_2(R_v) \backslash \mathbb{X}_v) = 2 \zeta_K(-1). \quad (15.7)$$

If $K = \mathbb{F}_q(Y)$ is the rational function field over \mathbb{F}_q and if we consider the valuation at infinity $v = v_\infty$ of K , then the Nagao lattice⁷ $\Gamma_v = \mathrm{PGL}_2(\mathbb{F}_q[Y])$ acts transitively on $\mathbb{P}^1(K)$. Its quotient graph of groups $\Gamma_v \backslash \mathbb{X}_v$ is the following *modular ray* (with associated edge-indexed graph)



where $\Gamma_{-1} = \mathrm{PGL}_2(\mathbb{F}_q)$, $\Gamma'_0 = \Gamma_0 \cap \Gamma_{-1}$ and, for every $n \in \mathbb{N}$,

$$\Gamma_n = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{F}_q[Y]) : a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[Y], \deg b \leq n + 1 \right\}.$$

Note that even though $\mathrm{PGL}_2(K_v)$ has inversions on \mathbb{X}_v , its subgroup $\Gamma_v = \mathrm{PGL}_2(R_v)$ acts without inversion on \mathbb{X}_v (see for instance [Ser3, II.1.3]). In particular, the quotient graph $\Gamma_v \backslash \mathbb{X}_v$ is then well defined.

⁵See for instance [BasL] for a profusion of geometrically infinite lattices in simplicial trees.

⁶See Section 2.7 for a definition.

⁷This lattice was studied by Nagao in [Nag], see also [Moz, BasL]. It is called the modular group in [Wei2].

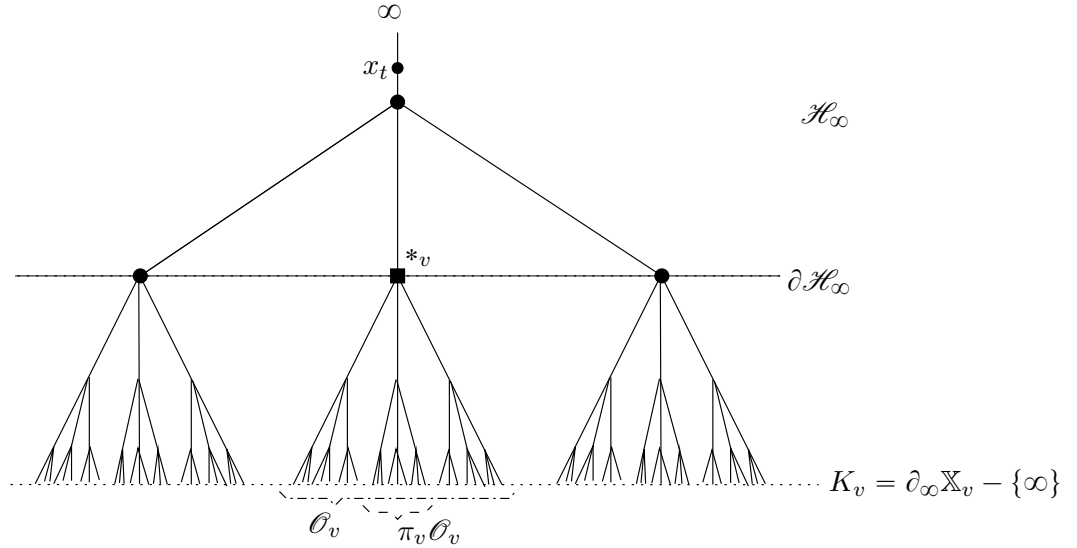
15.3 Computations of measures for Bruhat-Tits trees

In this Section, we compute explicit expressions for the skinning measures of horoballs and geodesic lines, and for the Bowen-Margulis measures, when considering lattices of Bruhat-Tits trees. See [PaP16a, Section 7] and [PaP16b, Section 4] for analogous computations in the real and complex hyperbolic spaces respectively, and [BrP1] for related computations in the tree case.

Let (K_v, v) be as in the beginning of Section 15.1. Let Γ be a lattice of the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) . Since \mathbb{X}_v is regular of degree $q_v + 1$, the critical exponent of Γ is

$$\delta_\Gamma = \ln q_v . \tag{15.8}$$

We normalise the Patterson density $(\mu_x)_{x \in V\mathbb{X}_v}$ of Γ as follows. Let \mathcal{H}_∞ be the horoball in \mathbb{X}_v centred at ∞ whose associated horosphere passes through $*_v$. Let $t \mapsto x_t$ be the geodesic ray in \mathbb{X}_v such that $x_0 = *_v$ and which converges to ∞ .



Hamenstädt’s measure⁸ associated with \mathcal{H}_∞

$$\mu_{\mathcal{H}_\infty} = \lim_{t \rightarrow +\infty} e^{\delta_\Gamma t} \mu_{x_t} = \lim_{t \rightarrow +\infty} q_v^t \mu_{x_t}$$

is a Radon measure on $\partial_\infty \mathbb{X}_v - \{\infty\} = K_v$, invariant under all isometries of \mathbb{X}_v preserving \mathcal{H}_∞ , since Γ is a lattice. Hence it is invariant under the translations by the elements of K_v . By the uniqueness property of Haar measures, $\mu_{\mathcal{H}_\infty}$ is a constant multiple of the chosen Haar measure⁹ of K_v , and we normalise the Patterson density $(\mu_x)_{x \in V\mathbb{X}_v}$ so that

$$\mu_{\mathcal{H}_\infty} = \mathrm{Haar}_{K_v} . \tag{15.9}$$

We summarise the various measure computations in the following result.

Proposition 15.2. *Let Γ be a lattice of the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) , with Patterson density normalised as above.*

⁸See Equation (7.5).

⁹Recall that we normalise the Haar measure of $(K_v, +)$ such that $\mathrm{Haar}_{K_v}(\mathcal{O}_v) = 1$.

(1) The outer/inner skinning measures of the singleton $\{*_v\}$ are given by

$$d\tilde{\sigma}_{\{*_v\}}^{\pm}(\rho) = d\mu_{*_v}(\rho_{\pm}) = \max\{1, |\rho_{\pm}|_v\}^{-2} d\text{Haar}_{K_v}(\rho_{\pm})$$

on the set of $\rho \in \partial_{\pm}^1\{*_v\}$ such that $\rho_{\pm} \neq \infty$.

(2) The total mass of the Patterson density is

$$\|\mu_x\| = \frac{q_v + 1}{q_v}$$

for all $x \in V\mathbb{X}_v$.

(3) The skinning measure of the horoball \mathcal{H}_{∞} is the projection of the Haar measure of K_v : For all $\rho \in \partial_{\pm}^1\mathcal{H}_{\infty}$, we have

$$d\tilde{\sigma}_{\mathcal{H}_{\infty}}^{\pm}(\rho) = d\mu_{\mathcal{H}_{\infty}}(\rho_{\pm}) = d\text{Haar}_{K_v}(\rho_{\pm}).$$

(4) If ∞ is a bounded parabolic fixed point of Γ , with Γ_{∞} its stabiliser in Γ , if $\mathcal{D} = (\gamma\mathcal{H}_{\infty})_{\gamma \in \Gamma/\Gamma_{\infty}}$, we have

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \text{Haar}_{K_v}(\Gamma_{\infty} \backslash K_v) = \text{Vol}(\Gamma_{\infty} \backslash \partial\mathcal{H}_{\infty}).$$

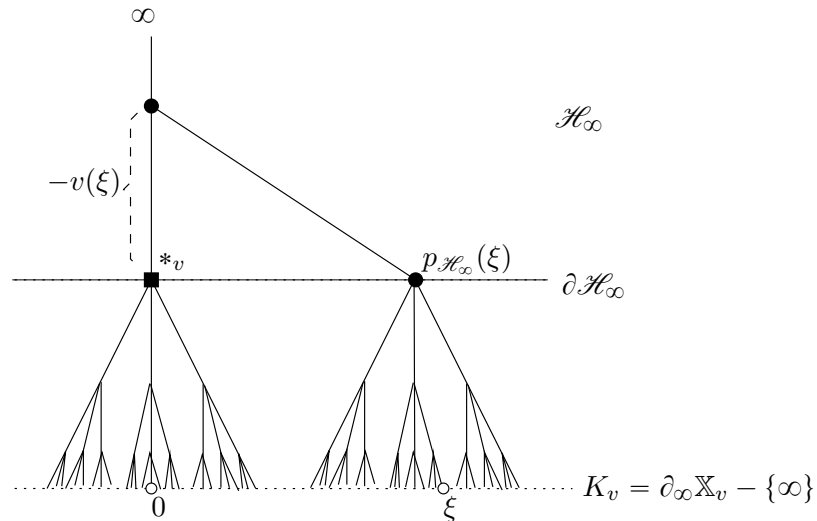
(5) Let L be a geodesic line in \mathbb{X}_v with endpoints $L_{\pm} \in K_v = \partial_{\infty}\mathbb{X}_v - \{\infty\}$. Then on the set of $\rho \in \partial_{+}^1 L$ such that $\rho_{+} \in K_v = \partial_{\infty}\mathbb{X}_v - \{\infty\}$ and $\rho_{+} \neq L_{\pm}$, the outer skinning measure of L is

$$d\tilde{\sigma}_L^{+}(\rho) = \frac{|L_{+} - L_{-}|_v}{|\rho_{+} - L_{-}|_v |\rho_{+} - L_{+}|_v} d\text{Haar}_{K_v}(\rho_{+}).$$

(6) Let L be a geodesic line in \mathbb{X}_v , let Γ_L be the stabiliser in Γ of L , and assume that $\Gamma_L \backslash L$ has finite length. Then with $\mathcal{D} = (\gamma L)_{\gamma \in \Gamma/\Gamma_L}$, we have

$$\|\sigma_{\mathcal{D}}^{\pm}\| = \frac{q_v - 1}{q_v} \text{Vol}(\Gamma_L \backslash L).$$

Proof. (1) For every $\xi \in K_v$, by the description of the geodesic lines in the Bruhat-Tits tree \mathbb{X}_v starting from ∞ given in Section 15.1, we have $\xi \in \mathcal{O}_v$ if and only if $P_{\mathcal{H}_{\infty}}(\xi) = *_v$.¹⁰



¹⁰Recall that $P_{\mathcal{H}_{\infty}} : \partial_v\mathbb{X}_v - \{\infty\} \rightarrow \partial\mathcal{H}_{\infty}$ is the closest point map to the horoball \mathcal{H}_{∞} , see Section 2.5.

For every $\xi \in K_v - \mathcal{O}_v$, by Equations (15.2) and (2.8), we have

$$|\xi|_v = d_{\mathcal{H}_\infty}(0, \xi)^{\ln q_v} = q_v^{\frac{1}{2} d(*_v, P_{\mathcal{H}_\infty}(\xi))}. \quad (15.10)$$

On the set of geodesic rays $\rho \in \partial_\pm^1 \{*_v\}$ such that $\rho_\pm \neq \infty$, by Equation (7.2), by the last claim of Proposition 7.2, by Equation (15.8),¹¹ since $P_{\mathcal{H}_\infty}(\rho_\pm)$ belongs to the geodesic ray $[*_v, \rho_\pm[$ (even when $\rho_\pm \in \mathcal{O}_v$), and by Equation (15.9), we have

$$\begin{aligned} d\tilde{\sigma}_{\{*_v\}}^\pm(\rho) &= d\mu_{*_v}(\rho_\pm) = e^{C_{\rho_\pm}(P_{\mathcal{H}_\infty}(\rho_\pm), *_v)} d\mu_{\mathcal{H}_\infty}(\rho_\pm) \\ &= e^{\delta_\Gamma \beta_{\rho_\pm}(P_{\mathcal{H}_\infty}(\rho_\pm), *_v)} d\mu_{\mathcal{H}_\infty}(\rho_\pm) \\ &= q_v^{-d(P_{\mathcal{H}_\infty}(\rho_\pm), *_v)} d\text{Haar}_{K_v}(\rho_\pm). \end{aligned}$$

Therefore, if $\rho \in \partial_\pm^1 \{*_v\}$ is such that

$$\rho_\pm \in \mathcal{O}_v = \{\xi \in K_v : |\xi|_v \leq 1\} = \{\xi \in K_v : P_{\mathcal{H}_\infty}(\xi) = *_v\},$$

then $d\tilde{\sigma}_{\{*_v\}}^\pm(\rho) = d\text{Haar}_{K_v}(\rho_\pm)$. If $\rho_\pm \in K_v - \mathcal{O}_v$, Equation (15.10) gives the claim.

(2) This Assertion follows from Assertion (1) by a geometric series argument, but we give a direct proof.

As Γ is a lattice, the family $(\mu_x)_{x \in V\mathbb{X}_v}$ is actually equivariant under $\text{Aut}(\mathbb{X}_v)$,¹² which acts transitively on the vertices of \mathbb{X}_v , and the stabiliser in $\text{Aut}(\mathbb{X}_v)$ of the standard base point $*_v$ acts transitively on the edges starting from $*_v$.

Since \mathbb{X}_v is $(q_v + 1)$ -regular, since the set of points at infinity of the geodesic rays starting from $*_v$, whose initial edge has endpoint $0 \in \text{lk}(*_v) = \mathbb{P}_1(k_v)$, is equal to $\pi_v \mathcal{O}_v$, since all geodesic lines from $\infty \in \partial_\infty \mathbb{X}_v$ to points of $\pi_v \mathcal{O}_v \subset \partial_\infty \mathbb{X}_v$ pass through $*_v$, and by the normalisation of the Patterson density and of the Haar measure, we have

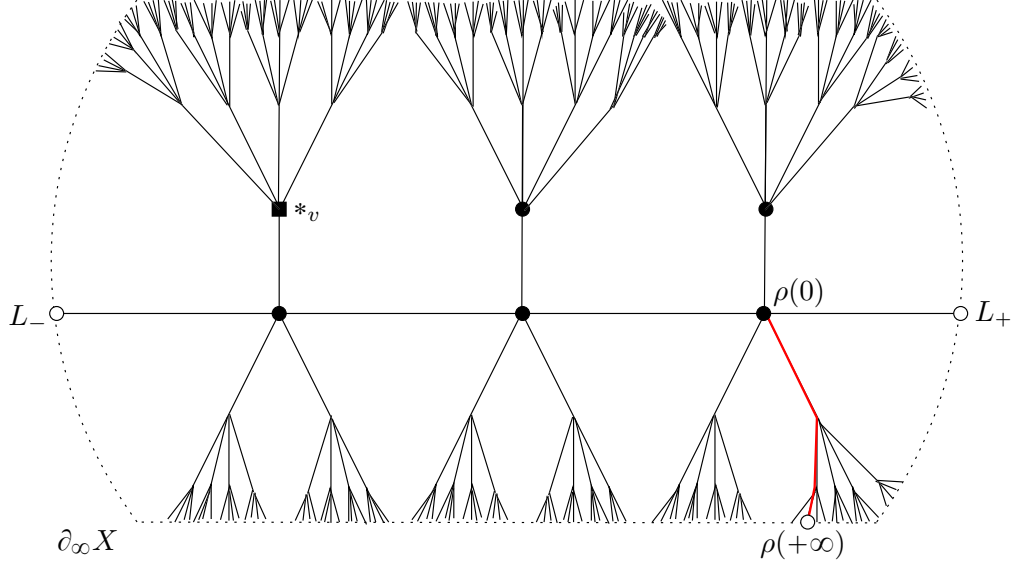
$$\begin{aligned} \|\mu_{*_v}\| &= (q_v + 1) \mu_{*_v}(\pi_v \mathcal{O}_v) = (q_v + 1) \mu_{\mathcal{H}_\infty}(\pi_v \mathcal{O}_v) = (q_v + 1) \text{Haar}_{K_v}(\pi_v \mathcal{O}_v) \\ &= \frac{q_v + 1}{q_v} \text{Haar}_{K_v}(\mathcal{O}_v) = \frac{q_v + 1}{q_v}. \end{aligned}$$

(3) This follows from Equation (7.4), and from the normalisation $\mu_{\mathcal{H}_\infty} = \text{Haar}_{K_v}$ of the Patterson density.

(4) This follows from Assertion (3) and from Equation (8.10) (where the normalisation of the Patterson density was different).

¹¹Since the potential is zero, the Gibbs cocycle is the critical exponent times the Busemann cocycle.

¹²See Proposition 4.14 (2).



(5) This follows from Lemma 8.6 applied with $\mathcal{H} = \mathcal{H}_\infty$, from Equations (15.9), (15.2) and (15.8).

(6) This follows from Assertion (2) and Equation (8.11) (where the normalisation of the Patterson density was different), since \mathbb{X}_v is $(q_v + 1)$ -regular:

$$\|\sigma_{\mathcal{D}}^\pm\| = \|\mu_{*v}\| \frac{q_v - 1}{q_v + 1} \text{Vol}(\Gamma_L \backslash L) = \frac{q_v - 1}{q_v} \text{Vol}(\Gamma_L \backslash L). \quad \square$$

We now turn to measure computations for arithmetic lattices Γ in \mathbb{X}_v in the function field case. We still assume that the Patterson density of Γ is normalised so that $\mu_{\mathcal{H}_\infty}(\mathcal{O}_v) = 1$, and we denote by m_{BM} the Bowen-Margulis measure of Γ associated with this choice of Patterson density.

Proposition 15.3. *Let K be a function field over \mathbb{F}_q and let v be a valuation of K . Let Γ be a finite index subgroup of $\Gamma_v = \text{PGL}_2(R_v)$, with Patterson density normalised such that $\mu_{\mathcal{H}_\infty} = \text{Haar}_{K_v}$.*

(1) *We have*

$$\|m_{\text{BM}}\| = \frac{(q_v + 1) [\Gamma_v : \Gamma]}{q_v} \text{Vol}(\Gamma_v \backslash \mathbb{X}_v) = \frac{2 (q_v + 1) \zeta_K(-1) [\Gamma_v : \Gamma]}{q_v},$$

and if $K = \mathbb{F}_q(Y)$ and $v = v_\infty$ is the valuation at infinity of $\mathbf{C} = \mathbb{P}_1$, then

$$\|m_{\text{BM}}\| = \frac{2 [\text{PGL}_2(\mathbb{F}_q[Y]) : \Gamma]}{q(q-1)^2}.$$

(2) *Let Γ_∞ be the stabiliser in Γ of $\infty \in \partial_\infty \mathbb{X}_v$, and let $\mathcal{D} = (\gamma \mathcal{H}_\infty)_{\gamma \in \Gamma/\Gamma_\infty}$. We have*

$$\|\sigma_{\mathcal{D}}^\pm\| = \frac{q^{g-1} [(\Gamma_v)_\infty : \Gamma_\infty]}{q-1}.$$

Proof. (1) Recall that $\Gamma_v = \mathrm{PGL}_2(R_v)$ acts without inversion on \mathbb{X}_v . By Equation (8.3) (which uses a different normalisation of the Patterson density of Γ), and by Proposition 15.2 (2), we have

$$\|m_{\mathrm{BM}}\| = \|\mu_{*v}\|^2 \frac{q_v}{q_v + 1} \mathrm{Vol}(\Gamma \backslash \mathbb{X}_v) = \frac{(q_v + 1) [\Gamma_v : \Gamma]}{q_v} \mathrm{Vol}(\Gamma_v \backslash \mathbb{X}_v).$$

The first claim of Assertion (1) hence follows from Equation (15.7).

If $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, then the second claim of Assertion (1) follows either from the first claim where the value of $\zeta_K(-1)$ is given by Equation (14.5), or from the fact that $q_v = q$ and that the covolume $\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{F}_q[Y]) \backslash \mathbb{X}_{v_\infty})$ of the Nagao lattice $\mathrm{PGL}_2(\mathbb{F}_q[Y])$ is

$$\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{F}_q[Y]) \backslash \mathbb{X}_{v_\infty}) = \frac{2}{(q-1)(q^2-1)}, \quad (15.11)$$

as an easy geometric series computation shows using the description of the modular ray in Section 15.2 (see also [BasL, Sect. 10.2]).

(2) Let us prove that

$$\mathrm{Haar}_{K_v}((\Gamma_v)_\infty \backslash K_v) = \frac{q^{g-1}}{q-1}. \quad (15.12)$$

The result then follows by Proposition 15.2 (4) since

$$\|\sigma_{\mathcal{D}}^\pm\| = \mathrm{Haar}_{K_v}(\Gamma_\infty \backslash K_v) = [(\Gamma_v)_\infty : \Gamma_\infty] \mathrm{Haar}_{K_v}((\Gamma_v)_\infty \backslash K_v).$$

The stabiliser of $\infty = [1 : 0]$ in Γ_v acts on K_v exactly by the set of transformations $z \mapsto az + b$ with $a \in (R_v)^\times$ and $b \in R_v$. Since $(R_v)^\times = (\mathbb{F}_q)^\times$ (see Equation (14.3)) acts freely by left translations on $(K_v - R_v)/R_v$, and by Lemma 14.4, we have

$$\mathrm{Haar}_{K_v}((\Gamma_v)_\infty \backslash K_v) = \frac{1}{q-1} \mathrm{Haar}_{K_v}(K_v/R_v) = \frac{q^{g-1}}{q-1}.$$

This proves Equation (15.12). □

15.4 Exponential decay of correlation and error terms for arithmetic quotients of Bruhat-Tits trees

As in the beginning of Section 15.1, let K_v be a non-Archimedean local field, with valuation v , valuation ring \mathcal{O}_v , choice of uniformiser π_v , and residual field k_v of order q_v . Let Γ be a lattice of the Bruhat-Tits tree \mathbb{X}_v of (PGL_2, K_v) . In this Section, we discuss the error terms estimates that we will use in Part III.

In part in order to simplify the references, we start by summarizing in the next statement the only results from the geometric Part II of this book, on geometric equidistribution and counting problems, that we will use in this algebraic Part III. We state it with the normalisation which will be useful there (see Section 15.3).

Theorem 15.4. *Let Γ be a lattice of \mathbb{X}_v whose length spectrum L_Γ is equal to $2\mathbb{Z}$. Assume that the Patterson density of Γ is normalised so that $\|\mu_x\| = \frac{q_v+1}{q_v}$ for every $x \in V\mathbb{X}_v$. Let \mathbb{D}^\pm*

be nonempty proper simplicial subtrees of \mathbb{X}_v with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ , such that the families $\mathcal{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite in \mathbb{X}_v . If the measure $\sigma_{\mathcal{D}^+}^-$ is nonzero and finite, then

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} \Delta_{\alpha_{e, \gamma}^-} = \tilde{\sigma}_{\mathbb{D}^-}^+,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X}_v$.

Furthermore, if Γ is geometrically finite, then for every $\beta \in]0, 1]$, there exists an error term for this equidistribution claim when evaluated on $\tilde{\psi} \in \mathcal{C}_c^\beta(\check{\mathcal{G}}\mathbb{X})$ of the form $O(\|\tilde{\psi}\|_\beta e^{-\kappa n})$ for some $\kappa > 0$.

As recalled at the end of Section 2.7, arithmetic lattices in $\text{PGL}_2(\mathbb{K}_v)$ are geometrically finite, see [Lub1]. We will hence be able to use the error term in Theorem 15.4 in particular when

- K_v is the completion of a function field K over \mathbb{F}_q with respect to a (normalised discrete) valuation v of K and Γ is a finite index subgroup of $\text{PGL}_2(R_v)$ with R_v the affine function ring associated with v ,¹³ as in Chapters 16 and 19, and in Sections 17.2 and 18.2;
- when $K_v = \mathbb{Q}_p$ and Γ is an arithmetic lattice in $\text{PGL}_2(K_v)$ derived from a quaternion algebra, see Sections 17.3 and 18.2.

Proof. In order to prove the first claim, we apply Corollary 11.11 with $\mathbb{X} = \mathbb{X}_v$ and $p = q = q_v$. Since $L_\Gamma = 2\mathbb{Z}$, the lattice Γ leaves invariant the partition of $V\mathbb{X}_v$ into vertices at even distance from a base point x_0 and vertices at odd distance from x_0 . Since the Patterson density is now normalised so that $\|\mu_{x_0}\| = \frac{q_v+1}{q_v}$ (instead of $\|\mu_{x_0}\| = \frac{q_v+1}{\sqrt{q_v}}$ in Corollary 11.11), the skinning measures $\sigma_{\mathcal{D}^\pm}^\mp$ are now $\frac{1}{\sqrt{q_v}}$ times the ones in the statement of Corollary 11.11. Hence the second assertion of Corollary 11.11 gives

$$\lim_{n \rightarrow +\infty} \frac{q_v^2 - 1}{2 q_v^2} \frac{\text{TVol}(\Gamma \backslash \mathbb{X}_v)}{\sqrt{q_v} \|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} \Delta_{\alpha_{e, \gamma}^-} = \sqrt{q_v} \tilde{\sigma}_{\mathbb{D}^-}^+.$$

By Equation (2.16), we have

$$\text{TVol}(\Gamma \backslash \mathbb{X}_v) = (q_v + 1) \text{Vol}(\Gamma \backslash \mathbb{X}_v).$$

The first claim follows.

The last claim concerning error terms follows from Remark (ii) following the statement of Theorem 12.17. \square

In the last four Chapters 16, 17, 18 and 19 of this book, we will need to push to infinity the measures appearing in the statement of Theorem 15.4. We regroup in the following two lemmas the necessary control tools for such a pushing.

The first one is a metric estimate on the extension of geodesic segments to geodesic rays.

Lemma 15.5. *Let X be a geodesically complete proper $\text{CAT}(-1)$ space, let $T \geq 1$, and let $\alpha \in \check{\mathcal{G}}X$ be a generalised geodesic line which is isometric exactly on $[0, T]$. For every generalised*

¹³See Section 14.2 for definitions

geodesic line $\rho \in \check{\mathcal{G}}X$ which is isometric exactly on $[0, +\infty[$, such that $\rho|_{[0,T]} = \alpha|_{[0,T]}$, we have

$$d(\alpha, \rho) = \frac{e^{-2T}}{4} < 1,$$

and hence for all $\beta \in]0, 1]$ and $\tilde{\psi} \in \mathcal{C}_b^\beta(\check{\mathcal{G}}X)$,

$$|\tilde{\psi}(\alpha) - \tilde{\psi}(\rho)| \leq \frac{e^{-2\beta T}}{4\beta} \|\tilde{\psi}\|_\beta.$$

With the notation of Theorem 15.4, we will use this result when $X = |\mathbb{X}_v|_1$ is the geometric realisation of the simplicial tree \mathbb{X}_v , $\alpha = \alpha_{e,\gamma}^-$ is¹⁴ the common perpendicular between \mathbb{D}^- and $\gamma\mathbb{D}^+$ for $\gamma \in \Gamma$ (when it exists), and $\rho = \rho_\gamma$ is any extension of α to a geodesic ray, or rather to a generalised geodesic line isometric exactly on $[0, +\infty[$. Under the assumptions of Theorem 15.4, we have

$$\frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} \Delta_{\rho_\gamma} \stackrel{*}{\sim} \tilde{\sigma}_{\mathbb{D}^-}^+, \quad (15.13)$$

with, if Γ is geometrically finite, an error term when evaluated on $\tilde{\psi} \in \mathcal{C}_c^\beta(\check{\mathcal{G}}\mathbb{X})$ of the form $O(\|\tilde{\psi}\|_\beta e^{-\kappa n})$ for some $\kappa > 0$ small enough (depending in particular on $\beta \in]0, 1]$).

Proof. By Equation (2.4) defining the distance on $\check{\mathcal{G}}X$, we have, since $d(\alpha(t), \rho(t)) = 0$ for all $t \in]-\infty, T]$ and $d(\alpha(t), \rho(t)) = t - T$ otherwise,

$$d(\alpha, \rho) = \int_T^{+\infty} (t - T) e^{-2t} dt = e^{-2T} \int_0^{+\infty} u e^{-2u} du = \frac{e^{-2T}}{4}.$$

The result follows. \square

The second lemma is a metric estimate on the map which associates to a geodesic ray its point at infinity. We start by giving some definitions.

Let X be a geodesically complete proper CAT(−1) space, and let D be a nonempty proper closed convex subset of X . The *distance-like map* $d_D : (\partial_\infty X - \partial_\infty D)^2 \rightarrow [0, +\infty[$ associated with D is defined in [HeP4, §2.2] as follows: For $\xi, \xi' \in \partial_\infty X - \partial_\infty D$, let $\xi_t, \xi'_t : [0, +\infty[\rightarrow X$ be the geodesic rays starting at the closest points $P_D(\xi), P_D(\xi')$ to ξ, ξ' on D and converging to ξ, ξ' as $t \rightarrow \infty$. Let

$$d_D(\xi, \xi') = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(\xi_t, \xi'_t) - t}. \quad (15.14)$$

The distance-like map d_D is invariant by the diagonal action of the isometries of X preserving D . If D consists of a single point x , then d_D is the visual distance¹⁵ d_x on $\partial_\infty X$ based at x . If D is a horoball with point at infinity ξ_0 , then d_D is Hamenstädt's distance¹⁶ on $\partial_\infty X - \{\xi_0\}$. As seen in [HeP4, §2.2, Ex. (4)], if X is a metric tree, then

$$d_D(\xi, \xi') = \begin{cases} e^{\frac{1}{2}d(P_D(\xi), P_D(\xi'))} > 1 & \text{if } P_D(\xi) \neq P_D(\xi') \\ d_x(\xi, \xi') = e^{-d(x, y)} \leq 1 & \text{if } P_D(\xi) = P_D(\xi') = x \\ & \text{and } [x, \xi[\cap [x, \xi'[= [x, y]. \end{cases}$$

¹⁴the generalised geodesic line isometric exactly on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$ paramtrising

¹⁵See Equation (2.1).

¹⁶See Equation (2.8).

In particular, although it is not an actual distance on its whole domain $\partial_\infty X - \partial_\infty D$, the map d_D is locally a distance, and we can define with the standard formula the β -Hölder-continuity of maps with values in $(\partial_\infty X - \partial_\infty D, d_D)$ and the β -Hölder-norm of a function defined on $(\partial_\infty X - \partial_\infty D, d_D)$. From now on, we endow $\partial_\infty X - \partial_\infty D$ with the distance-like map d_D .

Proposition 15.6. *Let \mathbb{X} be a locally finite simplicial tree without terminal vertices, and let \mathbb{D} be a proper nonempty simplicial subtree of \mathbb{X} . The homeomorphism $\partial^+ : \partial_+^1 \mathbb{D} \rightarrow (\partial_\infty \mathbb{X} - \partial_\infty \mathbb{D})$ defined by $\rho \mapsto \rho_+$ is $\frac{1}{2}$ -Hölder-continuous, and for all $\beta \in]0, 1]$ and $\psi \in \mathcal{C}_b^\beta(\partial_\infty \mathbb{X} - \partial_\infty \mathbb{D})$, the map $\psi \circ \partial^+ : \partial_+^1 \mathbb{D} \rightarrow \mathbb{R}$ is bounded and $\frac{\beta}{2}$ -Hölder-continuous, with*

$$\|\psi \circ \partial^+\|_{\frac{\beta}{2}} \leq (1 + 2^{\frac{\beta}{2}+1}) \|\psi\|_\beta .$$

With the notation of Theorem 15.4, using the claim following the statement of Lemma 15.5, we will use this result when $\mathbb{X} = \mathbb{X}_v$ and $\mathbb{D} = \mathbb{D}^-$. Under the assumptions of Theorem 15.4, with ρ_γ any extension to a geodesic ray of the common perpendicular $\alpha_{e, \gamma}^-$ between \mathbb{D}^- and $\gamma \mathbb{D}^+$ for $\gamma \in \Gamma$, since pushing forward measures on $\partial_+^1 \mathbb{D}^-$ by the homeomorphism ∂^+ is continuous, we have by Equation (15.13)

$$\frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \Delta_{(\rho_\gamma)_+} \stackrel{*}{\rightarrow} (\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ . \quad (15.15)$$

If Γ is geometrically finite, for all $\beta \in]0, 1]$ and $\psi \in \mathcal{C}_c^\beta(\partial_\infty \mathbb{X}_v - \partial_\infty \mathbb{D}^-)$, using the error term in Equation (15.13) with regularity $\frac{\beta}{2}$ when evaluated on $\tilde{\psi} = \psi \circ \partial^+ \in \mathcal{C}_c^{\frac{\beta}{2}}(\tilde{\mathcal{G}} \mathbb{X}_v)$, we have by the last claim of Proposition 15.6 an error term in Equation (15.15) evaluated on ψ of the form $O(\|\psi\|_\beta e^{-\kappa n})$ for some $\kappa > 0$ small enough.

Proof. Let us prove that for every $\rho, \rho' \in \partial_+^1 \mathbb{D}$, if $d(\rho, \rho') \leq 1$, then $\rho(0) = \rho'(0)$, and

$$d_{\mathbb{D}}(\rho_+, \rho'_+) = \sqrt{2} d(\rho, \rho')^{\frac{1}{2}} . \quad (15.16)$$

This proves that the map ∂^+ is $\frac{1}{2}$ -Hölder-continuous. We may assume that $\rho \neq \rho'$.

Let $\rho, \rho' \in \partial_+^1 \mathbb{D}$. If $\rho(0) \neq \rho'(0)$, then the images of ρ and ρ' are disjoint and their connecting segment in the tree \mathbb{X} joins $\rho(0)$ and $\rho'(0)$; hence for every $t \in [0, +\infty[$, we have

$$d(\rho(t), \rho'(t)) = d(\rho(0), \rho'(0)) + d(\rho(t), \rho(0)) + d(\rho'(t), \rho'(0)) \geq 1 + 2t .$$

Thus

$$\begin{aligned} d(\rho, \rho') &= \int_{-\infty}^0 d(\rho(0), \rho'(0)) e^{2t} dt + \int_0^{+\infty} d(\rho(t), \rho'(t)) e^{-2t} dt \\ &\geq \int_{-\infty}^0 e^{2t} dt + \int_0^{+\infty} (1 + 2t) e^{-2t} dt > 2 \int_0^{+\infty} e^{-2t} dt = 1 . \end{aligned}$$

Assume that $x = \rho(0) = \rho'(0)$ and let n be the length of the intersection of ρ and ρ' . Then

$$d_{\mathbb{D}}(\rho_+, \rho'_+) = d_x(\rho_+, \rho'_+) = \lim_{t \rightarrow +\infty} e^{\frac{1}{2} d(\rho(t), \rho'(t)) - t} = e^{-n} .$$

Furthermore, since $\rho(t) = \rho'(t)$ for $t \leq n$ and $d(\rho(t), \rho'(t)) = 2(t - n)$ otherwise, we have, using the change of variables $u = 2(t - n)$,

$$d(\rho, \rho') = \int_n^{+\infty} 2(t - n) e^{-2t} dt = e^{-2n} \int_0^{+\infty} u e^{-u} \frac{du}{2} = \frac{e^{-2n}}{2}.$$

This proves Equation (15.16).

Let $\beta \in]0, 1]$ and $\psi \in \mathcal{C}_b^\beta(\partial_\infty \mathbb{X} - \partial_\infty \mathbb{D})$. We have $\|\psi \circ \partial^+\|_\infty = \|\psi\|_\infty$ since ∂^+ is a homeomorphism, and, by Equation (15.16),

$$\begin{aligned} \|\psi \circ \partial^+\|'_{\frac{\beta}{2}} &= \sup_{\rho, \rho' \in \partial_+^1 \mathbb{D}, 0 < d(\rho, \rho') \leq 1} \frac{|\psi \circ \partial^+(\rho) - \psi \circ \partial^+(\rho')|}{d(\rho, \rho')^{\frac{\beta}{2}}} \\ &\leq \sup_{\rho, \rho' \in \partial_+^1 \mathbb{D}, 0 < d(\rho, \rho') \leq \frac{1}{2}} \frac{|\psi \circ \partial^+(\rho) - \psi \circ \partial^+(\rho')|}{d(\rho, \rho')^{\frac{\beta}{2}}} + \frac{2\|\psi \circ \partial^+\|_\infty}{2^{-\frac{\beta}{2}}} \\ &\leq \sup_{\substack{\xi, \xi' \in \partial_\infty \mathbb{X} - \partial_\infty \mathbb{D} \\ 0 < d_{\mathbb{D}}(\xi, \xi') \leq 1}} \frac{|\psi(\xi) - \psi(\xi')|}{2^{-\frac{\beta}{2}} d_{\mathbb{D}}(\xi, \xi')^\beta} + 2^{\frac{\beta}{2}+1} \|\psi\|_\infty \\ &\leq 2^{\frac{\beta}{2}+1} \|\psi\|_\beta. \end{aligned}$$

Since $\|\psi \circ \partial^+\|_{\frac{\beta}{2}} = \|\psi \circ \partial^+\|_\infty + \|\psi \circ \partial^+\|'_{\frac{\beta}{2}}$, this proves the last claim of Proposition 15.6. \square

We conclude Section 15.4 by giving a purely algebraic control of error terms, under the stronger regularity requirement on functions of being locally constant. We assume until the end of this Section that the lattice Γ is contained in $G_v = \mathrm{PGL}_2(K_v)$.

The group G_v acts (on the left) on the complex vector space of maps ψ from $\Gamma \backslash G_v$ to \mathbb{R} , by right translation on the source: For every $g \in G_v$, we have $g\psi : x \mapsto \psi(xg)$. A function $\psi : \Gamma \backslash G_v \rightarrow \mathbb{R}$ is *algebraically locally constant* if there exists a compact-open subgroup U of $\mathrm{PGL}_2(\mathcal{O}_v)$ which leaves ψ invariant:

$$\forall g \in U, \quad g\psi = \psi,$$

or equivalently, if ψ is constant on each orbit of U under the right action of G_v on $\Gamma \backslash G_v$. Note that ψ is then continuous, since the orbits of U are compact-open subsets. We define

$$d_\psi = \dim(\mathrm{Vect}_{\mathbb{R}}(\mathrm{PGL}_2(\mathcal{O}_v)\psi))$$

as the dimension of the complex vector space generated by the images of ψ under the elements of $\mathrm{PGL}_2(\mathcal{O}_v)$, which is finite, and even satisfies

$$d_\psi \leq [\mathrm{PGL}_2(\mathcal{O}_v) : U].$$

We define the *alc-norm* of every bounded algebraically locally constant map $\psi : \Gamma \backslash G_v \rightarrow \mathbb{R}$ by

$$\|\psi\|_{\mathrm{alc}} = \sqrt{d_\psi} \|\psi\|_\infty.$$

Though the alc-norm does not satisfy the triangle inequality, we have $\|\lambda\psi\|_{\mathrm{alc}} = |\lambda| \|\psi\|_{\mathrm{alc}}$ for every $\lambda \in \mathbb{R}$. We denote by $\mathrm{alc}(\Gamma \backslash G_v)$ the vector space of bounded algebraically locally constant maps ψ from $\Gamma \backslash G_v$ to \mathbb{R} .

For every $n \in \mathbb{N}$, let U_n be the compact-open subgroup of $\mathrm{PGL}_2(\mathcal{O}_v)$ which is the kernel of the morphism $\mathrm{PGL}_2(\mathcal{O}_v) \rightarrow \mathrm{PGL}_2(\mathcal{O}_v/\pi_v^n \mathcal{O}_v)$ of reduction modulo π_v^n . Note that any compact-open subgroup U of $\mathrm{PGL}_2(\mathcal{O}_v)$ contains U_n for some $n \in \mathbb{N}$. Hence $\psi : \Gamma \backslash G_v \rightarrow \mathbb{R}$ is algebraically locally constant if and only if there exists $n \in \mathbb{N}$ such that ψ is constant on each right orbit of U_n . For every $n \in \mathbb{N}$, since the order of $\mathrm{PGL}_2(\mathcal{O}_v/\pi_v^n \mathcal{O}_v)$ is at most the order of $(\mathcal{O}_v/\pi_v^n \mathcal{O}_v)^4$, which is q_v^{4n} , if $\psi : \Gamma \backslash G_v \rightarrow \mathbb{R}$ is constant on each right orbit of U_n , then

$$\|\psi\|_{\mathrm{alc}} \leq q_v^{2n} \|\psi\|_{\infty}. \quad (15.17)$$

The next result is an algebraic version of the error term statement in Theorem 15.4 (assuming for simplicity that $L_\Gamma = \mathbb{Z}$), which uses¹⁷ a stronger assumption on Γ , and obtains a weaker regularity (locally constant instead of Hölder-continuous, see Section 3.1 for definitions and notation). We will not use it in this book, but its version with $L_\Gamma = 2\mathbb{Z}$ is used in the announcement [BrPP] which only considers the locally constant regularity.

Theorem 15.7. *Let K_v be the completion of a function field K over \mathbb{F}_q with respect to a valuation v of K and let Γ be a nonuniform lattice of G_v with $L_\Gamma = \mathbb{Z}$. Then there exists $\kappa > 0$ such that for every $\epsilon \in]0, 1]$ and every ϵ -locally constant map $\tilde{\psi} : \tilde{\mathcal{G}}_{\mathbb{X}_v} \rightarrow \mathbb{R}$, we have, as $n \rightarrow +\infty$,*

$$\begin{aligned} & \frac{(q_v - 1)}{(q_v + 1) q_v} \frac{\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \tilde{\psi}(\alpha_{\epsilon, \gamma}^-) \\ &= \int_{\tilde{\mathcal{G}}_{\mathbb{X}_v}} \tilde{\psi} d\tilde{\sigma}_{\mathbb{D}^-}^+ + O(\|\tilde{\psi}\|_{\epsilon \mathrm{lc}, \ln q_v} e^{-\kappa n}). \end{aligned}$$

Proof. This result follows by replacing in the proof of Theorem 12.17 (or rather Remark (ii) following its statement) the use of the exponential decay of β -Hölder correlations given by Corollary 9.6 by the following result of decay of correlations under locally constant regularity (which does follow from Corollary 9.6 by Remark 3.2).

Proposition 15.8. *Let K_v be the completion of a function field K over \mathbb{F}_q with respect to a valuation v of K and let Γ be a nonuniform lattice of G_v with $L_\Gamma = \mathbb{Z}$. Then there exist $C, \kappa > 0$ such that for every $\epsilon \in]0, 1]$, for all ϵ -locally constant maps $\phi, \psi : \Gamma \backslash \mathcal{G}_{\mathbb{X}_v} \rightarrow \mathbb{R}$ and $n \in \mathbb{Z}$, we have*

$$\begin{aligned} & \left| \int_{\Gamma \backslash \mathcal{G}_{\mathbb{X}_v}} \phi \circ \mathbf{g}^{-n} \psi dm_{\mathrm{BM}} - \frac{1}{\|m_{\mathrm{BM}}\|} \int_{\Gamma \backslash \mathcal{G}_{\mathbb{X}_v}} \phi dm_{\mathrm{BM}} \int_{\Gamma \backslash \mathcal{G}_{\mathbb{X}_v}} \psi dm_{\mathrm{BM}} \right| \\ & \leq C e^{-\kappa |n|} \|\phi\|_{\epsilon \mathrm{lc}, \ln q_v} \|\psi\|_{\epsilon \mathrm{lc}, \ln q_v}. \end{aligned}$$

Proof. Recall¹⁸ that we have a natural homeomorphism $\Xi : \Gamma g \underline{A}(\mathcal{O}_v) \mapsto \Gamma g \ell^*$ between $\Gamma \backslash G_v / \underline{A}(\mathcal{O}_v)$ and $\Gamma \backslash \mathcal{G}_{\mathbb{X}_v}$. We denote by $p_{\mathcal{G}} : \Gamma \backslash G_v \rightarrow \Gamma \backslash \mathcal{G}_{\mathbb{X}_v}$ the composition map of the canonical projection $(\Gamma \backslash G_v) \rightarrow (\Gamma \backslash G_v / \underline{A}(\mathcal{O}_v))$ and of Ξ . By Equation (15.4), for every $x \in \Gamma \backslash G_v$, we have

$$p_{\mathcal{G}}(x a_v^n) = \mathbf{g}^n p_{\mathcal{G}}(x). \quad (15.18)$$

¹⁷Note that the existence of a nonuniform lattice in $G_v = \mathrm{PGL}_2(K_v)$ forces the characteristic to be positive, see for instance [Lub1].

¹⁸See Section 15.1.

Lemma 15.9. *For every $\epsilon \in]0, 1]$, for every ϵ -locally constant function $\psi : \Gamma \backslash \mathcal{GX}_v \rightarrow \mathbb{R}$, if $n = \lceil -\frac{1}{2} \ln \epsilon \rceil$, then the map $\psi \circ p_{\mathcal{G}} : \Gamma \backslash G_v \rightarrow \mathbb{R}$ is U_n -invariant and*

$$\|\psi \circ p_{\mathcal{G}}\|_{\text{alc}} \leq q_v^2 \|\psi\|_{\epsilon \text{lc}, \ln q_v} . \quad (15.19)$$

Proof. Let ϵ, ψ, n be as in the statement. Let us first prove that if $\ell, \ell' \in \mathcal{GX}_v$ satisfy $\ell_{[-n, +n]} = \ell'_{[-n, +n]}$, then $d(\ell, \ell') \leq \epsilon$.

If $\ell_{[-n, +n]} = \ell'_{[-n, +n]}$, then $d(\ell(t), \ell'(t)) = 0$ for $t \in [-n, n]$ and by the triangle inequality $d(\ell(t), \ell'(t)) \leq 2(|t| - n)$ if $|t| \geq n$, hence

$$\begin{aligned} d(\ell, \ell') &\leq 2 \int_n^{+\infty} 2(t-n) e^{-2t} dt = 2 e^{-2n} \int_0^{+\infty} u e^{-u} \frac{du}{2} = e^{-2n} \\ &\leq e^{-2(-\frac{1}{2} \ln \epsilon)} = \epsilon , \end{aligned}$$

as wanted.

In order to prove that $\psi \circ p_{\mathcal{G}} : \Gamma \backslash G_v \rightarrow \mathbb{R}$ is U_n -invariant, let $x, x' \in \Gamma \backslash G_v$ be such that $x' \in x U_n$. Since U_n acts by the identity map on the ball of radius n in the Bruhat-Tits tree \mathbb{X}_v , the geodesic lines $p_{\mathcal{G}}(x)$ and $p_{\mathcal{G}}(x')$ in $\Gamma \backslash \mathcal{GX}_v$ coincide (at least) on $[-n, n]$. Hence, as we saw in the beginning of the proof, we have $d(p_{\mathcal{G}}(x), p_{\mathcal{G}}(x')) \leq \epsilon$. Therefore $\psi(p_{\mathcal{G}}(x)) = \psi(p_{\mathcal{G}}(x'))$ since ψ is ϵ -locally constant.

Now, using Equation (15.17), we have

$$\begin{aligned} \|\psi \circ p_{\mathcal{G}}\|_{\text{alc}} &\leq q_v^{2n} \|\psi \circ p_{\mathcal{G}}\|_{\infty} \\ &\leq q_v^{2(1-\frac{1}{2} \ln \epsilon)} \|\psi\|_{\infty} = q_v^2 \epsilon^{-\ln q_v} \|\psi\|_{\infty} = q_v^2 \|\psi\|_{\epsilon \text{lc}, \ln q_v} . \quad \square \end{aligned}$$

Now, in order to prove Proposition 15.8, we will use an algebraic result of exponential decay of correlations, Theorem 15.10 (see for instance [AtGP]). We first recall some definitions and notation, useful for its statement.

Recall that the left action of the locally compact unimodular group G_v on the locally compact space \mathcal{GX}_v is continuous and transitive, and that its stabilisers are compact hence unimodular. Since Γ is a lattice, the (Borel, positive, regular) Bowen-Margulis measure \tilde{m}_{BM} on \mathcal{GX}_v is G_v -invariant (see Proposition 4.14 (2)). Hence by [Wei1] (see also [GoP, Lem. 5]), there exists a unique Haar measure on G_v , which disintegrates by the evaluation map $\tilde{p}_{\mathcal{G}} : G_v \rightarrow \mathcal{GX}_v$ defined by $g \mapsto g\ell_*$, with conditional measure on the fiber over $\ell = g\ell_* \in \mathcal{GX}_v$ the probability Haar measure on the stabiliser $g\mathcal{A}(\mathcal{O}_v)g^{-1}$ of ℓ under G_v . Hence, taking the quotient under Γ and normalising in order to have probability measures, if μ_v is the right G_v -invariant probability measure on $\Gamma \backslash G_v$, we have

$$(p_{\mathcal{G}})_* \mu_v = \frac{m_{\text{BM}}}{\|m_{\text{BM}}\|} . \quad (15.20)$$

For every $g \in G_v$, we denote by $R_g : \Gamma \backslash G_v \rightarrow \Gamma \backslash G_v$ the right action of g , and for all bounded continuous functions $\tilde{\psi}, \tilde{\psi}'$ on $\Gamma \backslash G_v$, we define

$$\text{cov}_{\mu_v, g}(\tilde{\psi}, \tilde{\psi}') = \int_{\Gamma \backslash G_v} \tilde{\psi} \circ R_g \tilde{\psi}' d\mu_v - \int_{\Gamma \backslash G_v} \tilde{\psi} d\mu_v \int_{\Gamma \backslash G_v} \tilde{\psi}' d\mu_v .$$

Note that by Equations (15.20) and (15.18), for all bounded continuous functions $\psi, \psi' : \Gamma \backslash \mathcal{G} \mathbb{X}_v \rightarrow \mathbb{R}$ and $n \in \mathbb{Z}$, we have¹⁹

$$\text{cov}_{\frac{m_{\text{BM}}}{\|m_{\text{BM}}\|}, n}(\psi, \psi') = \text{cov}_{\mu_v, a_v^n}(\psi \circ p_{\mathcal{G}}, \psi' \circ p_{\mathcal{G}}). \quad (15.21)$$

Recall that the adjoint representation of $G_v = \text{PGL}_2(K_v)$ is the continuous morphism $G_v \rightarrow \text{GL}(\mathcal{M}_2(K_v))$ defined by $[h] \mapsto \{x \mapsto hxh^{-1}\}$, which is independent of the choice of the representative $h \in \text{GL}_2(K_v)$ of $[h] \in \text{PGL}_2(K_v)$. For every $g \in G_v$, we denote by $|g|_v$ the operator norm of the adjoint representation of g . For instance, recalling that $a_v = \begin{bmatrix} 1 & 0 \\ 0 & \pi_v^{-1} \end{bmatrix}$, we have, for all $n \in \mathbb{Z}$,

$$|a_v^n|_v = q_v^{|n|}. \quad (15.22)$$

We refer for instance to [AtGP] for the following result of exponential decay of correlations.

Theorem 15.10. *Let Γ be a nonuniform lattice of G_v . There exist $C', \kappa' > 0$ such that, for all bounded locally constant functions $\tilde{\psi}, \tilde{\psi}' : \Gamma \backslash G_v \rightarrow \mathbb{R}$ and $g \in G_v$,*

$$|\text{cov}_{\mu_v, g}(\tilde{\psi}, \tilde{\psi}')| \leq C' \|\tilde{\psi}\|_{\text{alc}} \|\tilde{\psi}'\|_{\text{alc}} |g|_v^{-\kappa'}. \quad \square \quad (15.23)$$

Proposition 15.8 follows from this result applied to $\tilde{\psi} = \psi \circ p_{\mathcal{G}}$, $\tilde{\psi}' = \psi' \circ p_{\mathcal{G}}$ and $g = a_v^n$ by using Equations (15.21), (15.19) and (15.22) and by taking $C = C' q_v^A$ and $\kappa = \kappa' \ln q_v$. \square

This concludes the proof of Theorem 15.7. \square

Remark. There is a similar relationship between locally constant functions on K_v in an algebraic sense and the ones in the metric sense.

The additive group $(K_v, +)$ acts on the complex vector space of functions from K_v to \mathbb{R} , by translations on the source: for all $y \in K_v$ and $\psi : K_v \rightarrow \mathbb{R}$, the function $y \cdot \psi$ is equal to $x \mapsto \psi(x + y)$. A map $\psi : K_v \rightarrow \mathbb{R}$ is *algebraically locally constant* if there exists $k \in \mathbb{N}$ such that ψ is invariant under the action of the compact-open subgroup $(\pi_v)^k \mathcal{O}_v$ of K_v , that is, if for all $x \in K_v$ and $y \in (\pi_v)^k \mathcal{O}_v$, we have $\psi(x + y) = \psi(x)$. Note that a locally constant map from K_v to \mathbb{R} is continuous.

For any locally constant function $\psi : K_v \rightarrow \mathbb{R}$, the complex vector space $\text{Vect}_{\mathbb{R}}(\mathcal{O}_v \cdot \psi)$ generated by the images of ψ under the elements of \mathcal{O}_v is finite dimensional. Its dimension d_ψ satisfies, with k as above,

$$d_\psi \leq [\mathcal{O}_v : (\pi_v)^k \mathcal{O}_v] = q_v^k.$$

We define the *alc-norm* of every bounded algebraically locally constant map $\psi : K_v \rightarrow \mathbb{R}$ by

$$\|\psi\|_{\text{alc}} = \sqrt{d_\psi} \|\psi\|_{\infty}.$$

Though the alc-norm does not satisfy the triangle inequality, we have $\|\lambda\psi\|_{\text{alc}} = |\lambda| \|\psi\|_{\text{alc}}$ for every $\lambda \in \mathbb{R}$, and the set of bounded algebraically locally constant maps from K_v to \mathbb{R} is a real vector space.

Actually, a function $\psi : K_v \rightarrow \mathbb{R}$ is algebraically locally constant if and only if it is locally constant. More precisely, for every $\epsilon \in]0, 1]$, since the closed balls of radius q_v^{-k} in K_v are the orbits by translations under $(\pi_v)^k \mathcal{O}_v$, every ϵ -locally constant function $\psi : K_v \rightarrow \mathbb{R}$ is constant under the additive action of $(\pi_v)^k \mathcal{O}_v$ for $k = \lceil \frac{-\ln \epsilon}{\ln q_v} \rceil$, hence²⁰

$$\|\psi\|_{\text{alc}} \leq \|\psi\|_{\epsilon \text{lc}, \frac{1}{2}}.$$

¹⁹See Section 9.2 for a definition of $\text{cov}_{\mu, n}$.

²⁰See Section 3.1 for the notation.

15.5 Geometrically finite lattices with infinite Bowen-Margulis measure

This Section is a digression from the theme of arithmetic applications, in which we use the Nagao lattice defined in Section 15.2 in order to construct a geometrically finite discrete group of automorphisms of a simplicial tree which has infinite Bowen-Margulis measure. This example was promised towards the end of Section 4.4.

We will equivariantly change the lengths of the edges of a simplicial tree \mathbb{X} endowed with a geometrically finite (nonuniform) lattice Γ in order to turn it into a metric tree in which the group Γ remains a geometrically finite lattice, but now has a geometrically finite subgroup with infinite Bowen-Margulis measure. This example is an adaptation of the negatively curved manifold example of [DaOP, §4]. The simplicial example is obtained as a modification of the metric tree example.

Theorem 15.11. *There exists a geometrically finite discrete group of automorphisms of a regular metric tree with infinite Bowen-Margulis measure.*

There exists a geometrically finite discrete group of automorphisms of a simplicial tree with uniformly bounded degrees whose Bowen-Margulis measure is infinite.

Proof. Let $\widehat{K} = \mathbb{F}_q((Y^{-1}))$, $\mathcal{O} = \mathbb{F}_q[[Y^{-1}]]$ and $R = \mathbb{F}_q[Y]$. Let \mathbb{X} be the Bruhat-Tits tree of $(\mathrm{PGL}_2, \widehat{K})$ with base point $* = [\mathcal{O} \times \mathcal{O}]$. Let $\Gamma = \mathrm{PGL}_2(R)$, which is a lattice of \mathbb{X} , with quotient the modular ray $\Gamma \backslash \mathbb{X}$ described in Section 15.2. We denote by $(y_i)_{i=-1,0,\dots}$ the ordered vertices along $\Gamma \backslash \mathbb{X}$ with vertex stabilisers $(\Gamma_i)_{i=-1,0,\dots}$, and by $(e_i)_{i \in \mathbb{N}}$ the ordered edges along $\Gamma \backslash \mathbb{X}$ (pointing away from the origin of the modular ray).

The subgroup

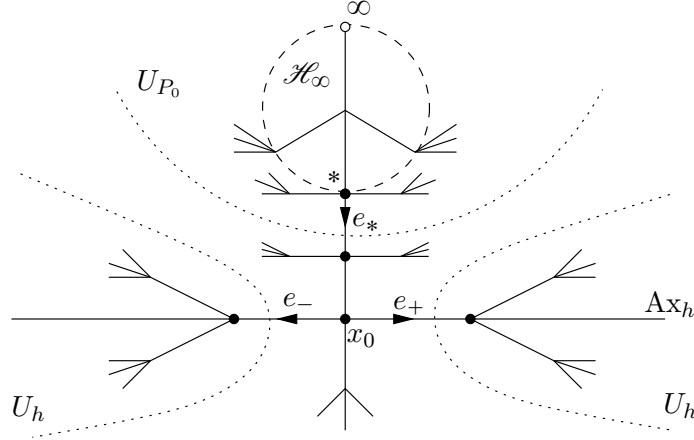
$$P = \bigcup_{i \geq 0} \Gamma_i = \left\{ \begin{bmatrix} a & Q \\ 0 & d \end{bmatrix} : Q \in \mathbb{F}_q[Y], a, d \in \mathbb{F}_q^\times \right\}$$

is the stabiliser in Γ of $\infty \in \partial_\infty \mathbb{X}$. Let P_0 be the finite index subgroup of P consisting of the elements $\begin{bmatrix} 1 & Q \\ 0 & 1 \end{bmatrix}$ with $Q(0) = 0$. Observing that $d(\gamma*, *) = 2(i+1)$ for any $\gamma \in \Gamma_i - \Gamma_{i-1}$ and that the cardinality of $(\Gamma_i - \Gamma_{i-1}) \cap P_0$ is $(q-1)q^{i+1}$, it is easy to see that the Poincaré series

$$\mathcal{Z}_{P,0,*,*}(s) = \sum_{\gamma \in P} e^{-s d(*, \gamma*)}$$

of the discrete (though elementary) subgroup P of $\mathrm{Isom}(\mathbb{X})$ is (up to a multiplicative constant) equal to $\sum_{i=0}^{\infty} q^i e^{-2si}$, which gives $\delta_P = \frac{\ln q}{2}$ for the critical exponent of P on \mathbb{X} .

Let h be a loxodromic element of Γ whose fixed points belong to the open subset $Y^{-1}\mathcal{O}$ of $\widehat{K} = \partial_\infty \mathbb{X} - \{\infty\}$. Hence the horoball \mathcal{H}_∞ centred at $\infty \in \partial_\infty \mathbb{X}$, whose horosphere contains $*$, is disjoint from the translation axis Ax_h of h . Note that the stabiliser of \mathcal{H}_∞ in Γ is P and that P_0 acts freely on the edges exiting \mathcal{H}_∞ . Let $x_0 \in V\mathbb{X}$ be the closest point on Ax_h to \mathcal{H}_∞ , let e_* be the edge with origin $*$ pointing towards x_0 , and let e_-, e_+ be the two edges with origin x_0 on Ax_h .



Let U_h be the set of points x in $V\mathbb{X} - \{x_0\}$ such that the geodesic segment from x_0 to x starts either by the edge e_- or by e_+ . Let U_{P_0} be the set of points y in $V\mathbb{X} - \{t(e_*)\}$ such that the geodesic segment from $t(e_*)$ to y starts by the edge \bar{e}_* . We have

- (1) $U_h \cap U_{P_0} = \emptyset$ and $x_0 \notin U_h \cup U_{P_0}$,
- (2) $h^k(V\mathbb{X} - U_h) \subset U_h$ for every $k \in \mathbb{Z} - \{0\}$ and $w(V\mathbb{X} - U_{P_0}) \subset U_{P_0}$ for every $w \in P_0 - \{\text{id}\}$,
- (3) $d(x, y) = d(x, x_0) + d(x_0, y)$ for all $x \in U_h$ and $y \in U_{P_0}$.

Let Γ' be the subgroup of Γ generated by P_0 and h . By a ping-pong argument, Γ' is a free product of P_0 and of the infinite cyclic group generated by h . Hence every element γ in $\Gamma' - \{e\}$ may be written uniquely $w_0 h^{n_0} w_1 h^{n_1} \dots w_k h^{n_k}$ with $k \in \mathbb{N}$, $w_i \in P_0$, $n_i \in \mathbb{Z}$ with $w_i \neq e$ if $i \neq 0$ and $n_i \neq 0$ if $i \neq k$. Using the above properties, we have by induction

$$d(x_0, w_0 h^{n_0} w_1 h^{n_1} \dots w_k h^{n_k} x_0) = \sum_{0 \leq i \leq k} d(x_0, h^{n_i} x_0) + \sum_{0 \leq i \leq k} d(x_0, w_i x_0). \quad (15.24)$$

Let $\lambda : E\mathbb{X} \rightarrow \mathbb{R}_+$ be the Γ' -invariant length map on the set of edges of \mathbb{X} such that for every $i \in \mathbb{N}$, the length of $e \in E\mathbb{X}$ is 1 if e is not contained in $\bigcup_{\gamma \in \Gamma'} \gamma \mathcal{H}_\infty$, and otherwise, if e maps to e_i or to \bar{e}_i under the canonical map $\mathbb{X} \rightarrow \Gamma \backslash \mathbb{X}$, then $\lambda(e) = 1 + \ln \frac{i+1}{i}$ if $i \geq 1$ and $\lambda(e) = 1$ if $i = 0$. Note that the distance in the metric graph $|\mathbb{X}|_\lambda$ from $*$ to the vertex on the geodesic ray from $*$ to ∞ originally at distance i from $*$ is now $i + \ln i$. The distances along the translation axis of h have not changed. Equation (15.24) remains valid with the new distance.

Let us now prove that the discrete subgroup Γ' of automorphisms of the regular metric tree (\mathbb{X}, λ) satisfies the first claim of Theorem 15.11.

By Γ' -invariance of λ , the group Γ' remains a subgroup of $\text{Aut}(\mathbb{X}, \lambda)$. The elements of $\Gamma' \infty$, or equivalently, the points at infinity of the horoballs in the Γ' -equivariant family of horoballs $(\gamma \mathcal{H}_\infty)_{\gamma \in \Gamma' / \Gamma'_\infty}$ with pairwise disjoint interiors in \mathbb{X} , remain bounded parabolic fixed points of Γ' , and the other limit points remain conical limit points of Γ' . Hence Γ' remains a geometrically finite discrete subgroup of $\text{Aut}(\mathbb{X}, \lambda)$.

The Poincaré series of the action of P_0 on (\mathbb{X}, λ) is (up to a multiplicative and additive constant) $\sum_{i=1}^{\infty} q^i e^{-2si} i^{-2s}$, which has the same critical exponent $\delta_P = \frac{\ln q}{2}$ as previously, but it is easy to see that the group P is now of convergence type if $q \geq 3$.

The computations of [DaOP, §4] now apply to our situation (with $C = 0$ in their notation, and we sum over P_0 instead of over the infinite cyclic group generated by the parabolic element p in their notation). Their argument shows that Γ' is of convergence type with critical exponent δ_P , up to replacing h by a big enough power. By Corollary 4.6, the Bowen-Margulis measure of Γ' is infinite.

In order to prove the second claim of Theorem 15.11, we first define a new length map $\lambda : E\mathbb{X} \rightarrow \mathbb{R}_+$ which coincides with the previous one on every edge e of \mathbb{X} , unless e maps to e_i or to \bar{e}_i under the canonical map $\mathbb{X} \rightarrow \Gamma \backslash \mathbb{X}$, in which case we set

$$\lambda(e) = 1 + [\ln(i + 1)] - [\ln i]$$

if $i \geq 1$ and $\lambda(e) = 1$ if $i = 0$ (where $[\cdot]$ is the largest previous integer map). This map λ now has values in $\{1, 2\}$, and we subdivide each edge of length 2 into two edges of length 1. The tree \mathbb{Y} thus obtained has uniformly bounded degrees (although it is no longer a uniform tree), and the group Γ' defines a geometrically finite discrete subgroup of $\text{Aut}(\mathbb{Y})$ with infinite Bowen-Margulis measure. \square

Chapter 16

Rational point equidistribution and counting in completed function fields

Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , and let R_v be the affine function ring associated with v . In this Chapter, we prove analogues of the classical results on the counting and equidistribution towards the Lebesgue measure on \mathbb{R} of the Farey fractions $\frac{p}{q}$ with $(p, q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ relatively prime (see for instance [Nev], as well as [PaP14b] for an approach using methods similar to the ones in this text). In particular, we prove various equidistribution results of locally finite families of elements of K towards the Haar measure on K_v , using the geometrical work on equidistribution of common perpendiculars done in Section 11.4 and recalled in Section 15.4.

16.1 Equidistribution of non-Archimedean Farey fractions

The first result of this Section is an analog in function fields of the equidistribution of Farey fractions to the Lebesgue measure in \mathbb{R} , see the Introduction, and for example [PaP14b, p. 978] for the precise statement and a geometric proof. For every $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$, let

$$m_{v, x_0, y_0} = \text{Card}\{a \in (R_v)^\times : \exists b \in x_0 R_v \cap y_0 R_v, (a-1)x_0 y_0 - b x_0 \in y_0^2 R_v\}.$$

For future use, note that by Equation (14.3)

$$m_{v, 1, 0} = q - 1. \tag{16.1}$$

For every $(a, b) \in R_v \times R_v$ and every subgroup H of $\text{GL}_2(R_v)$, let $H_{(a,b)}$ be the stabiliser of (a, b) for the linear action of H on $R_v \times R_v$.

Theorem 16.1. *Let G be a finite index subgroup of $\text{GL}_2(R_v)$, and let $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$. Then as $s \rightarrow +\infty$, if*

$$c = \frac{(q_v^2 - 1) (q_v + 1) \zeta_K(-1) m_{v, x_0, y_0} (N\langle x_0, y_0 \rangle)^2 [\text{GL}_2(R_v) : G]}{(q - 1) q^{g-1} q_v^3 [\text{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}]},$$

then

$$c s^{-2} \sum_{(x, y) \in G(x_0, y_0), |y|_v \leq s} \Delta_{\frac{x}{y}} \xrightarrow{*} \text{Haar}_{K_v}.$$

For every $\beta \in]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every β -Hölder-continuous function $\psi : K_v \rightarrow \mathbb{R}$ with compact support,¹ as for instance if $\psi : K_v \rightarrow \mathbb{R}$ is locally constant with compact support (see Remark 3.2), there is an error term in the equidistribution claim of Theorem 16.1 evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

It is remarkable that due to the general nature of our geometrical tools, we are able to work with any finite index subgroup G of $\mathrm{GL}_2(R_v)$, and not only with its congruence subgroups. In this generality, the usual techniques (for instance involving analysis of Eisenstein series) are not likely to apply. Also note that the Hölder regularity for the error term is a much weaker assumption than the locally constant one that is usually obtained by analytic number theory methods.

Theorem 1.13 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ (so that $g = 0$), $v = v_\infty$ and $(x_0, y_0) = (1, 0)$, and by using Equations (14.5) and (16.1) in order to simplify the constant.

Before proving Theorem 16.1, let us give a counting result which follows from this equidistribution result by considering the locally constant characteristic function of a closed and open fundamental domain of K_v modulo the action by translations of a finite index additive subgroup of R_v , and by using Lemma 14.4 with $I = R_v$.

The additive group R_v acts on $R_v \times R_v$ by the horizontal shears (transvections):

$$\forall k \in R_v, \forall (x, y) \in R_v \times R_v, \quad k \cdot (x, y) = (x + ky, y),$$

and this action preserves the absolute value $|y|_v$ of the vertical coordinate y . We may then define a *counting function* Ψ_{G, x_0, y_0} of elements in K in an orbit by homographies under a finite index subgroup G of $\mathrm{GL}_2(R_v)$, as

$$\Psi_{G, x_0, y_0}(s) = \mathrm{Card} \ R_{v, G} \setminus \{(x, y) \in G(x_0, y_0), |y|_v \leq s\},$$

where $R_{v, G}$ is the finite index additive subgroup of R_v consisting of the elements $x \in R_v$ such that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G$. Note that $R_{v, G} = R_v$ if $G = \mathrm{GL}_2(R_v)$.

Corollary 16.2. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$. Then there exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,*

$$\begin{aligned} \Psi_{G, x_0, y_0}(s) \\ = \frac{(q-1) q^{2g-2} q_v^3 [\mathrm{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}] [R_v : R_{v, G}]}{(q_v^2 - 1) (q_v + 1) \zeta_K(-1) m_{v, x_0, y_0} (N\langle x_0, y_0 \rangle)^2 [\mathrm{GL}_2(R_v) : G]} s^2 + O(s^{2-\kappa}). \quad \square \end{aligned}$$

Let us fix some notation for this Section. For every subgroup H of $\mathrm{GL}_2(R_v)$, we denote by \overline{H} its image in $\Gamma_v = \mathrm{PGL}_2(R_v)$. Let \mathbb{X}_v be the Bruhat-Tits tree of (PGL_2, K_v) , which is regular of degree $q_v + 1$. Let

$$r = \frac{x_0}{y_0} \in K \cup \{\infty\}.$$

If $y_0 = 0$, let $g_r = \mathrm{id} \in \mathrm{GL}_2(K)$, and if $y_0 \neq 0$, let

$$g_r = \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(K).$$

¹where K_v is endowed with the distance $(x, y) \mapsto |x - y|_v$

In the proof of Theorem 16.1, we apply Theorem 15.4 with $\Gamma := \overline{G}$, $\mathbb{D}^- := \mathcal{H}_\infty$ and $\mathbb{D}^+ := g_r \mathcal{H}_\infty$. Recall that \mathcal{H}_∞ is the horoball in \mathbb{X}_v centred at ∞ whose boundary contains $*_v$ (see Section 15.3).

Proof of Theorem 16.1. Note that Γ has finite index in Γ_v and, in particular, it is a lattice of \mathbb{X}_v . By [Ser3, II.1.2, Coro.], for all $x \in V\mathbb{X}_v$ and $\gamma \in \mathrm{GL}_2(R_v)$, the distance $d(x, \gamma x)$ is even since $v(\det \gamma) = 0$. Hence by the equivalence in Equation (4.13), the length spectrum L_{Γ_v} of Γ_v is $2\mathbb{Z}$. The length spectrum of Γ is also $2\mathbb{Z}$, since it is contained in L_{Γ_v} .

Note that \mathbb{D}^+ is a horoball in \mathbb{X}_v centred at $r = \frac{x_0}{y_0} \in \partial_\infty \mathbb{X}_v$, by Lemma 15.1. The stabiliser $\Gamma_{\mathbb{D}^-}$ of \mathbb{D}^- (respectively $\Gamma_{\mathbb{D}^+}$ of \mathbb{D}^+) coincides with the fixator Γ_∞ of $\infty \in \partial_\infty \mathbb{X}_v$ (respectively the fixator Γ_r of r) in Γ . Note that the families $\mathcal{D}^\pm = (\gamma \mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite, since Γ_v , and hence its finite index subgroup Γ , is geometrically finite,² and since ∞ and $r \in K$ are bounded parabolic limit points of Γ_v , hence of its finite index subgroup Γ .

For every $\gamma \in \Gamma/\Gamma_r$ such that \mathbb{D}^- and $\gamma \mathbb{D}^+$ are disjoint, let ρ_γ be the geodesic ray starting from $\alpha_{e, \gamma}^-(0)$ and ending at the point at infinity $\gamma \cdot r$ of $\gamma \mathbb{D}^+$. Note that ρ_γ and $\alpha_{e, \gamma}^-$ coincide on $[0, d(\mathbb{D}^-, \gamma \mathbb{D}^+)]$.

Since the Patterson densities of lattices of \mathbb{X}_v have total mass $\frac{q_v+1}{q_v}$ by Proposition 15.2 (2), they are normalized as in Theorem 15.4. Then by Equation (15.15), we have

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_r \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \Delta_{(\rho_\gamma)_+} = (\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ . \quad (16.2)$$

Furthermore, for every $\beta \in]0, \frac{1}{\ln q_v}]$, by the comment following Equation (15.15), we have an error term of the form $O(\|\psi\|_{\beta \ln q_v} e^{-\kappa n})$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathcal{C}_c^{\beta \ln q_v}(\partial \mathbb{X}_v - \{\infty\})$, where $\partial \mathbb{X}_v - \{\infty\}$ is endowed with Hamenstädt's distance $d_{\mathcal{H}_\infty}$. Hence we have an error term $O(\|\psi\|_{\beta} e^{-\kappa n})$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathcal{C}_c^\beta(K_v)$, where $K_v = \partial \mathbb{X}_v - \{\infty\}$ is endowed with the distance $(x, y) \mapsto |x - y|_v$, see Equation (15.3).

By Proposition 15.2 (3), we have

$$(\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ = \mathrm{Haar}_{K_v} .$$

Hence Equation (16.2) gives, with the appropriate error term,

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_r \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \Delta_{\gamma \cdot r} = \mathrm{Haar}_{K_v} . \quad (16.3)$$

Let $g \in \mathrm{GL}_2(K)$ be such that $g\infty \neq \infty$. This condition is equivalent to asking that the $(2, 1)$ -entry c of g is nonzero. By Lemma 15.1, the signed distance between the horospheres \mathcal{H}_∞ and $g\mathcal{H}_\infty$ is

$$d(\mathcal{H}_\infty, g\mathcal{H}_\infty) = -2 v(c) = 2 \frac{\ln |c|_v}{\ln q_v} . \quad (16.4)$$

If $y_0 \neq 0$, then $(x, y) = g(x_0, y_0)$ if and only if $(\frac{x}{y_0}, \frac{y}{y_0}) = gg_r(1, 0)$, and the $(2, 1)$ -entry of gg_r is $\frac{y}{y_0}$. If $y_0 = 0$ (which implies that $g_r = \mathrm{id}$ and $x_0 \neq 0$), then $(x, y) = g(x_0, y_0)$ if and only if

²See Section 15.2

$(\frac{x}{x_0}, \frac{y}{x_0}) = g(1, 0)$, and the $(2, 1)$ -entry of $g = gg_r$ is $\frac{y}{x_0}$. Let

$$z_0 = \begin{cases} y_0 & \text{if } y_0 \neq 0 \\ x_0 & \text{otherwise.} \end{cases}$$

By Equation (16.4), the signed distance between $\mathbb{D}^- = \mathcal{H}_\infty$ and $g\mathbb{D}^+ = g g_r \mathcal{H}_\infty$ is

$$d(\mathbb{D}^-, g\mathbb{D}^+) = \frac{2}{\ln q_v} \ln \left| \frac{y}{z_0} \right|_v.$$

By discreteness, there are only finitely many double classes $[g] \in G_{(1,0)} \backslash G / G_{(x_0, y_0)}$ such that $\mathbb{D}^- = \mathcal{H}_\infty$ and $g\mathbb{D}^+ = g g_r \mathcal{H}_\infty$ are not disjoint. Let $Z(G)$ be the centre of G , which is finite. Since $Z(G)$ acts trivially on $\mathbb{P}_1(K_v)$, the map $G/G_{(x_0, y_0)} \rightarrow \Gamma/\Gamma_r$ induced by the canonical map $\text{GL}_2(R_v) \mapsto \text{PGL}_2(R_v)$ is $|Z(G)|$ -to-1. Using the change of variable

$$s = |z_0|_v q_v^{\frac{n}{2}},$$

so that $q_v^{-n} = |z_0|_v^2 s^{-2}$, Equation (16.3) gives, with the appropriate error term,

$$\lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1) |z_0|_v^2 \text{Vol}(\Gamma \backslash \mathbb{X}_v)}{2 q_v^3 |Z(G)|} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} s^{-2} \sum_{(x, y) \in G_{(x_0, y_0)}, |y|_v \leq s} \Delta_{\frac{x}{y}} = \text{Haar}_{K_v}. \quad (16.5)$$

The order of the centre $Z(\text{GL}_2(R_v)) = (R_v)^\times \text{id}$ is $q - 1$ by Equation (14.3). The map $\text{GL}_2(R_v)/G \rightarrow \Gamma_v/\Gamma$ induced by the canonical map $\text{GL}_2(R_v) \rightarrow \text{PGL}_2(R_v)$ is hence $\frac{q-1}{|Z(G)|}$ -to-1. By Equation (15.7), we hence have

$$\begin{aligned} \text{Vol}(\Gamma \backslash \mathbb{X}_v) &= [\Gamma_v : \Gamma] \text{Vol}(\Gamma_v \backslash \mathbb{X}_v) = 2 \zeta_K(-1) [\Gamma_v : \Gamma] \\ &= \frac{2}{q-1} \zeta_K(-1) |Z(G)| [\text{GL}_2(R_v) : G]. \end{aligned} \quad (16.6)$$

Theorem 16.1 follows from Equations (16.5) and (16.6) and from Lemma 16.3 below. \square

Lemma 16.3. *We have*

$$\|\sigma_{\mathcal{D}^+}^-\| = \frac{q^{g-1} |z_0|_v^2 [\text{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}]}{m_{v, x_0, y_0} (N\langle x_0, y_0 \rangle)^2}.$$

Proof. Let γ_r be the image of g_r in $\text{PGL}_2(K)$. Let us define $\Gamma' = \gamma_r^{-1} \Gamma \gamma_r$, which is a finite index subgroup in $\Gamma'_v = \gamma_r^{-1} \Gamma_v \gamma_r$ and a lattice of \mathbb{X}_v . Since γ_r maps ∞ to r , the point ∞ is a bounded parabolic limit point of Γ' , and we have $(\Gamma')_\infty = \gamma_r^{-1} \Gamma_r \gamma_r$. Since the canonical map $\text{GL}_2(R_v) \rightarrow \text{PGL}_2(R_v)$ is injective on the stabiliser $\text{GL}_2(R_v)_{(x_0, y_0)}$, we have

$$[(\Gamma'_v)_\infty : (\Gamma')_\infty] = [(\Gamma_v)_r : \Gamma_r] = [\text{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}].$$

Since the Patterson density of a lattice does not depend on the lattice (see Proposition 4.14 (1)), the skinning measures $\tilde{\sigma}_{\mathcal{H}}^\pm$ of a given horoball \mathcal{H} do not depend on the lattice. Thus

$$\gamma_* \tilde{\sigma}_{\mathcal{H}}^\pm = \tilde{\sigma}_{\gamma \mathcal{H}}^\pm$$

for every $\gamma \in \text{Aut}(\mathbb{X}_v)$. Let $\mathcal{D}_1^+ = (\gamma' \mathcal{H}_\infty)_{\gamma' \in \Gamma'/\Gamma'_\infty}$, which is a locally finite Γ' -equivariant family of horoballs. We hence have, using Proposition 15.2 (4) for the third equality,

$$\begin{aligned} \|\sigma_{\mathcal{D}^+}^-\| &= \|\sigma_{\gamma_r \mathcal{D}_1^+}^-\| = \|\sigma_{\mathcal{D}_1^+}^-\| = \text{Haar}_{K_v}((\Gamma')_\infty \backslash K_v) \\ &= [(\Gamma'_v)_\infty : (\Gamma')_\infty] \text{Haar}_{K_v}((\Gamma'_v)_\infty \backslash K_v) \\ &= [\text{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}] \text{Haar}_{K_v}((\Gamma'_v)_\infty \backslash K_v). \end{aligned} \quad (16.7)$$

Every element in the stabiliser of ∞ in $\text{PGL}_2(K_v)$ can be uniquely written in the form $\alpha = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with $(a, b) \in (K_v)^\times \times K_v$. Note that

$$\begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix} = \begin{pmatrix} br + 1 & ar - br^2 - r \\ b & a - br \end{pmatrix}.$$

When $x_0 = 0$ or $y_0 = 0$, we have $\alpha \in \Gamma'_v$ if and only if

$$b \in R_v \quad \text{and} \quad a \in (R_v)^\times.$$

When $x_0, y_0 \neq 0$, we have $\alpha \in \Gamma'_v$ if and only if $\gamma_r \alpha \gamma_r^{-1} \in \Gamma_v$, hence if and only if

$$b \in R_v \cap \frac{1}{r} R_v, \quad a \in (R_v)^\times, \quad ar - br^2 - r \in R_v.$$

Let U'_∞ be the kernel of the map from $(\Gamma'_v)_\infty$ to $(K_v)^\times$ sending $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ to a , and let m_v be its index in $(\Gamma'_v)_\infty$. If $x_0 = 0$ or $y_0 = 0$, then m_v is equal to $|(R_v)^\times|$, so that, by Equation (14.3),

$$m_v = |(R_v)^\times| = |(\mathbb{F}_q)^\times| = q - 1.$$

If $x_0, y_0 \neq 0$, we have

$$m_v = \text{Card} \{a \in (R_v)^\times : \exists b \in R_v \cap \frac{1}{r} R_v, \quad ar - br^2 - r \in R_v\}.$$

Note that the notation m_v coincides with the constant m_{v, x_0, y_0} defined before the statement of Theorem 16.1 in both cases.

If $I_{(x_0, y_0)}$ is the nonzero fractional ideal

$$I_{(x_0, y_0)} = \begin{cases} R_v & \text{if } x_0 = 0 \text{ or } y_0 = 0, \\ R_v \cap \frac{1}{r} R_v \cap \frac{1}{r^2} R_v & \text{otherwise,} \end{cases}$$

then

$$U'_\infty = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in I_{(x_0, y_0)} \right\}.$$

Therefore by Lemma 14.4,

$$\text{Haar}_{K_v}((\Gamma'_v)_\infty \backslash K_v) = \frac{\text{Haar}_{K_v}(I_{(x_0, y_0)} \backslash K_v)}{[(\Gamma'_v)_\infty : U'_\infty]} = \frac{q^{g-1} N(I_{(x_0, y_0)})}{m_v}. \quad (16.8)$$

Let $(x_0) = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(x_0)}$ and $(y_0) = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(y_0)}$ be the prime decompositions of the principal ideals (x_0) and (y_0) . By the formulas of the prime decompositions of intersections, sums and products of ideals in Dedekind rings (see for instance [Nar, §1.1]), we have

$$(x_0^2) \cap (x_0 y_0) \cap (y_0^2) = (x_0^2) \cap (y_0^2) = \prod_{\mathfrak{p}} \mathfrak{p}^{2 \max\{\nu_{\mathfrak{p}}(x_0), \nu_{\mathfrak{p}}(y_0)\}}$$

and

$$\langle x_0, y_0 \rangle = \prod_{\mathfrak{p}} \mathfrak{p}^{\min\{\nu_{\mathfrak{p}}(x_0), \nu_{\mathfrak{p}}(y_0)\}} .$$

By the definition of the ideal $I_{(x_0, y_0)}$, by the multiplicativity of the norm, and by Equation (14.4), we hence have if $x_0 \neq 0$ and $y_0 \neq 0$

$$\frac{N(I_{(x_0, y_0)}) (N\langle x_0, y_0 \rangle)^2}{|y_0|_v^2} = N\left(\left((x_0^2) \cap (x_0 y_0) \cap (y_0^2)\right) \langle x_0, y_0 \rangle^2 (x_0)^{-2} (y_0)^{-2}\right) = 1 . \quad (16.9)$$

If $x_0 = 0$ or $y_0 = 0$, then

$$N(I_{(x_0, y_0)}) = N(R_v) = 1 . \quad (16.10)$$

Lemma 16.3 follows from Equations (16.7), (16.8) and (16.9) if $x_0 \neq 0$ and $y_0 \neq 0$ or (16.10) if $x_0 = 0$ or $y_0 = 0$. \square

Let us state one particular case of Theorem 16.1 in an arithmetic setting, using a congruence subgroup.

Theorem 16.4. *Let I be a nonzero ideal of R_v . Then as $t \rightarrow +\infty$, we have*

$$\frac{(q_v^2 - 1) (q_v + 1) \zeta_K(-1) N(I) \prod_{\mathfrak{p}|I} \left(1 + \frac{1}{N(\mathfrak{p})}\right)}{q^{g-1} q_v} (q_v)^{-2t} \sum_{\substack{(x, y) \in R_v \times I \\ \langle x, y \rangle = R_v, v(y) \geq -t}} \Delta_{\frac{x}{y}} \stackrel{*}{\sim} \text{Haar}_{K_v} ,$$

where the product ranges over the prime factors \mathfrak{p} of the ideal I . Furthermore, if

$$\Psi(t) = \text{Card } R_v \setminus \{(x, y) \in R_v \times I : \langle x, y \rangle = R_v, v(y) \geq -t\} ,$$

then there exists $\kappa > 0$ such that, as $t \rightarrow +\infty$,

$$\Psi(t) = \frac{q^{2g-2} q_v}{(q_v^2 - 1) (q_v + 1) \zeta_K(-1) N(I) \prod_{\mathfrak{p}|I} \left(1 + \frac{1}{N(\mathfrak{p})}\right)} q_v^{2t} + O(q_v^{(2-\kappa)t}) .$$

For every $\beta \in]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^\beta(K_v)$ there is an error term in the above equidistribution claim evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

Proof. The counting claim is deduced from the equidistribution claim in the same way that Corollary 16.2 is deduced from Theorem 16.1, noting that the action of R_v by horizontal shears preserves $R_v \times I$.

In order to prove the equidistribution claim, we apply Theorem 16.1 with $(x_0, y_0) = (1, 0)$ and with G the Hecke congruence subgroup

$$G_I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R_v) : c \in I \right\} , \quad (16.11)$$

which is the preimage of the upper triangular subgroup of $\mathrm{GL}_2(R_v/I)$ by reduction modulo I . In this case, the constant m_{v,x_0,y_0} appearing in the statement of Theorem 16.1 is equal to $q - 1$ by Equation (16.1). The group G_I has finite index in $\mathrm{GL}_2(R_v)$. The following result is well-known to arithmetic readers (see for instance [Shi, page 24] when R_v is replaced by \mathbb{Z}), we only give a sketch of proof (indicated to us by J.-B. Bost) for the sake of the geometer readers.

Lemma 16.5. *We have*

$$[\mathrm{GL}_2(R_v) : G_I] = N(I) \prod_{\mathfrak{p}|I} \left(1 + \frac{1}{N(\mathfrak{p})}\right).$$

where the product ranges over the prime factors \mathfrak{p} of the ideal I .

Proof. In this proof, we denote by $|E|$ the cardinality of a finite set E . For every commutative ring A with finite group of invertible elements A^\times , we have

$$\mathrm{GL}_2(A) = \bigcup_{a \in A^\times} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(A),$$

Hence $[\mathrm{GL}_2(A) : \mathrm{SL}_2(A)] = |A^\times|$. Since $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ belongs to G_I for all $a \in (R_v)^\times$, we have

$$G_I = \bigcup_{a \in (R_v)^\times} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} G_I \cap \mathrm{SL}_2(R_v),$$

so that $[\mathrm{GL}_2(R_v) : G_I] = [\mathrm{SL}_2(R_v) : G_I \cap \mathrm{SL}_2(R_v)]$.

The group morphism of reduction modulo I from $\mathrm{SL}_2(R_v)$ to $\mathrm{SL}_2(R_v/I)$ is onto, by an argument of further reduction to the various prime power factors of I and of lifting elementary matrices. The order of the upper triangular subgroup of $\mathrm{SL}_2(R_v/I)$ is $|(R_v/I)^\times| |R/I|$, where $(R_v/I)^\times$ is the group of invertible elements of the ring R_v/I (that we will see again below). Hence

$$\begin{aligned} [\mathrm{GL}_2(R_v) : G_I] &= [\mathrm{SL}_2(R_v) : G_I \cap \mathrm{SL}_2(R_v)] = \frac{|\mathrm{SL}_2(R_v/I)|}{|(R_v/I)^\times| |R/I|} \\ &= \frac{|\mathrm{GL}_2(R_v/I)|}{|(R_v/I)^\times|^2 |R/I|}. \end{aligned} \tag{16.12}$$

By the multiplicativity of the norm and by the Chinese remainder theorem,³ one reduces the result to the case when $I = \mathfrak{p}^n$ is the n -th power of a fixed prime ideal \mathfrak{p} with norm $N(\mathfrak{p}) = N$, where $n \in \mathbb{N}$. Note that since R_v/\mathfrak{p} is a field, we have

$$|\mathrm{GL}_1(R_v/\mathfrak{p})| = |(R_v/\mathfrak{p})^\times| = |R_v/\mathfrak{p}| - 1 = N - 1$$

and

$$|\mathrm{GL}_2(R_v/\mathfrak{p})| = (|R_v/\mathfrak{p}|^2 - 1)(|R_v/\mathfrak{p}|^2 - |R_v/\mathfrak{p}|) = N^2(N - 1)^2 \left(1 + \frac{1}{N}\right).$$

³saying that the rings R_v/I and $\prod_{\mathfrak{p}} R_v/\mathfrak{p}^{v_{\mathfrak{p}}(I)}$ are isomorphic, see for instance [Nar, page 11]

For $k = 1$ or $k = 2$, the kernel of the morphism of reduction modulo \mathfrak{p} from $\mathrm{GL}_k(R_v/I) = \mathrm{GL}_k(R_v/\mathfrak{p}^n)$ to $\mathrm{GL}_k(R_v/\mathfrak{p})$ has order $N^{k^2(n-1)}$. Hence

$$|\mathrm{GL}_2(R_v/I)| = N^{4(n-1)}N^2(N-1)^2\left(1 + \frac{1}{N}\right),$$

and

$$|(R_v/I)^\times| = N^{n-1}(N-1).$$

Therefore, by Equation (16.12), we have

$$[\mathrm{GL}_2(R_v) : G_I] = \frac{N^{4(n-1)}N^2(N-1)^2\left(1 + \frac{1}{N}\right)}{N^{2(n-1)}(N-1)^2N^n} = N^n\left(1 + \frac{1}{N}\right).$$

This proves the result. \square

We can now conclude the proof of Theorem 16.4. Note that $\mathrm{GL}_2(R_v)_{(1,0)} = (G_I)_{(1,0)}$. The result then follows from Theorem 16.1 and its Corollary 16.2, using the change of variables $s = (q_v)^t$, since

$$G_I(1,0) = \{(x,y) \in R_v \times I : \langle x,y \rangle = R_v\}. \quad \square$$

The following result is a particular case of Theorem 16.4.

Corollary 16.6. *Let P_0 be a nonzero element of the polynomial ring $R = \mathbb{F}_q[Y]$ over \mathbb{F}_q , and let $P_0 = a_0 \prod_{i=1}^k (P_i)^{n_i}$ be the prime decomposition of P_0 . Then as $t \rightarrow +\infty$,*

$$\frac{(q+1) \prod_{i=1}^k q^{n_i \deg P_i} (1 + q^{-\deg P_i})}{q-1} q^{-2t} \sum_{\substack{(P,Q) \in R \times (P_0 R) \\ PR+QR=R, \deg Q \leq t}} \Delta_{\frac{P}{Q}} \stackrel{*}{\sim} \mathrm{Haar}_{\mathbb{F}_q((Y^{-1}))}.$$

For every $\beta \in]0, \frac{1}{\ln q}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^\beta(\mathbb{F}_q((Y^{-1})))$ there is an error term in the above equidistribution claim evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

Proof. In this statement, we use the standard convention that $k = 0$ if P_0 is constant, $a_0 \in (\mathbb{F}_q)^\times$ and $P_i \in R$ is monic.

We apply the first claim of Theorem 16.4 with $K = \mathbb{F}_q(T)$ and $v = v_\infty$ so that $g = 0$, $q_v = q$ and $R_v = R$, and with $I = P_0 R$, so that $N(I) = \prod_{i=1}^k q^{n_i \deg P_i}$. The result follows from Equation (14.5). \square

16.2 Mertens's formula in function fields

In this Section, we recover the function field analogue of Mertens's classical formula on the average order of the Euler function. We begin with a more general counting and equidistribution result.

Let \mathfrak{m} be a (nonzero) fractional ideal of R_v , with norm $N(\mathfrak{m})$. Note that the action of the additive group R_v on $K_v \times K_v$ by the horizontal shears $k \cdot (x,y) = (x + ky, y)$ preserves $\mathfrak{m} \times \mathfrak{m}$. We consider the counting function $\psi_{\mathfrak{m}} : [0, +\infty[\rightarrow \mathbb{N}$ defined by

$$\psi_{\mathfrak{m}}(s) = \mathrm{Card}\left(R_v \setminus \{(x,y) \in \mathfrak{m} \times \mathfrak{m} : 0 < N(\mathfrak{m})^{-1}N(y) \leq s, \langle x,y \rangle = \mathfrak{m}\}\right).$$

Note that $\psi_{\mathfrak{m}}$ depends only on the ideal class of \mathfrak{m} and thus we can assume in the computations that \mathfrak{m} is integral.

Corollary 16.7. *There exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,*

$$\psi_{\mathbf{m}}(s) = \frac{(q-1) q^{2g-2} q_v^3}{(q_v^2-1)(q_v+1)\zeta_K(-1)m_{v,x_0,y_0}} s^2 + O(s^{2-\kappa}),$$

where $\mathbf{m} = \langle x_0, y_0 \rangle$. Furthermore, as $s \rightarrow +\infty$,

$$\frac{(q_v^2-1)(q_v+1)\zeta_K(-1)m_{v,x_0,y_0}}{(q-1)q^{g-1}q_v^3} s^{-2} \sum_{\substack{(x,y) \in \mathbf{m} \times \mathbf{m} \\ N(\mathbf{m})^{-1}N(y) \leq s, \langle x,y \rangle = \mathbf{m}}} \Delta_{\frac{x}{y}} \overset{*}{\sim} \text{Haar}_{K_v}.$$

For every $\beta \in]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^\beta(K_v)$ there is an error term in the above equidistribution claim evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

Theorem 1.14 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ (so that $g = 0$) and $v = v_\infty$. In order to simplify the constant, we use Equation (14.5) and the fact that the ideal class number of K , that equals the number of orbits of $\text{PGL}_2(\mathbb{F}_q[Y])$ on $\mathbb{P}^1(\mathbb{F}_q(Y))$, is 1. Thus, if $\mathbf{m} = \langle x_0, y_0 \rangle$ then the constant m_{v,x_0,y_0} is equal to $m_{v,1,0}$, which is $q-1$ by Equation (16.1).

Proof. Every nonzero ideal I in R_v is of the form $I = xR_v + yR_v$ for some $(x, y) \in R_v \times R_v - \{(0, 0)\}$, see for instance [Nar, page 10]. For all (x, y) and (z, w) in $R_v \times R_v$, we have $xR_v + yR_v = zR_v + wR_v$ if and only if $(z, w) \in \text{GL}_2(R_v)(x, y)$. The ideal class group of K corresponds bijectively to the set $\text{PGL}_2(R_v) \backslash \mathbb{P}^1(K)$ of cusps of the quotient graph of groups $\text{PGL}_2(R_v) \backslash \mathbb{X}_v$ (where \mathbb{X}_v is the Bruhat-Tits tree of (PGL_2, K_v)), by the map induced by $I = xR_v + yR_v \mapsto [x : y] \in \mathbb{P}^1(K)$.

Given a fixed ideal \mathbf{m} in R_v , we apply Theorem 16.1 with $G = \text{GL}_2(R_v)$ and $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$ a fixed pair such that $x_0R_v + y_0R_v = \mathbf{m}$. Using therein the change of variable $s \mapsto N(\mathbf{m})s$, the result follows from Theorem 16.1 and its Corollary 16.2. \square

As already encountered in the proof of Lemma 16.5, the Euler function φ_{R_v} of R_v is defined on the set of (nonzero, integral) ideals I of R_v by setting⁴

$$\varphi_{R_v}(I) = \text{Card}((R_v/I)^\times),$$

and we denote $\varphi_{R_v}(y) = \varphi_{R_v}(yR_v)$ for every $y \in R_v$. Thus, by the definition of the action of R_v on $R_v \times R_v$ by shears, we have

$$\begin{aligned} \psi_{R_v}(s) &= \sum_{y \in R_v, 0 < N(y) \leq s} \text{Card}\{x \in R_v/yR_v : \langle x, y \rangle = R_v\} \\ &= \sum_{y \in R_v, 0 < N(y) \leq s} \varphi_{R_v}(y). \end{aligned} \tag{16.13}$$

As a particular application of Corollary 16.7, we get a well-known asymptotic result on the number of relatively prime polynomials in $\mathbb{F}_q[Y]$. The Euler function of the ring of polynomials $R = \mathbb{F}_q[Y]$ is then the map $\phi_q : R - \{0\} \rightarrow \mathbb{N}$ defined by

$$\phi_q(Q) = |(R/QR)^\times| = \text{Card}\{P \in R : \langle P, Q \rangle = R, \deg P < \deg Q\}.$$

Note that $\phi_q(\lambda Q) = \phi_q(Q)$ for every $\lambda \in (\mathbb{F}_q)^\times$.

⁴See for example [Ros, §1].

Corollary 16.8 (Mertens’s formula for polynomials). *We have*

$$\lim_{n \rightarrow +\infty} \frac{1}{q^{2n}} \sum_{Q \in \mathbb{F}_q[X], \deg Q \leq n} \phi_q(Q) = \frac{(q-1)q}{q+1}.$$

Proof. We apply the first claim of Corollary 16.7, in the special case when $K = \mathbb{F}_q(T)$ and $v = v_\infty$ so that $g = 0$, $q_v = q$ and $R_v = R$, and with $\mathfrak{m} = R_v$, so that $m_{v, x_0, y_0} = q - 1$, in order to obtain the asymptotic value of $\psi_{R_v}(s)$ with the change of variable $s = q^n$. The result follows from Equations (16.13) and (14.5). \square

The above result is an analog of Mertens’s formula when K is replaced by \mathbb{Q} and R_v by \mathbb{Z} , see [HaW, Thm. 330]. See also [Grot, Satz 2], [Cos, §4.3], as well as [PaP14b] and [PaP16b, §5] for further developments.

A much more precise result than Corollary 16.8 can be obtained by purely number theoretical means as follows. The average value of ϕ_q is computed in [Ros, Prop. 2.7]: For $n \geq 1$,

$$\sum_{\deg f = n, f \text{ monic}} \phi_q(f) = q^{2n} \left(1 - \frac{1}{q}\right).$$

This gives $\sum_{\deg f = n} \phi_q(f) = q^{2n} \frac{(q-1)^2}{q}$, so that

$$\sum_{0 < \deg f \leq n} \phi_q(f) = \sum_{k=1}^n q^{2k} \frac{(q-1)^2}{q} = q(q-1)^2 \frac{q^{2n} - 1}{q^2 - 1} = \frac{q(q-1)(q^{2n} - 1)}{(q+1)},$$

from which Corollary 16.8 easily follows.

Chapter 17

Equidistribution and counting of quadratic irrational points in non-Archimedean local fields

Let K_v be a non-Archimedean local field, with valuation v , valuation ring \mathcal{O}_v , choice of uniformiser π_v , and residual field k_v of order q_v . Let \mathbb{X}_v be the Bruhat-Tits tree of (PGL_2, K_v) .¹ In this Chapter, we give counting and equidistribution results in $K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$ of an orbit under a lattice of $\mathrm{PGL}_2(K_v)$ of a fixed point of a loxodromic element of this lattice. We use these results to deduce equidistribution and counting results of quadratic irrational elements in non-Archimedean local fields.

When \mathbb{X}_v is replaced by a real hyperbolic space, or by a more general simply connected complete Riemannian manifold with negative sectional curvature, there are numerous quantitative results on the density of such an orbit, see the works of Patterson, Sullivan, Hill, Velani, Stratmann, Hersonsky-Paulin, Parkkonen-Paulin. See for instance [PaP16a] for references. The arithmetic applications when \mathbb{X}_v is replaced by the upper halfspace model of the real hyperbolic space of dimension 2, 3 or 5 are counting and equidistribution results of quadratic irrational elements in \mathbb{R} , \mathbb{C} and the Hamiltonian quaternions. See for instance [PaP12, Coro. 3.10] and [PaP14b].

17.1 Counting and equidistribution of loxodromic fixed points

An element $\gamma \in \mathrm{PGL}_2(K_v)$ is said to be *loxodromic* if it is loxodromic on the (geometric realisation of the) simplicial tree \mathbb{X}_v .² Its translation length is

$$\lambda(\gamma) = \min_{x \in V\mathbb{X}_v} d(x, \gamma x) > 0,$$

and the subset

$$Ax_\gamma = \{x \in V\mathbb{X}_v : d(x, \gamma x) = \lambda(\gamma)\}$$

is the image of a (discrete) geodesic line in \mathbb{X}_v , which we call the (*discrete*) *translation axis* of γ . The points at infinity of Ax_γ are denoted by γ^- and γ^+ chosen so that γ translates away

¹See Section 15.1.

²See Section 2.2.

from γ^- and towards γ^+ on Ax_γ . Note that for every $\gamma' \in \text{PGL}_2(K_v)$, we have

$$\gamma' \text{Ax}_\gamma = \text{Ax}_{\gamma'\gamma(\gamma')^{-1}} \quad \text{and} \quad \gamma' \gamma^\pm = (\gamma'\gamma(\gamma')^{-1})^\pm.$$

If Γ is a discrete subgroup of $\text{PGL}_2(K_v)$ and if α is one of the two fixed points of a loxodromic element of Γ , we denote the other fixed point of this element by α^σ . Since Γ is discrete, the translation axes of two loxodromic elements of Γ coincide if they have a common point at infinity. Hence α^σ is uniquely defined. We define the *complexity* $h(\alpha)$ of the loxodromic fixed point α by

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|_v} \tag{17.1}$$

if $\alpha, \alpha^\sigma \neq \infty$, and by $h(\alpha) = 0$ if α or α^σ is equal to ∞ . We define $\iota_\alpha = 2$ if there exists an element $\gamma \in \Gamma$ such that $\gamma \cdot \alpha = \alpha^\sigma$, and $\iota_\alpha = 1$ otherwise.³

Following [Ser3, II.1.2], we denote by $\text{PGL}_2(K_v)^+$ the kernel of the group morphism $\text{PGL}_2(K_v) \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by $\gamma = [g] \mapsto v(\det g) \pmod{2}$. The definition does not depend on the choice of a representative $g \in \text{GL}_2(K_v)$ of an element $\gamma \in \text{PGL}_2(K_v)$, since

$$v\left(\det \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 2v(\lambda)$$

is even for every $\lambda \in (K_v)^\times$. Note that when K_v is as in Section 14.2 the completion of a function field over \mathbb{F}_q endowed with a valuation v , with associated affine function ring R_v , the group $\Gamma_v = \text{PGL}_2(R_v)$ is contained in $\text{PGL}_2(K_v)^+$, see [Ser3, II.1.2]: For every $g \in \text{GL}_2(R_v)$, since $\det g \in (R_v)^\times = (\mathbb{F}_q)^\times$,⁴ we have $v(\det g) = 0$.

The following result proves the equidistribution in K_v of the loxodromic fixed points with complexity at most s in a given orbit by homographies under a lattice in $\text{PGL}_2(K_v)$ as $s \rightarrow +\infty$, and its associated counting result. If $\xi \in \partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$ and Γ is a subgroup of $\text{PGL}_2(K_v)$, we denote by Γ_ξ the stabiliser in Γ of ξ .

Theorem 17.1. *Let Γ be a lattice in $\text{PGL}_2(K_v)^+$, and let $\gamma_0 \in \Gamma$ be a loxodromic element of Γ . Then as $s \rightarrow +\infty$,*

$$\frac{(q_v + 1)^2 \text{Vol}(\Gamma \backslash \mathbb{X}_v)}{2 q_v^2 \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0})} s^{-1} \sum_{\alpha \in \Gamma \cdot \gamma_0^-, h(\alpha) \leq s} \Delta_\alpha \stackrel{*}{\sim} \text{Haar}_{K_v}$$

and

$$\text{Card}\{\alpha \in (\Gamma \cdot \gamma_0^-) \cap \mathcal{O}_v : h(\alpha) \leq s\} \sim \frac{2 q_v^2 \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0})}{(q_v + 1)^2 \text{Vol}(\Gamma \backslash \mathbb{X}_v)} s.$$

When Γ is geometrically finite, there is an error term of the form $O(s^{1-\kappa})$ for some $\kappa = \kappa_\Gamma > 0$ in the counting claim and, for every $\beta \in]0, \frac{1}{\ln q_v}]$, an error term of the form $O(s^{-\kappa} \|\psi\|_\beta)$ for some $\kappa > 0$ in the equidistribution claim evaluated on any β -Hölder-continuous function $\psi : \mathcal{O}_v \rightarrow \mathbb{C}$.

Proof. The second result follows from the first one by integrating on the characteristic function of the compact-open subset \mathcal{O}_v , whose Haar measure is 1.

³The groups $\text{GL}_2(K_v)$ and $\text{PGL}_2(K_v)$ act on $\mathbb{P}^1(K_v) = K_v \cup \{\infty\}$ by homographies. See Section 15.1.

⁴See Equation (14.3).

In order to prove the equidistribution result, we apply Theorem 15.4 with $\mathbb{D}^- := \{*_v\}$ and $\mathbb{D}^+ := \text{Ax}_{\gamma_0}$. The families $\mathcal{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite, since Γ is discrete and the stabiliser $\Gamma_{\mathbb{D}^+}$ of \mathbb{D}^+ acts cocompactly on \mathbb{D}^+ . Furthermore, $\|\sigma_{\mathcal{D}^+}^-\|$ is finite and nonzero by Equation (8.11). Since Γ is contained in $\text{PGL}_2(K_v)^+$, the length spectrum L_Γ of Γ is contained in $2\mathbb{Z}$ by [Ser3, II.1.2, Coro.]. Hence, it is equal to $2\mathbb{Z}$ by the equivalence given by Equation (4.13).

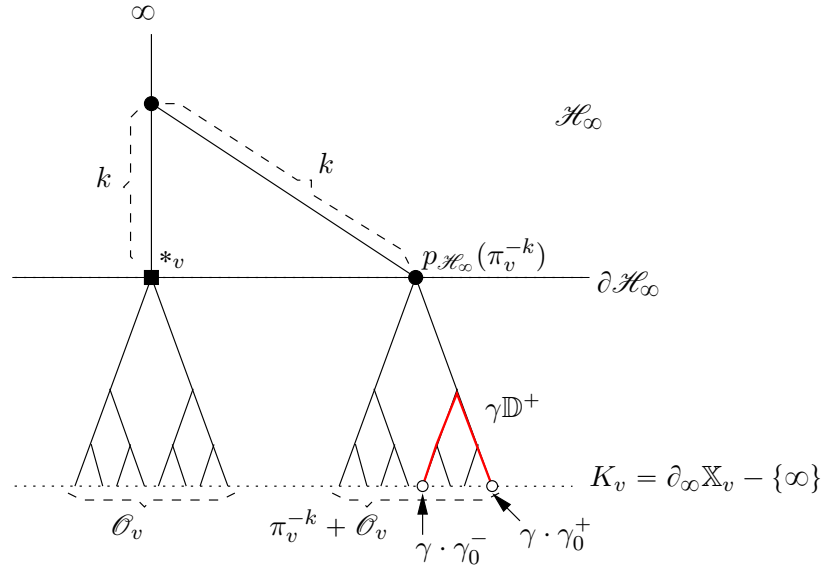
For every $\gamma \in \Gamma$ such that $d(\mathbb{D}^-, \gamma\mathbb{D}^+) > 0$,⁵ let ρ_γ be the geodesic ray starting at time 0 from the origin of $\alpha_{e, \gamma}^-$ (which is $*_v$) with point at infinity $\gamma \cdot \gamma_0^-$. Since \mathbb{X}_v is a tree and $\gamma \cdot \gamma_0^-$ is one of the two endpoints of $\gamma\mathbb{D}^+$, the geodesic segment $\alpha_{e, \gamma}^-|_{[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]}$ is an initial subsegment of ρ_γ .⁶ Therefore, by Equation (15.15), for the weak-star convergence of measures on $\partial_+^1 \mathbb{D}^-$, we have

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} \Delta_{\gamma \cdot \gamma_0^-} = (\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+. \quad (17.2)$$

Furthermore, for every $\beta \in]0, \frac{1}{\ln q_v}]$, by the comment following Equation (15.15), we have an error term of the form $O(\|\psi\|_\beta e^{-\kappa n})$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathcal{C}_c^\beta(\partial_\infty \mathbb{X}_v)$, where $\partial_\infty \mathbb{X}_v$ is endowed with the visual distance d_{*_v} . Note that on $x, y \in \mathcal{O}_v$, the visual distance d_{*_v} and the distance $(x, y) \mapsto |x - y|_v$ are related by

$$|x - y|_v = d_{\mathcal{H}_\infty}(x, y)^{\ln q_v} = d_{*_v}(x, y)^{\ln q_v},$$

see Equation (15.3). Hence we have an error term $O(\|\psi\|_\beta e^{-\kappa n})$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathcal{C}_c^\beta(\mathcal{O}_v)$, where \mathcal{O}_v is endowed with the distance $(x, y) \mapsto |x - y|_v$.



Let us fix for the moment $k \in \mathbb{N}$. For every $\xi \in \pi_v^{-k} + \mathcal{O}_v$, we have $|\xi|_v = q_v^{-v(\xi)} = q_v^k$ if $k \geq 1$ and $|\xi|_v \leq 1$ if $k = 0$. By restricting the measures to the compact-open subset $\pi_v^{-k} + \mathcal{O}_v$

⁵that is, such that $*_v \notin \gamma\mathbb{D}^+$

⁶It connects $*_v$ to its closest point $P_{\gamma\mathbb{D}^+}(*_v)$ on $\gamma\mathbb{D}^+$.

and by Proposition 15.2 (1), we have, with the appropriate error term when Γ is geometrically finite and $k = 0$,

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ \gamma \cdot \gamma_0^- \in \pi_v^{-k} + \mathcal{O}_v \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \Delta_{\gamma \cdot \gamma_0^-} = q_v^{-2k} \text{Haar}_{K_v |_{(\pi_v)^{-k} + \mathcal{O}_v}} . \quad (17.3)$$

If $\beta \in \Gamma$ is loxodromic and satisfies $\beta^- \in \pi_v^{-k} + \mathcal{O}_v$ and $\beta^+ \notin \pi_v^{-k} + \mathcal{O}_v$, then the translation axis of β passes at distance at most $2k$ from $*_v$, since it passes through $P_{\mathcal{H}_\infty}(\pi_v^{-k})$ which is the closest point on \mathcal{H}_∞ to π_v^{-k} . If $\beta \in \Gamma$ is loxodromic and satisfies $\beta^-, \beta^+ \in \pi_v^{-k} + \mathcal{O}_v$, then

$$d(*_v, \text{Ax}_\beta) = 2k + d(\mathcal{H}_\infty, \text{Ax}_\beta) .$$

Furthermore, we have, by Equations (15.2) and (2.8)

$$|\beta^- - \beta^+|_v = d_{\mathcal{H}_\infty}(\beta^-, \beta^+) \ln q_v = q_v^{-d(\mathcal{H}_\infty, \text{Ax}_\beta)} .$$

Therefore by the definition of the complexity in Equation (17.1), we have for these elements

$$h(\beta^-) = \frac{1}{|\beta^- - \beta^+|_v} = q_v^{d(\mathcal{H}_\infty, \text{Ax}_\beta)} = q_v^{d(*_v, \text{Ax}_\beta) - 2k} . \quad (17.4)$$

Since the family $\mathcal{D}^+ = (\gamma \mathbb{D}^+)_{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+}}$ is locally finite, there are only finitely many elements $\gamma \in \Gamma/\Gamma_{\mathbb{D}^+}$ such that $\gamma \mathbb{D}^+ = \text{Ax}_{\gamma \gamma_0 \gamma^{-1}}$ is at distance at most $2k$ from $*_v$. Hence for all but finitely many $\gamma \in \Gamma/\Gamma_{\mathbb{D}^+}$ such that $\gamma \cdot \gamma_0^- = (\gamma \gamma_0 \gamma^{-1})^- \in \pi_v^{-k} + \mathcal{O}_v$, we have $\gamma \cdot \gamma_0^+ = (\gamma \gamma_0 \gamma^{-1})^+ \in \pi_v^{-k} + \mathcal{O}_v$ and, using Equation (17.4) with $\beta = \gamma \gamma_0 \gamma^{-1}$,

$$h(\gamma \cdot \gamma_0^-) = q_v^{d(\mathbb{D}^-, \gamma \mathbb{D}^+) - 2k} .$$

Therefore, using the change of variable $s = q_v^{n-2k}$, Equation (17.3) becomes

$$\lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} s^{-1} \sum_{\substack{\gamma \in \Gamma/\Gamma_{\mathbb{D}^+} \\ \gamma \cdot \gamma_0^- \in \pi_v^{-k} + \mathcal{O}_v \\ 0 < h(\gamma \cdot \gamma_0^-) \leq s}} \Delta_{\gamma \cdot \gamma_0^-} = \text{Haar}_{K_v |_{(\pi_v)^{-k} + \mathcal{O}_v}} . \quad (17.5)$$

Note that the stabiliser $\Gamma_{\gamma_0^-}$ of γ_0^- in Γ has index $\iota_{\gamma_0^-}$ in $\Gamma_{\mathbb{D}^+}$ by the definition of $\iota_{\gamma_0^-}$ and that $\Gamma/\Gamma_{\gamma_0^-}$ identifies with $\Gamma \cdot \gamma_0^-$ by the map $\gamma \Gamma_{\gamma_0^-} \mapsto \gamma \cdot \gamma_0^-$. Since $((\pi_v)^{-k} + \mathcal{O}_v)_{k \in \mathbb{N}}$ is a countable family of pairwise disjoint compact-open subsets covering K_v , and since the support of any continuous function with compact support is contained in finitely many elements of this family, we have

$$\lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3 \iota_{\gamma_0^-}} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} s^{-1} \sum_{\substack{\alpha \in \Gamma \cdot \gamma_0^- \\ 0 < h(\alpha) \leq s}} \Delta_\alpha = \text{Haar}_{K_v} , \quad (17.6)$$

with the appropriate error term when Γ is geometrically finite.

Recall that by Equation (8.11), if the Patterson measures are normalised to be probability measures, then

$$\|\sigma_{\mathcal{D}^+}^-\| = \frac{q_v - 1}{q_v + 1} \text{Vol}(\Gamma_{\mathbb{D}^+} \backslash \mathbb{D}^+).$$

Hence if instead the Patterson densities are normalised to have total mass $\frac{q_v+1}{q_v}$ as in Proposition 15.2 (2), then

$$\|\sigma_{\mathcal{D}^+}^-\| = \frac{q_v - 1}{q_v} \text{Vol}(\Gamma_{\mathbb{D}^+} \backslash \mathbb{D}^+).$$

Note that, since $\iota_{\gamma_0^-} = [\Gamma_{\text{Ax}\gamma_0} : \Gamma_{\gamma_0^-}]$,

$$\text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}\gamma_0) = \iota_{\gamma_0^-} \text{Vol}(\Gamma_{\text{Ax}\gamma_0} \backslash \text{Ax}\gamma_0).$$

Equation (17.6) thus gives the equidistribution result in Theorem 17.1. \square

In the following two Sections, we use Theorem 17.1 to deduce counting and equidistribution results of elements of non-Archimedean local fields that are quadratic irrational over appropriate subfields, when an appropriate algebraic complexity tends to infinity.

17.2 Counting and equidistribution of quadratic irrationals in positive characteristic

Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , and let R_v be the affine function ring associated with v . We assume in this Section that the characteristic of K is different from 2.⁷

An element $\beta \in K_v$ is *quadratic irrational* over K if $\beta \notin K$ and β is a root of a quadratic polynomial $a\beta^2 + b\beta + c$ for some $a, b, c \in K$ with $a \neq 0$. The *Galois conjugate* β^σ of β is the other root of the same polynomial. Let $\text{tr}(\beta) = \beta + \beta^\sigma$ and $\mathbf{n}(\beta) = \beta\beta^\sigma$ be the relative *trace* and relative *norm* of β . It is easy to check that $\beta^\sigma \neq \beta$ since the characteristic of K is different from 2. The next proposition gives a characterisation of quadratic irrationals over K .

Proposition 17.2. *Let $\beta \in K_v$. The following assertions are equivalent:*

- (1) β is quadratic irrational over K ,
- (2) β is a fixed point of a loxodromic element of $\text{PGL}_2(R_v)$.

Proof. The fact that (2) implies (1) is immediate since $\text{PGL}_2(R_v)$ acts by homographies. The converse is classical, see for instance [PaP11b, Lem. 6.2] in the Archimedean case and [BerN] above its Section 5 when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. \square

If $\beta \in K_v$ is quadratic irrational over K , its Galois conjugate β^σ is the other fixed point of a loxodromic element of $\text{PGL}_2(R_v)$ fixing β , hence the notations β^σ in this Section and in Section 17.1 coincide.

The actions by homographies of the groups $\text{GL}_2(R_v)$ and $\text{PGL}_2(R_v)$ on $K_v \cup \{\infty\}$ preserve the set of quadratic irrationals over K . Contrary to the case of rational points, both groups act with infinitely many orbits.

⁷This is equivalent to q being odd.

The *complexity* of a quadratic irrational $\alpha \in K_v$ over K is

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|_v},$$

see for instance [HeP4, §6] for motivations and results when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. Note that this complexity is invariant under the action of the stabiliser $\mathrm{GL}_2(R_v)_\infty$ of ∞ in $\mathrm{GL}_2(R_v)$, which is its upper triangular subgroup. In particular, it is invariant under the action of R_v by translations.⁸ In [PaP12], where K and $|\cdot|_v$ are replaced by \mathbb{Q} and its Archimedean absolute value, there was, for convenience, an extra factor 2 in the numerator of the complexity, which is not needed here. We refer for instance to [PaP12, Lem. 4.2] for the connection of this complexity to the standard height, and to [PaP12, §4.2, 4.4] and [PaP12, §6.1] for studies using this complexity.

The complexity $h(\cdot)$ satisfies the following elementary properties, giving in particular its behaviour under the action of $\mathrm{PGL}_2(R_v)$ by homographies on the quadratic irrationals in K_v over K . We also give the well-known computation of the Jacobian of the Haar measure for the change of variables given by homographies, and prove (using the properties of the complexity $h(\cdot)$) the invariance of a measure which will be useful in Section 18.1.

For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$ and $z \in K_v$ such that $g \cdot z \neq \infty$, let

$$jg(z) = \frac{|\det g|_v}{|cz + d|_v^2}.$$

Proposition 17.3. *Let $\alpha \in K_v$ be a quadratic irrational over K .*

(1) *We have $h(\alpha) = \frac{1}{\sqrt{|\mathrm{tr}(\alpha)^2 - 4\mathbf{n}(\alpha)|_v}}$.*

(2) *For every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$ with $|\det g|_v = 1$, we have*

$$h(g \cdot \alpha) = |\mathbf{n}(d + c\alpha)|_v h(\alpha).$$

(3) *If $Q_\alpha : R_v \times R_v \rightarrow [0, +\infty[$ is the map $(x, y) \mapsto |\mathbf{n}(x - y\alpha)|_v$, then for every $g \in \mathrm{GL}_2(R_v)$, we have*

$$Q_{g \cdot \alpha} = \frac{h(\alpha)}{h(g \cdot \alpha)} Q_\alpha \circ g^{-1}.$$

In particular, if $g \in \mathrm{GL}_2(R_v)$ fixes α , then

$$Q_\alpha \circ g = Q_\alpha.$$

(4) *For all $x, y, z \in K_v$ and $g \in \mathrm{GL}_2(K_v)$ such that $g \cdot x, g \cdot y, g \cdot z \neq \infty$, we have*

$$|g \cdot x - g \cdot y|_v^2 = |x - y|_v^2 jg(x) jg(y)$$

and

$$jg(z) = \frac{d(g^{-1})_* \mathrm{Haar}_{K_v}}{d \mathrm{Haar}_{K_v}}(z).$$

⁸This is a particular case of Proposition 17.3 (2) below.

(5) *The measure*

$$d\mu(z) = \frac{d \text{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v}$$

on $K_v - \{\alpha, \alpha^\sigma\}$ is invariant under the stabiliser of α in $\text{PGL}_2(R_v)$

Proof. (1) This follows from the formula $(\alpha - \alpha^\sigma)^2 = (\alpha + \alpha^\sigma)^2 - 4\alpha\alpha^\sigma$.

(2) Since g has rational coefficients (that is, coefficients in K), we have

$$\begin{aligned} g \cdot \alpha - (g \cdot \alpha)^\sigma &= g \cdot \alpha - g \cdot \alpha^\sigma = \frac{a\alpha + b}{c\alpha + d} - \frac{a\alpha^\sigma + b}{c\alpha^\sigma + d} \\ &= \frac{(ad - bc)(\alpha - \alpha^\sigma)}{(c\alpha + d)(c\alpha^\sigma + d)} = \frac{(\det g)(\alpha - \alpha^\sigma)}{\mathfrak{n}(d + c\alpha)}. \end{aligned}$$

Taking absolute values and inverses, this gives Assertion (2).

(3) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R_v)$. Note that $g^{-1} \cdot \alpha = \frac{d\alpha - b}{a - c\alpha}$. For all $x, y \in R_v$, we hence have

$$\begin{aligned} \mathfrak{n}((ax + by) - (cx + dy)\alpha) &= \mathfrak{n}(x(a - c\alpha) - y(d\alpha - b)) \\ &= \mathfrak{n}(x(a - c\alpha) - y(a - c\alpha)g^{-1} \cdot \alpha) \\ &= \mathfrak{n}(a - c\alpha) \mathfrak{n}(x - yg^{-1} \cdot \alpha). \end{aligned}$$

Taking absolute values and using Assertion (2), we have

$$Q_\alpha \circ g = \frac{h(g^{-1} \cdot \alpha)}{h(\alpha)} Q_{g^{-1} \cdot \alpha}.$$

Assertion (3) follows by replacing g by its inverse.

(4) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(K_v)$. As seen in the proof of Assertion (2), we have

$$g \cdot x - g \cdot y = \frac{(\det g)(x - y)}{(cx + d)(cy + d)}.$$

Taking absolute values and squares, this gives the first claim of Assertion (4).

Recall that a homography $z \mapsto \frac{az+b}{cz+d}$ is holomorphic⁹ on $K_v - \{-\frac{d}{c}\}$, with derivative $z \mapsto \frac{ad-bc}{(cz+d)^2}$. Hence infinitesimally close to z , the homography acts (up to translations which leave the Haar measure invariant) by a homothety of ratio $\frac{ad-bc}{(cz+d)^2}$. By Equation (14.6), this proves that

$$d\text{Haar}_{K_v}(g \cdot z) = \frac{|\det g|_v}{|cz + d|_v^2} d\text{Haar}_{K_v}(z),$$

as wanted.

⁹We refer for instance to [Ser4] for background on holomorphic functions on non-Archimedean local fields.

(5) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R_v)$ fixing α . Note that an element of $\mathrm{GL}_2(R_v)$ which fixes α also fixes α^σ . By Assertion (4), we have

$$\begin{aligned} d\mu(g \cdot z) &= \frac{d\mathrm{Haar}_{K_v}(g \cdot z)}{|g \cdot z - \alpha|_v |g \cdot z - \alpha^\sigma|_v} = \frac{d\mathrm{Haar}_{K_v}(g \cdot z)}{|g \cdot z - g \cdot \alpha|_v |g \cdot z - g \cdot \alpha^\sigma|_v} \\ &= \frac{jg(z) d\mathrm{Haar}_{K_v}(z)}{|z - \alpha|_v \sqrt{jg(z) jg(\alpha)} |z - \alpha^\sigma|_v \sqrt{jg(z) jg(\alpha^\sigma)}} \\ &= \frac{1}{\sqrt{jg(\alpha) jg(\alpha^\sigma)}} d\mu(z). \end{aligned}$$

Again by Assertion (4), since $|\det g|_v = 1$ as $\det g \in (R_v)^\times = (\mathbb{F}_q)^\times$ by Equation (14.3), by Assertion (2) and since g fixes α , we have

$$\begin{aligned} \frac{1}{\sqrt{jg(\alpha) jg(\alpha^\sigma)}} &= |c\alpha + d|_v |c\alpha^\sigma + d|_v = |\mathbf{n}(c\alpha + d)|_v \\ &= \frac{h(g \cdot \alpha)}{h(\alpha)} = 1. \end{aligned}$$

The result follows. \square

Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$. We say that a quadratic irrational $\beta \in K_v$ over K is G -reciprocal (simply reciprocal if $G = \mathrm{GL}_2(R_v)$) if some element of G maps β to β^σ . We define the G -reciprocity index $i_G(\beta)$ as 2 if β is G -reciprocal and 1 otherwise. Similarly, we say that a loxodromic element γ of G is G -reciprocal (simply reciprocal if $G = \mathrm{GL}_2(R_v)$) if there exists an element in G that switches the two fixed points of γ .

Proposition 17.4. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let γ be a loxodromic element of G . The following assertions are equivalent:*

- (1) γ is conjugate in G to $\gamma'\gamma^{-1}$ for some $\gamma' \in G$ pointwise fixing Ax_γ ,
- (2) the loxodromic element γ is G -reciprocal,
- (3) the quadratic irrational γ^- is G -reciprocal.

When $G = \mathrm{GL}_2(R_v)$, Assertions (1), (2) and (3) are also equivalent to

- (4) the image of $\gamma''\gamma$ in $\mathrm{PGL}_2(R_v)$, for some $\gamma'' \in G$ pointwise fixing Ax_γ , is conjugate to the image in $\mathrm{PGL}_2(R_v)$ of ${}^t\gamma$.

Proof. Most of the proofs are similar to the ones when R_v , K and v are replaced by \mathbb{Z} , \mathbb{Q} and its Archimedean absolute value, see for instance [PaP12]. We only give hints for the sake of completeness. Let $\alpha = \gamma^-$.

If α is G -reciprocal, then let $\beta \in G$ be such that $\beta \cdot \alpha = \alpha^\sigma$. Since $R_v \subset K$, we have $\beta \cdot \alpha^\sigma = \alpha$. Hence $\beta\gamma\beta^{-1}$ is a loxodromic element of G fixing α and α^σ , having the same translation length as γ , but translating in the opposite direction on Ax_γ . Hence $\gamma' = \beta\gamma\beta^{-1}\gamma$ fixes pointwise Ax_γ . Therefore (3) implies (1).

If $\beta \in G$ conjugates γ to $\gamma'\gamma^{-1}$ for some $\gamma' \in G$ pointwise fixing Ax_γ , then β preserves the set $\{\alpha, \alpha^\sigma\}$. Hence, it preserves the translation axis of γ but it switches α and α^σ since γ and $\gamma'\gamma^{-1}$ translate in opposite directions on Ax_γ . Therefore (1) implies (2).

The fact that (2) implies (3) is immediate, since $\alpha^\sigma = \gamma^+$.

The equivalence between (1) and (4) when $G = \text{GL}_2(R_v)$ follows from the fact that the stabiliser of Ax_γ normalises the pointwise stabiliser of Ax_γ , and from the formula

$${}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

which is valid over any field. □

The following result says that any orbit of a given quadratic irrational in K_v over K , by homographies under a given finite index subgroup of the modular group $\text{PGL}_2(R_v)$, equidistributes to the Haar measure on K_v . Again, note that we are not assuming the finite index subgroup to be a congruence subgroup.

Theorem 17.5. *Let G be a finite index subgroup of $\text{GL}_2(R_v)$. Let $\alpha_0 \in K_v$ be a quadratic irrational over K . Then, as $s \rightarrow +\infty$,*

$$\frac{(q_v + 1)^2 \zeta_K(-1) m_0 [\text{GL}_2(R_v) : G]}{2 q_v^2 (q - 1) |v(\text{tr } g_0)|} s^{-1} \sum_{\alpha \in G \cdot \alpha_0 : h(\alpha) \leq s} \Delta_\alpha \xrightarrow{*} \text{Haar}_{K_v},$$

where $g_0 \in G$ fixes α_0 with $v(\text{tr } g_0) \neq 0$, and where m_0 is the index of $g_0^{\mathbb{Z}}$ in the stabiliser of α_0 in G . Furthermore, there exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,

$$\text{Card}\{\alpha \in (G \cdot \alpha_0) \cap \mathcal{O}_v : h(\alpha) \leq s\} = \frac{2 q_v^2 (q - 1) |v(\text{tr } g_0)|}{(q_v + 1)^2 \zeta_K(-1) m_0 [\text{GL}_2(R_v) : G]} s + O(s^{1-\kappa}).$$

For every $\beta \in]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^\beta(K_v)$ there is an error term in the above equidistribution claim evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

Proof. We apply Theorem 17.1 with Γ the image of G in $\text{PGL}_2(R_v)$ and with γ_0 the image in $\text{PGL}_2(R_v)$ of the element g_0 introduced in the statement. Note that Γ , which is contained in Γ_v , is indeed contained in $\text{PGL}_2(K_v)^+$.

By Equation (15.6), the translation length of g in \mathbb{X}_v is $2 |v(\text{tr } g)|$, and $g \in \text{GL}_2(R_v)$ is loxodromic if and only if $v(\text{tr } g) \neq 0$. This implies that g_0 exists, since G has finite index in $\text{GL}_2(R_v)$, and such an element exists in $\text{GL}_2(R_v)$ by Proposition 17.2. Furthermore

$$\lambda(\gamma_0) = 2 |v(\text{tr } g_0)|.$$

Since the centre of $\text{GL}_2(K_v)$ acts trivially by homographies, we have

$$G \cdot \alpha_0 = \Gamma \cdot \alpha_0.$$

For every $\alpha \in G \cdot \alpha_0$, the complexities $h(\alpha)$, when α is considered as a quadratic irrational or when α is considered as a loxodromic fixed point, coincide.

Since the centre $Z(G)$ of G acts trivially by homographies, by the definition of m_0 in the statement, we have

$$[\Gamma_{\gamma_0^-} : \gamma_0^{\mathbb{Z}}] = \frac{[G_{\alpha_0} : g_0^{\mathbb{Z}}]}{|Z(G)|} = \frac{m_0}{|Z(G)|}.$$

Therefore,

$$\begin{aligned} \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0}) &= \frac{1}{[\Gamma_{\gamma_0^-} : \gamma_0^{\mathbb{Z}}]} \text{Vol}(\gamma_0^{\mathbb{Z}} \backslash \text{Ax}_{\gamma_0}) = \frac{\lambda(\gamma_0)}{[\Gamma_{\gamma_0^-} : \gamma_0^{\mathbb{Z}}]} \\ &= \frac{2 |v(\text{tr } g_0)| |Z(G)|}{m_0}. \end{aligned} \tag{17.7}$$

Theorem 17.5 now follows from Theorem 17.1 using Equations (16.6) and (17.7). \square

Example 17.6. (1) Theorem 1.13 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, and by using Equation (14.5) in order to simplify the constant.

(2) Let G_I be the Hecke congruence subgroup associated with a nonzero ideal I of R_v , see Equation (16.11). By Lemma 16.5, we have, as $s \rightarrow +\infty$,

$$\frac{(q_v + 1)^2 \zeta_K(-1) m_0 N(I) \prod_{\mathfrak{p}|I} (1 + \frac{1}{N(\mathfrak{p})})}{q_v^2 (q - 1) |v(\text{tr } g_0)|} s^{-1} \sum_{\alpha \in G_I \cdot \alpha_0 : h(\alpha) \leq s} \Delta_\alpha \xrightarrow{*} \text{Haar}_{K_v}.$$

We conclude this Section by a characterisation of quadratic irrationals and reciprocal quadratic irrationals in the field of formal Laurent series $\mathbb{F}_q((Y^{-1}))$ in terms of continued fractions. When $\mathbb{F}_q[Y]$, $\mathbb{F}_q(Y)$ and v_∞ are replaced by \mathbb{Z} , \mathbb{Q} and its Archimedean absolute value, we refer for instance to [Sarn] and [PaP12, Prop. 4.3] for characterisations of reciprocal quadratic irrationals.

Recall that Artin's *continued fraction expansion* of $f \in \mathbb{F}_q((Y^{-1})) - \mathbb{F}_q(Y)$ is the sequence $(a_i = a_i(f))_{i \in \mathbb{N}}$ in $\mathbb{F}_q[Y]$ with $\deg a_i > 0$ if $i > 0$ such that

$$f = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}}.$$

See for instance the surveys [Las, Sch2], and [Pau1] for a geometric interpretation. We say that the continued fraction expansion of f is *eventually periodic* if there exist $n \in \mathbb{N}$ and $N \in \mathbb{N} - \{0\}$ such that $a_{n+i} = a_{n+N+i}$ for every $i \in \mathbb{N}$, and we write

$$f = [a_0, \dots, a_{n-1}, \overline{a_n, \dots, a_{n+N-1}}].$$

Such a sequence a_n, \dots, a_{n+N-1} is called a *period* of f , and if of minimal length, it is well defined up to cyclic permutation.

Two elements $\beta, \beta' \in \mathbb{F}_q((Y^{-1}))$ are in the same $\text{PGL}_2(\mathbb{F}_q[Y])$ -orbit if and only if their continued fraction expansions have equal tails up to an invertible element of $\mathbb{F}_q[Y]$ by [BerN, Theo. 1]. More precisely, $\beta, \beta' \in K_v$ are in the same $\text{PGL}_2(\mathbb{F}_q[Y])$ -orbit if and only if there exist $m, n \in \mathbb{N}$ and $x \in \mathbb{F}_q^\times$ such that for every $k \in \mathbb{N}$, we have $a_{n+k}(\beta') = x^{(-1)^k} a_{m+k}(\beta)$.

Proposition 17.7. *Let $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.*

- (1) *An element $\alpha \in K_v - K$ is quadratic irrational over K if and only if its continued fraction expansion of β is eventually periodic, and if and only if it is a fixed point of a loxodromic element of $\text{PGL}_2(\mathbb{F}_q[Y])$.*

- (2) A quadratic irrational $\alpha \in K_v$ is reciprocal if and only if the period a_0, \dots, a_{N-1} of the continued fraction expansion of α is palindromic up to cyclic permutation and invertible elements, in the sense that there exist $x \in \mathbb{F}_q^\times$ and $p \in \mathbb{N}$ such that for $k = 0, \dots, N-1$, we have $a_{k+p} = x^{(-1)^k} a_{N-k-1}$ (with indices modulo N).

Proof. (1) The equivalence of being quadratic irrational and having an eventually periodic continued fraction expansion is well-known, see for instance the survey [Las, Theo. 3.1]. The rest of the claim follows from Proposition 17.2.

(2) The proof is similar to the Archimedean case in [Per, §23].¹⁰ For every quadratic irrational $f \in \mathbb{F}_q((Y^{-1}))$, up to the action of $\mathrm{GL}_2(\mathbb{F}_q[[Y]])$, we may assume that $f, (f^\sigma)^{-1} \in Y^{-1}\mathbb{F}_q[[Y^{-1}]]$ and $f = [0, \overline{a_1, a_2, \dots, a_n}]$. Then we may define by induction quadratic irrationals $f_2, \dots, f_n \in \mathbb{F}_q((Y^{-1}))$ over $\mathbb{F}_q(Y)$ such that

$$\frac{1}{f} = a_1 + f_2, \quad \frac{1}{f_2} = a_2 + f_3, \quad \dots, \quad \frac{1}{f_{n+1}} = a_{n+1} + f_n, \quad \frac{1}{f_n} = a_n + f.$$

Passing to the Galois conjugates, we have

$$\frac{1}{f^\sigma} = a_1 + f_2^\sigma, \quad \frac{1}{f_2^\sigma} = a_2 + f_3^\sigma, \quad \dots, \quad \frac{1}{f_n^\sigma} = a_n + f^\sigma.$$

Taking these equations in the reverse order, we have

$$\frac{1}{-\frac{1}{f^\sigma}} = a_n - \frac{1}{f_n^\sigma}, \quad \frac{1}{-\frac{1}{f_n^\sigma}} = a_{n-1} - \frac{1}{f_{n-1}^\sigma}, \quad \dots, \quad \frac{1}{-\frac{1}{f_2^\sigma}} = a_1 - \frac{1}{f^\sigma},$$

so that, since $-\frac{1}{f^\sigma} \in Y^{-1}\mathbb{F}_q[[Y^{-1}]]$, we have

$$-\frac{1}{f^\sigma} = [0, \overline{a_n, \dots, a_2, a_1}].$$

Therefore $f^\sigma = [\overline{-a_n, \dots, -a_2, -a_1}]$. Thus, if f and f^σ are in the same orbit, the periods are palindromic by [BerN, Theo. 1]. \square

17.3 Counting and equidistribution of quadratic irrationals in \mathbb{Q}_p

There are interesting arithmetic (uniform) lattices of $\mathrm{PGL}_2(\mathbb{Q}_p)$ constructed using quaternion algebras. In this Section, we study equidistribution properties of loxodromic fixed points elements of these lattices. We use [Vig] as our standard reference on quaternion algebras.

Let F be a field and let $a, b \in F^\times$. Let $D = (\frac{a,b}{F})$ be the quaternion algebra over F with basis $1, i, j, k$ as a F -vector space such that $i^2 = a$, $j^2 = b$ and $ij = ji = -k$. If $x = x_0 + x_1i + x_2j + x_3k \in D$, then its *conjugate* is

$$\bar{x} = x_0 - x_1i - x_2j - x_3k,$$

¹⁰See also [BerN, Coro. 1] by relating, using twice the period, what the authors call the $-$ continued fraction expansion to the standard expansion.

its (reduced) *norm* is

$$\mathbf{N}(x) = x\bar{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$$

and its (reduced) *trace* is

$$\mathbf{Tr}(x) = x + \bar{x} = 2x_0.$$

Let us fix two negative rational integers a, b and let $D = \left(\frac{a, b}{\mathbb{Q}}\right)$. For every field extension E of \mathbb{Q} , we denote by D_E the quaternion algebra $D \otimes_{\mathbb{Q}} E$ over E , and we say that D *splits over* E if the E -algebra $D \otimes_{\mathbb{Q}} E$ is isomorphic to $M_2(E)$. The assumption that a, b are negative implies that D does not split over \mathbb{R} . Furthermore, when $p \in \mathbb{N}$ is an odd prime, D splits over \mathbb{Q}_p if and only if the equation $ax^2 + by^2 = 1$ has a solution in \mathbb{Q}_p , see [Fig, page 32].

The *reduced discriminant* of D is

$$\mathbf{Disc}_D = \prod_{q \in \text{Ram}(D)} q.$$

where $\text{Ram}(D)$ is the finite set of primes p such that D does not split over \mathbb{Q}_p .

For instance, the quaternion algebra $D = \left(\frac{-1, -1}{\mathbb{Q}}\right)$ splits over \mathbb{Q}_p if and only if $p \neq 2$, hence it has reduced discriminant 2.

Assume from now on that $p \in \mathbb{N}$ is a positive rational prime such that D splits over \mathbb{Q}_p and, for simplicity, that \mathbb{Q}_p contains square roots \sqrt{a} and \sqrt{b} of a and b . For example, if $a = b = -1$, this is satisfied if $p \equiv 1 \pmod{4}$. We then have an isomorphism of \mathbb{Q}_p -algebras $\theta = \theta_{a, b} : D_{\mathbb{Q}_p} \rightarrow M_2(\mathbb{Q}_p)$ defined by

$$\theta(x_0 + x_1i + x_2j + x_3k) = \begin{pmatrix} x_0 + x_1\sqrt{a} & \sqrt{b}(x_2 + \sqrt{a}x_3) \\ \sqrt{b}(x_2 - \sqrt{a}x_3) & x_0 - x_1\sqrt{a} \end{pmatrix}, \quad (17.8)$$

so that

$$\det(\theta(x)) = \mathbf{N}(x) \quad \text{and} \quad \text{tr}(\theta(x)) = \mathbf{Tr}(x).$$

If the assumption on the existence of the square roots in \mathbb{Q}_p is not satisfied, we can replace \mathbb{Q}_p by an appropriate finite extension, and prove equidistribution results in this extension.

Let \mathcal{O} be a $\mathbb{Z}\left[\frac{1}{p}\right]$ -order in $D_{\mathbb{Q}_p}$, that is, a finitely generated $\mathbb{Z}\left[\frac{1}{p}\right]$ -submodule of $D_{\mathbb{Q}_p}$ generating $D_{\mathbb{Q}_p}$ as a \mathbb{Q}_p -vector space, which is a subring of $D_{\mathbb{Q}_p}$. Let \mathcal{O}^1 be the group of elements of norm 1 in \mathcal{O} . Then the image $\Gamma_{\mathcal{O}}^1$ of $\theta(\mathcal{O}^1)$ in $\text{PGL}_2(\mathbb{Q}_p)$ is a cocompact lattice, see for instance [Fig, Sect. IV.1]. In fact, this lattice is contained in $\text{PSL}_2(\mathbb{Q}_p)$, hence in $\text{PGL}_2(\mathbb{Q}_p)^+$. We denote by \mathbb{X}_p the Bruhat-Tits tree of $(\text{PSL}_2, \mathbb{Q}_p)$, which is $(p+1)$ -regular.

The next result computes the covolume of this lattice.¹¹

Proposition 17.8. *Let D be a quaternion algebra over \mathbb{Q} which splits over \mathbb{Q}_p and does not split over \mathbb{R} . If \mathcal{O}_{\max} is a maximal $\mathbb{Z}\left[\frac{1}{p}\right]$ -order in $D_{\mathbb{Q}_p}$ containing \mathcal{O} , then*

$$\text{Vol}(\Gamma_{\mathcal{O}}^1 \backslash \mathbb{X}_p) = [\mathcal{O}_{\max}^1 : \mathcal{O}^1] \frac{p}{12} \prod_{q | \text{Disc}_D} (q-1).$$

Proof. We refer to [Fig, page 53] for the (common) definition of the *discriminant* $\text{Disc}(\mathbb{Q}_p)$ of the local field \mathbb{Q}_p and $\text{Disc}(D_{\mathbb{Q}_p})$ of the quaternion algebra $D_{\mathbb{Q}_p}$ over the local field \mathbb{Q}_p . We will only use the facts that $\text{Disc}(\mathbb{Q}_p) = 1$ as it easily follows from the definition, and that

$$\text{Disc}(D_{\mathbb{Q}_p}) = \text{Disc}(\mathbb{Q}_p)^4 N(p\mathbb{Z}_p)^2 = p^2 \quad (17.9)$$

¹¹The index q ranges over the primes dividing Disc_D , that is, over the elements of $\text{Ram}(D)$.

which follows by [Vig, Lem. 4.7, page 53] and [Vig, Cor. 1.7, page 35] for the first equality and $N(p\mathbb{Z}_p) = \text{Card}(\mathbb{Z}_p/p\mathbb{Z}_p) = \text{Card}(\mathbb{Z}/p\mathbb{Z}) = p$ for the second one.

We refer to [Vig, Sect. II.4, page 52] for the definition of the *Tamagawa measure* μ_T on X^\times when $X = D_{\mathbb{Q}_p}$ or $X = \mathbb{Q}_p$. It is a Haar measure of the multiplicative locally compact group X^\times , and understanding its explicit normalisation is the main point of this proposition. By [Vig, Lem. 4.6, page 52],¹² with dx the Haar measure on the additive group X ,¹³ with $\|x\|$ the *module* of the left multiplication by $x \in X^\times$ on the additive group X ,¹⁴ we have

$$d\mu_T(x) = \frac{1}{\sqrt{\text{Disc}(X)} \|x\|} dx .$$

By [Vig, Lem. 4.3, page 49], identifying $D_{\mathbb{Q}_p}$ to $M_2(\mathbb{Q}_p)$ by θ , the measure of $\text{GL}_2(\mathbb{Z}_p)$ for the measure $\frac{1}{(1-p^{-1}) \|x\|} dx$ is $1 - p^{-2}$. Hence, by scaling and by Equation (17.9), we have

$$\mu_T(\text{GL}_2(\mathbb{Z}_p)) = \frac{(1-p^{-2})(1-p^{-1})}{\sqrt{\text{Disc}(D_{\mathbb{Q}_p})}} = \frac{(p^2-1)(p-1)}{p^4} .$$

Again by [Vig, Lem. 4.3, page 49], the mass of \mathbb{Z}_p^\times for the measure $\frac{1}{(1-p^{-1}) \|x\|} dx$ on \mathbb{Q}_p^\times is 1, hence by scaling

$$\mu_T(\mathbb{Z}_p^\times) = \frac{1-p^{-1}}{\sqrt{\text{Disc}(\mathbb{Q}_p)}} = \frac{p-1}{p} .$$

By [Vig, page 54], since we have an exact sequence

$$1 \longrightarrow \text{SL}_2(\mathbb{Q}_p) \longrightarrow \text{GL}_2(\mathbb{Q}_p) \xrightarrow{\det} \mathbb{Q}_p^\times \longrightarrow 1 ,$$

the Tamagawa measure of $\text{GL}_2(\mathbb{Q}_p)$ disintegrates by the determinant over the Tamagawa measure of \mathbb{Q}_p^\times with conditional measures the translates of a measure on $\text{SL}_2(\mathbb{Q}_p)$, called the *Tamagawa measure* of $\text{SL}_2(\mathbb{Q}_p)$ and again denoted by μ_T . Thus,

$$\mu_T(\text{SL}_2(\mathbb{Z}_p)) = \frac{\mu_T(\text{GL}_2(\mathbb{Z}_p))}{\mu_T(\mathbb{Z}_p^\times)} = \frac{p^2-1}{p^3}$$

By Example 3 on page 108 of [Vig], since the $\mathbb{Z}[\frac{1}{p}]$ -order \mathcal{O}_{\max} is maximal, we have, with $G = \theta(\mathcal{O}_{\max}^1)$,

$$\mu_T(G \backslash \text{SL}_2(\mathbb{Q}_p)) = \frac{1}{24} (1-p^{-2}) \prod_{q | \text{Disc}_D} (q-1) .$$

Since $\text{GL}_2(\mathbb{Q}_p)$ acts transitively on $V\mathbb{X}_p$ with stabiliser of the base point $* = [\mathbb{Z}_p \times \mathbb{Z}_p]$ the maximal compact subgroup $\text{GL}_2(\mathbb{Z}_p)$,¹⁵ and by the centred equation mid-page 116 of [Ser3], we have

$$\begin{aligned} \text{Vol}(G \backslash \mathbb{X}_p) &= \sum_{[x] \in G \backslash V\mathbb{X}_p} \frac{1}{|G_x|} = \frac{\mu_T(G \backslash \text{GL}_2(\mathbb{Q}_p))}{\mu_T(\text{GL}_2(\mathbb{Z}_p))} = \frac{\mu_T(G \backslash \text{SL}_2(\mathbb{Q}_p))}{\mu_T(\text{SL}_2(\mathbb{Z}_p))} \\ &= \frac{p}{24} \prod_{q | \text{Disc}_D} (q-1) . \end{aligned}$$

¹²See also the top of page 55 in loc. cit.

¹³with a normalisation that does not need to be made precise

¹⁴so that $(M_x)_* dx = \|x\| dx$ where $M_x : y \mapsto xy$ is the left multiplication by x on X

¹⁵See Section 15.1.

The natural homomorphism $G = \theta(\mathcal{O}_{\max}^1) \rightarrow \Gamma_{\mathcal{O}_{\max}}^1$ is 2-to-1 and $[\Gamma_{\mathcal{O}_{\max}}^1 : \Gamma_{\mathcal{O}}^1] = [\mathcal{O}_{\max}^1 : \mathcal{O}^1]$, so that

$$\text{Vol}(\Gamma_{\mathcal{O}_{\max}}^1 \backslash \mathbb{X}_p) = 2 \text{Vol}(G \backslash \mathbb{X}_p).$$

Proposition 17.8 follows. \square

Note that the fixed points z for the action on $\mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$ by homographies of the elements in the image of $\theta(D)$ are quadratic over $\mathbb{Q}(\sqrt{a}, \sqrt{b})$. More precisely, $\frac{z}{\sqrt{b}}$ is quadratic over $\mathbb{Q}(\sqrt{a})$. An immediate application of Theorem 17.1, using Proposition 17.8, gives the following result of equidistribution of quadratic elements in \mathbb{Q}_p over $\mathbb{Q}(\sqrt{a}, \sqrt{b})$.

Theorem 17.9. *Let Γ be a finite index subgroup of $\Gamma_{\mathcal{O}}^1$, and let $\gamma_0 \in \Gamma$ be a loxodromic element of Γ . Then as $s \rightarrow +\infty$,*

$$\frac{(p+1)^2 \prod_{q|\text{Disc}_D} (q-1) [\mathcal{O}_{\max}^1 : \mathcal{O}^1] [\Gamma_{\mathcal{O}}^1 : \Gamma]}{24 p \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0})} s^{-1} \sum_{\alpha \in \Gamma \cdot \gamma_0^-, h(\alpha) \leq s} \Delta_{\alpha} \xrightarrow{*} \text{Haar}_{\mathbb{Q}_p},$$

where \mathcal{O}_{\max} is a maximal $\mathbb{Z}[\frac{1}{p}]$ -order in $D_{\mathbb{Q}_p}$ containing \mathcal{O} , and there exists $\kappa > 0$ such that as $s \rightarrow +\infty$

$$\begin{aligned} & \text{Card}\{\alpha \in (\Gamma \cdot \gamma_0^-) \cap \mathbb{Z}_p : h(\alpha) \leq s\} \\ &= \frac{24 p \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0})}{(p+1)^2 \prod_{q|\text{Disc}_D} (q-1) [\mathcal{O}_{\max}^1 : \mathcal{O}^1] [\Gamma_{\mathcal{O}}^1 : \Gamma]} s + O(s^{1-\kappa}). \quad \square \end{aligned}$$

Assume furthermore that the positive rational prime $p \in \mathbb{N}$ is such that $p \equiv 1 \pmod{4}$ and that the integer $\frac{p^2-1}{4}$ is not of the form $4^a(8b+7)$ for $a, b \in \mathbb{N}$ (for instance $p = 5$). By Legendre's three squares theorem (see for instance [Gros]), there exist $x'_1, x'_2, x'_3 \in \mathbb{Z}$ such that $\frac{p^2-1}{4} = x_1'^2 + x_2'^2 + x_3'^2$. Hence there are $x_1, x_2, x_3 \in 2\mathbb{Z}$ such that $p^2 - 1 = x_1^2 + x_2^2 + x_3^2$.

A standard consequence of Hensel's theorem says that when p is odd, a number $n \in \mathbb{Z}$ has a square root in \mathbb{Z}_p if n is relatively prime to p and has a square root modulo p , see for instance [Kna, page 351]. Thus, $1 - p^2$ has a square root in \mathbb{Z}_p , that we denote by $\sqrt{1 - p^2}$. As noticed above, since $p \equiv 1 \pmod{4}$, the element -1 has a square root in \mathbb{Q}_p , that we denote by ε . The element

$$\alpha_0 = \frac{\varepsilon x_1 + \sqrt{1 - p^2}}{x_3 + \varepsilon x_2}$$

is a quadratic irrational in \mathbb{Q}_p over $\mathbb{Q}(\varepsilon)$.

The following result is a counting and equidistribution result of quadratic irrationals over $\mathbb{Q}(\varepsilon)$ in \mathbb{Q}_p . We denote by α^σ the Galois conjugate of a quadratic irrational α in \mathbb{Q}_p over $\mathbb{Q}(\varepsilon)$, and by

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|_p}$$

the *complexity* of α .

Theorem 17.10. Let $D = \left(\frac{-1, -1}{\mathbb{Q}}\right)$ be Hamilton's quaternion algebra over \mathbb{Q} . Let $p \in \mathbb{N}$ be a positive rational prime with $p \equiv 1 \pmod{4}$ such that $\frac{p^2-1}{4}$ is not of the form $4^a(8b+7)$ for $a, b \in \mathbb{N}$ and let \mathcal{O} be the $\mathbb{Z}\left[\frac{1}{p}\right]$ -order¹⁶

$$\mathcal{O} = \left\{x \in \mathbb{Z}\left[\frac{1}{p}\right] + \mathbb{Z}\left[\frac{1}{p}\right]i + \mathbb{Z}\left[\frac{1}{p}\right]j + \mathbb{Z}\left[\frac{1}{p}\right]k : x \equiv 1 \pmod{2}\right\}$$

in $D_{\mathbb{Q}_p}$. Let Γ be a finite index subgroup of $\Gamma_{\mathcal{O}}^1$. Then as $s \rightarrow +\infty$,

$$\frac{(p+1)^2 [\Gamma_{\mathcal{O}}^1 : \Gamma]}{2 p^2 k_{\Gamma}} s^{-1} \sum_{\alpha \in \Gamma \cdot \alpha_0, h(\alpha) \leq s} \Delta_{\alpha} \stackrel{*}{\sim} \text{Haar}_{\mathbb{Q}_p},$$

where k_{Γ} is the smallest positive integer such that $\begin{bmatrix} 1 + \varepsilon x_1 & -x_3 + \varepsilon x_2 \\ x_3 + \varepsilon x_2 & 1 - \varepsilon x_1 \end{bmatrix}^{k_{\Gamma}} \in \Gamma$. Furthermore, there exists $\kappa > 0$ such that as $s \rightarrow +\infty$

$$\text{Card}\{\alpha \in (\Gamma \cdot \alpha_0) \cap \mathbb{Z}_p : h(\alpha) \leq s\} = \frac{2 p^2 k_{\Gamma}}{(p+1)^2 [\Gamma_{\mathcal{O}}^1 : \Gamma]} s + O(s^{1-\kappa}).$$

Proof. The group \mathcal{O}^{\times} of invertible elements of \mathcal{O} is

$$\mathcal{O}^{\times} = \{x \in \mathcal{O} : \mathbb{N}(x) \in p^{\mathbb{Z}}\}.$$

The centre of \mathcal{O}^{\times} is $Z(\mathcal{O}^{\times}) = \{\pm p^n : n \in \mathbb{Z}\}$ and the centre of \mathcal{O}^1 is $Z(\mathcal{O}^1) = \{\pm 1\}$. We identify $\mathcal{O}^1/Z(\mathcal{O}^1)$ with its image in $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$. The quotient group $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$ is a free group on $s = \frac{p+1}{2}$ generators $\gamma_1, \gamma_2, \dots, \gamma_s$, which are the images modulo $Z(\mathcal{O}^{\times})$ of some elements of \mathcal{O} of norm p , see for instance [Lub2, Coro. 2.1.11].¹⁷

Since $\mathbb{N}(p) = p^2$, any reduced word of even length in $S = \{\gamma_1^{\pm}, \gamma_2^{\pm}, \dots, \gamma_s^{\pm}\}$ belongs to $\mathcal{O}^1/Z(\mathcal{O}^1)$. Two distinct elements in S differ by a reduced word of length 2, and γ_1 does not belong to $\mathcal{O}^1/Z(\mathcal{O}^1)$. Hence $\{1, \gamma_1\}$ is a system of left coset representatives of $\mathcal{O}^1/Z(\mathcal{O}^1)$ in $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$, and the index of $\mathcal{O}^1/Z(\mathcal{O}^1)$ in $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$ is

$$[\mathcal{O}^{\times}/Z(\mathcal{O}^{\times}) : \mathcal{O}^1/Z(\mathcal{O}^1)] = 2. \tag{17.10}$$

Let

$$g_0 = \begin{pmatrix} \frac{1+\varepsilon x_1}{p} & \frac{-x_3+\varepsilon x_2}{p} \\ \frac{x_3+\varepsilon x_2}{p} & \frac{1-\varepsilon x_1}{p} \end{pmatrix}.$$

By the definition of the isomorphism θ in Equation (17.8) (with $\sqrt{a} = \sqrt{b} = \varepsilon$) and of the integers x_1, x_2, x_3 , the element g_0 belongs to $\theta(\mathcal{O})$ since x_1, x_2, x_3 are even (and p is odd), and $\det g_0 = 1$. Hence $g_0 \in \theta(\mathcal{O}^1)$. Its fixed points for its action by homography on $\mathbb{P}^1(\mathbb{Q}_p)$ are, by an easy computation,

$$\frac{\varepsilon x_1 \pm \sqrt{1-p^2}}{x_3 + \varepsilon x_2}.$$

¹⁶This order plays an important role in the construction of Ramanujan graphs by Lubotzky, Phillips and Sarnak [LuPS1, LuPS2] (see also [Lub2, §7.4]), and in the explicit construction of free subgroups of $\text{SO}(3)$ in order to construct Hausdorff-Banach-Tarsky paradoxical decompositions of the 2-sphere, see for instance [Lub2, page 11].

¹⁷The group $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$ is denoted by $\Lambda(2)$ in [Lub2, page 11].

In particular, α_0 is one of these two fixed points. Note that $\text{tr } g_0 = \frac{2}{p}$, hence $|v_p(\text{tr } g_0)| = 1$, and the image $[g_0]$ of g_0 in $\text{PGL}_2(\mathbb{Q}_p)$ is a primitive loxodromic element of $\Gamma_{\mathcal{O}}^1$.

Let us define

$$\gamma_0 = [g_0]^{\epsilon k_{\Gamma}}$$

where $\epsilon \in \{\pm 1\}$ is chosen so that $\gamma_0^- = \alpha_0$ and where k_{Γ} is defined in the statement of Theorem 17.10. Since Γ has finite index in $\Gamma_{\mathcal{O}}^1$, some power of $[g_0]$ does belong to Γ , hence k_{Γ} exists (and note that $k_{\Gamma} = 1$ if $\Gamma = \Gamma_{\mathcal{O}}^1$). By the minimality of k_{Γ} , the element γ_0 is a primitive loxodromic element of Γ . We will apply Theorem 17.1 to this γ_0 .

The algebra isomorphism θ induces a group isomorphism from $\mathcal{O}^{\times}/Z(\mathcal{O}^{\times})$ onto its image in $\text{PGL}_2(\mathbb{Q}_p)$, that we denote by $\Gamma_{\mathcal{O}}^{\times}$.¹⁸ By [Lub2, Lem. 7.4.1], the group $\Gamma_{\mathcal{O}}^{\times}$ acts simply transitively on the vertices of the Bruhat-Tits tree \mathbb{X}_p .

In particular, $\Gamma_{\mathcal{O}}^1$ acts freely on \mathbb{X}_p , and by Equation (17.10), we have

$$\begin{aligned} \text{Vol}(\Gamma_{\mathcal{O}}^1 \backslash \mathbb{X}_p) &= [\Gamma_{\mathcal{O}}^{\times} : \Gamma_{\mathcal{O}}^1] \text{Vol}(\Gamma_{\mathcal{O}}^{\times} \backslash \mathbb{X}_p) \\ &= [\mathcal{O}^{\times}/Z(\mathcal{O}^{\times}) : \mathcal{O}^1/Z(\mathcal{O}^1)] \text{Card}(\Gamma_{\mathcal{O}}^{\times} \backslash V\mathbb{X}_p) = 2. \end{aligned} \quad (17.11)$$

Again since $\Gamma_{\mathcal{O}}^1$ (hence Γ) acts freely on \mathbb{X}_p , and since γ_0 is primitive loxodromic in Γ , we have

$$\begin{aligned} \text{Vol}(\Gamma_{\gamma_0^-} \backslash \text{Ax}_{\gamma_0}) &= \text{Card}(\Gamma_{\gamma_0^-} \backslash V\text{Ax}_{\gamma_0}) = \lambda(\gamma_0) \\ &= k_{\Gamma} \lambda([g_0]) = 2 k_{\Gamma} |v_p(\text{tr } g_0)| = 2 k_{\Gamma}. \end{aligned} \quad (17.12)$$

Using Equations (17.11) and (17.12), the result now follows from Theorem 17.1. \square

¹⁸This group is denoted by $\Gamma(2)$ in [Lub2, page 95].

Chapter 18

Counting and equidistribution of crossratios

We use the same notation as in Chapter 17: K_v is a non-Archimedean local field, with valuation v , valuation ring \mathcal{O}_v , choice of uniformiser π_v , residual field k_v of order q_v , and \mathbb{X}_v is the Bruhat-Tits tree of (PGL_2, K_v) . Let Γ be a lattice in $\mathrm{PGL}_2(K_v)$.

In this Chapter, we give counting and equidistribution results in $K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$ of orbit points under Γ , using a complexity defined using crossratios, which is different from the one in Chapter 17. We refer to [PaP14b] for the development when K_v is \mathbb{R} or \mathbb{C} with its standard absolute value.

Recall that the *crossratio* of four pairwise distinct points a, b, c, d in $\mathbb{P}_1(K_v) = K_v \cup \{\infty\}$ is

$$[a, b, c, d] = \frac{(c-a)(d-b)}{(c-b)(d-a)} \in (K_v)^\times,$$

with the standard conventions when one of the points is ∞ . Adopting Ahlfors's terminology in the complex case, the *absolute crossratio* of four pairwise distinct points $a, b, c, d \in \mathbb{P}_1(K_v) = K_v \cup \{\infty\}$ is

$$|a, b, c, d|_v = |[a, b, c, d]|_v = \frac{|c-a|_v |d-b|_v}{|c-b|_v |d-a|_v},$$

with conventions analogous to the previous ones when one of the points is ∞ . As in the classical case, the crossratio and the absolute crossratio are invariant under the diagonal projective action of $\mathrm{GL}_2(K_v)$ on the set of quadruples of pairwise distinct points in $\mathbb{P}_1(K_v)$.

18.1 Counting and equidistribution of crossratios of loxodromic fixed points

Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ . Recall that $\alpha^\sigma, \beta^\sigma$ is the other fixed point of a loxodromic element of Γ fixing α, β , respectively. The *relative height* of β with respect to α is¹

$$h_\alpha(\beta) = \frac{1}{|\alpha - \alpha^\sigma|_v |\beta - \beta^\sigma|_v} \max \{ |\beta - \alpha|_v |\beta^\sigma - \alpha^\sigma|_v, |\beta - \alpha^\sigma|_v |\beta^\sigma - \alpha|_v \}.$$

¹The factor $|\alpha - \alpha^\sigma|_v$ in the denominator, that did not appear in [PaP14b] in the analogous definition for the case when K_v is \mathbb{R} or \mathbb{C} , is there in order to simplify the statements below.

When $\beta \notin \{\alpha, \alpha^\sigma\}$, we have

$$h_\alpha(\beta) = \max\{|\alpha, \beta, \beta^\sigma, \alpha^\sigma|_v, |\alpha, \beta^\sigma, \beta, \alpha^\sigma|_v\} = \frac{1}{\min\{|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v, |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v\}}. \quad (18.1)$$

We will use the relative height as a complexity when β varies in a given orbit of Γ (and α is fixed).

The following properties of relative heights are easy to check using the definitions and the invariance properties of the crossratio.

Lemma 18.1. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ . Then*

- (1) $h_{\alpha^\rho}(\beta^\tau) = h_\alpha(\beta)$ for all $\rho, \tau \in \{\text{id}, \sigma\}$.
- (2) If $\beta \in \{\alpha, \alpha^\sigma\}$, then $h_\alpha(\beta) = 1$.
- (3) $h_{\gamma \cdot \alpha}(\gamma \cdot \beta) = h_\alpha(\beta)$ for every $\gamma \in \Gamma$.
- (4) $h_\alpha(\gamma \cdot \beta) = h_\alpha(\beta)$ for every $\gamma \in \text{Stab}_\Gamma(\{\alpha, \alpha^\sigma\})$. □

The following result relates the relative height of two loxodromic fixed points with the distance between the two translation axes.

Proposition 18.2. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ such that $\beta \notin \{\alpha, \alpha^\sigma\}$. Then*

$$h_\alpha(\beta) = q_v^{d([\alpha, \alpha^\sigma], [\beta, \beta^\sigma])}.$$

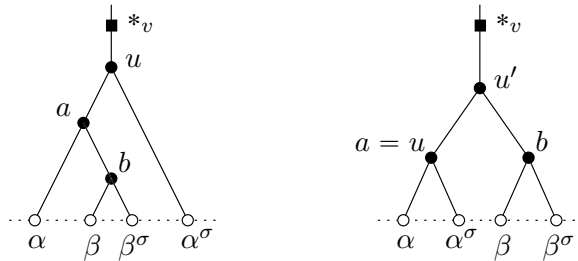
In particular, we have $h_\alpha(\beta) > 1$ if and only if the geodesic lines $]\alpha, \alpha^\sigma[$ and $]\beta, \beta^\sigma[$ in \mathbb{X}_v are disjoint, and $h_\alpha(\beta) = 1$ otherwise (using Lemma 18.1 (2) when $\beta \in \{\alpha, \alpha^\sigma\}$).

Proof. Up to replacing $\alpha, \beta, \alpha^\sigma, \beta^\sigma$ by their images under a big enough power γ of a loxodromic element in Γ with attracting fixed point in \mathcal{O}_v , we may assume that these four points belong to \mathcal{O}_v . Note that γ exists since $\Lambda\Gamma = \partial_\infty \mathbb{X}_v$, and it preserves the relative height by Lemma 18.1 (3) as well as the distances between translation axes.

Let $A =]\alpha, \alpha^\sigma[$ and $B =]\beta, \beta^\sigma[$. Let u be the closest point to $*_v$ on A , so that

$$v(\alpha - \alpha^\sigma) = d(u, *_v).$$

We will consider five configurations.



Case 1. First assume that A and B are disjoint. Let $[a, b]$ be the common perpendicular from A to B , with $a \in A$, so that

$$d(A, B) = d(a, b).$$

First assume that $u \neq a$. Up to exchanging α, α^σ (which does not change $d(A, B)$ or $h_\alpha(\beta)$ by Lemma 18.1 (1)), we may assume that $a \in [u, \alpha[$. Then (see the picture on the left above),

$$v(\beta - \beta^\sigma) = d(b, *_v), \quad v(\alpha - \beta) = v(\alpha - \beta^\sigma) = d(a, *_v)$$

and

$$v(\alpha^\sigma - \beta) = v(\alpha^\sigma - \beta^\sigma) = d(u, *_v) .$$

Therefore

$$\begin{aligned} |\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v &= |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{d(a, *_v)-d(b, *_v)} = q_v^{-d(a, b)} = q_v^{-d(A, B)} , \end{aligned}$$

which proves the result by Equation (18.1).

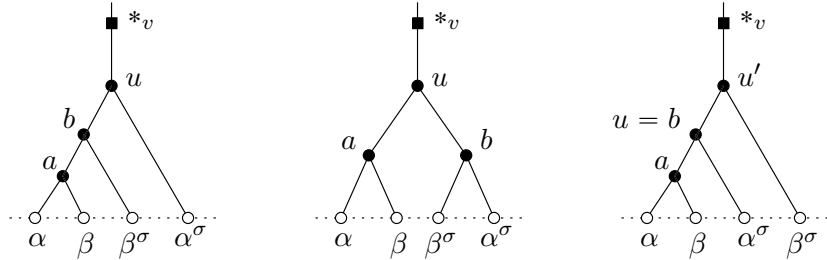
Assume on the contrary that $u = a$. Let $u' \in V\mathbb{X}_v$ be such that $[a, *_v] \cap [a, b] = [a, u']$. Note that $u' \in [*_v, b]$ since $\beta, \beta^\sigma \in \mathcal{O}_v$. Then (see the picture on the right above),

$$v(\beta - \beta^\sigma) = d(b, *_v), \quad v(\alpha - \beta) = v(\alpha - \beta^\sigma) = v(\alpha^\sigma - \beta) = v(\alpha^\sigma - \beta^\sigma) = d(u', *_v) .$$

Therefore

$$\begin{aligned} |\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v &= |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{2d(u', *_v)-d(a, *_v)-d(b, *_v)} = q_v^{-d(a, b)} = q_v^{-d(A, B)} , \end{aligned}$$

which proves the result by Equation (18.1).



Case 2. Now assume that A and B are not disjoint, so that

$$d(A, B) = 0 .$$

Since $\beta \notin \{\alpha, \alpha^\sigma\}$, the intersection $A \cap B$ is a compact segment $[a, b]$ (possibly with $a = b$) in \mathbb{X}_v . Up to exchanging α and α^σ , as well as β and β^σ (which does not change $d(A, B)$ nor $h_\alpha(\beta)$ by Lemma 18.1 (1)), we may assume that $\alpha, a, b, \alpha^\sigma$ and $\beta, a, b, \beta^\sigma$ are in this order on A and B respectively, and that $a \in [u, \alpha[$.

Assume first that $b \in]u, \alpha[$. Then (see the picture on the left above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha - \beta^\sigma) = v(\beta - \beta^\sigma) = d(b, *_v)$$

and

$$v(\beta - \alpha^\sigma) = v(\beta^\sigma - \alpha^\sigma) = d(u, *_v) .$$

Therefore

$$|\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^{d(a, *_v)-d(b, *_v)} = q_v^{d(a,b)} \geq 1 ,$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} ,$$

which proves the result by Equation (18.1).

Assume that $b \in]u, \alpha^\sigma[$. Then (see the picture in the middle above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha^\sigma - \beta^\sigma) = d(b, *_v)$$

and

$$v(\alpha - \beta^\sigma) = v(\beta - \alpha^\sigma) = v(\beta - \beta^\sigma) = d(u, *_v) .$$

Therefore

$$\begin{aligned} |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v &= q_v^{v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{d(a, *_v)+d(b, *_v)-2d(u, *_v)} = q_v^{d(a,b)} \geq 1 , \end{aligned}$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} ,$$

which proves the result by Equation (18.1).

Assume at last that $b = u$. Let $u' \in V\mathbb{X}_v$ be such that $[b, *_v] \cap [b, \beta^\sigma[= [b, u']$. Then (see the picture on the right above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha^\sigma - \beta) = d(u, *_v)$$

and

$$v(\alpha - \beta^\sigma) = v(\beta - \beta^\sigma) = v(\alpha^\sigma - \beta^\sigma) = d(u', *_v) .$$

Therefore

$$\begin{aligned} |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v &= q_v^{v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{d(a, *_v)-d(u, *_v)} = q_v^{d(a,b)} \geq 1 , \end{aligned}$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} ,$$

which proves the result by Equation (18.1). □

The next result says that the relative height is an appropriate complexity on a given orbit under Γ of a loxodromic fixed point, and that the counting function we will study is well defined. We denote by Γ_ξ the stabiliser in Γ of a point $\xi \in \partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$.

Lemma 18.3. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ . Then for every $s > 1$, the set*

$$E_s = \{\beta' \in \Gamma_\alpha \backslash \Gamma \cdot \beta : 1 < h_\alpha(\beta') \leq s\}$$

is finite.

Proof. The set E_s is well defined by Lemma 18.1 (4). Recall that a loxodromic fixed point is one of the two points at infinity of a unique translation axis. By local finiteness, there are, up to the action of the stabiliser of a fixed translation axis A , only finitely many images under Γ of another translation axis B at distance at most $\frac{\ln s}{\ln q_v}$ from A . Since the stabiliser of A contains the stabiliser of either of its points at infinity with index at most 2, the result then follows from Proposition 18.2. \square

We now state our main counting and equidistribution result of orbits of loxodromic fixed points, when the complexity is the relative height with respect to a fixed loxodromic fixed point.

Theorem 18.4. *Let Γ be a lattice in $\mathrm{PGL}_2(K_v)^+$. Let $\alpha_0, \beta_0 \in K_v$ be loxodromic fixed points of Γ . Then for the weak-star convergence of measures on $K_v - \{\alpha_0, \alpha_0^\sigma\}$, as $s \rightarrow +\infty$,*

$$\frac{(q_v + 1)^2 \mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{2 q_v^2 |\alpha_0 - \alpha_0^\sigma|_v \mathrm{Vol}(\Gamma_{\beta_0} \backslash]\beta_0, \beta_0^\sigma[)} s^{-1} \sum_{\beta \in \Gamma \cdot \beta_0 : h_{\alpha_0}(\beta) \leq s} \Delta_\beta \xrightarrow{*} \frac{d \mathrm{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v}.$$

Furthermore, as $s \rightarrow +\infty$,

$$\mathrm{Card} \Gamma_{\alpha_0} \backslash \{\beta \in \Gamma \cdot \beta_0 : h_{\alpha_0}(\beta) \leq s\} \sim \frac{2 q_v \mathrm{Vol}(\Gamma_{\alpha_0} \backslash]\alpha_0, \alpha_0^\sigma[) \mathrm{Vol}(\Gamma_{\beta_0} \backslash]\beta_0, \beta_0^\sigma[)}{(q_v + 1) \mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)} s.$$

If Γ is geometrically finite, for every $\beta' \in]0, 1]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^{\beta'}(K_v - \{\alpha_0, \alpha_0^\sigma\})$, where $K_v - \{\alpha_0, \alpha_0^\sigma\}$ is endowed with the distance-like map $d_{] \alpha_0, \alpha_0^\sigma [}$,² there is an error term in the equidistribution claim of Theorem 18.4 when evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_{\beta'})$. This result applies for instance if $\psi : K_v - \{\alpha_0, \alpha_0^\sigma\} \rightarrow \mathbb{R}$ is locally constant with compact support, see Remark 3.2.

Proof. The proof of the equidistribution claim is similar to the one of Theorem 17.1. We now apply Theorem 15.4 with $\mathbb{D}^- :=]\alpha_0, \alpha_0^\sigma[$ and $\mathbb{D}^+ :=]\beta_0, \beta_0^\sigma[$. Since Γ is contained in $\mathrm{PGL}_2(K_v)^+$, the length spectrum L_Γ of Γ is equal to $2\mathbb{Z}$. The families $\mathcal{D}^\pm = (\gamma \mathbb{D}^\pm)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^\pm}}$ are locally finite, and $\|\sigma_{\mathcal{D}^+}^-\|$ is finite and nonzero.

Arguing as in the proof of Theorem 17.1,³ we have

$$\lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2 q_v^3} \frac{\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \mathbb{D}^+) \leq n}} \Delta_{\gamma \cdot \beta_0} = (\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ \quad (18.2)$$

for the weak-star convergence of measures on $\partial_\infty \mathbb{X}_v - \partial_\infty \mathbb{D}_-$. When Γ is geometrically finite, for every $\beta' \in]0, 1]$, there exists $\kappa > 0$ such that for every β -Hölder-continuous function $\psi \in \mathcal{C}_c^{\beta'}(\partial_\infty \mathbb{X}_v - \partial_\infty \mathbb{D}_-)$, where $\partial_\infty \mathbb{X}_v - \partial_\infty \mathbb{D}_-$ is endowed with the distance-like map $d_{\mathbb{D}^-}$, there is an error term in the equidistribution claim of Theorem 18.4 when evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_{\beta'})$.

By Proposition 18.2, we have

$$h_{\alpha_0}(\gamma \cdot \beta_0) = q_v^{d(\mathbb{D}^-, \gamma \mathbb{D}^+)}.$$

²See Equation (15.14).

³See Equation (17.2).

By Proposition 15.2 (5), we have

$$(\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ = \frac{|\alpha_0 - \alpha_0^\sigma|_v}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v} d\text{Haar}_{K_v}(z)$$

on the full measure subset $K_v - \{\alpha_0, \alpha_0^\sigma\}$ of $\partial_\infty \mathbb{X}_v$. Hence, using the change of variable $s = q_v^n$, we have, with the appropriate error term when Γ is geometrically finite,

$$\lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{|\alpha_0 - \alpha_0^\sigma|_v \|\sigma_{\mathcal{D}^+}^-\|} s^{-1} \sum_{\substack{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+} \\ 1 < h_{\alpha_0}(\gamma \cdot \beta_0) \leq s}} \Delta_{\gamma \cdot \beta_0} = \frac{d\text{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v} .$$

We again denote by ι_{α_0} the index

$$\iota_{\alpha_0} = [\Gamma_{\{\alpha_0, \alpha_0^\sigma\}} : \Gamma_{\alpha_0}] ,$$

and similarly for β_0 . Since the stabiliser Γ_{β_0} of β_0 in Γ has index ι_{β_0} in $\Gamma_{\mathbb{D}^+}$ and $\Gamma / \Gamma_{\beta_0}$ identifies with $\Gamma \cdot \beta_0$ by the map $\gamma \mapsto \gamma \cdot \beta_0$, we have, with the appropriate error term when Γ is geometrically finite,

$$\lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{|\alpha_0 - \alpha_0^\sigma|_v \|\sigma_{\mathcal{D}^+}^-\| \iota_{\beta_0}} s^{-1} \sum_{\substack{\beta \in \Gamma \cdot \beta_0 \\ 1 < h_{\alpha_0}(\beta) \leq s}} \Delta_\beta = \frac{d\text{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v} .$$

As in the end of the proof of Theorem 17.1, we have

$$\|\sigma_{\mathcal{D}^+}^-\| = \frac{q_v - 1}{q_v \iota_{\beta_0}} \text{Vol}(\Gamma_{\beta_0} \backslash]\beta_0, \beta_0^\sigma[) .$$

This proves the equidistribution claim, and its error term when Γ is geometrically finite.

In order to obtain the counting claim, we note that since $\tilde{\sigma}_{\mathbb{D}^-}^+$ is invariant under the stabiliser in Γ of \mathbb{D}^- , hence under Γ_{α_0} , the measures on both sides of the equidistribution claim in Theorem 18.4 are invariant under Γ_{α_0} , see Proposition 17.3 (5) for the invariance of the right hand side. By Proposition 15.2 (5) and (6), and by the definition of ι_{α_0} , we have

$$\begin{aligned} \int_{\Gamma_{\alpha_0} \backslash (K_v - \{\alpha_0, \alpha_0^\sigma\})} \frac{d\text{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v} &= \frac{\iota_{\alpha_0}}{|\alpha_0 - \alpha_0^\sigma|_v} \int_{\Gamma_{\mathbb{D}^-} \backslash \partial_{\mathbb{D}^-}^+ \mathbb{D}^-} d\tilde{\sigma}_{\mathbb{D}^-}^+ \\ &= \frac{(q_v + 1) \iota_{\alpha_0} \text{Vol}(\Gamma_{\mathbb{D}^-} \backslash \mathbb{D}^-)}{q_v |\alpha_0 - \alpha_0^\sigma|_v} \\ &= \frac{(q_v + 1) \text{Vol}(\Gamma_{\alpha_0} \backslash]\alpha_0, \alpha_0^\sigma[)}{q_v |\alpha_0 - \alpha_0^\sigma|_v} . \end{aligned} \quad (18.3)$$

The counting claim follows by evaluating the equidistribution claim on the characteristic function ψ of a compact open fundamental domain for the action of Γ_{α_0} on $K_v - \{\alpha_0, \alpha_0^\sigma\}$. This characteristic function is locally constant, hence β' -Hölder-continuous for the distance-like function $d_{\mathbb{D}^-}$, as seen end of Section 3.2. \square

18.2 Counting and equidistribution of crossratios of quadratic irrationals

In this Section, we give two arithmetic applications of Theorem 18.4.

Let us first consider an application in positive characteristic. Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , and let R_v be the affine function ring associated with v . We assume that the characteristic of K is different from 2. Given two quadratic irrationals $\alpha, \beta \in K_v$ over K , with Galois conjugates $\alpha^\sigma, \beta^\sigma$ respectively, such that $\beta \notin \{\alpha, \alpha^\sigma\}$, we define the *relative height* of β with respect to α by

$$h_\alpha(\beta) = \frac{1}{\min\{|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v, |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v\}}. \quad (18.4)$$

The following result says that the orbit of any quadratic irrational in K_v over K , by homographies under a given finite index subgroup of the modular group $\mathrm{PGL}_2(R_v)$, equidistributes, when its complexity is given by the relative height with respect to another fixed quadratic irrational α_0 . The limit measure is absolutely continuous with respect to the Haar measure on K_v and it is invariant under the stabiliser of α_0 in $\mathrm{PGL}_2(R_v)$ by Proposition 17.3 (5).

Theorem 18.5. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$. Let $\alpha_0, \beta_0 \in K_v$ be quadratic irrationals over K . Then, as $s \rightarrow +\infty$,*

$$\frac{(q_v + 1)^2 \zeta_K(-1) n_0 [\mathrm{GL}_2(R_v) : G]}{2 q_v^2 (q - 1) |\alpha_0 - \alpha_0^\sigma|_v |v(\mathrm{tr} h_0)|} s^{-1} \sum_{\beta \in G \cdot \beta_0 : h_{\alpha_0}(\beta) \leq s} \Delta_\beta \xrightarrow{*} \frac{d \mathrm{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v},$$

and there exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,

$$\mathrm{Card} \Gamma_{\alpha_0} \setminus \{\beta \in G \cdot \beta_0 : h_{\alpha_0}(\beta) \leq s\} = \frac{4 q_v (q - 1) |v(\mathrm{tr} g_0)| |v(\mathrm{tr} h_0)| |Z(G)|}{(q_v + 1) \zeta_K(-1) m_0 n_0 [\mathrm{GL}_2(R_v) : G]} s + O(s^{1-\kappa}).$$

Here $g_0, h_0 \in G$ fixes α_0, β_0 with $v(\mathrm{tr} g_0), v(\mathrm{tr} h_0) \neq 0$, and m_0, n_0 is the index of $g_0^\mathbb{Z}, h_0^\mathbb{Z}$ in the stabiliser of α_0, β_0 in G respectively.

Proof. This follows, as in the proof of Theorem 17.5, from Theorem 18.4 using Equations (16.6) and (17.7), as well as Equation (18.3) for the counting claim. \square

Example 18.6. (1) Theorem 1.16 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, and by using Equation (14.5) in order to simplify the constant.

(2) If G_I is the Hecke congruence subgroup associated with a nonzero ideal I of R_v (see Equation (16.11)), using Lemma 16.5, we have, as $s \rightarrow +\infty$,

$$\frac{(q_v + 1)^2 \zeta_K(-1) n_0 N(I) \prod_{\mathfrak{p}|I} (1 + \frac{1}{N(\mathfrak{p})})}{2 q_v^2 (q - 1) |\alpha_0 - \alpha_0^\sigma|_v |v(\mathrm{tr} h_0)|} s^{-1} \sum_{\beta \in G_I \cdot \beta_0 : h_{\alpha_0}(\beta) \leq s} \Delta_\beta \xrightarrow{*} \frac{d \mathrm{Haar}_{K_v}(z)}{|z - \alpha_0|_v |z - \alpha_0^\sigma|_v}.$$

The second arithmetic application of Theorem 18.4 is in \mathbb{Q}_p . We use the notation of Section 17.3.

Let $p \in \mathbb{N}$ be a positive rational prime with $p \equiv 1 \pmod{4}$ such that $\frac{p^2-1}{4}$ is not of the form $4^a(8b+7)$ for $a, b \in \mathbb{N}$ (for instance $p = 5$). Let ε be a square root of -1 in \mathbb{Q}_p . Let $x_1, x_2, x_3 \in 2\mathbb{Z}$ be such that $p^2 - 1 = x_1^2 + x_2^2 + x_3^2$. We again consider

$$\alpha_0 = \frac{\varepsilon x_1 + \sqrt{1 - p^2}}{x_3 + \varepsilon x_2},$$

which is a quadratic irrational in \mathbb{Q}_p over $\mathbb{Q}(\varepsilon)$. We denote by α^σ the Galois conjugate of a quadratic irrational α in \mathbb{Q}_p over $\mathbb{Q}(\varepsilon)$, and by

$$h_\alpha(\beta) = \frac{1}{\min\{|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_p, |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_p\}} \quad (18.5)$$

the *relative height* of a quadratic irrational β in \mathbb{Q}_p over $\mathbb{Q}(\varepsilon)$ with respect to α , such that $\beta \notin \{\alpha, \alpha^\sigma\}$. We again consider Hamilton's quaternion algebra $D = \left(\frac{-1, -1}{\mathbb{Q}}\right)$ over \mathbb{Q} and its $\mathbb{Z}\left[\frac{1}{p}\right]$ -order

$$\mathcal{O} = \left\{x \in \mathbb{Z}\left[\frac{1}{p}\right] + \mathbb{Z}\left[\frac{1}{p}\right] i + \mathbb{Z}\left[\frac{1}{p}\right] j + \mathbb{Z}\left[\frac{1}{p}\right] k : x \equiv 1 \pmod{2}\right\}.$$

The following result says that the orbit of α_0 in \mathbb{Q}_p by homographies under a given finite index subgroup of the arithmetic group $\Gamma_{\mathcal{O}}^1$ equidistributes, when its complexity is given by the relative height with respect to α_0 , to a measure absolutely continuous with respect to the Haar measure on \mathbb{Q}_p .

Theorem 18.7. *With the above notation, let Γ be a finite index subgroup of $\Gamma_{\mathcal{O}}^1$. Then, as $s \rightarrow +\infty$,*

$$\frac{(p+1)^2 [\Gamma_{\mathcal{O}}^1 : \Gamma]}{2 p^2 k_\Gamma |\alpha_0 - \alpha_0^\sigma|_p} s^{-1} \sum_{\alpha \in \Gamma \cdot \alpha_0 : h_{\alpha_0}(\alpha) \leq s} \Delta_\alpha \stackrel{*}{\rightarrow} \frac{d\text{Haar}_{\mathbb{Q}_p}(z)}{|z - \alpha_0|_p |z - \alpha_0^\sigma|_p},$$

where k_Γ is the smallest positive integer such that $\begin{bmatrix} 1 + \varepsilon x_1 & -x_3 + \varepsilon x_2 \\ x_3 + \varepsilon x_2 & 1 - \varepsilon x_1 \end{bmatrix}^{k_\Gamma} \in \Gamma$. Furthermore, there exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,

$$\text{Card } \Gamma_{\alpha_0} \setminus \{\alpha \in \Gamma \cdot \alpha_0 : h_{\alpha_0}(\alpha) \leq s\} = \frac{4 p (k_\Gamma)^2}{(p+1) [\Gamma_{\mathcal{O}}^1 : \Gamma]} s + O(s^{1-\kappa}).$$

Proof. This follows, as in the proof of Theorem 17.10, from Theorem 18.4 using Equations (17.11) and (17.12), as well as Equation (18.3) for the counting claim. \square

Chapter 19

Counting and equidistribution of integral representations by quadratic norm forms

In the final Chapter of this text, we give another equidistribution and counting result of rational elements in non-Archimedean local fields of positive characteristic, again using our equidistribution and counting results of common perpendiculars in trees summarized in Section 15.4. In this Chapter, we use a complexity defined using the norm forms associated with fixed quadratic irrationals. In particular, the complexity in this Chapter is different from that used in the Mertens type of results in Section 16.1. We refer for instance to [PaP14b, §5.3] for motivations and results in the Archimedean case, and also to [GoP] for higher dimensional norm forms.

Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , and let R_v be the affine function ring associated with v . Let $\alpha \in K_v$ be a quadratic irrational over K . The *norm form* \mathfrak{n}_α associated with α is the quadratic form $K \times K \rightarrow K$ defined by

$$(x, y) \mapsto \mathfrak{n}(x - y\alpha) = (x - y\alpha)(x - y\alpha^\sigma) = x^2 - xy \operatorname{tr}(\alpha) + y^2 \mathfrak{n}(\alpha).$$

See Proposition 17.3 for elementary transformation properties under elements of $\operatorname{GL}_2(R_v)$ of this norm form.

A pair $(x, y) \in R_v \times R_v$ is an *integral representation* of an element $z \in K$ by the quadratic norm form \mathfrak{n}_α if $\mathfrak{n}_\alpha(x, y) = z$. The following result describes the projective equidistribution as $s \rightarrow +\infty$ of the integral representations by \mathfrak{n}_α of elements with absolute value at most s . For every $(x_0, y_0) \in R_v \times R_v$, let $H_{(x_0, y_0)}$ be the stabiliser of (x_0, y_0) for the linear action of any subgroup H of $\operatorname{GL}_2(R_v)$ on $R_v \times R_v$. We use the notation $N\langle x_0, y_0 \rangle$ for the norm of the ideal $\langle x_0, y_0 \rangle$ generated by x_0, y_0 (see Section 14.2) and the notation m_{v, x_0, y_0} introduced above Theorem 16.1.

Theorem 19.1. *Let G be a finite index subgroup of $\operatorname{GL}_2(R_v)$, let $\alpha \in K_v$ be a quadratic irrational over K , and let $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$. Let*

$$c' = \frac{(q_v - 1)(q_v + 1)^2 \zeta_K(-1) m_{v, x_0, y_0} (N\langle x_0, y_0 \rangle)^2 [\operatorname{GL}_2(R_v) : G]}{q_v^3 (q - 1) q^{g-1} [\operatorname{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}}.$$

Then for the weak-star convergence of measures on $K_v - \{\alpha, \alpha^\sigma\}$, we have

$$\lim_{s \rightarrow +\infty} c' s^{-1} \sum_{(x,y) \in G(x_0, y_0), |\mathfrak{n}(x-y\alpha)|_v \leq s} \Delta_{\frac{x}{y}} = \frac{d \text{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v}.$$

For every $\beta \in]0, 1]$, there exists $\kappa > 0$ such that for every $\psi \in \mathcal{C}_c^\beta(K_v - \{\alpha, \alpha^\sigma\})$, where $K_v - \{\alpha, \alpha^\sigma\}$ is endowed with the distance-like map $d_{|\alpha, \alpha^\sigma|}$ ¹ as for instance if $\psi : K_v - \{\alpha, \alpha^\sigma\} \rightarrow \mathbb{R}$ is locally constant with compact support (see Remark 3.2), there is an error term in the equidistribution claim of Theorem 19.1 when evaluated on ψ , of the form $O(s^{-\kappa} \|\psi\|_\beta)$.

Examples 19.2. (1) Let $(x_0, y_0) = (1, 0)$, $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. Theorem 1.17 in the Introduction follows from Theorem 19.1, using Equations (14.5) and (16.1) to simplify the constant c' .

(2) Let $(x_0, y_0) = (1, 0)$ and let $G = G_I$ be the Hecke congruence subgroup of $\text{GL}_2(R_v)$ defined in Equation (16.11). The index in $[\text{GL}_2(R_v) : G_I]$ is given by Lemma 16.5 and G_I satisfies $(G_I)_{(1,0)} = \text{GL}_2(R_v)_{(1,0)}$. For every nonzero ideal I of R_v , for the weak-star convergence of measures on $K_v - \{\alpha, \alpha^\sigma\}$, we have

$$\lim_{s \rightarrow +\infty} c_I s^{-1} \sum_{(x,y) \in R_v \times I, \langle x, y \rangle = R_v, |\mathfrak{n}(x-y\alpha)|_v \leq s} \Delta_{\frac{x}{y}} = \frac{d \text{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v},$$

where

$$c_I = \frac{(q_v - 1) (q_v + 1)^2 \zeta_K(-1) N(I) \prod_{\mathfrak{p}|I} (1 + \frac{1}{N(\mathfrak{p})})}{q_v^3 q^{g-1}}.$$

(3) This example is interesting when the ideal class number is larger than 1. Given any fractional ideal \mathfrak{m} of R_v , taking $(x_0, y_0) \in R_v \times R_v$ such that the fractional ideals $\langle x_0, y_0 \rangle$ and \mathfrak{m} have the same ideal class and $G = \text{GL}_2(R_v)$, using the change of variables $s \mapsto sN(\mathfrak{m})^2$ in the statement of Theorem 19.1, for the weak-star convergence of measures on $K_v - \{\alpha, \alpha^\sigma\}$, with the same error term as for Theorem 19.1, we have

$$\lim_{s \rightarrow +\infty} c_{\mathfrak{m}} s^{-1} \sum_{(x,y) \in \mathfrak{m} \times \mathfrak{m}, \langle x, y \rangle = \mathfrak{m}, N(\mathfrak{m})^{-2} |\mathfrak{n}(x-y\alpha)|_v \leq s} \Delta_{\frac{x}{y}} = \frac{d \text{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v},$$

where

$$c_{\mathfrak{m}} = \frac{(q_v - 1) (q_v + 1)^2 \zeta_K(-1) m_{v, x_0, y_0}}{q_v^3 (q - 1) q^{g-1}}.$$

Before proving Theorem 19.1, let us give a counting result which follows from it. Any subgroup of G acts on the left on any orbit of G . Furthermore, the stabiliser G_α of α in G preserves the map $(x, y) \mapsto |\mathfrak{n}(x - y\alpha)|_v$, by Proposition 17.3 (3). We may then define a counting function $\Psi'(s) = \Psi'_{G, \alpha, x_0, y_0}(s)$ of elements in $R_v \times R_v$ in a linear orbit under a finite index subgroup G of $\text{GL}_2(R_v)$ on which the absolute value of the norm form associated with α is at most s , as

$$\Psi'(s) = \text{Card } G_\alpha \setminus \{(x, y) \in G(x_0, y_0), |\mathfrak{n}(x - y\alpha)|_v \leq s\}.$$

¹See Equation (15.14).

Corollary 19.3. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, let $\alpha \in K_v$ be a quadratic irrational over K , and let $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$. Let $g_0 \in G_\alpha$ with $v(\mathrm{tr} g_0) \neq 0$ and let m_0 be the index of $g_0^{\mathbb{Z}}$ in G_α . Let*

$$c'' = \frac{2 q_v^2 (q-1) q^{g-1} |Z(G)| |v(\mathrm{tr} g_0)| [\mathrm{GL}_2(R_v)_{(x_0, y_0)} : G_{(x_0, y_0)}]}{(q_v^2 - 1) \zeta_K(-1) |\alpha - \alpha^\sigma|_v m_0 m_{v, x_0, y_0} (N\langle x_0, y_0 \rangle)^2 [\mathrm{GL}_2(R_v) : G]} .$$

Then there exists $\kappa > 0$ such that, as $s \rightarrow +\infty$,

$$\Psi'(s) = c'' s + O(s^{1-\kappa}) .$$

Proof. Using Equation (18.3) (with Γ the image of G in $\mathrm{PGL}_2(R_v)$) and Equation (17.7), we have

$$\int_{G_\alpha \backslash (K_v - \{\alpha, \alpha^\sigma\})} \frac{d \mathrm{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v} = \frac{2 (q_v + 1) |Z(G)| |v(\mathrm{tr} g_0)|}{q_v |\alpha - \alpha^\sigma|_v m_0} .$$

The corollary then follows by applying the equidistribution claim in Theorem 19.1 to the characteristic function of a compact-open fundamental domain of $K_v - \{\alpha, \alpha^\sigma\}$ modulo the action by homographies of G_α . \square

Example 19.4. Let $(x_0, y_0) = (1, 0)$, $K = \mathbb{F}_q(Y)$, $v = v_\infty$ and $G = \mathrm{GL}_2(\mathbb{F}_q[Y])$. Using Equations (14.5) and (16.1), Proposition 17.3 (1), and the fact that $|Z(G)| = q - 1$ to simplify the constant c'' of Corollary 19.3, and recalling the expression of the absolute value at ∞ in terms of the degree from Section 14.2, we get the following counting result: For every integral quadratic irrational $\alpha \in \mathbb{F}_q((Y))$ over $\mathbb{F}_q(Y)$, there exists $\kappa > 0$ such that, as $t \rightarrow +\infty$,

$$\begin{aligned} \mathrm{Card} \ \mathrm{GL}_2(\mathbb{F}_q[Y])_\alpha \backslash \left\{ (x, y) \in \mathbb{F}_q[Y] \times \mathbb{F}_q[Y] : \begin{array}{l} \langle x, y \rangle = \mathbb{F}_q[Y], \\ \deg(x^2 - xy \mathrm{tr}(\alpha) + y^2 \mathfrak{n}(\alpha)) \leq t \end{array} \right\} \\ = \frac{2}{m_0} (q-1)^4 \deg(\mathrm{tr} g_0) q^{1-\frac{1}{2} \deg(\mathrm{tr}(\alpha)^2 - 4\mathfrak{n}(\alpha))} q^t + O(q^{t-\kappa}) , \end{aligned}$$

where $g_0 \in \mathrm{GL}_2(\mathbb{F}_q[Y])$ fixes α with $\deg(\mathrm{tr} g_0) \neq 0$ and m_0 is the index of $g_0^{\mathbb{Z}}$ in the stabiliser $\mathrm{GL}_2(\mathbb{F}_q[Y])_\alpha$ of α in $\mathrm{GL}_2(\mathbb{F}_q[Y])$.

Proof of Theorem 19.1. The proof is similar to that of Theorem 16.1. Let $r = \frac{x_0}{y_0} \in K \cup \{\infty\}$. If $y_0 = 0$, let $g_r = \mathrm{id} \in \mathrm{GL}_2(K)$, and if $y_0 \neq 0$, let

$$g_r = \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(K) .$$

We apply Theorem 15.4 with $\Gamma := \overline{G}$ the image of G in $\mathrm{PGL}_2(R_v)$, $\mathbb{D}^- :=]\alpha, \alpha^\sigma[$ the (image of any) geodesic line in \mathbb{X}_v with points at infinity α and α^σ , and $\mathbb{D}^+ := \gamma_r \mathcal{H}_\infty$, where γ_r is the image of g_r in $\mathrm{PGL}_2(R_v)$.

We have $L_{\Gamma_v} = 2\mathbb{Z}$ and the family $\mathcal{D}^+ = (\gamma \mathbb{D}^+)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+}}$ is locally finite, as seen in the beginning of the proof of Theorem 16.1. The family $\mathcal{D}^- = (\gamma \mathbb{D}^-)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^-}}$ is locally finite as seen in the beginning of the proof of Theorem 17.1.

By Proposition 15.2 (5), we have (on the full measure subset $K_v - \{\alpha, \alpha^\sigma\}$ of $\partial_\infty \mathbb{X}_v$)

$$(\partial^+)_* \tilde{\sigma}_{\mathbb{D}^-}^+ = \frac{|\alpha - \alpha^\sigma|_v}{|z - \alpha|_v |z - \alpha^\sigma|_v} d \mathrm{Haar}_{K_v}(z) .$$

For every $\gamma \in \Gamma/\Gamma_r$ such that \mathbb{D}^- and $\gamma\mathbb{D}^+$ are disjoint, let ρ_γ be the geodesic ray starting from $\alpha_{e,\gamma}^- (0)$ and ending at the point at infinity $\gamma \cdot r$ of $\gamma\mathbb{D}^+$.

Hence, as in order to obtain Equation (16.2), we have, with an error term for every $\beta \in]0, 1]$ of the form $O(s^{-\kappa} \|\psi\|_\beta)$ for some $\kappa > 0$ when evaluated on $\psi \in \mathcal{C}_c^\beta(\partial_\infty \mathbb{X}_v - \partial_\infty \mathbb{D}^-)$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)}{2q_v^3} \frac{\text{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} q_v^{-n} \sum_{\substack{\gamma \in \Gamma/\Gamma_r \\ 0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq n}} \Delta_{\gamma \cdot r} \\ = \frac{|\alpha - \alpha^\sigma|_v}{|z - \alpha|_v |z - \alpha^\sigma|_v} d\text{Haar}_{K_v}(z). \end{aligned} \quad (19.1)$$

We use the following Lemma to switch from counting over elements $\gamma \in \Gamma/\Gamma_r$ for which $0 < d(\mathbb{D}^-, \gamma\mathbb{D}^+) \leq t$ to counting over integral representations with bounded value of the norm form. See [PaP11a, page 1054] for the analogous result for the real hyperbolic 3-space and indefinite binary Hermitian forms.

Lemma 19.5. *Let $g \in \text{GL}_2(R_v)$ and let γ be the image of g in $\text{PGL}_2(K)$. Let $z_0 = y_0$ if $y_0 \neq 0$ and $z_0 = x_0$ otherwise. Let $(x, y) = g(x_0, y_0)$. If $d(\mathbb{D}^-, \gamma\mathbb{D}^+) > 0$, then*

$$d(\mathbb{D}^-, \gamma\mathbb{D}^+) = \frac{1}{\ln q_v} \ln \left(|\mathbf{n}(x - y\alpha)|_v \frac{h(\alpha)}{|z_0|_v^2} \right).$$

Proof. We start by showing that

$$g g_r(1, 0) = \left(\frac{x}{z_0}, \frac{y}{z_0} \right).$$

Indeed, if $y_0 \neq 0$, we have

$$g g_r(1, 0) = g(r, 1) = \frac{1}{y_0} g(x_0, y_0)$$

and otherwise

$$g g_r(1, 0) = g(1, 0) = \frac{1}{x_0} g(x_0, 0) = \frac{1}{x_0} g(x_0, y_0).$$

In particular,

$$(g g_r)^{-1} = \begin{pmatrix} * & * \\ -\frac{y}{z_0} & \frac{x}{z_0} \end{pmatrix}.$$

Note that $g g_r \in \text{GL}_2(K)$ and $|\det(g g_r)|_v = |\det g|_v |\det g_r|_v = 1$ since $g \in \text{GL}_2(R_v)$. By Proposition 17.3 (2), we hence have

$$h((g g_r)^{-1} \cdot \alpha) = \left| \mathbf{n} \left(\frac{x}{z_0} - \frac{y}{z_0} \alpha \right) \right|_v h(\alpha) = |\mathbf{n}(x - y\alpha)|_v \frac{h(\alpha)}{|z_0|_v^2}. \quad (19.2)$$

We use the signed distance $d(L, H) = \min_{x \in L} \beta_\xi(x, x_H)$ between a geodesic line L and a horoball H centred at $\xi \neq L^\pm$, where x_H is any point of the boundary of H . Now, by Equations (15.2) and (2.8), we have

$$\begin{aligned} d(\mathbb{D}^-, \gamma\mathbb{D}^+) &= d([\alpha, \alpha^\sigma], \gamma\gamma_r \mathcal{H}_\infty) = d([(g g_r)^{-1} \cdot \alpha, (g g_r)^{-1} \cdot \alpha^\sigma], \mathcal{H}_\infty) \\ &= v((g g_r)^{-1} \cdot \alpha - (g g_r)^{-1} \cdot \alpha^\sigma) \\ &= \frac{-\ln |(g g_r)^{-1} \cdot \alpha - (g g_r)^{-1} \cdot \alpha^\sigma|_v}{\ln q_v} = \frac{\ln h((g g_r)^{-1} \cdot \alpha)}{\ln q_v}. \end{aligned} \quad (19.3)$$

Combining Equations (19.2) and (19.3) gives the result. \square

By discreteness, there are only finitely many double classes $[g] \in G_\alpha \backslash G / G_{(x_0, y_0)}$ such that $\mathbb{D}^- =]\alpha, \alpha^\sigma[$ and $g\mathbb{D}^+ = g g_r \mathcal{H}_\infty$ are not disjoint. Let $Z(G)$ be the centre of G , which is finite. Since $Z(G)$ acts trivially on $\mathbb{P}_1(K_v)$, the map $G / G_{(x_0, y_0)} \rightarrow \Gamma / \Gamma_r$ induced by the canonical map $\mathrm{GL}_2(R_v) \rightarrow \mathrm{PGL}_2(R_v)$ is $|Z(G)|$ -to-1. Using the change of variable

$$s = \frac{|z_0|_v^2}{h(\alpha)} q_v^n,$$

by using Lemma 19.5, Equation (19.1) gives

$$\begin{aligned} \lim_{s \rightarrow +\infty} \frac{(q_v^2 - 1)(q_v + 1)|z_0|_v^2}{2q_v^3|Z(G)|} \frac{\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)}{\|\sigma_{\mathcal{D}^+}^-\|} s^{-1} & \sum_{(x, y) \in G_{(x_0, y_0)}, |\mathfrak{n}(x - y\alpha)|_v \leq s} \Delta_{\frac{x}{y}} \\ & = \frac{d\mathrm{Haar}_{K_v}(z)}{|z - \alpha|_v |z - \alpha^\sigma|_v}, \end{aligned}$$

with the appropriate error term. Replacing $\mathrm{Vol}(\Gamma \backslash \mathbb{X}_v)$ and $\|\sigma_{\mathcal{D}^+}^-\|$ by their values respectively given by Equation (16.6) and Lemma 16.3, the claim of Theorem 19.1 follows. \square

Appendix A

A weak Gibbs measure is the unique equilibrium, by J. Buzzi

Abstract

For a transitive topological Markov shift endowed with a Hölder-continuous potential, we prove that a weak Gibbs measure is the unique equilibrium measure.

A.1 Introduction

Let $\sigma : \Sigma \rightarrow \Sigma$ be a topological Markov shift (possibly one- or two-sided), see for instance Section 5.1. More precisely, we consider the one-sided and two-sided vertex-shifts defined by a countable oriented graph G with set of vertices V_G and set of arrows $A_G \subset V_G \times V_G$. We assume that Σ is transitive, that is, that G is connected (as an oriented graph).

We denote by $\mathbb{P}(\Sigma)$ the set of σ -invariant probability measures on Σ and by $\mathbb{P}_{\text{erg}}(\Sigma)$ the subset of ergodic ones. Recall that, for all $n \in \mathbb{N}$, the n -cylinders are the following subsets of Σ , where x varies in Σ :

$$C_n(x) = [x_0 \dots x_{n-1}] = \{y \in \Sigma : \forall k \in \{0, \dots, n-1\}, y_k = x_k\},$$

so that the 1-cylinders are $[v] = \{y \in \Sigma : y_0 = v\}$ for all $v \in V_G$. The points of Σ admitting $n \in \mathbb{N}$ as period under the shift σ form the set

$$\text{Fix}_n(\Sigma) = \{x \in \Sigma : \sigma^n x = x\}.$$

We fix a *potential* on Σ , that is, a continuous function $\phi : \Sigma \rightarrow \mathbb{R}$. We do not assume that ϕ is bounded. We define $\phi^- = \max\{-\phi, 0\}$ and, for all $n \in \mathbb{N} - \{0\}$,

$$\text{var}_n(\phi) = \sup_{x, y \in \Sigma, \forall k \in \{0, \dots, n-1\}, x_k = y_k} |\phi(y) - \phi(x)|$$

if (Σ, σ) is one-sided and otherwise

$$\text{var}_n(\phi) = \sup_{x, y \in \Sigma, \forall k \in \{-n+1, \dots, n-1\}, x_k = y_k} |\phi(y) - \phi(x)|$$

We say that ϕ has *summable variations* if $\sum_{n \geq 1} \text{var}_n(\phi) < \infty$. Let $S_n \phi = \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ \sigma^i$ for all $n \in \mathbb{N}$.

Definition A.1. A weak Gibbs measure for the potential ϕ is a σ -invariant Borel probability measure m on Σ such that there exists a number $c(m) \in \mathbb{R}$ such that for every $v \in V_G$, there exists $C \geq 1$ with

$$\forall n \geq 1, \forall x \in \text{Fix}_n(\Sigma) \cap [v], \quad C^{-1} \leq \frac{m(C_n(x))}{\exp(S_n\phi(x) - c(m)n)} \leq C. \quad (\text{A.1})$$

Note that $c(m)$ is then unique, called the *Gibbs constant* of m . Let us stress that we do not assume the so-called *Big Image Property* [Sar1] and hence using the above weakened Gibbs property (that is, allowing C to depend on v) is necessary.

Note that if Σ is locally compact, that is, if every vertex of G has finite degree (the number of arrows arriving or leaving from the given vertex), then the above condition is equivalent to the fact that for any nonempty compact subset K in Σ , there exists $C \geq 1$ with

$$\forall n \geq 1, \forall x \in \text{Fix}_n(\Sigma) \cap K, \quad C^{-1} \leq \frac{m(C_n(x))}{\exp(S_n\phi(x) - c(m)n)} \leq C.$$

The *pressure* $P(\phi, \nu)$ of an element $\nu \in \mathbb{P}(\Sigma)$ such that $\int \phi^- d\nu < +\infty$ is

$$P(\phi, \nu) = h_\nu(\sigma) + \int \phi d\nu.$$

An *equilibrium measure* μ_{eq} for (Σ, ϕ) is an element $\mu_{eq} \in \mathbb{P}(\Sigma)$ such that $\int \phi^- d\mu_{eq} < +\infty$ and

$$P(\phi, \mu_{eq}) = \sup\{P(\phi, \nu) : \nu \in \mathbb{P}(\Sigma) \text{ and } \int \phi^- d\nu < +\infty\}.$$

The *Gurevič pressure* is

$$P_G(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \sum_{x \in \text{Fix}_n(\Sigma) \cap [v]} e^{S_n\phi(x)}$$

for any vertex $v \in V_G$. Note that the Gurevič pressure does not depend on v . Let us recall a few results on the above notions.

Theorem A.2 (Iommi-Jordan [IJ, Theorem 2.2]). *The following variational principle holds:*

$$P_G(\phi) = \sup\{P(\phi, \nu) : \nu \in \mathbb{P}(\Sigma) \text{ and } \int \phi^- d\nu < +\infty\}. \quad \square$$

Theorem A.3 (Buzzi-Sarig [BuS, Theorem 1.1]). *If $P_G(\phi) < \infty$, then there exists at most one equilibrium measure.*

If there exists an equilibrium measure μ , then $d\mu = h d\nu$ where $h : \Sigma \rightarrow \mathbb{R}$ is a continuous, positive function and ν is a positive measure on Σ such that

- $L_\phi h = e^{P_G(\phi)} h$, and $L_\phi^* \nu = e^{P_G(\phi)} \nu$ where L_ϕ is the transfer operator defined by $L_\phi u(x) = \sum_{y \in \sigma^{-1}x} e^{\phi(y)} u(y)$.
- ν is finite on each cylinder. □

We note that [BuS] assumed $\sup \phi < \infty$, but this was only used to justify the variational principle and so this condition can be removed by using Theorem A.2.

We now state the main result of this appendix.

Theorem A.4. *Let (Σ, σ) be a one-sided transitive topological Markov shift and let $\phi : \Sigma \rightarrow \mathbb{R}$ be a potential with summable variations. Let m be an invariant probability measure of Σ such that $\int \phi^- dm < +\infty$.*

Then m is a weak Gibbs measure if and only if it is an equilibrium measure. In this case, the Gibbs constant $c(m)$ is equal to the Gurevič pressure and the equilibrium measure is unique.

By a classical argument that follows, this result extends to two-sided topological Markov shifts (up to a slight strengthening of the regularity assumption).

Corollary A.5. *Let (Σ, σ) be a two-sided transitive topological Markov shift and let $\phi : \Sigma \rightarrow \mathbb{R}$ be a potential with $\sum_{n \geq 1} n \operatorname{var}_n(\phi) < \infty$. Let m be an invariant probability measure of Σ such that $\int \phi^- dm < +\infty$.*

Then m is a weak Gibbs measure if and only if it is an equilibrium measure. In this case, the Gibbs constant $c(m)$ is equal to the Gurevič pressure and the equilibrium measure is unique.

Remark. The case of the full shift $\mathbb{N}^{\mathbb{Z}}$ has been treated in [PeSZ, Sec. 3]. More generally, assuming the Big Image Property, the above result follows from [Sar1] and [BuS] along the lines of [PeSZ].

Proof of Corollary A.5. Let (Σ, σ) , ϕ , and m be as in the statement of this Corollary. Let $\pi : \Sigma \rightarrow \Sigma_+$ with $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n \in \mathbb{N}}$ be the obvious factor map onto the one-sided topological Markov shift (Σ_+, σ_+) defined by the same graph G as for (Σ, σ) , called the *natural extension*.

First, we replace ϕ by a potential $\bar{\phi}$ depending only on future coordinates. The proof of [Bowe2, Lemma 1.6] applies to our non-compact setting without changes. To be more precise, for each vertex $a \in V_G$, choose $z^a \in \Sigma$ with $z^a_0 = a$, and define $r : \Sigma \rightarrow \Sigma$ by $r(x) = y$ with $y_n = x_n$ for $n \geq 0$ and $y_n = z^{x_0}_n$ for $n \leq 0$ and let

$$u(x) = \sum_{k \geq 0} (\phi \circ \sigma^k - \phi \circ \sigma^k \circ r)(x) .$$

This defines a bounded real function on Σ since $|\phi \circ \sigma^k - \phi \circ \sigma^k \circ r| \leq \operatorname{var}_{k+1}(\phi)$ and ϕ has summable variations. Moreover, u itself has summable variations since, given $x, y \in \Sigma$ with $x_k = y_k$ for $|k| < n$, we have

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{0 \leq k \leq \lfloor n/2 \rfloor} \left(|\phi(\sigma^k x) - \phi(\sigma^k y)| + |\phi(\sigma^k(rx)) - \phi(\sigma^k(ry))| \right) \\ &\quad + \sum_{k > \lfloor n/2 \rfloor} \left(|\phi \circ \sigma^k(x) - \phi \circ \sigma^k(rx)| + |\phi \circ \sigma^k(y) - \phi \circ \sigma^k(ry)| \right) \\ &\leq 4 \sum_{k \geq \lfloor n/2 \rfloor} \operatorname{var}_{k+1}(\phi) , \end{aligned}$$

so that

$$\sum_{n \geq 1} \operatorname{var}_n(u) \leq 4 \sum_{k \geq 1} (2k - 2) \operatorname{var}_k(\phi) < \infty .$$

Now define $\bar{\phi} : \Sigma \rightarrow \mathbb{R}$ by

$$\bar{\phi} = \phi + u \circ \sigma - u .$$

The function $\bar{\phi}$ is continuous with summable variations. Following [Bowe2], let us prove that $\bar{\phi} = \bar{\phi} \circ r$. We have

$$\begin{aligned}\bar{\phi} &= \phi + \sum_{k \geq 0} (\phi \circ \sigma^{k+1} - \phi \circ \sigma^k \circ r \circ \sigma) - \sum_{k \geq 0} (\phi \circ \sigma^k - \phi \circ \sigma^k \circ r) \\ &= \phi - \phi - \sum_{k \geq 0} (\phi \circ \sigma^k \circ r \circ \sigma - \phi \circ \sigma^k \circ r) \\ &= \sum_{k \geq 0} (\phi \circ \sigma^k \circ r - \phi \circ \sigma^k \circ r \circ \sigma) .\end{aligned}$$

Now, $r^2 = r$ and $r \circ \sigma \circ r = r \circ \sigma$. Hence $\bar{\phi} \circ r = \bar{\phi}$ as claimed. Thus, $\bar{\phi}$ induces on the one-sided shift a function $\tilde{\phi} : \Sigma_+ \rightarrow \mathbb{R}$ defined by

$$\tilde{\phi} : (x_n)_{n \in \mathbb{N}} \mapsto \bar{\phi}(\dots z_{-2}^{x_0} z_{-1}^{x_0} x_0 x_1 \dots) ,$$

satisfying $\bar{\phi} = \tilde{\phi} \circ \pi$.

To conclude, observe that $\phi - \bar{\phi}$ is bounded and that cylinders defined by the same finite words have the same measure for an invariant probability measure m on the two-sided shift (Σ, σ) and for its image $\pi_* m$ on the one-sided shift (Σ_+, σ_+) . Therefore m is a weak Gibbs measure for ϕ if and only if $\pi_* m$ is a weak Gibbs measure for $\tilde{\phi}$.

By construction $\pi_* m(\tilde{\phi}) = m(\bar{\phi}) = m(\phi)$ since m is invariant. As it is well-known, the natural extension π preserves the entropy. Thus, the measure m is an equilibrium measure with respect to ϕ if and only if $\pi_* m$ is an equilibrium measure with respect to $\tilde{\phi}$.

The reduction to one-sided topological Markov shifts is thus complete. \square

A.2 Proof of the main result Theorem A.4

The uniqueness of the equilibrium state is given by Theorem A.3. We need to prove that weak Gibbs measures and equilibrium measures coincide under the integrability assumption on ϕ^- and that the number $c(m)$ is equal to the pressure.

Step 1. If m is an equilibrium measure, then it is a weak Gibbs measure.

This is a routine consequence of Theorem A.3. Our definition of an equilibrium measure m enforces $\int \phi^- dm < +\infty$ (hences excludes the concomitance of $h_m(\sigma) = +\infty$ and $\int \phi dm = -\infty$).

Recall from Theorem A.3 that $dm = h d\nu$. For $v \in V_G$ and $x \in \text{Fix}_n(\Sigma) \cap [v]$, we have

$$m(C_n(x)) = \int h \mathbb{1}_{C_n(x)} d\nu = e^{-n P_G(\phi)} \int L_\phi^n(h) \mathbb{1}_{C_n(x)} d\nu .$$

By definition, $L_\phi^n(h) \mathbb{1}_{C_n(x)}(z) = \exp S_n \phi(x_0 \dots x_{n-1} z) h(x_0 \dots x_{n-1} z)$ for all $z \in \sigma^n(C_n(x)) = [v]$ (and $L_\phi^n(h) \mathbb{1}_{C_n(x)}(z) = 0$ otherwise). Hence

$$m(C_n(x)) = e^{-n P_G(\phi)} \exp \left(S_n \phi(x) \pm \sum_{k=1}^n \text{var}_k(\phi) \right) \int_{[v]} h d\nu .$$

As $0 < \int_{[v]} h d\nu < +\infty$ and $\sum_{k=1}^{+\infty} \text{var}_k(\phi) < +\infty$, the measure m is a weak Gibbs measure for ϕ with Gibbs constant $c(m) = P_G(\phi)$.

We now turn to the converse implication. Let m be a weak Gibbs measure for ϕ such that $\int \phi^- dm < +\infty$.

The weak Gibbs condition only controls the cylinders that start and end with the same symbol. Passing to an induced system (that is, considering a first return map on a 1-cylinder) will remove this restriction. More precisely, let $a \in V_G$ be a vertex of G and let μ be an invariant probability measure on (Σ, σ) with $\mu([a]) > 0$. The *induced system* on the 1-cylinder $[a] = \{x \in \Sigma : x_0 = a\}$ is the map $\bar{\sigma} : [a] \rightarrow [a]$ defined as follows:

- let $\tau(x) = \inf\{n \geq 1 : \sigma^n x \in [a]\}$ be the *first-return time* in $[a]$, that we also denote by $\tau_{[a]}(x)$ when we want to emphasize $[a]$;
- let $\bar{\sigma}(x) = \sigma^{\tau(x)}(x)$ if $\tau(x) < \infty$;
- let $\bar{\mu}(B) = \mu(B \cap [a])/\mu([a])$ for every Borel subset B of Σ be the restriction of μ to $[a]$ normalized to be a probability measure.

We also define $\tau^0(x) = 0$ and by induction $\tau^{n+1}(x) = \tau(x) + \tau^n(\bar{\sigma}x)$ for every $n \in \mathbb{N}$. Note that $\bar{\sigma}$ can only be iterated on the subset

$$\{x \in [a] : \forall n \geq 1, \tau^n(x) < \infty\}.$$

By Poincaré's recurrence theorem, this is a full measure subset of $[a]$, hence the distinction will be irrelevant for our purposes.

The *induced partition* is

$$\beta = \{[a\xi_1 \dots \xi_{n-1}a] \neq \emptyset : n \geq 1, \xi_i \neq a\}.$$

We note that $\bar{\sigma} : [a] \rightarrow [a]$ is topologically Bernoulli with respect to the partition β (that is, $\bar{\sigma} : b \rightarrow [a]$ is a homeomorphism for each $b \in \beta$). For every integer $N \geq 1$, we define the N -th *iterated partition* β^N of β by

$$\beta^N = \{b_0 \cap \bar{\sigma}^{-1}b_1 \cap \dots \cap \bar{\sigma}^{-N+1}b_{N-1} \neq \emptyset : b_0, \dots, b_{N-1} \in \beta\}$$

and we write $\beta^N(x)$ for the element of the partition β^N that contains x .

Step 2. The topological Markov shift may be assumed to be topologically mixing.

This follows from the spectral decomposition for topological Markov shifts, see for instance [BuS, Lem. 2.2].

Step 3. The Gibbs property implies full support and ergodicity.

Let A be an invariant ($\sigma^{-1}(A) = A$) measurable subset of Σ with $m(A) > 0$ and let us prove that $m(A) = 1$.

Observe that the Gibbs property, together with the transitivity of Σ , implies that any cylinder has positive measure for m , hence that m has full support. Let $a \in V_G$ be such that $m(A \cap [a]) > 0$.

As $m([a]) > 0$, we may consider the induced system on $[a]$. Let $N \geq 1$. When f is a homeomorphism between topological spaces, let f^* denote the pushforwards of measures

by f^{-1} . First note that, for almost every $x \in [a]$ and every $N \in \mathbb{N} - \{0\}$, since $\bar{\sigma}^N$ is an homeomorphism from $\beta^N(x)$ onto $[a]$, we have

$$\frac{m(A \cap [a])}{m([a])} = \frac{m(\bar{\sigma}^N(A \cap \beta^N(x)))}{m(\bar{\sigma}^N(\beta^N(x)))} = \frac{\int_{\beta^N(x) \cap A} \frac{d(\bar{\sigma}^N)^* m}{dm} dm}{\int_{\beta^N(x)} \frac{d(\bar{\sigma}^N)^* m}{dm} dm}.$$

Now, observe¹ that for m -almost every $y \in \beta^N(x)$:

$$\frac{d(\bar{\sigma}^N)^* m}{dm}(y) = \lim_{n \rightarrow \infty} \frac{m(\sigma^{\tau^N(y)}[y_0 \dots y_n])}{m([y_0 \dots y_n])} = C^{\pm 2} e^{-S_{\tau^N(y)} \phi(y) + \tau^N(y) c(m)}.$$

Hence,

$$\frac{m(A \cap [a])}{m([a])} = C^{\pm 4} \frac{m(A \cap \beta^N(x))}{m(\beta^N(x))}.$$

By Doob's increasing martingale convergence theorem (see for instance [Pet]), for m -almost every $x \in [a] - A$, the ratio on the right hand side converges to 0 as $N \rightarrow \infty$. Thus $[a]$ is contained in A modulo m . Therefore $A = \bigcup_{a \in W} [a]$ modulo m for some subset W of V_G .

Since Σ is topologically mixing, for any vertex b , the intersection $[a] \cap \sigma^{-i}[b] \cap \sigma^{-j}[a]$ is not empty for some integers $0 < i < j$. Pick some point x in that set. By invariance, $m([b]) \geq m(\sigma^i(C_j(x))) \geq m(C_j(x))$. But this last number is positive by the weak Gibbs property. Thus $[b]$ is contained in A modulo m . Hence $m(A) = 1$, proving the ergodicity of m .

Step 4. The Gurevič pressure $P_G(\phi)$ is equal to $c(m)$, hence is finite. Furthermore $h_m(\sigma) < \infty$ and $\phi \in \mathbb{L}^1(m)$.

Fix $v \in V_G$ and let $K = \{x \in \Sigma : x_0 = v\}$. Note that $m(K) > 0$. The ergodicity of m gives a Cesaro convergence: as $n \rightarrow \infty$, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} m(K \cap \sigma^{-k} K) \longrightarrow m(K)^2 > 0.$$

The Gibbs property implies that, for all $n \geq 1$,

$$\begin{aligned} m(K \cap \sigma^{-n} K) &= C^{\pm 1} \sum_{x \in \text{Fix}_n(\Sigma) \cap K} e^{S_n \phi(x) - c(m)n} \\ &= C^{\pm 1} \left(\sum_{x \in \text{Fix}_n(\Sigma) \cap K} e^{S_n \phi(x)} \right) e^{-c(m)n}. \end{aligned} \quad (\text{A.2})$$

If we write Z_n for the term between the parenthesis, we have by the definition of the Gurevič pressure:

$$P_G(\phi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln Z_n.$$

As the value of the left hand side of Equation (A.2) is less than one, we see that $c(m) \geq P_G(\phi)$. If this was a strict inequality, then the left hand side of Equation (A.2) would converge to zero, contradicting its Cesaro convergence to $m(K)^2 > 0$. Therefore $P_G(\phi) = c(m)$.

Since $c(m)$ is finite, so is $P_G(\phi)$. Hence Theorem A.2 implies that, for any $\nu \in \mathbb{P}(\Sigma)$ with $\int \phi^- d\nu < +\infty$, we have $h_\nu(\sigma) < \infty$ and ϕ is ν -integrable. In particular, this holds for $\nu = m$, which finishes the proof of Step 4.

¹using, for all $u, v, c > 0$ and $n \in \mathbb{N}$, the notation $u = c^{\pm n} v$ if $\frac{1}{c^n} v \leq u \leq c^n v$

Step 5. If the mean entropy $H_{\bar{\mu}}(\beta) = -\sum_{b \in \beta} \bar{\mu}(b) \ln \bar{\mu}(b)$ is finite, then

$$h_{\bar{\mu}}(\bar{\sigma}) = - \int \ln \frac{d\bar{\mu}}{d\bar{\sigma}^* \bar{\mu}} d\bar{\mu} ,$$

where $\bar{\sigma}^* \bar{\mu}$ is the measure on Σ defined by $B \mapsto \sum_{b \in \beta} \bar{\mu}(\bar{\sigma}(B \cap b))$, with respect to which μ is absolutely continuous: $\mu \ll \bar{\sigma}^* \bar{\mu}$.

This is a classical formula which follows from the computation of the entropy in terms of the information function

$$h_{\bar{\mu}}(\bar{\sigma}) = - \int \sum_{b \in \beta} \mathbb{1}_b(x) \lim_{n \rightarrow \infty} \ln \mathbb{E}_{\bar{\mu}}(\mathbb{1}_b | \bar{\sigma}^{-1} \beta \vee \dots \vee \bar{\sigma}^{-n} \beta)(x) d\bar{\mu}(x)$$

and from the identity, for $x \in b$,

$$\mathbb{E}_{\bar{\mu}}(\mathbb{1}_b | \bar{\sigma}^{-1} \beta \vee \dots \vee \bar{\sigma}^{-n} \beta)(x) = \frac{\bar{\mu}(\beta^{n+1}(x))}{\bar{\mu}(\beta^n(\bar{\sigma}x))} = \frac{\bar{\mu}(\beta^{n+1}(x))}{\bar{\sigma}^* \bar{\mu}(\beta^{n+1}(x))} .$$

The absolute continuity follows from a direct computation and ensures that the integral above is well-defined.

Step 6. For all $a \in V_G$, $N \geq 1$ and $\mu \in \mathbb{P}_{\text{erg}}(\Sigma)$ with $\int \phi^- d\mu < +\infty$, we have

$$h_{\mu}(\sigma) = -\mu([a]) \int_{[a]} \frac{1}{N} \ln \frac{d\bar{\mu}}{d((\bar{\sigma}^N)^* \bar{\mu})} d\bar{\mu} \tag{A.3}$$

We use arguments from the proof of [BuS, Theorem 1.1]: the key is to see that the partition β of $[a]$ has finite mean entropy for the induced measure $\bar{\mu}$ using a Bernoulli approximation.

Let us consider the Bernoulli measure $\bar{\mu}_B$ for $([a], \bar{\sigma})$ defined by

$$\bar{\mu}_B \left(\bigcap_{i=0}^{n-1} \bar{\sigma}^{-i} B_i \right) = \prod_{i=0}^{n-1} \bar{\mu}(B_i)$$

for all $B_i \in \beta$. We construct from it an invariant and ergodic measure μ_B on (Σ, σ) : For every Borel subset A , let

$$\mu_B(A) = \mu([a]) \int_{[a]} \sum_{i=0}^{\tau_{[a]}-1} \mathbb{1}_A \circ \sigma^i d\bar{\mu}_B .$$

Define $\bar{\phi} = \sum_{i=0}^{\tau_{[a]}-1} \phi \circ \sigma^i$. Using the assumption that ϕ has the summable variations, we have

$$\int \phi d\mu_B \geq \mu([a]) \int_{[a]} \bar{\phi} d\bar{\mu} - C = \int \phi d\mu - C > -\infty .$$

Therefore, the last two lines of the proof of Step 4 apply to $\nu = \mu_B$ and $h_{\mu_B}(\sigma) < +\infty$. Since μ_B is ergodic, Abramov's formula yields

$$h_{\mu_B}(\sigma) = \mu([a]) h_{\bar{\mu}_B}(\bar{\sigma}) .$$

Since $\bar{\mu}_B$ is Bernoulli, the right hand side of this equality is equal to

$$\mu([a]) H_{\bar{\mu}_B}(\beta) = \mu([a]) H_{\bar{\mu}}(\beta)$$

which is hence proven to be finite. Thus, Step 5 applies:

$$h_{\bar{\mu}}(\bar{\sigma}) = - \int_{[a]} \ln \frac{d\bar{\mu}}{d(\bar{\sigma}^* \bar{\mu})} d\bar{\mu} .$$

This formula extends to $h_{\bar{\mu}}(\bar{\sigma}^N)$ for all integers $N \geq 1$. Using Abramov's formula this time for μ and $\bar{\mu}$ (since μ is ergodic), we have

$$h_{\mu}(\sigma) = \frac{\mu([a])}{N} h_{\bar{\mu}}(\bar{\sigma}^N) = -\mu([a]) \int_{[a]} \frac{1}{N} \ln \frac{d\bar{\mu}}{d((\bar{\sigma}^N)^* \bar{\mu})} d\bar{\mu} ,$$

as claimed.

Step 7. The entropy of m is equal to $c(m) - \int \phi dm$.

In order to prove this, we apply Step 6 with $\mu = m$ (which is possible, since m has been proven to be ergodic in Step 3). As in the proof of Step 3, the Radon-Nikodym derivative is almost everywhere

$$\frac{d\bar{\mu}}{d((\bar{\sigma}^N)^* \bar{\mu})}(x) = \lim_{n \rightarrow \infty} \frac{\bar{\mu}(\beta^n(x))}{\bar{\mu}(\bar{\sigma}^N(\beta^n(x)))} = C^{\pm 2} \exp(-c(m)\tau^N(x) + S_{\tau^N(x)}\phi(x)) .$$

Therefore, using Step 6 and the fact that $\mu|_{[a]} = \mu([a]) \bar{\mu}$, we have

$$h_{\mu}(\sigma) = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\int_{[a]} (c(m)\tau^N(x) - S_{\tau^N(x)}\phi(x)) d\mu \pm 2 \ln C \right) . \quad (\text{A.4})$$

Note that $\tau^N(x)$ can be seen as a Birkhoff sum for the induced system on $[a]$ and the function τ and that, by Kac's theorem (see for instance [Pet, Sect. 2.4]),

$$\int_{[a]} \tau d\bar{\mu} = \mu([a])^{-1} .$$

Therefore, Birkhoff's ergodic theorem yields, with convergence in $\mathbb{L}^1(\bar{\mu})$,

$$\lim_{N \rightarrow \infty} \frac{c(m)\tau^N(x)}{N} = \frac{c(m)}{\mu([a])} .$$

To analyze the second term in Equation (A.4), let $\hat{\phi}(x) = \sum_{k=0}^{\tau(x)-1} \phi(\sigma^k x)$ and observe that, by a variation of the proof of Kac's theorem, $\hat{\phi} \in \mathbb{L}^1(\bar{\mu})$ with $\bar{\mu}(\hat{\phi}) = \mu([a])^{-1} \mu(\phi)$. Indeed, passing to the natural extension, one can assume the system to be invertible and use the partition modulo μ given by

$$\bigcup_{n \geq 1, 0 \leq k < n} \sigma^k(\{x \in [a] : \tau(x) = n\}) .$$

Since $S_{\tau^N(x)}\phi(x)$ coincides with the Birkhoff sum $\bar{S}_N \hat{\phi}(x)$ for the induced system, Birkhoff's ergodic theorem yields, with convergence in $\mathbb{L}^1(\bar{\mu})$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} S_{\tau^N(x)}\phi(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \bar{S}_N \hat{\phi}(x) = \mu([a])^{-1} \mu(\phi) .$$

The claim follows.

Step 8. Conclusion: any weak Gibbs measure is an equilibrium measure and $c(m) = P_G(\phi)$.

Steps 4 and 7 prove that $h_m(\sigma) + \int \phi dm$ is well-defined and equal to $c(m)$, which by Step 4 is equal to $P_G(\phi)$, which is equal to $\sup\{P(\phi, \nu) : \nu \in \mathbb{P}(\Sigma) \text{ and } \int \phi^- d\nu < +\infty\}$ by Theorem A.2, so that m is an equilibrium measure. This completes the proof of Theorem A.4. \square

List of Symbols

∞	standard point at infinity $[1 : 0]$ of a projective plane	228
$\mathbb{1}_A$	characteristic function of a subset A	25
$\sim = \sim_{\mathcal{D}}$	equivalence relation on index set of an equivariant family \mathcal{D}	109
$\ f\ _{\alpha}$	α -Hölder norm of $f \in \mathcal{C}^{\alpha}(Z)$	42
$\ \psi\ _{\ell}$	Sobolev $W^{\ell,2}$ -norm of $\psi \in \mathcal{C}^{\ell}(N)$	122
$ \cdot _v$	(normalised) absolute value associated to a valuation v	223
$*_v$	base point $*_v = [\mathcal{O}_v \times \mathcal{O}_v]$ of the Bruhat-Tits tree \mathbb{X}_v	227
$\underline{A}(\mathcal{O}_v)$	maximal compact-open diagonal subgroup of $\mathrm{PGL}_2(K_v)$	229
$\mathrm{Aut}(\mathbb{X}, \lambda)$	automorphism group (edge-preserving, without inversion) of a metric tree (\mathbb{X}, λ)	36
$\mathrm{Aut} \mathbb{X}$	automorphism group (without inversion) of a simplicial tree \mathbb{X}	36
$B(x, r)$	closed ball of center x and radius r in a metric space	25
$B^{\pm}(w, \eta')$	Hamenstädt's ball of radius $\eta' > 0$ with center any geodesic line extension of $w \in \mathcal{G}_{\pm} X$	31
C	geometrically connected smooth projective curve over \mathbb{F}_q	223
${}^c A$	complementary set of a subset A	25
$c(g)$	period for a system of conductances c of a closed orbit g for the geodesic flow	215
$\mathcal{C}_c(Z)$	space of real-valued continuous maps with compact support on Z	25
$\mathcal{C}_b^{\alpha}(Z)$	space of bounded α -Hölder-continuous real-valued functions on Z	42
$\mathcal{C}_b^{k, \alpha}(Z)$	space of real-valued functions on Z with bounded α -Hölder-continuous derivatives of order at most k along the flow	131
$\mathcal{C}_c^{\alpha}(Z)$	space of α -Hölder-continuous real-valued functions with compact support on Z	42
$\mathcal{C}_c^{k, \alpha}(Z)$	space of real-valued functions on Z with bounded α -Hölder-continuous derivatives of order at most k along the flow and compact support	132
$\mathcal{C}_c^{\ell}(N)$	space of real-valued \mathcal{C}^{ℓ} -smooth functions with compact support on a smooth manifold N	122
$\mathrm{codeg}_{\mathbb{D}}(x)$	codegree of a vertex x with respect to a subtree \mathbb{D}	116
$\mathrm{codeg}_{\mathcal{D}}(x)$	codegree of a vertex x with respect to a family of subtrees \mathcal{D}	117

$\text{cov}_{m,n}(\phi, \psi)$	n -th correlation coefficient of ϕ, ψ for the measure m under a transformation	122
$\text{cov}_{\mu,t}(\psi, \psi')$	correlation coefficient of ψ, ψ' under a flow at time t for the measure μ	132
$\text{cov}_{\mu_v,g}$	correlation coefficient for $g \in G_v$ and a measure μ_v on $\Gamma \backslash G_v$	242
$\mathcal{C}\Lambda\Gamma$	convex hull in X of the limit set $\Lambda\Gamma$ of Γ	26
\bar{x}	conjugate of a quaternion x	267
$[a, b, c, d]$	crossratio of pairwise distinct points a, b, c, d	273
$ a, b, c, d _v$	absolute crossratio of pairwise distinct points a, b, c, d for a valuation v	273
$\partial_\infty X$	boundary at infinity of X	26
$\partial_e \mathbb{X}$	set of points at infinity of the geodesic rays whose initial (oriented) edge is e	37
$\partial V\mathbb{D}$	boundary of set of vertices of a simplicial subtree \mathbb{D}	116
$\partial\mathbb{D}$	maximal subgraph with set of vertices $\partial V\mathbb{D}$	116
$\partial_-^1 D$	inner unit normal bundle of a closed convex subset D	32
$\partial_+^1 D$	outer unit normal bundle of a closed convex subset D	32
$\text{deg } v$	degree of valuation v , equal to $\dim_{\mathbb{F}_q} k_v$	223
$\delta = \delta_{\Gamma, F}$	critical exponent of (Γ, F)	45
Δ_{c^\pm}	Laplacian operator associated to a system of conductances c^\pm	92
Δ_x	unit Dirac mass at a point x	25
$\text{Disc } D$	reduced discriminant of a quaternion algebra D over \mathbb{Q}	268
d_D	distance-like map on $\partial_\infty X - \partial_\infty D$ associated with closed convex subset D	238
$d = d_{\mathcal{G}X}$	distance on space of generalised geodesic lines	28
$d = d_{T^1 X}$	distance on space of germs of geodesic lines	29
$d_{\mathcal{H}}$	Hamenstädt's distance at infinity associated to an horoball \mathcal{H}	31
d_x	visual distance on $\partial_\infty X$ seen from $x \in X$	27
$d_{W^\pm(w)}$	Hamenstädt's distance on the strong stable/unstable ball of $w \in \mathcal{G}_\pm X$	30
$E\mathbb{X}$	set of edges of a graph \mathbb{X}	36
φ_{R_v}	Euler function of fonction ring R_v	255
f^-	negative part of a real-valued map f	25
f_D^\pm	fibration over $\partial_\pm^1 D$ with fibers the stable/unstable leaves	33
\tilde{F}	potential on $T^1 X$	43
F	potential on $\Gamma \backslash T^1 X$	43
\mathbb{F}_q	finite field of order a prime power q	223
g	genus of the smooth projective curve \mathbf{C}	223
$\tilde{\mathcal{G}}X$	space of generalised geodesic lines in X	28
$\tilde{\mathcal{G}}\mathbb{X}$	space of generalised discrete geodesic lines in a simplicial tree \mathbb{X}	36
$\tilde{\mathcal{G}}_{\text{even}} \mathbb{X}$	space of generalised discrete geodesic lines ℓ in \mathbb{X} with $d(\ell(0), x_0)$ even	65

$\mathcal{G}_{\text{even}}\mathbb{X}$	space of discrete geodesic lines ℓ in \mathbb{X} with $\ell(0)$ at even distance from x_0	65
$\mathcal{G}_{\pm}X$	space of generalised positive/negative geodesic rays in X	29
$\mathcal{G}_{\pm,0}X$	space of generalised geodesic lines in X isometric exactly on $\pm[0, +\infty[$	29
$(\mathbf{g}^t)_{t \in \mathbb{R}}$	(continuous time) geodesic flow on space of generalised geodesics $\check{\mathcal{G}}X$	28
$(\mathbf{g}^t)_{t \in \mathbb{Z}}$	(discrete time) geodesic flow on space of generalised geodesics $\check{\mathcal{G}}\mathbb{X}$	37
$h_m(T)$	metric entropy of a transformation T with respect to a probability measure m	85
$h_m(\phi_1)$	metric entropy of a flow $(\phi_t)_{t \in \mathbb{R}}$ with respect to a probability measure m	87
$h(\alpha)$	complexity of loxodromic fixed point α	258
$h(\alpha)$	complexity of quadratic irrational α in K_v	262
$h(\alpha)$	complexity of quadratic irrational α in \mathbb{Q}_p	270
$h_{\alpha}(\beta)$	relative height of loxodromic fixed point β with respect to α	273
$h_{\alpha}(\beta)$	relative height of quadratic irrational β with respect to α	279
Haar_{K_v}	normalised Haar measure of $(K_v, +)$	225
$\mathcal{H}[t]$	horoball contained in \mathcal{H} whose boundary is at distance t from the boundary of \mathcal{H}	28
$HB_+(w)$	stable horosphere of $w \in \mathcal{G}_+X$	31
$HB_-(w)$	unstable horosphere of $w \in \mathcal{G}_-X$	31
$H_+(w)$	stable horosphere of $w \in \mathcal{G}_+X$	31
$H_-(w)$	unstable horosphere of $w \in \mathcal{G}_-X$	31
ι	antipodal map $w \mapsto \{t \mapsto w(-t)\}$ of $\check{\mathcal{G}}X$	28
$\iota_G(\beta)$	G -reciprocity index of a quadratic irrational β	264
$\text{Isom}(X)$	isometry group of X	26
\mathcal{I}_v	set of classes of fractional ideals of R_v	231
K	global function field over \mathbb{F}_q	223
K_v	completion of function field K for the valuation v	223
k_v	residual field of the valuation v on K	223
$\lambda(\gamma)$	translation length of an isometry γ of X	26
$\Lambda\Gamma$	limit set of a discrete group of isometries Γ of X	26
$\Lambda_c\Gamma$	conical limit set of a discrete group of isometries Γ of X	26
$\mathbb{L}^2(\mathbb{Y}, G_*)$	Hilbert space of square integrable maps on $V\mathbb{Y}$ for the measure $\text{vol}_{(\mathbb{Y}, G_*)}$	38
lk	links of vertices in simplicial trees	227
L_{Γ}	length spectrum of action of Γ on X	65
\ln	natural logarithm (with $\ln(e) = 1$)	25
ℓ_{\pm}	positive/negative endpoint of geodesic line ℓ	28
ℓ^*	standard basepoint in space of geodesic lines $\mathcal{G}\mathbb{X}_v$	229

$m_{\mathcal{D}}(x)$	multiplicity of a vertex x with respect to an equivariant family \mathcal{D}	116
\tilde{m}_F	Gibbs measure on the space of geodesic lines $\mathcal{G}X$	55
m_F	Gibbs measure on the quotient space of geodesic lines $\Gamma \backslash \mathcal{G}X$	55
\tilde{m}_F	Gibbs measure on the space of discrete geodesic lines $\mathcal{G}\mathbb{X}$	62
m_F	Gibbs measure on the quotient space of discrete geodesic lines $\Gamma \backslash \mathcal{G}\mathbb{X}$	62
\overline{m}_F	renormalised Gibbs measure $m_F/ m_F $ on $\Gamma \backslash \mathcal{G}X$	121
\overline{m}_c	renormalised Gibbs measure $m_c/ m_c $ on $\Gamma \backslash \mathcal{G}\mathbb{X}$	123
$(\mu_x^{\pm})_{x \in X}$	(normalised) Patterson density for the pair $(\Gamma, \tilde{F}^{\pm})$	53
$\mu_{W^{\pm}(w)}$	skinning measures on the strong stable or strong unstable leaf $W^{\pm}(w)$	106
$(\mu_x^{\text{Haus}})_{x \in X}$	Hausdorff measures of the visual distances d_x on $\Lambda\Gamma$	64
$N(I)$	(absolute) norm of a nonzero ideal I in a Dedekind ring	225
N_w^{\pm}	homeomorphism between stable/unstable leaves and inner/outer normal bundles of horoballs	32
$\mathcal{N}_{\epsilon}A$	closed ϵ -neighbourhood of a subset A of a metric space	25
$\mathbf{n}(\beta)$	relative norm of quadratic irrational β	261
$\mathbf{N}(x)$	reduced norm of a quaternion x	268
ν_w^{\mp}	conditional measure on the (weak) stable/unstable leaf $W^{0\pm}(w)$ of $w \in \mathcal{G}_{\pm}X$	107
$\mathcal{O}_x A$	shadow of a subset A of X seen from $x \in X \cup \partial_{\infty}X$	26
\mathcal{O}_v	valuation ring of v in K_v	223
π	footpoint projection $w \mapsto w(0)$ on (generalised) geodesic lines	28
π_v	uniformizer of a valuation v of a function field K over \mathbb{F}_q	223
$\pi(\mathbb{Y}, G_*)$	fundamental groupoid of a graph of groups (\mathbb{Y}, G_*)	196
P_D	closest point map to a convex subset D	32
P_D^{\pm}	closest point map homeomorphism from $\partial_{\infty}X - \partial_{\infty}D$ to outer/inner normal bundle of D	32
$\tilde{\phi}_{\mu}^{\pm}$	total mass function of Patterson density $(\mu_x^{\pm})_{x \in X}$	61
P_{ϕ}	pressure of a potential ϕ under a transformation	85
P_{ψ}	pressure of a potential ψ under a flow	9, 87
$P_{\phi}(m)$	metric pressure for a potential ϕ of a probability measure m invariant under a transformation	85
$P_{\psi}(m)$	metric pressure for a potential ψ of a flow-invariant probability measure m	9, 87
$Q = Q_{\Gamma, F, x, y}$	Poincaré series of (Γ, F)	45
q_v	order of residual field k_v	223
R_v	affine algebra of the affine curve $\mathbf{C} - \{v\}$	224

β^σ	Galois conjugate of a quadratic irrational β in K_v	261
α^σ	Galois conjugate of a quadratic irrational α in \mathbb{Q}_p	270
$\tilde{\sigma}_D^\pm$	skinning measure on outer/inner normal bundle of convex subset D	103
$\sigma_{\mathcal{D}}^-$	inner skinning measure on $\Gamma \backslash \check{\mathcal{G}}X$ of a family of closed convex subsets \mathcal{D}	110
$\sigma_{\mathcal{D}}^+$	outer skinning measure on $\Gamma \backslash \check{\mathcal{G}}X$ of a family of closed convex subsets \mathcal{D}	110
$\tilde{\sigma}_{\mathcal{D}}^\pm$	outer/inner skinning measure on $\check{\mathcal{G}}X$ of a family of closed convex subsets \mathcal{D}	109
$\tilde{\sigma}_\Omega^\pm$	outer/inner skinning measure of a family $\Omega = (\Omega_i)_{i \in I}$ of subsets of $(\partial_\pm^1 D_i)_{i \in I}$	109
$T\pi$	first edge map of a geodesic line	37
$T^1 X$	space of germs at $t = 0$ of geodesic lines in X	29
$\mathbf{tr}(\beta)$	relative trace of quadratic irrational β	261
$\mathbf{Tr}(x)$	reduced trace of a quaternion x	268
$\mathbf{Tvol}_{(\mathbb{Y}, G_*)}$	volume form on the set of edges of a graph of finite groups (\mathbb{Y}, G_*)	38
$\mathbf{Tvol}(\mathbb{Y}, G_*, \lambda)$	volume form of the set of edges of a metric graph of finite groups $(\mathbb{Y}, G_*, \lambda)$	38
$\mathbf{TVol}(\mathbb{Y}, G_*)$	total volume of the set of edges of a graph of finite groups (\mathbb{Y}, G_*)	38
$\mathbf{TVol}(\mathbb{Y}, G_*, \lambda)$	total volume of the set of edges of a metric graph of finite groups $(\mathbb{Y}, G_*, \lambda)$	39
\mathcal{U}_D^\pm	domain of the fibration f_D^\pm	32
$V_{w, \eta, \eta'}^\pm$	dynamical neighbourhoods of a point $w \in \mathcal{G}_\pm X$	33
$\mathcal{V}_{\eta, \eta'}^\pm(\Omega^\mp)$	dynamical neighbourhood of a subset Ω^\mp of $\mathcal{G}_\pm X$	33
v_ℓ	germ at $t = 0$ of geodesic line ℓ	29
v_∞	valuation at infinity of $\mathbb{F}_q(Y)$	222
$\mathbf{vol}_{(\mathbb{Y}, G_*)}$	volume form on the set of vertices of a graph of finite groups (\mathbb{Y}, G_*)	38
$\mathbf{Vol}(\mathbb{Y}, G_*)$	volume of a graph of finite groups (\mathbb{Y}, G_*)	38
$V\mathbb{X}$	set of vertices of a graph \mathbb{X}	36
w_\pm	positive/negative endpoint of generalised geodesic line w	28
$W^+(w)$	strong stable leaf of $w \in \mathcal{G}_+ X$	30
$W^{0+}(w)$	stable leaf of $w \in \mathcal{G}_+ X$	31
$W^-(w)$	strong unstable leaf of $w \in \mathcal{G}_- X$	30
$W^{0-}(w)$	unstable leaf of $w \in \mathcal{G}_- X$	31
$ \mathbb{X} _\lambda$	geometric realization of a metric tree (\mathbb{X}, λ)	36
\mathbb{X}_v	Bruhat-Tits tree of (PGL_2, K_v)	227
$\zeta_K(s)$	Dedekind's zeta function of a function field K	225

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