Division algebra codes achieve MIMO block fading channel capacity within a constant gap

Laura Luzzi, Roope Vehkalahti

Division algebra codes achieve MIMO block fading channel capacity within a constant gap

Laura Luzzi
Laboratoire ETIS, CNRS - ENSEA - UCP
Cergy-Pontoise, France
laura.luzzi@ensea.fr

Roope Vehkalahti
Department of Mathematics and Statistics, University of Turku
Finland
roiive@utu.fi

***Abstract*—This work addresses the question of achieving capacity with lattice codes in multi-antenna block fading channels when the number of fading blocks tends to infinity. In contrast to the standard approach in the literature which employs random lattice ensembles, the existence results in this paper are derived from number theory. It is shown that a multiblock construction based on division algebras achieves rates within a constant gap from block fading capacity both under maximum likelihood decoding and naive lattice decoding. First the gap to capacity is shown to depend on the discriminant of the chosen division algebra; then class field theory is applied to build families of algebras with small discriminants. The key element in the construction is the choice of a sequence of division algebras whose centers are number fields with small root discriminants.**

***Index Terms*—MIMO, block fading, number theory, division algebras**

I. Introduction

The closed-form expression of the capacity of ergodic multiple-input multiple-output (MIMO) channels was given in [1] and [2]. In this work we consider the question of achieving MIMO capacity with lattice codes and we prove that there exists a family of so-called multi-block division algebra codes [3, 4] that achieve a constant gap to capacity over the block fading MIMO channel when the number of fading blocks tends to infinity.

Our constructions are based on two results from classical class field theory. First we choose the center K of the algebra from an ensemble of Hilbert class fields having small root discriminant and then we prove the existence of a K-central division algebra with small discriminant. Our lattices belong to a very general family of division algebra codes introduced in [3, 4, 5], and developed further in [6] and [7]. We will use the most general form presented in [8].

While we discuss specific lattice codes from division algebras, our proofs do work for any ensemble of matrix lattices with asymptotically good normalized minimum determinant. The larger this value is, the smaller the gap to the capacity.

This work suggests that capacity questions in fading channels are naturally linked to problems in the mathematical research area of *geometry of numbers*. Unlike our previous work in the single antenna case [9], many of the questions that arise have not been actively studied by the mathematical community.

We note that, while studying diversity-multiplexing gain tradeoff (DMT) of multiblock codes in [4], H.-f. Lu conjectured that these codes might approach MIMO capacity. Our work confirms that conjecture; however, we point out that it is unlikely that DMT-optimality alone is enough to approach capacity. Instead one should pick the code very carefully by maximizing the normalized minimum determinant.

II. Preliminaries

A. Channel model

We consider a MIMO system with n transmit and n_r receive antennas, where transmission takes place over k quasi-static Rayleigh fading blocks of delay $T = n$. Each multi-block codeword $X \in M_{n\times nk}(\mathbb{C})$ has the form $(X_1, X_2, \ldots, X_k)$, where the submatrix $X_i \in M_n(\mathbb{C})$ is sent during the i-th block. The received signals are given by

$$Y_i = H_i X_i + W_i, \qquad i \in \{1, \ldots, k\} \tag{1}$$

where $H_i \in M_{n_r\times n}(\mathbb{C})$ and $W_i \in M_{n_r\times T}(\mathbb{C})$ are the channel and noise matrices. The coefficients of H_i and W_i are modeled as circular symmetric complex Gaussian with zero mean and unit variance per complex dimension, and the fading blocks H_i are independent. We assume that perfect channel state information is available at the receiver, and that decoding is performed after all k blocks have been received. We will call such a channel an (n, n_r, k)-multiblock channel.

A multi-block code $\mathcal{C}$ in a (n, n_r, k)-channel is a set of matrices in $M_{n\times nk}(\mathbb{C})$. In particular we will concentrate on finite codes that are drawn from lattices. Let R denote the code rate in bits per complex channel use; equivalently, $|\mathcal{C}| = 2^{Rkn^2}$. We assume that every matrix X in a finite code $\mathcal{C} \subset M_{n\times nk}(\mathbb{C})$ satisfies the average power constraint

$$\frac{1}{nk}\|X\|^2 \leq P. \tag{2}$$

B. Lattices and spherical shaping

Definition 2.1: A *matrix lattice* $L \subseteq M_{n\times T}(\mathbb{C})$ has the form $L = \mathbb{Z}B_1 \oplus \mathbb{Z}B_2 \oplus \cdots \oplus \mathbb{Z}B_s$, where the matrices $B_1, \ldots, B_s$ are linearly independent over $\mathbb{R}$, i.e., form a lattice basis, and s is called the *rank* or the *dimension* of the lattice.

In the following we will use the notations $\mathbb{R}(L)$ for the linear space which is generated by the basis elements of the

lattice L, and $\mathrm{Vol}(L)$ for the volume of a fundamental region of L according to the Lebesgue measure in $\mathbb{R}(L)$.

Lemma 2.2: [10] Let us suppose that L is a lattice in $M_{n\times kn}(\mathbb{C})$ and S is a Jordan measurable bounded subset of $\mathbb{R}(L)$. Then there exists $X \in M_{n\times kn}(\mathbb{C})$ such that

$$|(L+X)\cap S| \geq \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(L)}.$$

C. Minimum determinant for the multiblock channel

Let us first assume that we have an l-dimensional square matrix lattice L in $M_{n\times n}(\mathbb{C})$. The minimum determinant of the lattice L is defined as

$$\det_{min}(L) = \inf_{X\neq\{\mathbf{0}\}} \{|\det(X)|\}.$$

The pairwise-error probability based determinant criterion by Tarokh *et al.* [11] motivates us to define the *normalized minimum determinant* $\delta(L)$, which is obtained by scaling the lattice L to have a unit volume fundamental parallelotope before taking the minimum determinant. A simple computation proves the following:

Lemma 2.3: Let $L \subset M_{n\times n}(\mathbb{C})$ be an l-dimensional matrix lattice. We then have that

$$\delta(L) = \det_{min}(L)/(\mathrm{Vol}(L))^{n/l}.$$

This concept generalizes to the multiblock case as follows.

Let us suppose that $L \subset M_{n\times kn}(\mathbb{C})$ is a multiblock code and that $X = (X_1, X_2, \ldots X_k)$ is a codeword in L. The received signal matrix

$$(H_1X_1, H_2X_2, \ldots, H_kX_k) + (W_1, W_2, \ldots, W_k),$$

can just as well be written in the form

$$(H_1, H_2, \cdots, H_k)\mathrm{diag}(X) + \mathrm{diag}(W_1, W_2, \ldots, W_k),$$

where the diag-operator places the i-th $n\times n$ entry in the i-th diagonal block of a matrix in $M_{kn\times kn}(\mathbb{C})$. This reveals that optimizing a code L for the (n, n_r, k)-multiblock channel is equivalent to optimizing $\mathrm{diag}(L)$ for the usual one shot $nk \times kn_r$ MIMO channel, where $\mathrm{diag}(L)$ is defined as $\{\mathrm{diag}(X) \mid X \in L\}$.

Definition 2.4: By abusing notation we define

$$\det_{min}(L) := \det_{min}(\mathrm{diag}(L)) \text{ and } \delta(L) := \delta(\mathrm{diag}(L)).$$

III. Lattices from division algebras

Let us now describe how lattice codes from division algebras are typically built.

Definition 3.1: Let K be an algebraic number field of degree m and assume that E/K is a cyclic Galois extension of degree n with Galois group $Gal(E/K) = \langle\sigma\rangle$. We can define an associative K-algebra

$$\mathcal{D} = (E/K, \sigma, \gamma) = E \oplus uE \oplus u^2E \oplus \cdots \oplus u^{n-1}E,$$

where $u \in \mathcal{D}$ is an auxiliary generating element subject to the relations $xu = u\sigma(x)$ for all $x \in E$ and $u^n = \gamma \in K^*$. We call the resulting algebra a *cyclic algebra*.

It is clear that the center of the algebra $\mathcal{D}$ is precisely the field K. That is, an element of $\mathcal{D}$ commutes with all other elements of $\mathcal{D}$ if and only if it lies in K.

Definition 3.2: We call $\sqrt{[\mathcal{D}:K]}$ the *degree* of the algebra $\mathcal{D}$. It is easily verified that the degree of $\mathcal{D}$ is equal to n.

Definition 3.3: A $\mathbb{Z}$*-order* Λ in $\mathcal{D}$ is a subring of $\mathcal{D}$ having the same identity element as $\mathcal{D}$, and such that Λ is a finitely generated module over $\mathbb{Z}$ which generates $\mathcal{D}$ as a linear space over $\mathbb{Q}$.

To every $\mathbb{Z}$-order Λ in $\mathcal{D}$ we can associate a non-zero integer $d(\Lambda/\mathbb{Z})$ called the $\mathbb{Z}$*-discriminant of* Λ [12, Chapter 2].

IV. Asymptotically good families of division algebra codes

A. Number fields with small root discriminants

The following theorem by Martinet [13] proves the existence of infinite sequences of number fields K with small discriminants d_K. As we will see, choosing such a field as the center of the algebra $\mathcal{D}$ is the key to obtaining a good normalized minimum determinant.

Theorem 4.1: There exists an infinite tower of totally complex number fields $\{K_k\}$ of degree $2k = 5\cdot 2^t$, such that

$$|d_{K_k}|^{\frac{1}{2k}} = G, \tag{3}$$

for $G \approx 92.368$, and an infinite tower of totally real number fields $\{F_k\}$ of degree $k = 2^t$ such that

$$|d_{F_k}|^{\frac{1}{k}} = G_1, \tag{4}$$

where $G_1 \approx 1058$.

B. Division algebra based $2kn^2$-dimensional codes in $M_{n\times nk}(\mathbb{C})$

Let us now suppose that we have a totally complex field K of degree $2k$ and a K-central division algebra $\mathcal{D}$ of degree n.

Proposition 4.2: [8, Proposition 5] Let Λ be a $\mathbb{Z}$-order in $\mathcal{D}$. Then there exists an injective mapping $\varphi : \mathcal{D} \mapsto M_{n\times nk}(\mathbb{C})$ such that $\varphi(\Lambda)$ is a $2kn^2$-dimensional lattice in $M_{n\times nk}(\mathbb{C})$ and

$$\det_{min}(\varphi(\Lambda)) = 1, \quad \mathrm{Vol}(\varphi(\Lambda)) = \sqrt{|d(\Lambda/\mathbb{Z})|\cdot 2^{-2kn^2}},$$
$$\delta(\varphi(\Lambda)) = \left(\frac{2^{2kn^2}}{|d(\Lambda/\mathbb{Z})|}\right)^{1/4n}.$$

We can now see that in order to maximize the minimum determinant of a multiblock code, we should minimize the $\mathbb{Z}$-discriminant of the corresponding $\mathcal{O}_K$-order Λ, given by

$$d(\Lambda/\mathbb{Z}) = N_{K/\mathbb{Q}}(d(\Lambda/\mathcal{O}_K))(d_K)^{n^2},$$

where $N_{K/\mathbb{Q}}$ is the algebraic norm in K.

Let P_1 and P_2 be some prime ideals of K with norms $N_{K/\mathbb{Q}}(P_1)$ and $N_{K/\mathbb{Q}}(P_2)$. According to [14, Theorem 6.14] there exists a degree n division algebra $\mathcal{D}$ having $\mathbb{Z}$-order Λ with discriminant

$$d(\Lambda/\mathbb{Z}) = (N_{K/\mathbb{Q}}(P_1)N_{K/\mathbb{Q}}(P_2))^{n(n-1)}(d_K)^{n^2}. \tag{5}$$

Let us now aim for building the families of (n,n,k) multiblock codes with as large as possible normalized minimum determinant.

A trivial observation is that every number field of degree $2k$ has prime ideals P_1 and P_2 such that

$$N_{K/\mathbb{Q}}(P_1) \leq 2^{2k} \text{ and } N_{K/\mathbb{Q}}(P_2) \leq 3^{2k}. \tag{6}$$

Armed with this observation, we have the following.

Proposition 4.3: Given n there exists a family of $2n^2k$-dimensional lattices $L_{n,k} \subset M_{n\times nk}(\mathbb{C})$, such that

$$\det_{min}(L_{n,k}) = 1, \quad \text{Vol}(L_{n,k}) \leq 6^{kn(n-1)} \left(\frac{G}{2}\right)^{n^2k}.$$

Proof: Suppose that K is a degree $2k$ field extension in the Martinet family of totally complex fields such that (3) holds. We know that this field K has some primes P_1 and P_2 such that $N_{K/\mathbb{Q}}(P_1) \leq 2^{2k}$ and $N_{K/\mathbb{Q}}(P_2) \leq 3^{2k}$. Then, there exists a central division algebra $\mathcal{D}$ of degree n over K, and a maximal order Λ of $\mathcal{D}$, such that

$$d(\Lambda/\mathbb{Z}) = (N_{k/\mathbb{Q}}(P_1)N_{K/\mathbb{Q}}(P_2))^{n(n-1)}(d_K)^{n^2} \leq$$
$$\leq (6^{2k})^{n(n-1)}(G^{2k})^{n^2}. \qquad \square$$

Let us now show how we can design multiblock codes $\mathcal{C}$ having rate R, and satisfying average power constraint P, from a scaled version $\alpha L_{n,k} \subseteq M_{n\times nk}(\mathbb{C})$ of the lattices defined in Proposition 4.3. Here α is a suitable energy normalization constant. We denote by $B(r)$ the set of matrices in $M_{n\times nk}(\mathbb{C})$ with Frobenius norm smaller or equal to r. According to Lemma 2.2, we can choose a constant shift $X_R \in M_{n\times nk}(\mathbb{C})$ such that for $\mathcal{C} = B(\sqrt{Pkn}) \cap (X_R + \alpha L_{n,k})$ we have

$$2^{Rnk} = |\mathcal{C}| \geq \frac{\text{Vol}(B(\sqrt{Pkn}))}{\text{Vol}(\alpha L_{n,k})} = \frac{C_{n,k}P^{n^2k}}{\alpha^{2n^2k}\,\text{Vol}(L_{n,k})},$$

where $C_{n,k} = \frac{(\pi nk)^{n^2k}}{(n^2k)!}$. We find the following condition for the scaling constant:

$$\alpha^2 = \frac{C_{n,k}^{\frac{1}{n^2k}}P}{2^{\frac{R}{n}}\,\text{Vol}(L_{n,k})^{\frac{1}{n^2k}}} \geq \frac{C_{n,k}^{\frac{1}{n^2k}}P}{2^{\frac{R}{n}}(G/2)6^{1-\frac{1}{n}}}. \tag{7}$$

C. Division algebra based kn^2-dimensional codes in $M_{n\times nk}(\mathbb{C})$

Let $K/\mathbb{Q}$ be a totally real number field of degree k and $\mathcal{D}$ a K-central division algebra of degree n. Then there exists an embedding $\varphi : \mathcal{D} \to M_{n\times nk}(\mathbb{C})$ [7] with the following properties.

Proposition 4.4: Let us suppose that Λ is a $\mathbb{Z}$-order in $\mathcal{D}$. Then $\varphi(\Lambda)$ is an n^2k-dimensional lattice in $M_{n\times nk}(\mathbb{C})$ and

$$\det_{min}(\varphi(\Lambda)) = 1, \ \text{Vol}(\varphi(\Lambda)) = \sqrt{d(\Lambda/\mathbb{Z})}, \text{ and}$$
$$\delta(\varphi(\Lambda)) = \left(\frac{1}{|d(\Lambda/\mathbb{Z})|}\right)^{1/2n}.$$

Next, we will focus on a particular instance of this family of lattices. We use the notation $\mathbf{H}$ for matrices of the form

$$\begin{pmatrix} c & -b^* \\ b & c^* \end{pmatrix}$$

and $M_{1\times k}(\mathbf{H})$ for matrices in $M_{2\times 2k}(\mathbb{C})$ such that each 2×2 block is of type $\mathbf{H}$.

Proposition 4.5: [15] Let K be a totally real number field of degree k. Then there exists a degree 2 K-central division algebra $\mathcal{D}$ and a $\mathbb{Z}$-order $\Lambda \subset \mathcal{D}$ such that $\varphi(\Lambda)$ is a $4k$-dimensional lattice in $M_{1\times k}(\mathbf{H}) \subseteq M_{2\times 2k}(\mathbb{C})$ and

$$\det_{min}(\varphi(\Lambda)) = 1, \ \text{Vol}(\varphi(\Lambda)) = |d_K|^2.$$

Assuming that the center K belongs to the family of real fields from Theorem 4.1, we have the following.

Corollary 4.6: For every $k = 2^m$, there exists a $4k$-dimensional lattice $L_{Alam,k} \subset M_{1\times k}(\mathbf{H})$ such that

$$\det_{min}(\varphi(\Lambda)) = 1, \ \text{Vol}(\varphi(\Lambda)) = G_1^{2k},$$

where $G_1 \approx 1058$.

Similarly to the previous section, we will produce codes $\mathcal{C}$ having rate R and satisfying average power constraint P from the lattices $L_{Alam,k} \subseteq M_{2\times 2k}(\mathbb{C})$. Let α be an energy normalization constant. According to Lemma 2.2, $\exists X_R \in M_{2\times 2k}(\mathbb{C})$ such that for $\mathcal{C} = B(\sqrt{2Pk}) \cap (X_R + \alpha L_{Alam,k})$ we have

$$2^{2Rk} = |\mathcal{C}| \geq \frac{\text{Vol}(B(\sqrt{Pk2}))}{\text{Vol}(\alpha L_{Alam,k})} = \frac{C_{Alam,k}P^{2k}}{\alpha^{4k}\,\text{Vol}(L_{Alam,k})},$$

where $C_{Alam,k} = \frac{(2k\pi)^{2k}}{(2k)!}$. Solving for α, we find

$$\alpha^2 = \frac{C_{Alam,k}^{\frac{1}{2k}}P}{2^R\,\text{Vol}(L_{n,k})^{\frac{1}{2k}}} \geq \frac{C_{Alam,k}^{\frac{1}{2k}}P}{2^R G_1}. \tag{8}$$

V. Achieving constant gap

A. The codes for the (n,n,k)-multiblock channel

Let us now consider the lattice codes $\mathcal{C}$ of section IV-B. Here the underlying lattice $L_{n,k} \subset M_{n\times nk}(\mathbb{C})$ is $2n^2k$-dimensional. We are considering the channel model (1) in the symmetric MIMO case where $n_r = n$. We will analyze the performance of these codes when the number of antennas n is fixed and the number of blocks k tends to infinity.
Let $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ denote the Digamma function. Then we have the following:

Proposition 5.1: Over the (n,n,k) multiblock channel, reliable communication is guaranteed when $k \to \infty$ for rates

$$R < n\left(\log\frac{P}{n}e^{\frac{1}{n}\sum_{i=1}^{n}\psi(i)} - \log n + \log\frac{\pi e}{2} - \log 6^{1-\frac{1}{n}}G\right)$$

when using the multiblock code construction in Section IV-B.

Remark 5.2: We can compare the achievable rate with the tight lower bound in [20, eq. (7)] for n transmit and receive antennas[1]:

$$C \geq n\log\left(1 + \frac{P}{n}e^{\frac{1}{n}\sum_{i=1}^{n}\psi(i)}\right)$$

[1]We note that the capacity (per channel use) of the block fading MIMO channel of finite block length T with perfect channel state information at the receiver is independent of T [21, eq. (9)]. So the bounds in [2] and [20] still hold in our case.

Proof of Proposition 5.1: Let d_H denote the minimum Euclidean distance in the received constellation:

$$d_H^2 = \min_{\substack{X,\bar{X}\in\mathcal{C}\\ X\neq\bar{X}}} \sum_{i=1}^{k} \left\| H_i(X_i - \bar{X}_i) \right\|^2.$$

Suppose that the receiver performs maximum likelihood decoding or "naive" lattice decoding (closest point search in the infinite lattice). For both, the error probability is bounded by

$$P_e \leq \mathbb{P}\left\{ \|W\|^2 \geq \left(\frac{d_H}{2}\right)^2 \right\},$$

where $W = (W_1, \ldots, W_k)$ is the multiblock noise. Note that

$$d_H^2 \geq \alpha^2 n \min_{X\in L_{n,k}\setminus\{0\}} \sum_{i=1}^{k} |\det(H_i X_i)|^{\frac{2}{n}} \geq$$

$$\geq \alpha^2 nk \min_{X\in L_{n,k}\setminus\{0\}} \prod_{i=1}^{k} |\det(H_i X_i)|^{\frac{2}{nk}} \geq \alpha^2 nk \prod_{i=1}^{k} |\det(H_i)|^{\frac{2}{nk}}$$

where the first step comes from the Minkowski inequality, the second step comes from the arithmetic mean - geometric mean inequality, and the third from observing that $\prod_{i=1}^{k} |\det(X_i)| \geq 1$ for all $X \in L_{n,k} \setminus \{0\}$. Therefore

$$P_e \leq \mathbb{P}\left\{ \frac{\|W\|^2}{kn^2} \geq \frac{\alpha^2}{4n} \prod_{i=1}^{k} |\det(H_i)|^{\frac{2}{nk}} \right\}$$

Given $\epsilon > 0$, we can bound the error probability by

$$\mathbb{P}\left\{ \frac{\|W\|^2}{kn^2} \geq 1+\epsilon \right\} + \mathbb{P}\left\{ \frac{\alpha^2}{4n} \prod_{i=1}^{k} |\det(H_i)|^{\frac{2}{nk}} < 1+\epsilon \right\} \quad (9)$$

Note that $2\|W\|^2 \sim \chi^2(2kn^2)$, and the tail of the chi-square distribution is bounded as follows for $\epsilon \in (0,1)$ [16]:

$$\mathbb{P}\left\{ \frac{\|W\|^2}{kn^2} \geq 1+\epsilon \right\} \leq 2e^{-\frac{kn^2\epsilon^2}{8}}. \quad (10)$$

Therefore the first term in (9) when $k \to \infty$. In order to upper bound the second term, we need to analyze the distribution of the random variable $\prod_{i=1}^{k} |\det(H_i)|^2$.

In the single block case, it is well-known [17, 18] that if H is an $n \times n$ matrix with i.i.d. complex Gaussian entries having variance per real dimension $1/2$, the random variable $2^n |\det(H)|^2$ is distributed as the product $V_n = Z_1 \cdots Z_n$ of n independent chi square random variables $Z_j \sim \chi^2(2j)$, $j \in \{1, \ldots, n\}$ with density $p_{Z_j}(x) = \frac{1}{2^j \Gamma(j)} x^{j-1} e^{-\frac{x}{2}}$. We have

$$\mathbb{E}[\ln Z_j] = \frac{1}{2^j\Gamma(j)} \int_0^\infty x^{j-1} e^{-\frac{x}{2}} \ln x \, dx = \psi(j) + \ln 2.$$

where $\psi(x)$ is the Digamma function. Let

$$M_n = \mathbb{E}[\ln V_n] = n \ln 2 + \sum_{j=1}^{n} \psi(j). \quad (11)$$

Observe that

$$\mathbb{E}[Z_j^{-v}] = \frac{1}{2^j\Gamma(i)} \int_0^\infty x^{j-1-v} e^{-\frac{x}{2}} dx = \frac{\Gamma(j-v)}{2^v \Gamma(j)}, \quad (12)$$

$$\mathbb{E}[Z_j^{-v} \ln Z_j] = \frac{1}{2^j\Gamma(j)} \int_0^\infty x^{j-1-v} e^{-\frac{x}{2}} \ln x \, dx =$$

$$= \frac{\Gamma(j-v)}{2^v\Gamma(j)} (\psi(j-v) + \ln 2). \quad (13)$$

Now let's turn to the multiblock case, and let $S_k = 2^{nk} \prod_{i=1}^{k} |\det(H_i)|^2$. We have $S_k = V_n^{(1)} \cdots V_n^{(k)}$, where $V_n^{(i)} = Z_1^{(i)} \cdots Z_n^{(i)}$ are i.i.d. products of n independent chi squared random variables $Z_j^{(i)} \sim \chi^2(2j)$. Note that $\mathbb{E}[\ln S_k] = k\mathbb{E}[\ln V_n] = kM_n$. Consider the zero-mean random variable

$$B_k = -\ln S_k + kM_n = -\sum_{i=1}^{k} \ln V_n^{(i)} + kM_n = \sum_{i=1}^{k} T_n^{(i)},$$

where $T_n^{(i)}$ are i.i.d. with distribution $T_n = -\sum_{j=1}^{n} \ln Z_j + M_n$. From the Chernoff bound [19] for B_k, given $\delta > 0$, $\forall v > 0$ we have

$$\mathbb{P}\{B_k \geq nk\delta\} \leq e^{-v\delta nk} \mathbb{E}[e^{vB_k}]. \quad (14)$$

The tightest bound in (14) is obtained for v_δ such that

$$\mathbb{E}[B_k e^{v_\delta B_k}] = \delta nk \mathbb{E}[e^{v_\delta B_k}]. \quad (15)$$

It is easy to see that

$$\mathbb{E}[e^{vT_n}] = e^{vM_n} \prod_{j=1}^{n} \mathbb{E}[Z_j^{-v}] = e^{v(M_n - n\ln 2)} \prod_{j=1}^{n} \frac{\Gamma(j-v)}{\Gamma(j)}$$

Recalling that the variables Z_j are independent, we find

$$\mathbb{E}[T_n e^{vT_n}] = \mathbb{E}\Big[\Big(-\sum_{j=1}^{n} \ln Z_j + M_n\Big) e^{vM_n} \Big(\prod_{l=1}^{n} Z_l^{-v}\Big)\Big] =$$

$$= e^{vM_n} \Big(\sum_{j=1}^{n} \mathbb{E}[-Z_j^{-v} \ln Z_j] \prod_{l\neq j} \mathbb{E}[Z_l^{-v}] + M_n \prod_{l=1}^{n} \mathbb{E}[Z_l^{-v}] \Big) =$$

$$= e^{v(M_n - n\ln 2)} \Big(\prod_{l=1}^{n} \frac{\Gamma(l-v)}{\Gamma(l)} \Big) \Big(M_n - n\ln 2 - \sum_{j=1}^{n} \psi(j-v) \Big).$$

We can finally compute

$$\mathbb{E}[e^{vB_k}] = \mathbb{E}\left[e^{v\sum_{i=1}^{k} T_n^{(i)}}\right] = \left(\mathbb{E}[e^{vT_n}]\right)^k =$$

$$= e^{vk(M_n - n\ln 2)} \prod_{j=1}^{n} \frac{\Gamma(j-v)^k}{\Gamma(j)^k}$$

Similarly,

$$\mathbb{E}[B_k e^{vB_k}] = \sum_{i=1}^{k} \mathbb{E}\left[T_n^{(i)} e^{vT_n^{(i)}}\right] \mathbb{E}\left[e^{\sum_{l\neq i} vT_n^{(l)}}\right] =$$

$$= ke^{kv(M_n - n\ln 2)} \Big(M_n - n\ln 2 - \sum_{j=1}^{n} \psi(j-v) \Big) \prod_{l=1}^{n} \frac{\Gamma(l-v)^k}{\Gamma(l)^k}$$

Thus, the tightest bound (15) is achieved for v_δ such that

$$n\delta = M_n - n\ln 2 - \sum_{j=1}^{n} \psi(j-v_\delta) = \sum_{j=1}^{n} (\psi(j) - \psi(j-v_\delta))$$

Clearly, for fixed n, $v_\delta \to 0$ when $\delta \to 0$. From (14) we get

$$\mathbb{P}\{S_k^{\frac{1}{nk}} \leq e^{\frac{M_n}{n} - \delta}\} = \mathbb{P}\{B_k \geq nk\delta\} \leq$$

$$\leq e^{k(v_\delta(-n\delta+M_n-n\ln 2)-\sum_{j=1}^{n}(\ln\Gamma(j)-\ln\Gamma(j-v_\delta)))} =$$

$$= e^{k(\ln\Gamma(1-v_\delta)+v_\delta\psi(1-v_\delta)+\sum_{i=2}^{n}(-\ln\Gamma(i)+\ln\Gamma(i-v_\delta)+v_\delta\psi(i-v_\delta)))}.$$

Recall that $\Gamma(x)$ is monotone decreasing for $0 < x < a_0 = 1.461632\ldots$ and monotone increasing for $x > a_0$. Using the mean value theorem for the function $\ln\Gamma(x)$ in the interval $[i-v_\delta, i]$ we get that for $i = 1$, $v_\delta\psi(1-v_\delta)+\ln\Gamma(1-v_\delta) \leq 0$, and for $i \geq 2$, $v_\delta\psi(i-v_\delta) \leq \ln\Gamma(i) - \ln\Gamma(i-v_\delta)$. Thus,

$$\mathbb{P}\left\{2\prod_{i=1}^{k}|\det(H_i)|^{\frac{2}{nk}} \leq e^{\frac{M_n}{n}-\delta}\right\} = \mathbb{P}\left\{S_k^{\frac{1}{nk}} \leq e^{\frac{M_n}{n}-\delta}\right\} \leq$$
$$\leq e^{-kK_{n,\delta}}$$

for some positive constant $K_{n,\delta}$. The second term in (9) vanishes when $k \to \infty$ provided that $\frac{8n(1+\epsilon)}{\alpha^2} < e^{\frac{M_n}{n}-\delta}$. Recalling the bound for α from (7), a sufficient condition is

$$\frac{4n(1+\epsilon)2^{\frac{R}{n}}6^{1-\frac{1}{n}}G}{(C_{n,k})^{\frac{1}{n^2k}}P} < e^{\frac{M_n}{n}-\delta}.$$

From Stirling's approximation, for large k we have $(C_{n,k})^{\frac{1}{n^2k}} \approx \pi e/(n(2\pi n^2 k)^{\frac{1}{2n^2k}})$. Thus, we find that any rate

$$R < n\left(\log P - \frac{1}{2n^2k}\log(2\pi n^2 k) + \log\frac{\pi e}{4(1+\epsilon)} + \right.$$
$$\left. + \log(e^{\frac{M_n}{n}-\delta}) - 2\log n - \log 6^{1-\frac{1}{n}}G\right), \quad (16)$$

where the logarithms are understood to be binary, is achievable asymptotically as $k \to \infty$. Note that $e^{\frac{M_n}{n}} = e^{\ln 2+\frac{1}{n}\sum_{i=1}^{n}\psi(i)} = 2e^{\frac{1}{n}\sum_{i=1}^{n}\psi(i)}$. Since (16) holds $\forall\delta > 0, \forall\epsilon > 0$, this concludes the proof. □

Remark 5.3: The number field towers we used are not the best known. In fact there exists a family of totally complex fields such that $G < 82.2$ [22], but this would add some notational complications. Just as well the estimate given in equation (6) is not optimal and it is likely that we can reduce the term $\log 6$ in the achievable rate formula.

B. The $(2,1,k)$-multiblock channel

Let us now consider the codes of Corollary 4.6 in the 2×1 block fading channel. Here the matrices H_i are simply vectors $[h_1, h_2]$ and we suppose that the delay is 2. The codewords in the lattice $L_{Alam,k}$ have block structure $X = [X_1, \ldots, X_k]$, where each $X_i \in \mathbf{H}$.

For these codes we can prove the following:

Proposition 5.4: Over the $(2,1,k)$ multiblock channel, reliable communication is guaranteed when $k \to \infty$ for

$$R < \log\left(\frac{Pe^{1-\gamma}}{2}\right) + \log\frac{\pi e}{4} - \log G_1$$

when using the multiblock code construction $L_{Alam,k}$.

The proof is similar to the one in the previous section and is omitted for lack of space. We can compare the achievable rate in Proposition 5.4 to the tight lower bound on ergodic capacity in [20, eq. (7)] for $n = 2$ and $n_r = 1$:

$$C \geq \log\left(1 + \frac{P}{2}e^{1-\gamma}\right).$$

ACKNOWLEDGEMENT

The research of R. Vehkalahti is supported by the Finnish Cultural Foundation.

REFERENCES

[1] G. Foschini and M. Gans, "On limits of wireless communications in a fading environment when using multiple antennas", *Wireless Personal Communications*, March 1998.

[2] E. Telatar, "Capacity of multi-antenna Gaussian channels", *Europ. Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, 1999.

[3] S. Yang and J.-C. Belfiore, "Optimal space-time codes for the MIMO amplify-and-forward cooperative channel", *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 647–663, Feb. 2007.

[4] H.-f. Lu, "Constructions of multi-block space-time coding schemes that achieve the diversity-multiplexing tradeoff", *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3790-3796, Aug. 2008.

[5] P. Elia, P. Vijay Kumar, "Approximately-Universal Space-Time Codes for the Parallel, Multi-Block and Cooperative-Dynamic-Decode-and-Forward Channels", available at http://arxiv.org/abs/0706.3502.

[6] C. Hollanti and H.-f. Lu, "Construction methods for asymmetric and multi-block space-time codes," *IEEE Trans. Inf. Theory*, vol. 55, no. 3, pp. 1086 – 1103, 2009.

[7] R. Vehkalahti, C. Hollanti, and F. Oggier, "Fast-decodable asymmetric space-time codes from division algebras", *IEEE Trans. Inf. Theory*, vol. 58, pp. 2362– 2384, April 2012.

[8] B. Linowitz, M. Satriano and R.Vehkalahti, "A non-commutative analogue of the Odlyzko bounds and bounds on performance for space-time lattice codes", *IEEE Trans. Inf. Theory*, vol. 61, pp. 1971– 1984, April 2015.

[9] R. Vehkalahti and L. Luzzi, "Number field lattices achieve Gaussian and Rayleigh channel capacity within a constant gap", in *IEEE Int. Symp. Inform. Theory* (ISIT), Hong Kong, China, June 2015

[10] P. M. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, Elsevier, Amsterdam, The Netherlands, 1987.

[11] V. Tarokh, N. Seshadri, and A.R. Calderbank, "Space-Time Codes for High Data Rate Wireless Communications: Performance Criterion and Code Construction", *IEEE Trans. Inf. Theory*, vol. 44, pp. 744–765, March 1998.

[12] I. Reiner, *Maximal Orders*, Academic Press, New York 1975.

[13] J. Martinet, "Tours de corps de classes et estimations de discriminants", *Invent. Math.* n. 44, 1978, pp. 65–73.

[14] R. Vehkalahti, C. Hollanti, J. Lahtonen, and K. Ranto, "On the densest MIMO lattices from cyclic division algebras," *IEEE Trans. Inf. Theory*, vol. 55, no. 8, pp. 3751–3780, Aug 2009.

[15] R. Vehkalahti, "Some properties of Alamouti-like MISO codes", in *IEEE Int. Symp. Inform. Theory*, Seoul, South Korea, 2009.

[16] B. Laurent, P. Massart, "Adaptive estimation of a quadratic functional by model selection", *Annals of Statistics*, vol. 28, pp. 1302–1338, 2000.

[17] N. R. Goodman, "The distribution of the determinant of a complex Wishart distributed matrix", *Ann. Math. Statist.*, vol. 34, pp. 178–180, 1963

[18] A. Edelman, "Eigenvalues and condition numbers of random matrices", Ph.D. Thesis, MIT 1989

[19] J. Proakis, *Digital communications*, 4th ed., McGraw-Hill 2001

[20] O. Oyman, R. Nabar, H. Bölcskei, and A. Paulraj, "Tight Lower Bounds on the Ergodic Capacity of Rayleigh Fading MIMO Channels", *IEEE GLOBECOM*, Nov. 2002, pp. 1172–1176.

[21] T. L. Marzetta, B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading", *IEEE Trans. Inf. Theory* vol 45 n.1, Jan 1999

[22] F. Hajir and C. Maire, "Asymptotically good towers of global fields", *Proc. European Congress of Mathematics*, pp. 207–218, Birkhäuser Basel, 2001.