Observer design for linear singular time-delay systems

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Abstract

This paper investigates the problem of observer design for a general class of linear singular time-delay systems, in which the time delays are involved in the state and its derivatives, the output and the known input (if there exists). The involvement of the delay could be multiple which however is rarely studied in the literature. Sufficient conditions are proposed which guarantees the existence of a Luenberger-like observer for the general system.

Keywords: Delayed singular systems, commensurate delays, observer.

1. Introduction

Singular systems (known also as Descriptor systems/Algebraic-Differential systems) enable us to model many physical, biological and economical systems Campbell (1982), which has been well discussed in Wang et al. (2014). For such systems, time delay can be also encountered Niculescu and Rasvan (2000). Thus, the singular time-delay systems have been extensively analyzed during last decade. Lots of researches are focused on the stabilization of singular time-delay systems, such as H_∞ control, observer-based control and so on Gu et al. (2013).

Concerning the concepts of observability and observer for the singular time-delay systems, a direct adaptation of existing results on observability and observers from regular systems to singular systems is not immediate due to the fact that they involve both differential and algebraic equations. Up to now, most of the existing works are based on the simple case with only one delay in the state, i.e. $E\dot{x}(t) = A_0x(t) + A_1x(t-h)$ where the input could be also involved. For this simple case, a general solution was derived in Wei (2013), based on which a sufficient condition for exact observability in finite time was deduced. When designing an observer for the mentioned simple linear singular time-delay systems, a few results can be found in the literature. In Feng et al. (2003), three kinds of observers (functional observer, reduced-order observer and full-order observer) are studied for the above simple case. In Ezzine et al. (2011), a functional observer for singular time-delay systems with unknown inputs was presented, and the existence condition of such observer and the gain implemented in the design are obtained by solving LMIs. A Luenberger-like observer is proposed in Khadhraoui et al. (2014) for the linear singular time-delay systems with unknown inputs not affected by time delays.

In this paper, we deal with a quite general linear singular time-delay system of the form: $\sum_{i=0}^{l} E_i\dot{x}(t-ih) = \sum_{i=0}^{k} A_ix(t-ih)$, which in fact covers different types of time-delay system (singular or not, with or without neutral term). For such a presentation, there exist lots of applications, such as LC electrical lines Brayton (1968) and so on. More concrete applications can be found in Niculescu (2001). For such a general form, observability analysis and observer design become more difficult. Recently, inspired by the well-known Silverman and Molinari algorithm (see Silverman (1969); Molinari (1976)) to analyze the observability for linear systems (with or without unknown input), a similar and checkable sufficient condition was proposed in Bejarano and Zheng (2016a) to analyze the observability for this general singular time-delay systems. As far as we know, the only work on the observer design for this general case is Perdon and Anderlucci (2006), where a Luenberger-like observer is proposed by using a virtual discrete time system with the matrices of the originally considered continuous-time system. In this method, a geometric notion of conditioned invariant submodule is introduced for this class of systems over a ring and a design procedure is presented. However, the sufficient condition deduced in that paper is difficult to be checked, since it highly depends on the assumption of the existence of Hurwitz polynomial. *The contributions of this paper are as follows*. Firstly, the class of the studied systems is quite general (we allow multiple delays both in the $x(t)$ and its derivative), which in fact can cover four different classes of systems. As far as we known, there exist some methods to eliminate (or reduce the degree of) the delay, such as Lee et al. (1982), Germani et al. (2001) and Garate-Garcia et al. (2011). It has been proven in Garate-Garcia et al. (2011) that the elimination or the reduction of delay degree via a bicausal transformation with

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the same dimension is possible if some conditions on $A(\delta)$ and $B(\delta)$ are satisfied. In other words, the elimination or the reduction of delay degree is not always possible. This issue has been highlighted in Zheng et al. (2015); Bejarano and Zheng (2016c, 2014, 2016b). Therefore, allowing multiple delays in the state, in the derivative, and in the output is an essential generalization. The second contribution of this paper is to deduce a checkable sufficient condition such that a Luenberger-like observer exists for such a general linear singular time-delay system. Moreover, it provides as well a constructive way to synthesize the proposed observer.

2. Notations and problem statement

In this paper, we consider the following class of linear systems with commensurate delays:

$$\begin{cases} \sum_{i=0}^{k_e} \tilde{E}_i \dot{x}(t-ih) = \sum_{i=0}^{k_a} \tilde{A}_i x(t-ih) \\ \tilde{y}(t) = \sum_{i=0}^{k_c} \tilde{C}_i x(t-ih) \end{cases} \tag{1}$$

where the vector $x(t) \in \mathbb{R}^n$, the system output vector $\tilde{y}(t) \in \mathbb{R}^p$, h represents the basic delay, the initial condition $\varphi(t)$ is a piece-wise continuous function $\varphi(t) : [-kh, 0] \to \mathbb{R}^n$ ($k = \max\{k_a, k_c, k_e\}$); thereby $x(t) = \varphi(t)$ on $[-kh, 0]$. $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{E}_i$ are matrices of appropriate dimension with entries in $\mathbb{R}$. It is worth noting that the studied system of the form (1) is quite general, and it covers different types of time-delay systems. More precisely:

- if $k_e = 0$ and $\tilde{E}_0 = \mathrm{I}$, then system (1) becomes the classical linear time-delay system of the form $\dot{x} = \sum_{i=0}^{k_a} \tilde{A}_i x(t-ih)$;
- if $k_e = 0$ and the rank of $\tilde{E}_i$ is not full, then system (1) is equivalent to a linear singular time-delay systems of the form $\tilde{E}_0 \dot{x}(t) = \sum_{i=0}^{k_a} \tilde{A}_i x(t-ih)$;
- when $k_e > 0$ and $\tilde{E}_0 = \mathrm{I}$, then system (1) can be written as $\dot{x}(t) = \sum_{i=0}^{k_a} \tilde{A}_i x(t-ih) - \sum_{i=1}^{k_e} \tilde{E}_i \dot{x}(t-ih)$ which is typically a linear time-delay neutral system;
- moreover, if $k_e > 0$ and the rank of $\tilde{E}_0$ is not full, system (1) represents a general linear singular time-delay neutral system of the form $\tilde{E}_0 \dot{x}(t) = \sum_{i=0}^{k_a} \tilde{A}_i x(t-ih) - \sum_{i=1}^{k_e} \tilde{E}_i \dot{x}(t-ih)$.

Nevertheless, the form (1) is so general that it might include as well advanced systems for which the existence of the solution cannot be always guaranteed. In order to exclude such ill-conditioned systems, it is assumed in this paper that system (1) admits at least one solution. It is worth noting that *the existence of a unique solution is not necessary* for the observability analysis and the observer design. Take any system with several solutions as an example, this issue can be easily identified if the output of this system is all the state. In this case, the system is always observable even if several solutions exist.

Remark 1. *For the special class of systems mentioned above, there exist some results on observer design in the literature, for example Darouach and Boutayeb (1995), Hou et al. (2002) and Ezzine et al. (2011), and lots of the existing works consider only one delay in $x(t)$, i.e. $k_e = k_c = 0$ and $k_a = 1$ in (1). However, for a linear system with commensurate delays of the general form (1) covering a large class of linear singular (or not) time-delay neutral (or not) systems, to the best of our knowledge, rare results on observer design have been reported in the literature. Therefore, the problem of designing an observer is still an open problem for the general form (1).*

Motivated by this fact, this paper proposes a Luenberger-like observer and sufficient conditions are deduced which guarantee the existence of such an observer.

In the following, for the purpose of simplifying the analysis, let us introduce the delay operator $\delta : x(t) \to x(t-h)$ with $\delta^k x(t) = x(t-kh)$, $k \in \mathbb{N}_0$. Then the following notations will be used in this paper. $\mathbb{R}$ is the field of real numbers. The set of positive integers is denoted by $\mathbb{N}$. I_n for $n \in \mathbb{N}$ means the identity matrix of order n. $\mathbb{R}[\delta]$ is the polynomial ring over the field $\mathbb{R}$. $\mathbb{R}^n[\delta]$ is the $\mathbb{R}[\delta]$-module whose elements are the vectors of dimension n and whose entries are polynomials. By $\mathbb{R}^{q\times s}[\delta]$ we denote the set of matrices of dimension $q \times s$, whose entries are in $\mathbb{R}[\delta]$. For a matrix $M(\delta)$, $\mathrm{rank}_{\mathbb{R}[\delta]} M(\delta)$ means the rank of the matrix $M(\delta)$ over $\mathbb{R}[\delta]$. $M(\delta) \sim N(\delta)$ means the similarity between two polynomial matrices $M(\delta)$ and $N(\delta)$ over $\mathbb{R}[\delta]$, i.e. there exist two unimodular matrices $U_1(\delta)$ and $U_2(\delta)$ over $\mathbb{R}[\delta]$ such that $M(\delta) = U_1(\delta)N(\delta)U_2(\delta)$.

After having introduced the delay operator δ, system (1) may be then represented in the following compact form:

$$\begin{cases} \tilde{E}(\delta)\dot{x}(t) &= \tilde{A}(\delta)x(t) \\ \tilde{y}(t) &= \tilde{C}(\delta)x(t) \end{cases} \tag{2}$$

where $\tilde{A}(\delta) \in \mathbb{R}^{\bar{n}\times n}[\delta]$, $\tilde{C}(\delta) \in \mathbb{R}^{p\times n}[\delta]$ and $\tilde{E}(\delta) \in \mathbb{R}^{\bar{n}\times n}[\delta]$ are matrices over the polynomial ring $\mathbb{R}[\delta]$, defined as $\tilde{A}(\delta) := \sum_{i=0}^{k_a} \tilde{A}_i \delta^i$, $\tilde{C}(\delta) := \sum_{i=0}^{k_c} \tilde{C}_i \delta^i$ and $\tilde{E}(\delta) := \sum_{i=0}^{k_e} \tilde{E}_i \delta^i$. Without loss of generality, it is thus assumed in this paper that $\bar{n} \le n$, $\mathrm{rank}_{\mathbb{R}[\delta]} \tilde{E}(\delta) = q \le \bar{n}$ and $\mathrm{rank}_{\mathbb{R}[\delta]} \tilde{C}(\delta) = p$.

Remark 2. *The above assumptions are not restrictive.* $\mathrm{rank}_{\mathbb{R}[\delta]} \tilde{E}(\delta) = q \le \bar{n} \le n$ *is due to the fact that the studied system might be singular or neutral. Moreover, if* $\mathrm{rank}_{\mathbb{R}[\delta]} \tilde{C}(\delta) = \tilde{p} < p$, *we can always eliminate the dependent outputs of (2). More precisely, if* $\mathrm{rank}_{\mathbb{R}[\delta]} \tilde{C}(\delta) = \tilde{p} < p$, *then there exists a polynomial matrix* $U(\delta) \in \mathbb{R}^{\tilde{p}\times p}[\delta]$ *such that* $U(\delta)\tilde{C}(\delta) \in \mathbb{R}^{\tilde{p}\times n}[\delta]$ *is of full row rank. In this case, we can rewrite the output as* $U(\delta)\tilde{y} = U(\delta)\tilde{C}(\delta)x$ *with* $\mathrm{rank}_{\mathbb{R}[\delta]} U(\delta)\tilde{C}(\delta) = \tilde{q}$.

3. Definition, assumptions and preliminary result

For each instant t, the available measurements are only $y(t)$ and its delayed values which can be used to estimate $x(t)$. We cannot in fact utilize the future value of $y(t)$, otherwise it is not

causal. Therefore, it is desired to use only the actual and the past information (not the future information) of the measurement to design an observer for time-delay systems due to the requirement of the causality. Thus the following definition of backward observability is given.

Definition 1. *System (2) is said to be backward observable on $[t_1,t_2]$ if and only if, for each $\tau \in [t_1,t_2]$ there exist $\bar{t}_1$ and $\bar{t}_2 \leq \tau$ such that $y(t) = 0$ for all $t \in [\bar{t}_1,\bar{t}_2]$ implies $x(\tau) = 0$.*

The above definition of backward observability is related to the final observability given Lee and Olbrot (1981), and it can be interpreted that $x(t)$ depends only on the previous values of $y(t)$, its time delayed values, and its derivatives.

Since we are going to analyze system (2) which is described by the polynomial matrices over $\mathbb{R}[\delta]$, let us firstly give the following definition of unimodularity over $\mathbb{R}[\delta]$.

Definition 2. *Antsaklis and Michel (2007) A polynomial matrix $A(\delta) \in \mathbb{R}^{n\times q}[\delta]$ is said to be left (or right) unimodular (or invertible) over $\mathbb{R}[\delta]$ if there exists $A_L^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ with $n \geq q$ (or $A_R^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ with $n \leq q$), such that $A_L^{-1}(\delta)A(\delta) = I_{q\times q}$ (or $A(\delta)A_R^{-1}(\delta) = I_{n\times n}$). A square matrix $A(\delta) \in \mathbb{R}^{n\times n}[\delta]$ is said to be unimodular (or invertible) over $\mathbb{R}[\delta]$ if $A_L^{-1}(\delta) = A_R^{-1}(\delta) = A^{-1}(\delta)$.*

It is well known that for any polynomial matrix $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ with $\text{rank}_{\mathbb{R}[\delta]}D(\delta) = r \leq \min\{p,m\}$, there exists a unimodular matrix $U(\delta)$ over $\mathbb{R}[\delta]$ such that $U(\delta)D(\delta) = \begin{bmatrix} \bar{D}(\delta) \\ 0 \end{bmatrix}$ where $\text{rank}_{\mathbb{R}[\delta]}\bar{D}(\delta) = \text{rank}_{\mathbb{R}[\delta]}D(\delta)$ and $\bar{D}(\delta)$ is of full row rank over $\mathbb{R}[\delta]$. Also, there exist two unimodular matrices $U(\delta)$ and $V(\delta)$ over $\mathbb{R}[\delta]$ such that $D(\delta)$ is reduced to its Smith form, i.e.,

$$U(\delta)D(\delta)V(\delta) = \begin{bmatrix} \mathscr{S}_{D(\delta)} & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

with $\mathscr{S}_{D(\delta)}$ being a diagonal matrix as $\mathscr{S}_{D(\delta)} = \text{diag}\{\psi^1_{D(\delta)},\cdots,\psi^r_{D(\delta)}\}$ where $\left\{\psi^i_{D(\delta)}\right\}$ are the invariant factors of $D(\delta)$. For simplicity, let us denote $\text{Inv}_S[D(\delta)] = \left\{\psi^i_{D(\delta)}\right\}_{1\leq i\leq r}$ as the set of its invariant factors of the Smith form of $D(\delta)$, the following result, adapted from the result on the left unimodularity stated in Hou et al. (2002), is obvious.

Lemma 1. *For any polynomial matrix $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$, it is left (or right) unimodular over $\mathbb{R}[\delta]$ if and only if the following conditions are satisfied:*

1. $\text{rank}_{\mathbb{R}[\delta]}D(\delta) = m \leq p$ *(or* $\text{rank}_{\mathbb{R}[\delta]}D(\delta) = p \leq m$ *);*
2. $\text{Inv}_S[D(\delta)] \subset \mathbb{R}$.

Concerning the definition of backward observability, and following the ideas of Silverman (1969) and Molinari (1976), define $\left\{\tilde{N}_k(\delta)\right\}$ the matrices generated by the following algorithm for the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$, proposed in Bejarano and Zheng (2016a):

- Initialization: Set $\tilde{\Delta}_1(\delta) \triangleq \tilde{C}(\delta)$ and $\tilde{N}_1(\delta) \triangleq \tilde{\Delta}_1(\delta)$;
- Iteration: For $k \geq 1$, there exists a unimodular matrix $\tilde{S}_k(\delta) = \begin{bmatrix} \tilde{S}_{k,1}(\delta) & \tilde{S}_{k,2}(\delta) \\ \tilde{S}_{k,3}(\delta) & \tilde{S}_{k,4}(\delta) \end{bmatrix}$ such that $\tilde{S}_k(\delta)\begin{bmatrix} \mathrm{I}_{\bar{n}} & \tilde{E}(\delta) \\ 0 & \tilde{N}_k(\delta) \end{bmatrix} \triangleq \begin{bmatrix} \tilde{S}_{k,1}(\delta) & \tilde{\Lambda}_k(\delta) \\ \tilde{S}_{k,3}(\delta) & 0 \end{bmatrix}$ with $\text{rank}_{\mathbb{R}[\delta]}\tilde{\Lambda}_k(\delta) = \text{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} \tilde{E}(\delta) \\ \tilde{N}_k(\delta) \end{bmatrix}$; then set $\tilde{\Delta}_{k+1}(\delta) = \tilde{S}_{k,3}(\delta)\tilde{A}(\delta)$ and define

$$\tilde{N}_{k+1}(\delta) = \begin{bmatrix} \tilde{N}_k(\delta) \\ \tilde{\Delta}_{k+1}(\delta) \end{bmatrix} \tag{4}$$

With the above algorithm, the following result was stated in Bejarano and Zheng (2016a).

Lemma 2. *Bejarano and Zheng (2016a) If there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, then system (2) is backward observable (or the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$ is backward observable).*

Due to the above result, the following assumption is imposed.

Assumption 1. *For system (2) with the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$, there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ defined in (4) is left unimodular over $\mathbb{R}[\delta]$.*

Let us highlight that the condition imposed in Assumption 1 is only sufficient but not necessary for system (2) to be backward observable. However, it becomes to be necessary and sufficient condition of observability for system (2) without delays.

Since $\text{rank}_{\mathbb{R}[\delta]}\tilde{E}(\delta) = q \leq \bar{n}$, then there exists a unimodular matrix $\Pi(\delta) = \begin{bmatrix} \Pi_1(\delta) \\ \Pi_2(\delta) \end{bmatrix}$ over $\mathbb{R}[\delta]$ such that

$$\begin{aligned} \Pi(\delta)\tilde{E}(\delta) &= \begin{bmatrix} E(\delta) \\ 0 \end{bmatrix} \\ \Pi(\delta)\tilde{A}(\delta) &= \begin{bmatrix} A(\delta) \\ \underline{A}(\delta) \end{bmatrix} \end{aligned} \tag{5}$$

with $E(\delta) \in \mathbb{R}^{q\times n}[\delta]$ of full row rank over $\mathbb{R}[\delta]$ which is equal to q. It is worth noting that the choice of $\Pi(\delta)$ is not unique, thus different choices of $\Pi(\delta)$ will give different $E(\delta)$ and $A(\delta)$. The influence of different choices of $\Pi(\delta)$ will be analyzed in Section 4. It is obvious that if $q = n$, then we can just choose $\Pi(\delta) = \mathrm{I}_n$ and $E(\delta) = \tilde{E}(\delta)$. By noting $C(\delta) = \begin{bmatrix} \underline{A}(\delta) \\ \tilde{C}(\delta) \end{bmatrix}$ and $y(t) = \begin{bmatrix} 0 \\ \tilde{y}(t) \end{bmatrix}$, system (2) can be rewritten as follows:

$$\begin{cases} E(\delta)\dot{x}(t) &= A(\delta)x(t) \\ y(t) &= C(\delta)x(t) \end{cases} \tag{6}$$

with $E(\delta) \in \mathbb{R}^{q\times n}[\delta]$, $A(\delta) \in \mathbb{R}^{q\times n}[\delta]$ and $C(\delta) \in \mathbb{R}^{(\bar{n}+p-q)\times n}[\delta]$.

Lemma 3. *If Assumption 1 is satisfied, then the triple $(E(\delta),A(\delta),C(\delta))$ for system (6) is backward observable.*

Proof. Suppose that there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ defined in (4) is left unimodular over $\mathbb{R}[\delta]$. Applying the algorithm for system (2) of the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$, when $k=1$, we have $\tilde{N}_1(\delta)=\tilde{\Delta}_1(\delta)=\tilde{C}(\delta)$. Due to the fact that

$$\begin{bmatrix} \mathrm{I}_{\bar{n}} & \tilde{E}(\delta) \\ 0 & \tilde{N}_1(\delta) \end{bmatrix} = \begin{bmatrix} \Pi^{-1}(\delta) & 0 \\ 0 & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} \Pi(\delta) & \Pi(\delta)\tilde{E}(\delta) \\ 0 & \tilde{N}_1(\delta) \end{bmatrix} = \begin{bmatrix} \Pi^{-1}(\delta) & 0 \\ 0 & \mathrm{I}_p \end{bmatrix} \begin{bmatrix} \Pi_1(\delta) & E(\delta) \\ \Pi_2(\delta) & 0 \\ 0 & \tilde{N}_1(\delta) \end{bmatrix}$$

then there exists a unimodular matrix $\bar{S}_1(\delta)=\begin{bmatrix} \bar{S}_{1,1}(\delta) & 0 & \bar{S}_{1,2}(\delta) \\ 0 & \mathrm{I}_{\bar{n}-q} & 0 \\ \bar{S}_{1,3}(\delta) & 0 & \bar{S}_{1,4}(\delta) \end{bmatrix}$ such that

$$\bar{S}_1(\delta)\begin{bmatrix} \Pi_1(\delta) & E(\delta) \\ \Pi_2(\delta) & 0 \\ 0 & \tilde{N}_1(\delta) \end{bmatrix} = \begin{bmatrix} \bar{S}_{1,1}(\delta)\Pi_1(\delta) & \tilde{\Lambda}_1(\delta) \\ \Pi_2(\delta) & 0 \\ \bar{S}_{1,3}(\delta)\Pi_1(\delta) & 0 \end{bmatrix}$$

with $\mathrm{rank}_{\mathbb{R}[\delta]}\tilde{\Lambda}_1(\delta)=\mathrm{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} \tilde{E}(\delta) \\ \tilde{N}_1(\delta) \end{bmatrix}$. Thus we have $\tilde{N}_2(\delta)=\begin{bmatrix} \tilde{\Delta}_1(\delta) \\ \Pi_2(\delta)\tilde{A}(\delta) \\ \bar{S}_{1,3}(\delta)\Pi_1(\delta)\tilde{A}(\delta) \end{bmatrix}=\begin{bmatrix} \tilde{C}(\delta) \\ \underline{A}(\delta) \\ \bar{S}_{1,3}(\delta)A(\delta) \end{bmatrix}$.

Applying the same algorithm for system (6) of the triple $(E(\delta),A(\delta),C(\delta))$, when $k=1$, we have $N_1(\delta)=\Delta_1(\delta)=C(\delta)=\begin{bmatrix} \underline{A}(\delta) \\ \tilde{C}(\delta) \end{bmatrix}$ thus there exists a unimodular matrix $\Upsilon_1(\delta)$ over $\mathbb{R}[\delta]$ such that $N_1(\delta)=\Upsilon_1(\delta)\begin{bmatrix} \tilde{N}_1(\delta) \\ \underline{A}(\delta) \end{bmatrix}$. Moreover, by applying the same matrix $\bar{S}_1(\delta)$ obtained above, we have

$$\bar{S}_1(\delta)\begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & C(\delta) \end{bmatrix}=\bar{S}_1(\delta)\begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & \underline{A}(\delta) \\ 0 & \tilde{C}(\delta) \end{bmatrix}=\begin{bmatrix} \bar{S}_{1,1}(\delta) & \bar{\Lambda}_1(\delta) \\ 0 & \underline{A}(\delta) \\ \bar{S}_{1,3}(\delta) & 0 \end{bmatrix}$$

with $\mathrm{rank}_{\mathbb{R}[\delta]}\bar{\Lambda}_1(\delta)=\mathrm{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} E(\delta) \\ \tilde{C}(\delta) \end{bmatrix}$. Thus there exists a unimodular matrix $S_1(\delta)$ over $\mathbb{R}[\delta]$ such that

$$S_1(\delta)\begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & C(\delta) \end{bmatrix} = \begin{bmatrix} S_{1,1}(\delta) & \Lambda_1(\delta) \\ S_{1,3}(\delta) & 0 \end{bmatrix} = \begin{bmatrix} S_{1,1}(\delta) & \Lambda_1(\delta) \\ S_1^A(\delta) & 0 \\ \bar{S}_{1,3}(\delta) & 0 \end{bmatrix}$$

with $\mathrm{rank}_{\mathbb{R}[\delta]}\Lambda_1(\delta)=\mathrm{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} E(\delta) \\ \underline{A}(\delta) \\ \tilde{C}(\delta) \end{bmatrix}$, which gives $N_2(\delta)=\begin{bmatrix} C(\delta) \\ S_1^A(\delta)A(\delta) \\ \bar{S}_{1,3}(\delta)A(\delta) \end{bmatrix}$. This means that there exists a unimodular matrix $\Upsilon_2(\delta)$ over $\mathbb{R}[\delta]$ such that $N_2(\delta)=\Upsilon_2(\delta)\begin{bmatrix} \tilde{N}_2(\delta) \\ S_1^A(\delta)A(\delta) \end{bmatrix}$. By iteration, we can find the following relation:

$$N_{k+1}(\delta)=\Upsilon_{k+1}(\delta)\begin{bmatrix} \tilde{N}_{k+1}(\delta) \\ S_k^A(\delta)A(\delta) \end{bmatrix}$$

where $\Upsilon_{k+1}(\delta)$ is a unimodular matrix over $\mathbb{R}[\delta]$. If there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, then $N_{k^*+1}(\delta)$ is also left unimodular over $\mathbb{R}[\delta]$, which implies that the triple $(E(\delta),A(\delta),C(\delta))$ for system (6) is backward observable. ■

Assumption 2. *For the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$ of system (2), it is assumed that the following condition:*

$$\mathrm{Inv}_S\begin{bmatrix} \tilde{E}(\delta) & \tilde{A}(\delta) \\ 0 & \tilde{E}(\delta) \\ 0 & \tilde{C}(\delta) \end{bmatrix}=\mathrm{Inv}_S\begin{bmatrix} \tilde{E}(\delta) & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \tag{7}$$

is satisfied.

Remark 3. *When the treated system (2) does not contain any delay, i.e. the triple $\left(\tilde{E}(\delta),\tilde{A}(\delta),\tilde{C}(\delta)\right)$ becomes a constant triple $\left(\tilde{E},\tilde{A},\tilde{C}\right)$. The set of invariant factors can be characterized by using the rank condition over $\mathbb{R}$. Therefore, the condition (7) imposed in the above assumption is equivalent to the following one:*

$$\mathrm{rank}_{\mathbb{R}}\begin{bmatrix} \tilde{E} & \tilde{A} \\ 0 & \tilde{E} \\ 0 & \tilde{C} \end{bmatrix}=\mathrm{rank}_{\mathbb{R}}\begin{bmatrix} \tilde{E} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}=n+\mathrm{rank}_{\mathbb{R}}\tilde{E}$$

It has been shown in Darouach and Boutayeb (1995) that the above condition is necessary for the existence of a Luenberger-like observer. In this sense, the required condition (7) in Assumption 2 is not restrictive.

Lemma 4. *The condition (7) in Assumption 2 is equivalent to the following one:*

$$\mathrm{Inv}_S\begin{bmatrix} E(\delta) & A(\delta) \\ 0 & E(\delta) \\ 0 & C(\delta) \end{bmatrix}=\mathrm{Inv}_S\begin{bmatrix} E(\delta) & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \tag{8}$$

Proof. With the decomposition by the unimodular matrix $\Pi(\delta)$ over $\mathbb{R}[\delta]$ defined in (5), we have the following similarity property:

$$\begin{bmatrix} \tilde{E}(\delta) & \tilde{A}(\delta) \\ 0 & \tilde{E}(\delta) \\ 0 & \tilde{C}(\delta) \end{bmatrix} \sim \begin{bmatrix} E(\delta) & A(\delta) \\ 0 & \underline{A}(\delta) \\ 0 & E(\delta) \\ 0 & \tilde{C}(\delta) \end{bmatrix} \sim \begin{bmatrix} E(\delta) & A(\delta) \\ 0 & E(\delta) \\ 0 & C(\delta) \end{bmatrix}$$

Following the same argument, we have:

$$\begin{bmatrix} \tilde{E}(\delta) & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \sim \begin{bmatrix} E(\delta) & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}$$

Therefore, the condition (7) in Assumption 2 is equivalent to (8). ■

Lemma 5. *For system (6), the matrix $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$ if and only if Assumption 2 is satisfied for system (2).*

Proof. According to Lemma 4, it is equivalent to prove that $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$ if and only if (8) is satisfied.

Since $E(\delta)$ is full row rank over $\mathbb{R}[\delta]$ which is equal to q, denoting by $\mathscr{S}_{\mathrm{E}(\delta)} = \mathrm{diag}\{\psi^1_{E(\delta)}, \cdots, \psi^q_{E(\delta)}\}$ the diagonal matrix of the invariant factors of $E(\delta)$, then (8) can be rewritten as:

$$\mathrm{Inv}_S \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & A(\delta) \\ 0 & E(\delta) \\ 0 & C(\delta) \end{bmatrix} = \mathrm{Inv}_S \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}$$

The above equation implies that

$$\mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & A(\delta) \\ 0 & E(\delta) \\ 0 & C(\delta) \end{bmatrix} = \mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \quad (9)$$

Therefore, we have:

$$\mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = n.$$

Denote $\mathscr{S}_{\mathrm{EC}(\delta)} = \mathrm{diag}\{\psi^1_{EC(\delta)}, \cdots, \psi^n_{EC(\delta)}\}$ as the diagonal matrix of the invariant factors of $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$, then (9) becomes

$$\mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & A(\delta) \\ 0 & \mathscr{S}_{\mathrm{EC}(\delta)} \end{bmatrix} = \mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} \mathscr{S}_{\mathrm{E}(\delta)} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}$$

The above equation implies that

$$\mathrm{Inv}_S \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} \subset \mathbb{R}.$$

According to Lemma 1, we can then state that $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$ if and only if Assumption 2 is satisfied. ■

Corollary 1. *If Assumption 2 is satisfied, then there exists a unimodular matrix $P(\delta) \in \mathbb{R}^{(\bar{n}+p)\times(\bar{n}+p)}[\delta]$ over $\mathbb{R}[\delta]$ such that*

$$P(\delta) \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_n \\ 0 \end{bmatrix}.$$

Proof. As stated in Lemma 5, if Assumption 2 is satisfied, then we have $\mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = n$. and $\mathrm{Inv}_S \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} \subset \mathbb{R}$. Thus, there exists a unimodular matrix $S_{EC}(\delta)$ over $\mathbb{R}[\delta]$ that transforms $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ into the following Hermite form:

$$S_{EC}(\delta) \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = \begin{bmatrix} \tilde{Q}(\delta) \\ 0 \end{bmatrix}$$

where $\mathrm{rank}_{\mathbb{R}[\delta]} \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = n$ and $\tilde{Q}(\delta) \in \mathbb{R}^{n\times n}[\delta]$ is of full rank over $\mathbb{R}[\delta]$. Since $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$, then $\tilde{Q}(\delta)$ is unimodular over $\mathbb{R}[\delta]$ as well, which implies that there exists $\tilde{Q}^{-1}(\delta)$ such that $\tilde{Q}^{-1}(\delta)\tilde{Q}(\delta) = \mathrm{I}_n$. Therefore, for the left unimodular matrix $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ over $\mathbb{R}[\delta]$, there exists a unimodular matrix $P(\delta) = \begin{bmatrix} P_1(\delta) & P_2(\delta) \\ P_3(\delta) & P_4(\delta) \end{bmatrix} = \begin{bmatrix} \tilde{Q}^{-1}(\delta) & 0 \\ 0 & \mathrm{I}_{\bar{n}+p-n} \end{bmatrix} S_{EC}(\delta)$ over $\mathbb{R}[\delta]$, with $P_1(\delta) \in \mathbb{R}^{n\times q}[\delta]$, $P_2(\delta) \in \mathbb{R}^{n\times(\bar{n}+p-q)}[\delta]$, $P_3(\delta) \in \mathbb{R}^{(\bar{n}+p-n)\times q}[\delta]$ and $P_4(\delta) \in \mathbb{R}^{(\bar{n}+p-n)\times(\bar{n}+p-q)}[\delta]$, such that $P(\delta) \begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_n \\ 0 \end{bmatrix}$, i.e., the following equations:

$$\begin{aligned} P_1(\delta)E(\delta) + P_2(\delta)C(\delta) &= \mathrm{I}_n \\ P_3(\delta)E(\delta) + P_4(\delta)C(\delta) &= 0 \end{aligned} \quad (10)$$

are always satisfied. ■

In the following, let us denote

$$\bar{A}(\delta) = P_1(\delta)A(\delta) \in \mathbb{R}^{n\times n}[\delta] \quad (11)$$

and

$$\bar{C}(\delta) = \begin{bmatrix} P_3(\delta)A(\delta) \\ C(\delta) \end{bmatrix} \in \mathbb{R}^{(2\bar{n}-n-q+2p)\times n}[\delta] \quad (12)$$

Then define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\bar{\mathscr{O}}_l(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{l-1}(\delta) \end{bmatrix} \in \mathbb{R}^{(2\bar{n}-n-q+2p)l\times n}[\delta] \quad (13)$$

where $l \in \mathbb{N}$, and we arrive to the following lemma.

Lemma 6. *Suppose Assumptions 1 and 2 are both satisfied for system (2), then there exists a least integer $l^* \in \mathbb{N}$ such that $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (13) is left unimodular over $\mathbb{R}[\delta]$.*

Proof. If Assumption 2 is satisfied, then according to Lemma 5 the matrix $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$. Moreover, if Assumption 1 is satisfied, i.e. there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ for (2) is left unimodular over $\mathbb{R}[\delta]$, then Lemma 3 ensures that there exists a least integer k^* such that $N_{k^*+1}(\delta)$ for (6) is left unimodular over $\mathbb{R}[\delta]$ when applying the proposed algorithm.

For system (6), note now $\Delta_1(\delta) = C(\delta)$, $N_1(\delta) = \Delta_1(\delta)$. Since $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$, then there exists a unimodular matrix $P(\delta) = \begin{bmatrix} P_1(\delta) & P_2(\delta) \\ P_3(\delta) & P_4(\delta) \end{bmatrix}$ over $\mathbb{R}[\delta]$ such that (10) is satisfied. Set $S_1(\delta) = P(\delta)$, we have

$$\begin{aligned} S_1(\delta) \begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & N_1(\delta) \end{bmatrix} &= S_1(\delta) \begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & C(\delta) \end{bmatrix} \\ &= \begin{bmatrix} P_1(\delta) & \mathrm{I}_n \\ P_3(\delta) & 0 \end{bmatrix} \end{aligned}$$

Then we have $\Delta_2(\delta) = P_3(\delta)A(\delta)$, and

$$N_2(\delta) = \begin{bmatrix} \Delta_1(\delta) \\ \Delta_2(\delta) \end{bmatrix} = \begin{bmatrix} C(\delta) \\ P_3(\delta)A(\delta) \end{bmatrix}$$

Since $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$, then $\begin{bmatrix} E(\delta) \\ C(\delta) \\ P_3(\delta)A(\delta) \end{bmatrix} = \begin{bmatrix} E(\delta) \\ N_2(\delta) \end{bmatrix}$ is also left unimodular over $\mathbb{R}[\delta]$, thus there exists a unimodular matrix $S_2(\delta) = \begin{bmatrix} P_1(\delta) & P_2(\delta) & 0 \\ P_3(\delta) & P_4(\delta) & 0 \\ P_3(\delta)A(\delta)P_1(\delta) & P_3(\delta)A(\delta)P_2(\delta) & -\mathrm{I}_{\bar{n}+p-n} \end{bmatrix}$ over $\mathbb{R}[\delta]$ such that

$$
\begin{aligned}
& S_2(\delta) \begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & N_2(\delta) \end{bmatrix} \\
&= \begin{bmatrix} P_1(\delta), P_2(\delta) & 0 \\ P_3(\delta), P_4(\delta) & 0 \\ P_3(\delta)A(\delta)\left[P_1(\delta), P_2(\delta)\right] & -\mathrm{I}_{\bar{n}+p-n} \end{bmatrix} \begin{bmatrix} \mathrm{I}_q & E(\delta) \\ 0 & C(\delta) \\ 0 & P_3(\delta)A(\delta) \end{bmatrix} \\
&= \begin{bmatrix} P_1(\delta) & \mathrm{I}_n \\ P_3(\delta) & 0 \\ P_3(\delta)A(\delta)P_1(\delta) & 0 \end{bmatrix}
\end{aligned}
$$

thus we have $\Delta_3(\delta) = \begin{bmatrix} P_3(\delta)A(\delta) \\ P_3(\delta)A(\delta)P_1(\delta)A(\delta) \end{bmatrix}$, and $N_3(\delta) = \begin{bmatrix} N_2(\delta) \\ \Delta_3(\delta) \end{bmatrix} = \begin{bmatrix} C(\delta) \\ P_3(\delta)A(\delta) \\ P_3(\delta)A(\delta)P_1(\delta)A(\delta) \end{bmatrix}$. By induction, for $k \geq 2$, if Assumption 2 is satisfied we have

$$
N_{k+1}(\delta) = \begin{bmatrix} C(\delta) \\ P_3(\delta)A(\delta) \\ \vdots \\ P_3(\delta)A(\delta)\left[P_1(\delta)A(\delta)\right]^{k-1} \end{bmatrix}
$$

Now, if there exists a least integer k^* such that $\tilde{N}_{k^*+1}(\delta)$ defined in (4) for (2) is left unimodular over $\mathbb{R}[\delta]$, it implies that, according to Lemma 3,

$$
N_{k^*+1}(\delta) = \begin{bmatrix} C(\delta) \\ P_3(\delta)A(\delta) \\ \vdots \\ P_3(\delta)A(\delta)\left[P_1(\delta)A(\delta)\right]^{k^*-1} \end{bmatrix}
$$

is left unimodular over $\mathbb{R}[\delta]$. Since

$$
\bar{\mathscr{O}}_{k^*}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{k^*-1}(\delta) \end{bmatrix} \sim \begin{bmatrix} N_{k^*}(\delta) \\ C(\delta)P_3(\delta)A(\delta) \\ \vdots \\ C(\delta)\left[P_3(\delta)A(\delta)\right]^{k^*} \end{bmatrix}
$$

thus $\bar{\mathscr{O}}_{k^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. ■

With the above lemma, we can state the following result.

Lemma 7. *If there exists a least integer $l^* \in \mathbb{N}$ such that $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (13) is left unimodular over $\mathbb{R}[\delta]$, then there exists a matrix $\Gamma(\delta)$ and a left unimodular matrix $T(\delta)$ over $\mathbb{R}[\delta]$ such that $T(\delta)\left[\bar{A}(\delta) - \Gamma(\delta)\bar{C}(\delta)\right]T_L^{-1}(\delta)$ is constant (independent on δ) and Hurwitz.*

Proof. The proof is based on the results stated in Hou et al. (2002) in which it has been proven that there exists a left unimodular matrix $T(\delta)$ over $\mathbb{R}[\delta]$ such that

$$
\begin{aligned}
& T(\delta)\bar{A}(\delta)T_L^{-1}(\delta) = A_0 + F(\delta)C_0 \\
& \bar{C}(\delta)T_L^{-1}(\delta) = C_0
\end{aligned}
\tag{14}
$$

where $F(\delta) = \left[F_1^T(\delta), \cdots, F_{l^*}^T(\delta)\right]^T$ and

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & \mathrm{I}_{2\bar{n}-n-q+2p} & 0 & \cdots & 0 \\ 0 & \cdots & \mathrm{I}_{2\bar{n}-n-q+2p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \mathrm{I}_{2\bar{n}-n-q+2p} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\
C_0 &= \begin{bmatrix} \mathrm{I}_{2\bar{n}-n-q+2p} & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}
\tag{15}
$$

if and only if Assumption 2 is satisfied. Moreover, the matrix $T(\delta)$ is given by:

$$
\begin{cases} T_1(\delta) = \bar{C}(\delta) \\ T_{i+1}(\delta) = T_i(\delta)\bar{A}(\delta) - F_i(\delta)\bar{C}(\delta), \text{ for } 1 \leq i \leq l^* - 1 \end{cases}
\tag{16}
$$

with $F_i(\delta)$ being determined through the following equation:

$$
\left[F_{l^*}(\delta), \cdots, F_1(\delta)\right] = \bar{C}(\delta)\bar{A}^{l^*}(\delta)\left[\bar{\mathscr{O}}_{l^*}(\delta)\right]_L^{-1}
\tag{17}
$$

Obviously, the pair (A_0, C_0) is observable, thus there exists a constant matrix Γ_0 such that $(A_0 - \Gamma_0 C_0)$ is Hurwitz. Therefore, if Assumption 2 is satisfied, we have

$$
T(\delta)\left[\bar{A}(\delta) - \Gamma(\delta)\bar{C}(\delta)\right]T_L^{-1}(\delta) = A_0 + F(\delta)C_0 - T(\delta)\Gamma(\delta)C_0
$$

By choosing $\Gamma(\delta) = T_L^{-1}(\delta)\left[F(\delta) + \Gamma_0\right]$, we obtain

$$
T(\delta)\left[\bar{A}(\delta) - \Gamma(\delta)\bar{C}(\delta)\right]T_L^{-1}(\delta) = A_0 - \Gamma_0 C_0
\tag{18}
$$

which is Hurwitz, independent of the time-delay. ■

4. Observer and design procedure

Based on the deduced result, we are ready to present our main result.

Theorem 1. *Suppose Assumptions 1 and 2 are both satisfied for system (2), then there exist the following matrices:*

$$
\begin{aligned}
N(\delta) &= P_1(\delta)A(\delta) - M(\delta)P_3(\delta)A(\delta) - H(\delta)C(\delta) \\
K(\delta) &= P_2(\delta) - M(\delta)P_4(\delta) \\
L(\delta) &= H(\delta) + N(\delta)K(\delta)
\end{aligned}
\tag{19}
$$

where $P_i(\delta)$ for $1 \leq i \leq 4$ are defined in (10) and

$$
\begin{bmatrix} H(\delta) & M(\delta) \end{bmatrix} = T_L^{-1}(\delta)\left[F(\delta) + \Gamma_0\right]
\tag{20}
$$

with $T_L^{-1}(\delta)$ and $F(\delta)$ being defined in (16) and (17), and Γ_0 being a constant matrix which makes $(A_0 - \Gamma_0 C_0)$ Hurwitz, such that the following dynamical system:

$$
\begin{cases} \dot{w} = N(\delta)w + L(\delta)y \\ \hat{x} = w + K(\delta)y \end{cases}
\tag{21}
$$

is an exponential observer for system (2).

Proof. As we have proved that, if Assumption 2 is satisfied, then there exists a matrix $P(\delta)$ such that (10) is satisfied. Denote $e = x - \hat{x}$, then

$$e = [\mathrm{I}_n - K(\delta)C(\delta)]x - w$$

which gives:

$$\dot{e} = [\mathrm{I}_n - K(\delta)C(\delta)]\dot{x} - \dot{w}$$

By choosing $K(\delta) = P_2(\delta) - M(\delta)P_4(\delta)$ where $M(\delta)$ is given in (20), then we have

$$\mathrm{I}_n - K(\delta)C(\delta) = [P_1(\delta) - M(\delta)P_3(\delta)]E(\delta)$$

Therefore we obtain

$$\begin{aligned}\dot{e} &= [P_1(\delta) - M(\delta)P_3(\delta)]E(\delta)\dot{x} - \dot{w} \\ &= [P_1(\delta) - M(\delta)P_3(\delta)]A(\delta)x - N(\delta)w - L(\delta)y \\ &= [P_1(\delta)A(\delta) - M(\delta)P_3(\delta)A(\delta) \\ &\quad -L(\delta)C(\delta) + N(\delta)K(\delta)C(\delta)]x - N(\delta)\hat{x}\end{aligned}$$

By choosing

$$L(\delta) = H(\delta) + N(\delta)K(\delta)$$

and

$$N(\delta) = P_1(\delta)A(\delta) - M(\delta)P_3(\delta)A(\delta) - H(\delta)C(\delta)$$

we have

$$\begin{aligned}\dot{e} &= [P_1(\delta)A(\delta) - M(\delta)P_3(\delta)A(\delta) - H(\delta)C(\delta)]x \\ &\quad -N(\delta)\hat{x} \\ &= \left[\bar{A}(\delta) - [H(\delta), M(\delta)]\bar{C}(\delta)\right]e = N(\delta)e\end{aligned} \tag{22}$$

According to Lemma 7, if Assumptions 1 and 2 are both satisfied for system (2), then there exists a left unimodular matrix $T(\delta)$ defined in (16) and a gain matrix $\Gamma(\delta) = T_L^{-1}(\delta)[F(\delta) + \Gamma_0]$ such that $T(\delta)\left[\bar{A}(\delta) - \Gamma(\delta)\bar{C}(\delta)\right]T_L^{-1}(\delta)$ is Hurwitz. By setting

$$\Gamma(\delta) = \left[\begin{array}{cc} H(\delta), & M(\delta) \end{array}\right] = T_L^{-1}(\delta)[F(\delta) + \Gamma_0]$$

then we obtain that $\dot{e} = N(\delta)e$ with $N(\delta) = \bar{A}(\delta) - \Gamma(\delta)\bar{C}(\delta)$. By noting $z = T(\delta)x$ and $e_z = T(\delta)e$, we arrive at

$$\begin{aligned}\dot{e}_z &= T(\delta)\dot{e} = T(\delta)N(\delta)e \\ &= T(\delta)N(\delta)T_L^{-1}(\delta)T(\delta)e \\ &= [A_0 - \Gamma_0 C_0]\,T(\delta)e \\ &= [A_0 - \Gamma_0 C_0]\,e_z\end{aligned}$$

where we used the equality (18). Since $[A_0 - \Gamma_0 C_0]$ is Hurwitz and independent of δ, thus we proved that $e_z = T(\delta)e$ is exponentially stable for any value of delay. Due to the fact that $T(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, we can conclude that the observation error $e = T_L^{-1}(\delta)e_z$ exponentially tends to zero. ■

Remark 4. *When designing observers for time-delay systems, the convergence of observation error dynamics might either depend or not depend on the delays. For the case depending on the delays, the most existing methods need to impose some conditions (like boundedness of time delay and its derivative) on the delay in order to prove the stability of the observation error dynamics Bhat and Koivo (1976); Fattouh et al. (1999); Sename (2001); Fridman and Shaked (2002); Richard (2003); Nguyen et al. (2016). However, the proposed method is based on the output injection (delayed) technique. Since the basic delay h is constant, thus all the delays involved in the studied system are constant. It can be seen that the observation error dynamics (22) is equivalent to $\dot{e}_z = [A_0 - \Gamma_0 C_0]e_z$. It is obvious that the convergence speed of e_z is independent of the delay, which implies that this method can be applied to any commensurate and constant delay. In other words, no limitations of delays are required for such a method, the only limitations are Assumptions 1 and 2, and these assumptions impose structural conditions which are independent of the delay size.*

Remark 5. *It is worth noticing that the stability of the observation error depends on neither the choice of the matrix Π nor that of the matrix P. This is clear since the previous theorem is valid for any selection of such matrices. Moreover, the convergence rate depends only on the choice of the matrix Γ_0, which modifies the eigenvalues of the closed-loop matrix $A_0 - \Gamma_0 C_0$. In this sense, we may say that matrices Π and P do not play an important role in the asymptotic behavior of the observation dynamics.*

Remark 6. *The proposed method can be easily extended to treat the linear singular time-delay system with known input of the following form:*

$$\left\{\begin{array}{c} E(\delta)\dot{x}(t) = A(\delta)x(t) + B(\delta)u(t) \\ y(t) = C(\delta)x(t) + D(\delta)u(t) \end{array}\right. \tag{23}$$

By applying the same procedure, it can be proven that, if Assumptions 1 and 2 are both satisfied for system (2), then the following dynamics:

$$\left\{\begin{array}{l} \dot{w} = N(\delta)w + L(\delta)y + Q(\delta)u \\ \hat{x} = w + K(\delta)y + R(\delta)u \end{array}\right. \tag{24}$$

with $N(\delta)$, $K(\delta)$ and $K(\delta)$ defined in (19), and

$$\begin{aligned} R(\delta) &= -K(\delta)D(\delta) \\ Q(\delta) &= P_1(\delta)B(\delta) - M(\delta)P_3(\delta)B(\delta) - L(\delta)D(\delta) \end{aligned}$$

is an exponential observer for system (2).

If all conditions of Theorem 1 are satisfied for (2), then the following summarizes the procedure to design the proposed observer for system (2):

Step 1: For the triple $\left(\tilde{E}(\delta), \tilde{A}(\delta), \tilde{C}(\delta)\right)$, calculate a unimodular matrix $\Pi(\delta)$ over $\mathbb{R}[\delta]$ such that $\Pi(\delta)\tilde{E}(\delta) = \left[\begin{array}{c} E(\delta) \\ 0 \end{array}\right]$ and $\Pi(\delta)\tilde{A}(\delta) = \left[\begin{array}{c} A(\delta) \\ \underline{A}(\delta) \end{array}\right]$. Then define $C(\delta) = \left[\begin{array}{c} \underline{A}(\delta) \\ \tilde{C}(\delta) \end{array}\right]$;

Step 2: For the deduced triple $(E(\delta), A(\delta), C(\delta))$, determine the unimodular matrix $P(\delta)$ such that $P(\delta)\left[\begin{array}{c} E(\delta) \\ C(\delta) \end{array}\right] = \left[\begin{array}{c} \mathrm{I}_n \\ 0 \end{array}\right]$;

Step 3: With the deduced $P(\delta)$, compute $\bar{A}(\delta) = P_1(\delta)A(\delta)$ and $\bar{C}(\delta) = \begin{bmatrix} P_3(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$;

Step 4: After having obtained $\bar{A}(\delta)$ and $\bar{C}(\delta)$, deduce $T(\delta)$ and $F(\delta)$ defined in (16) and (17) respectively;

Step 5: With A_0 and C_0 defined in (15), calculate Γ_0 such that $(A_0 - \Gamma_0 C_0)$ is Hurwitz, and get $M(\delta)$ and $H(\delta)$ according to (20);

Step 6: Following (19), compute $N(\delta)$, $L(\delta)$ and $K(\delta)$ based on which the observer of the form (24) can be achieved.

5. Illustrative example

Consider the following linear singular time-delay neutral system of form (1):

$$\begin{cases} \dot{x}_1(t) = -\dot{x}_2(t-h) + x_1(t-h) + x_4(t-h) \\ \dot{x}_2(t) = -\dot{x}_3(t-h) + x_1(t) + x_4(t) \\ \dot{x}_3(t) = -\dot{x}_1(t-h) + x_1(t) + x_2(t) \\ 0 = x_1(t) + x_2(t) + x_3(t) \\ y_1(t) = x_3(t-h) \\ y_2(t) = x_4(t) \end{cases}$$

which can be written into the form (2) with

$$\tilde{E}(\delta) = \begin{bmatrix} 1 & \delta & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 1 & \delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{A}(\delta) = \begin{bmatrix} \delta & 0 & 0 & \delta \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$\tilde{C}(\delta) = \begin{bmatrix} 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 1:
It is clear that we can choose $\Pi(\delta) = \mathrm{I}_4$ such that

$$E(\delta) = \begin{bmatrix} 1 & \delta & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 1 & \delta & 0 \end{bmatrix}, \; A(\delta) = \begin{bmatrix} \delta & 0 & 0 & \delta \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and

$$C(\delta) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2:
It is easy to check that for the given triple $(\tilde{E}(\delta), \tilde{A}(\delta), \tilde{C}(\delta))$ Assumption 2 is satisfied, thus $\begin{bmatrix} E(\delta) \\ C(\delta) \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$. In fact we can find

$$P(\delta) = \begin{bmatrix} P_1(\delta) & P_2(\delta) \\ P_3(\delta) & P_4(\delta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\delta & 0 & \delta & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -\delta & 1 & \delta^2 & 0 & -\delta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \delta - 1 & -1 & \delta - 1 - \delta^2 & 1 & 1 - \delta + \delta^2 & 0 \\ \delta^2 & -\delta & -\delta^3 & 0 & 1 + \delta^3 & 0 \end{bmatrix}$$

Step 3:
We have

$$\bar{A}(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{C}(\delta) = \begin{bmatrix} P_3(\delta)A(\delta) \\ C(\delta) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & -1 \\ -\delta & -\delta & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can check that

$$\bar{\mathcal{O}}_1(\delta) = \begin{bmatrix} -2 & -\delta & 1 & 0 & 0 & -1 & -\delta & 2 & \delta & 0 \\ -1 & -\delta & 1 & 0 & 0 & 0 & 0 & 1 & \delta & 0 \\ 0 & 0 & 1 & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & -\delta & 1 & 0 & 0 \end{bmatrix}^T$$

which is left unimodular over $\mathbb{R}[\delta]$.
Step 4:
Then according to (17) we have

$$[F_2(\delta), F_1(\delta)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which yields the following transformation

$$T(\delta) = \begin{bmatrix} -2 & -\delta & 1 & 0 & 0 & -1 & -\delta & 0 & -\delta & 0 \\ -1 & -\delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & -\delta & 0 & -\delta & 0 \end{bmatrix}^T$$

Step 5:
Finally we get

$$A_0 = T(\delta)\bar{A}(\delta)T_L^{-1}(\delta) = \begin{bmatrix} 0_{5\times5} & \mathrm{I}_5 \\ 0_{5\times5} & 0_{5\times5} \end{bmatrix}$$

and

$$C_0 = C(\delta)T_L^{-1}(\delta) = \begin{bmatrix} \mathrm{I}_5 & 0_{5\times5} \end{bmatrix}$$

It is clear that the pair (A_0, C_0) is observable. By choosing

$$\Gamma_0 = \begin{bmatrix} 55 & 0 & 0 & 0 & 0 & 250 & 0 & 0 & 0 & 0 \\ 0 & 70 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 & 0 & 0 & 200 & 0 & 0 \\ 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 250 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 & 0 & 0 & 0 & 125 \end{bmatrix}^T$$

we assign $(-5, -5, -10, -10, -20, -20, -25, -25, -50, -50)$ as the eigenvalues to the Hurwitz matrix $A_0 - \Gamma_0 C_0$. After having determined Γ_0, we can then follow (19) to calculate all necessary matrices $N(\delta)$, $K(\delta)$ and $L(\delta)$ for the observer:

$$
\begin{aligned}
N(\delta) &= \begin{bmatrix} 250-30\delta & 250-30\delta & 250 & 0 \\ 30\delta-444 & 30\delta-445 & -445 & 1 \\ 197 & 197 & 196 & -30 \\ 30\delta & 30\delta & 0 & 0 \end{bmatrix}^T \\
K(\delta) &= \begin{bmatrix} 0 & 30\delta^3+d+30 & 0 \\ 0 & -30\delta^3-31 & 0 \\ 0 & -\delta^2 & 0 \\ 0 & -30\delta^3-30 & 1 \end{bmatrix} \\
L(\delta) &= \begin{bmatrix} -250 & -280\delta^2+280\delta-250 & 0 \\ 445 & 475\delta^2-474\delta+445 & 1 \\ -196 & 900\delta^3-196\delta^2+197\delta+703 & 0 \\ 0 & 30\delta(\delta-1) & 0 \end{bmatrix}
\end{aligned}
$$

The corresponding simulation results are depicted in the following figure.

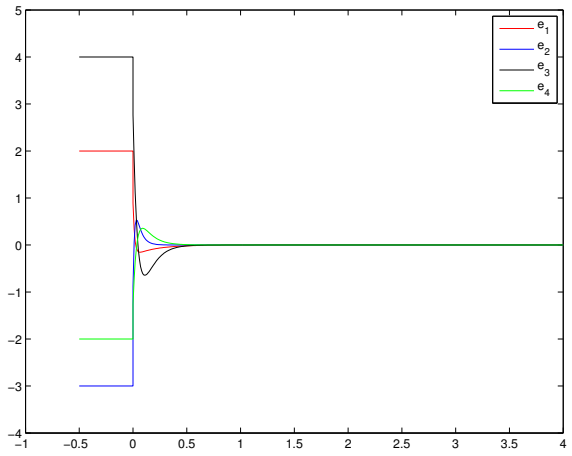

Figure 1: Estimation errors for x_1, x_2, x_3 and x_4 with $h = 0.5s$.

6. Conclusion

For linear singular time-delay systems, the most existing results focus on the simple case, i.e. $E\dot{x}(t) = A_0x(t) + A_1x(t-\tau)$. Few results have been stated for the general linear singular time-delay systems of the form $\sum_{i=0}^{l} E_i\dot{x}(t-i\tau) = \sum_{i=0}^{k} A_ix(t-i\tau)$, which covers also neutral delay systems. For such a general case, this paper deduced sufficient conditions, with which a simple Luenberger-like observer can be designed in order to exponentially estimate the states.

References

Antsaklis, P. J., Michel, A. N., 2007. A Linear Systems Primer, 1st Edition. Birkhäuser Basel.

Bejarano, F. J., Zheng, G., 2014. Observability of linear systems with commensurate delays and unknown inputs. Automatica 50 (8), 2077–2083.

Bejarano, F. J., Zheng, G., 2016a. Observability of singular systems with commensurate time-delays and neutral terms. to appear, Automatica.

Bejarano, F. J., Zheng, G., 2016b. Observability of singular time-delay systems with unknown inputs. Systems and Control Letters 89, 55–60.

Bejarano, F. J., Zheng, G., 2016c. Unknown Input Functional Observability of Descriptor Systems with Delays. In: Proc. IEEE Conf. Decision Control. Las Vegas, United States.

Bhat, K., Koivo, H., 1976. An observer theory for time delay systems. IEEE Transactions on Automatic Control 21 (2), 266–269.

Brayton, R. K., 1968. Small-signal stability criterion for electrical networks containing lossless transmission lines. IBM Journal of Research and Development 12 (6), 431–440.

Campbell, S., 1982. Singular systems of differential equations II. Pitman.

Darouach, M., Boutayeb, M., July 1995. Design of observers for descriptor systems. IEEE Transactions on Automatic Control 40 (7), 1323–1327.

Ezzine, M., Darouach, M., Ali, H. S., Messaoud, H., 2011. Unknown inputs functional observers designs for descriptor systems with constant time delay. In: in Proc. IFAC World Congress.

Fattouh, A., Sename, O., Jean-Michel, D., 1999. An unknown input observer design for linear time-delay systems. In: IEEE Conference on Decision and Control. Vol. 4. pp. 4222–4227.

Feng, J., Zhu, S., Cheng, Z., 2003. Observer design for linear singular time-delay systems. In: in Proc. IEEE Conf. Decision Control.

Fridman, E., Shaked, U., 2002. A descriptor system approach to h control of linear time-delay systems. IEEE Transactions on Automatic Control 47 (2), 253–270.

Garate-Garcia, A., Marquez-Martinez, L., Moog, C., 2011. Equivalence of linear time-delay systems. IEEE Transactions on Automatic Control 56 (3), 666–670.

Germani, A., Manes, C., Pepe, P., 2001. An asymptotic state observer for a class of nonlinear delay systems. Kybernetika 3 (7), 459–478.

Gu, Z., Su, H., Shi, P., Chu, J., 2013. Analysis and Synthesis of Singular Systems with Time-Delays. Lecture notes in control and information science. Springer.

Hou, M., Zitek, M. H., Patton, R., 2002. An observer design for linear time-delay systems. IEEE Transactions on Automatic Control 47, 121–125.

Khadhraoui, M., Ezzine, M., Messaoud, H., Darouach, M., 2014. Design of full order unknown input observers for delayed singular systems with state variable time delay. In: in Symposium on Communications, Control and Signal Processing.

Lee, E., Neftci, S., Olbrot, A., 1982. Canonical forms for time delay system. IEEE Transactions on Automatic Control 27 (1), 128–132.

Lee, E., Olbrot, A., 1981. Observability and related structural results for linear hereditary systems. International Journal of Control 34 (6), 1061–1078.

Molinari, B., October 1976. A strong controllability and observability in linear multivariable control. IEEE Transactions on Automatic Control 21 (5), 761–764.

Nguyen, M. C., Trinh, H., Nam, P. T., 2016. Non-linear observer design for a class of singular time-delay systems with lipschitz non-linearities. IMA Journal of Mathematical Control and Information.

Niculescu, S., Rasvan, V., 2000. Delay-independent stability in lossless propagation models with applications. In: in Proc. MTNS.

Niculescu, S.-I., 2001. Delay effects on stability: a robust control approach. Vol. 269. Springer Science & Business Media.

Perdon, A., Anderlucci, M., 2006. An unknown input observer for singular time-delay systems. In: in Proc. Mediterranean Conference on Control and Automation.

Richard, J.-P., 2003. Time-delay systems: an overview of some recent advances and open problems. automatica 39 (10), 1667–1694.

Sename, O., 2001. New trends in design of observers for time-delay systems. Kybernetika 37 (4), 427–458.

Silverman, L., 1969. Inversion of multivariable linear systems. IEEE Transactions on Automatic Control 14 (3), 270–276.

Wang, G., Zhang, Q., Yan, X., 2014. Analysis and Design of Singular Markovian Jump Systems. SpringerLink : Bücher. Springer International Publishing.

Wei, G., 2013. General solution and observability of singular differential. Abstract and Applied Analysis 2.

Zheng, G., Bejarano, F. J., Perruquetti, W., Richard, J.-P., 2015. Unknown input observer for linear time-delay systems. Automatica 61, 35 – 43.