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Diffusion approximation for random parabolic operators with oscillating coefficients

Marina Kleptsyna[‡] Andrey Piatnitski[§] and Alexandre Popier[‡]

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Abstract

We consider Cauchy problem for a divergence form second order parabolic operator with rapidly oscillating coefficients that are periodic in spatial variables and random stationary ergodic in time. As was proved in [31] and [15] in this case the homogenized operator is deterministic. The paper focuses on the diffusion approximation of solutions in the case of non-diffusive scaling, when the oscillation in spatial variables is faster than that in temporal variable. Our goal is to study the asymptotic behaviour of the normalized difference between solutions of the original and the homogenized problems.

1 Introduction

In this work we consider the asymptotic behaviour of solutions to the following Cauchy problem

$$(1) \quad \begin{aligned} \frac{\partial}{\partial t} u^\varepsilon &= \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^\varepsilon \right] && \text{in } \mathbb{R}^d \times (0, T] \\ u^\varepsilon(x, 0) &= \varphi(x). \end{aligned}$$

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Here ε is a small positive parameter that tends to zero, α satisfies the inequality $0 < \alpha < 2$, $a(z, s)$ is a positive definite matrix whose entries are periodic in z variable and random stationary ergodic in s .

It is known (see [31, 15]) that this problem admits homogenization and that the homogenized operator is deterministic and has constant coefficients. The homogenized Cauchy problem takes the form

$$(2) \quad \begin{aligned} \frac{\partial}{\partial t} u^0 &= \operatorname{div}(a^{\text{eff}} \nabla u^0) \\ u^0(x, 0) &= \varphi(x). \end{aligned}$$

The formula for the effective matrix a^{eff} is given in (6) in Section 2 (see also [15]).

The goal of this paper is to study the limit behaviour of the difference $u^\varepsilon - u^0$, as $\varepsilon \rightarrow 0$.

In the existing literature there is a number of works devoted to homogenization of random parabolic problems. The results obtained in [18] and [25] for random divergence form elliptic operators also apply to the parabolic case. In the presence of large lower order terms the limit dynamics might remain random and show diffusive or even more complicated behaviour. The papers [6], [26], [17] focus on the case of time dependent parabolic operators with periodic in spatial variables and random in time coefficients. The fully random case has been studied in [27], [3], [4], [11].

One of the important aspects of homogenization theory is estimating the rate of convergence. For random operators the first estimates have been obtained in [13]. Further important progress in this direction was achieved in the recent works [10], [9].

Problem (1) in the case of diffusive scaling $\alpha = 2$ was studied in our previous work [16]. It was shown that, under proper mixing conditions, the difference $u^\varepsilon - u^0$ is of order ε , and that the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$ after subtracting an appropriate corrector, converges in law to a solution of some limit SPDE.

In the present paper we consider the case $0 < \alpha < 2$. In other words, bearing in mind the diffusive scaling, we assume that the oscillation in spatial variables is faster than that in time. In this case the principal part of the asymptotics of $u^\varepsilon - u^0$ consists of a finite number of correctors, the oscillating part of each of them being a solution of an elliptic PDE with periodic in spatial variable coefficients. The number of correctors increases as α approaches

2. After subtracting these correctors, the resulting expression divided by $\varepsilon^{\alpha/2}$ converges in law to a solution of the limit SPDE.

In contrast with the diffusive scaling, for $\alpha < 2$ the interplay between the scalings in spatial variables and time and the necessity to construct higher order correctors results in additional regularity assumptions on the coefficients. Indeed, each corrector is introduced as a solution of some elliptic equation in which time is a parameter, thus this corrector has the same regularity in time as the coefficients of the equation. When we construct the next term of the expansion, this corrector is differentiated in time. This reduces the regularity. The result mentioned in the previous paragraph holds if the coefficients $a^{ij}(z, s)$ in (1) are smooth enough functions.

We also consider in the paper the special case of diffusive dependence on time. Namely, we assume in this case that $a(z, s) = a(z, \xi_s)$, where ξ is a stationary diffusion process in \mathbb{R}^n and $a(z, y)$ is a periodic in z smooth deterministic function. It should be emphasized that in the said diffusive case Theorem 1 does not apply because the coefficients a^{ij} do not possess the required regularity in time. That is why in the diffusive case we have to use a different approach and provide another proof of convergence which relies on the Itô and Malliavin calculus and estimates based on anticipating stochastic integration as well as on a number of estimates of the fundamental solution of divergence form second order parabolic equations. The latter estimates are of independent interest. We consider the generic divergence form second order parabolic equation with coefficients that are regular in the spatial variables and measurable in time, and show that the derivatives of its fundamental solution admit upper bounds that only depend on the ellipticity constants and the L^∞ norm of the gradient of the coefficients in spatial variables (see Lemma 3). To our best knowledge, these estimates are new.

The case $\alpha > 2$ will be considered elsewhere.

The paper is organized as follows. In Section 2 we introduce the studied problem and provide all the assumptions. Then we formulate the main results of the paper.

In Section 3 we outline the scheme of the proof, consider a number of auxiliary problems and define the higher order terms of the asymptotics of solution.

Section 4 focuses on the proof of Theorem 1.

In Section 5 we consider the special case of diffusive dependence on time.

In Section 6 we obtain an estimate for a solution of parabolic equation with a diffusion on the right-hand side. Here we use anticipating calculus and

the properties of the fundamental solution of a stochastic parabolic equation with random coefficients.

Finally, in the Appendix a number of estimates for the fundamental solutions of the studied parabolic equations are proved.

2 Problem setup and main results

In this section we provide all the assumptions on the data of problem (1), introduce some notations and formulate the main results.

For the studied Cauchy problem (1), where ε is a small positive parameter, we assume that the following conditions hold true:

- a1.** the matrix $a(z, s) = \{a^{ij}(z, s)\}_{i,j=1}^d$ is symmetric and satisfies uniform ellipticity conditions

$$\lambda|\zeta|^2 \leq a(z, s)\zeta \cdot \zeta \leq \lambda^{-1}|\zeta|^2, \quad \zeta \in \mathbb{R}^d, \quad \lambda > 0;$$

- a2.** $\varphi \in C_0^\infty(\mathbb{R}^d)$. In fact, this condition can be essentially relaxed, see Remark 2.

In this paper we consider two different settings. In the first setting it is assumed that the coefficients of matrix a are smooth functions that have good mixing properties in time variable. The smoothness is important because our approach relies on auxiliary elliptic equations that depend on time as a parameter, and we have to differentiate these equations w.r.t. time.

In the second setting it is assumed that the coefficients of matrix a are diffusion processes in time. In this case even for smooth functions $a(z, y)$ the coefficients of matrix $a(z, \xi_s)$ are just Hölder continuous in and not differentiable in time, and the method used in the smooth case fails to work. Here we use the approach based on the Itô and Malliavin calculus and Aronson type estimates for the fundamental solution of parabolic operators. In particular, we show that the spatial and Malliavin derivatives of the fundamental solution admit upper bounds that do not depend on the regularity of the coefficients with respect to time.

In the case of smooth coefficients our assumptions read

- h1.** The coefficients $a^{ij}(z, s)$ are periodic in z with the period $[0, 1]^d$ and random stationary ergodic in s . Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$

with an ergodic dynamical system τ_s , we assume that $a^{ij}(z, s, \omega) = \mathbf{a}^{ij}(z, \tau_s \omega)$, where $\{\mathbf{a}^{ij}(z, \omega)\}_{i,j=1}^d$ is a collection of random periodic in z functions that satisfy the above uniform ellipticity conditions.

h2. The realizations $a^{ij}(z, s)$ are smooth. For any $N \geq 1$ and $k \geq 1$ there exist $C_{N,k}$ such that

$$\mathbf{E} \|a^{ij}\|_{C^N(\mathbb{T}^d \times [0, T])}^k \leq C_{N,k};$$

here and in what follows we identify periodic functions with functions on the torus \mathbb{T}^d , \mathbf{E} stands for the expectation.

h3. Mixing condition. The strong mixing coefficient $\gamma(r)$ of $a(z, \cdot)$ satisfies the inequality

$$\int_0^\infty (\gamma(r))^{1/2} dr < \infty.$$

For the reader's convenience we recall here the definition of strong mixing coefficient. Let $\mathcal{F}_{\leq s}$ and $\mathcal{F}_{\geq s}$ be the σ -algebras generated by $\{a(z, t) : z \in \mathbb{T}^d, t \leq s\}$ and $\{a(z, t) : z \in \mathbb{T}^d, t \geq s\}$, respectively. We set

$$\gamma(r) = \sup |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|,$$

where the supremum is taken over all $A \in \mathcal{F}_{\leq 0}$ and $B \in \mathcal{F}_{\geq r}$.

In our second setting we assume that the matrix $a(z, s)$ has the form

$$(3) \quad a(z, s) = \mathbf{a}(z, \xi_s),$$

where ξ_s is a stationary diffusion process in \mathbb{R}^n with a generator

$$\mathcal{L} = \frac{1}{2} \text{Tr}[q(y)D^2] + b(y) \cdot \nabla$$

(∇ stands for the gradient, D^2 for the Hessian matrix). We still assume that Conditions **a1** and **a2** hold. Moreover we suppose that the matrix-functions $\mathbf{a}(z, y)$, $q(y)$ and vector-function $b(y)$ possess the following properties:

c1. $\mathbf{a} = \mathbf{a}(z, y)$ is periodic in z and smooth in both variables z and y . Moreover, for each $N > 0$ there exists $C_N > 0$ such that

$$\|\mathbf{a}\|_{C^N(\mathbb{T}^d \times \mathbb{R}^n)} \leq C_N.$$

c2. The matrix $q = q(y)$ satisfies the uniform ellipticity conditions: there exist $\lambda > 0$ such that

$$\lambda^{-1}|\zeta|^2 \leq q(y)\zeta \cdot \zeta \leq \lambda|\zeta|^2, \quad y, \zeta \in \mathbb{R}^n.$$

Moreover there exists a matrix $\sigma = \sigma(y)$ such that $q(y) = \sigma^*(y)\sigma(y)$.

c3. The matrix function σ and vector-function b are smooth, for each $N > 0$ there exists $C_N > 0$ such that

$$\|\sigma\|_{C^N(\mathbb{R}^n)} \leq C_N, \quad \|b\|_{C^N(\mathbb{R}^n)} \leq C_N.$$

c4. The following inequality holds for some $R > 0$ and $C_0 > 0$ and $p > -1$:

$$b(y) \cdot y \leq -C_0|y|^p \quad \text{for all } y \in \{y \in \mathbb{R}^n : |y| \geq R\}.$$

Remark 1 *We would like to emphasize that even for a smooth matrix-function $a(z, y)$ the coefficients of the matrix $a(z, \xi_s)$ are just Hölder continuous and need not be differentiable in s .*

We say that

- Condition **(H)** holds if **a1**, **a2** and **h1** – **h3** are fulfilled.
- Condition **(C)** holds if **a1**, **a2** and **c1** – **c4** are satisfied. This case is called the *diffusive case*.

According to [15], under **(H)** or **(C)**, the sequence u^ε converges in probability, as $\varepsilon \rightarrow 0$, to a solution u^0 of problem (2). For the reader convenience we provide here the definition of the effective matrix a^{eff} . If **(H)** holds, we solve the following auxiliary problem

$$(4) \quad \operatorname{div}(a(z, s, \omega)\nabla\chi^0(z, s, \omega)) = -\operatorname{div} a(z, s, \omega), \quad z \in \mathbb{T}^d;$$

here s and ω are parameters, and χ^0 is an unknown vector function: $\chi^0 = (\chi^{0,1}, \dots, \chi^{0,d})$. In what follows *we usually do not indicate explicitly the dependence of ω* . Due to ellipticity of the matrix a equation (4) has a unique, up to an additive constant vector, periodic solution, $\chi^0 \in (L^\infty(\mathbb{T}^d) \cap H^1(\mathbb{T}^d))^d$. This constant vector is chosen in such a way that

$$(5) \quad \int_{\mathbb{T}^d} \chi^0(z, s) dz = 0 \quad \text{for all } s \text{ and } \omega.$$

Then we define the effective matrix a^{eff} by

$$(6) \quad a^{\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} (\mathbf{I} + a(z, s)) \nabla \chi^0(z, s) dz,$$

where \mathbf{I} stands for the unit matrix, and $\{\nabla \chi^0(z, s)\}^{ij} = \frac{\partial}{\partial z_i} \chi^{0,j}$. If **(C)** holds, $\chi^0 = \chi^0(z, y)$ is a periodic solution of the equation

$$(7) \quad \text{div}_z (a(z, y) \nabla_z \chi^0(z, y)) = -\text{div}_z a(z, y);$$

here $y \in \mathbb{R}^n$ is a parameter. We choose an additive constant in such a way that $\int_{\mathbb{T}^d} \chi^0(z, y) dz = 0$. Let us emphasize that it follows from (4) and (7) that the zero order correctors χ^0 coincide in both settings: $\chi^0(z, s) = \chi^0(z, \xi_s)$. The effective matrix is again given by (6):

$$a^{\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} (\mathbf{I} + a(z, \xi_s)) \nabla_z \chi^0(z, \xi_s) dz.$$

It is known that the matrix a^{eff} is positive definite (see, for instance, [15]). Therefore, problem (2) is well posed, and function u^0 is uniquely defined. Under assumption **a2** the function u^0 is smooth and satisfies the estimates

$$(8) \quad \left| (1 + |x|)^N \frac{\partial^{\mathbf{k}} u^0(x, t)}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_d^{k_d}} \right| \leq C_{N, \mathbf{k}}$$

for all $N > 0$ and all multi index $\mathbf{k} = (k_0, k_1, \dots, k_d)$, $k_i \geq 0$.

2.1 The case of smooth coefficients with good mixing properties. Main result

Here we assume that condition **(H)** holds. In order to formulate the main results we need a number of auxiliary functions and quantities. For $j = 1, 2, \dots, J^0$ with $J^0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$, the higher order correctors are introduced as periodic solutions to the equations

$$(9) \quad \text{div}(a(z, s) \nabla \chi^j(z, s)) = \partial_s \chi^{j-1}(z, s),$$

where $\lfloor \cdot \rfloor$ stands for the integer part. Due to (5) for $j = 1$ this equation is solvable in the space of periodic in z functions. A solution χ^1 is uniquely

defined up to an additive constant vector. Choosing the constant vector in a proper way yields

$$\int_{\mathbb{T}^d} \chi^1(z, s) dz = 0 \quad \text{for all } s \text{ and } \omega$$

and thus the solvability of the equation for χ^2 . Iterating this procedure, we define all χ^j , $j = 1, 2, \dots, J^0$.

Next, we introduce the functions $u^j = u^j(x, t)$, $j = 1, \dots, J^0$. They solve the following problems:

$$(10) \quad \begin{aligned} \frac{\partial}{\partial t} u^j &= \operatorname{div}(a^{\text{eff}} \nabla u^j) + \sum_{k=1}^j \{a^{k, \text{eff}}\}^{im} \frac{\partial^2}{\partial x_i \partial x_m} u^{j-k} \\ u^j(x, 0) &= 0 \end{aligned}$$

with

$$(11) \quad a^{k, \text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} a(z, s) \nabla \chi^k(z, s) dz;$$

here and later on we assume summation from 1 to d over repeated indices.

To characterize the diffusive term in the limit equation we introduce the matrix

$$\Xi(s) = \int_{\mathbb{T}^d} \left[(a(z, s) + a(z, s) \nabla \chi^0(z, s)) - \mathbf{E} \{a(z, s) + a(z, s) \nabla \chi^0(z, s)\} \right] dz.$$

By construction the matrix function Ξ is stationary and its entries satisfy condition **h3** (mixing condition). Denote

$$\Lambda = \frac{1}{2} \int_0^\infty \mathbf{E} \left(\Xi(s) \otimes \Xi(0) + \Xi(0) \otimes \Xi(s) \right) ds, \quad \Lambda = \{\Lambda^{ijkl}\},$$

where $(\Xi(s) \otimes \Xi(0))^{ijkl} = \Xi^{ij}(s) \Xi^{kl}(0)$. Under condition **h3** the integral on the right-hand side converges.

The first main result of this paper is

Theorem 1 *Let Condition (H) be fulfilled, and assume that $\alpha < 2$. Then the functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left(u^\varepsilon - u^0 - \sum_{j=1}^{J_0} \varepsilon^{j(2-\alpha)} u^j \right)$$

converge in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ to the unique solution of the following SPDE

$$(12) \quad \begin{aligned} dv^0 &= \operatorname{div}(a^{\text{eff}} \nabla v^0) dt + (\Lambda^{1/2})^{ijkl} \frac{\partial^2}{\partial x_i \partial x_j} u^0 dW_t^{kl} \\ v^0(x, 0) &= 0; \end{aligned}$$

where $W. = \{W^{ij}\}$ is the standard d^2 -dimensional Brownian motion.

Remark 2 The regularity assumption on φ given in condition **a2** can be weakened. Namely, the statement of Theorem 1 holds if φ is $J^0 + 1$ times continuously differentiable and the corresponding partial derivatives decay at infinity sufficiently fast.

2.2 Diffusive case

In this part we formulate our result under the assumption that condition **(C)** is fulfilled. As before we introduce several correctors and auxiliary quantities.

First let us recall that according to [27] under conditions **c2** and **c4** a diffusion process ξ . with the generator \mathcal{L} has an invariant measure in \mathbb{R}^n that has a smooth density $\rho = \rho(y)$. For any $N > 0$ it holds

$$(1 + |y|)^N \rho(y) \leq C_N$$

with some constant C_N . The function ρ is the unique up to a multiplicative constant bounded solution of the equation $\mathcal{L}^* \rho = 0$; here $*$ denotes the formally adjoint operator. We assume that the process ξ_t is stationary and distributed with the density ρ . The effective matrix can be written here as follows:

$$a^{\text{eff}} = \int_{\mathbb{R}^n} \int_{\mathbb{T}^d} \left(a(z, y) + a(z, y) \nabla_z \chi^0(z, y) \right) \rho(y) dz dy.$$

Higher order correctors are defined as periodic solutions of the equations

$$(13) \quad \operatorname{div}_z (a(z, y) \nabla_z \chi^j(z, y)) = -\mathcal{L}_y \chi^{j-1}(z, y), \quad j = 1, 2, \dots, J^0.$$

Notice that $\int_{\mathbb{T}^d} \chi^{j-1}(z, y) dz = 0$ for all $j = 1, 2, \dots, J^0$, thus the compatibility condition is satisfied and the equations are solvable.

Remark 3 We have already mentioned that according to (4) and (7) the zero order correctors coincide in both studied cases. It is interesting to compare the correctors defined in (13) with the ones given by (9) and to observe that the higher order correctors need not coincide.

We introduce the matrices

$$a^{k,\text{eff}} = \int_{\mathbb{R}^n} \int_{\mathbb{T}^d} [\mathbf{a}(z, y) \nabla_z \chi^k(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^k(z, y))] \rho(y) dz dy, \quad k = 1, 2, \dots,$$

and matrix valued functions

$$\begin{aligned} \widehat{\mathbf{a}}^0(z, y) &= \mathbf{a}(z, y) + \mathbf{a}(z, y) \nabla_z \chi^0(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^0(z, y)), \\ \widehat{\mathbf{a}}^k(z, y) &= \mathbf{a}(z, y) \nabla_z \chi^k(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^k(z, y)), \quad k = 1, 2, \dots, \\ \langle \mathbf{a} \rangle^0(y) &= \int_{\mathbb{T}^d} (\widehat{\mathbf{a}}^0(z, y) - a^{\text{eff}}) dz, \\ \langle \mathbf{a} \rangle^k(y) &= \int_{\mathbb{T}^d} (\widehat{\mathbf{a}}^k(z, y) - a^{k,\text{eff}}) dz, \quad k = 1, 2, \dots \end{aligned}$$

The functions $u^j = u^j(x, t)$ are defined as solutions of problems

$$(14) \quad \begin{aligned} \frac{\partial}{\partial t} u^j &= \text{div}(a^{\text{eff}} \nabla u^j) + \sum_{k=1}^j \{a^{k,\text{eff}}\}^{im} \frac{\partial^2}{\partial x_i \partial x_m} u^{j-k} \\ u^j(x, 0) &= 0 \end{aligned}$$

Since for each $j = 1, 2, \dots$ problem (14) has a unique solution, the functions u^j are uniquely defined. Finally, we consider the equation

$$(15) \quad \mathcal{L}Q^0(y) = \langle \mathbf{a} \rangle^0(y).$$

According to [28], Theorems 1 and 2, this equation has a unique up to an additive constant solution of at most polynomial growth. Denote

$$(16) \quad \Lambda = \{\Lambda^{ijml}\} = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial y_{r_1}} (Q^0)^{ij}(y) \right] q^{r_1 r_2}(y) \left[\frac{\partial}{\partial y_{r_2}} (Q^0)^{ml}(y) \right] \rho(y) dy.$$

The matrix Λ is non-negative. Consequently its square root $\Lambda^{1/2}$ is well defined.

Now we introduce one more assumption on the process ξ_s . In order to formulate it we define the functions

$$(17) \quad \tilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \quad \tilde{\sigma}_{i,j}^l(s) = (\partial_{x_j} \sigma_{i,l})(\xi_s),$$

which appear when computing the Malliavin derivative of ξ (see Lemma 6.1). The notation \mathbb{S}^{d-1} stands for the unit sphere in \mathbb{R}^d .

(S). There exist $\mathfrak{p} \geq 2$ and $\mathfrak{c} > 0$ such that a.s. for any $t \geq 0$ and any $\theta \in \mathbb{S}^{d-1}$

$$\mathcal{Q}(t, \theta) + \frac{\mathfrak{p}}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \theta)^2 \leq -\mathfrak{c}$$

where

$$(18) \quad \mathcal{Q}(t, \theta) = (\tilde{b}(t)\theta, \theta) + \frac{1}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \tilde{\sigma}^l(t)\theta) - \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \theta)^2.$$

This assumption plays an important role in obtaining upper bounds for the Malliavin derivative of ξ (see Lemma 6.3 and the discussion after it). It is not clear to us if it can be relaxed. As an example, let us consider the multidimensional Ornstein-Uhlenbeck process:

$$d\xi_t = U\xi_t dt + \Sigma dB_t,$$

where U and Σ are two $d \times d$ matrices. Here $\tilde{b}(t) = U$ and $\tilde{\sigma}^l(t) = 0$. Therefore, condition (S) is reduced to $(U\theta, \theta) \leq -\mathfrak{c}$. Since θ remains in the sphere which is compact, it is sufficient to assume that $(U\theta, \theta) < 0$ for any θ .

In the diffusive case the following result holds:

Theorem 2 *Under Conditions (C) and (S), the normalized functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left(u^\varepsilon - u^0 - \sum_{j=1}^{J_0} \varepsilon^{j(2-\alpha)} u^j \right)$$

converge in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ to the unique solution of (12) with the standard d^2 -dimensional Brownian motion W . and Λ defined in (16)

Note that Remark 2 on φ still applies in this case.

3 Scheme of the proofs

In both cases the beginning of the proofs is the same. We write down the following ansatz

$$V^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x, t) - \sum_{k=0}^{J^0} \varepsilon^{k\delta} \left(u^k(x, t) + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^k(x, t) \right) \right\},$$

here and in what follows the symbol δ stands for $2 - \alpha$. In the diffusive case,

$$\chi^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) = \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right).$$

Then we substitute V^ε for u^ε in (1) and we obtain for V^ε

- a PDE with random coefficients when **(H)** is in force;
- a SPDE in the diffusive case.

We prove that V^ε converges in law in the suitable functional space to the solution of (12). In the case of smooth coefficients we combine the definition of correctors, formula (10) and the Central Limit Theorem for stationary mixing processes. After some manipulations this yields the desired convergence (see Section 4).

In the diffusive case in order to follow the same strategy as in the proof of Theorem 1 we should obtain suitable uniform in ε estimates for the solution of auxiliary problems. To this end we express these solutions in the mild form (Equation (50)) with the fundamental solution Γ . Since Γ is an anticipating process, following [23] and [1], we use the Malliavin calculus in order to obtain the mild form of the solution (see section 6.1). For a fixed $\varepsilon > 0$, we use Aronson's estimates for Γ and its Malliavin derivative to obtain an important intermediate estimate (Lemma 6.2). In estimate (61) of Lemma 6.2, the Malliavin derivative of the process ξ on the interval $[0, T/\varepsilon^\alpha]$ appears. Thanks to hypothesis **(S)** we get a uniform in ε estimate on this Malliavin derivative (see Lemma 6.3).

Auxiliary problems

We begin by considering problem (4). This equation has a unique up to an additive constant vector periodic solution. Since $\chi^0(\cdot, s)$ only depends on

$a(\cdot, s)$, the solution with zero average is stationary and the strong mixing coefficient of the pair $(a(\cdot, s), \chi^0(\cdot, s))$ coincides with that for $a(\cdot, s)$. The same statement is valid for any finite collection $(a(\cdot, s), \chi^0(\cdot, s), \chi^1(\cdot, s), \dots)$. By the classical elliptic estimates, under our standing assumptions we have

$$(19) \quad \|\chi^0\|_{L^\infty(\mathbb{T}^d \times [0, T])} \leq C, \quad \mathbf{E}\|\chi^0\|_{C^k(\mathbb{T}^d \times [0, T])}^N \leq C_{k, N}.$$

Indeed, multiplying equation (4) by χ^0 , using the Schwartz and Poincaré inequalities and considering (5), we conclude that $\|\chi^0(\cdot, s)\|_{H^1(\mathbb{T}^d)} \leq C$ for all $s \in \mathbb{R}$. The first estimate in (19) then follows from [8, Theorem 8.4]. The second estimate follows from the Schauder estimates, see [8, Chapter 6]

By the similar arguments, the solutions χ^j of equations (9) are stationary, satisfy strong mixing condition with the same coefficient $\gamma(r)$, and the following estimates hold: for any $N \geq 1$ and $k \geq 0$

$$(20) \quad \mathbf{E}\|\chi^j\|_{C^k(\mathbb{T}^d \times [0, T])}^N \leq C_{k, N}, \quad j = 0, 1, \dots, J_0.$$

The solutions χ^j defined by (13) satisfy the same estimate: for any $N > 0$ there exists C_N such that

$$\|\chi^j\|_{C^N(\mathbb{T}^d \times \mathbb{R}^n)} \leq C_N.$$

Solutions u^0 and u^j of problems (2), (10) and (14) are smooth functions. Moreover, for any $\mathbf{k} = (k_0, k_1, \dots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k}, N}$ such that

$$(21) \quad |D^{\mathbf{k}}u^j| \leq C_{\mathbf{k}, N}(1 + |x|)^{-N},$$

where $D^{\mathbf{k}}f(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \dots \frac{\partial^{k_d}}{\partial x_d^{k_d}} f(x, t)$.

4 Proof of Theorem 1

In this section we prove Theorem 1. For the sake of brevity we use the following notational conventions

$$(22) \quad \begin{aligned} \partial_{z_j} &= \frac{\partial}{\partial z_j}, \quad \partial_t = \frac{\partial}{\partial t}, \\ (\partial_{x_j} f)\left(\frac{x}{\varepsilon}\right) &= \partial_{z_j} f(z)\Big|_{z=x/\varepsilon}, \quad (\partial_t f)\left(\frac{t}{\varepsilon^\alpha}\right) = \partial_s f(s)\Big|_{s=t/\varepsilon^\alpha}. \end{aligned}$$

Denote

$$\begin{aligned}\widehat{a}^{0,ij}(z, s) &= a^{ij}(z, s) + a^{im}(z, s)\partial_{z_m}\chi^{0,j}(z, s) + \partial_{z_m}(a^{mi}(z, s)\chi^{0,j}(z, s)), \\ \widehat{a}^{k,ij}(z, s) &= a^{im}(z, s)\partial_{z_m}\chi^{k,j}(z, s) + \partial_{z_m}(a^{mi}(z, s)\chi^{k,j}(z, s)), \quad k = 1, 2, \dots, \\ \text{and from (11)}\end{aligned}$$

$$a^{k,\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} [\widehat{a}^k(z, s)] dz, \quad k = 1, 2, \dots$$

Substituting V^ε for u^ε in (1) yields

$$\begin{aligned}(23) \quad \partial_t V^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V^\varepsilon \right] &= -\varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta} \left[\partial_t u^k \right. \\ &+ \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)} (\partial_t \chi^j) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^k + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_t \nabla u^k \left. \right] \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta-1} \left[(\operatorname{div} a) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta} (\operatorname{div}(a \nabla \chi^j)) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \right] \nabla u^k \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \widehat{a}^{j,im} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2}{\partial x_i \partial x_m} u^k \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta+1} (a^{im} \chi^{j,l}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^3}{\partial x_i \partial x_m \partial x_l} u^k, \\ V^\varepsilon(x, 0) &= \sum_{j=0}^{J^0} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, 0 \right) \nabla u^0(x, 0)\end{aligned}$$

Due to (4) and (9),

$$\begin{aligned}& - \sum_{k=0}^{J^0} \varepsilon^{k\delta} \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)} (\partial_t \chi^j) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^k \\ & + \sum_{k=0}^{J^0} \varepsilon^{k\delta-1} \left[(\operatorname{div} a) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta} (\operatorname{div}(a \nabla \chi^j)) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \right] \nabla u^k \\ & = -\varepsilon^{(J^0+1)\delta-1} \sum_{k=0}^{J^0} (\partial_t \chi^{J^0-k}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^k.\end{aligned}$$

Considering our choice of J^0 we have: $(J^0 + 1)\delta - 1 > 1 + \alpha/2$. Therefore, with the help of (2) and (10) the first relation in (23) can be rearranged as

follows

$$\begin{aligned}
& \partial_t V^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V^\varepsilon \right] = \\
(24) \quad & -\varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta} \partial_t u^k + \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \widehat{a}^{j,im} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2 u^k}{\partial x_i \partial x_m} + \mathcal{R}^\varepsilon(x, t) \\
& = \varepsilon^{-\frac{\alpha}{2}} \sum_{j=0}^{J^0} \sum_{k=0}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{a}^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{j,\text{eff}} \right]^{im} \frac{\partial^2 u^k}{\partial x_i \partial x_m} + \mathcal{R}^\varepsilon(x, t),
\end{aligned}$$

where we identify $a^{0,\text{eff}}$ with a^{eff} , and \mathcal{R}^ε is the sum of all the terms on the right-hand side in (23) that are multiplied by a positive power of ε . One can easily check that

$$(25) \quad \mathcal{R}^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{N_0} \varepsilon^{1+j\delta} \theta^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \Phi^j(x, t),$$

where $\theta^j(z, s)$ are periodic in z , stationary in s and satisfy the estimates

$$(26) \quad \mathbf{E}(\|\theta^j\|_{C(\mathbb{T}^d \times [0, T])}^k) \leq C_k;$$

Φ^j are smooth functions such that

$$(27) \quad |D^{\mathbf{k}} \Phi^j| \leq C_{\mathbf{k}, N} (1 + |x|)^{-N},$$

and N_0 is a finite number; we do not specify these quantities explicitly because we do not need this. We represent V^ε as the sum $V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon$, where V_1^ε and V_2^ε solve the following problems:

$$(28) \quad \begin{cases} \partial_t V_1^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_1^\varepsilon \right] \\ = \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left[\widehat{a}^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{j,\text{eff}} \right]^{im} \frac{\partial^2}{\partial x_i \partial x_m} u^k, \\ V_1^\varepsilon(x, 0) = 0, \end{cases}$$

and

$$(29) \quad \begin{cases} \partial_t V_2^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_2^\varepsilon \right] = \mathcal{R}^\varepsilon(x, t), \\ V_2^\varepsilon(x, 0) = V^\varepsilon(x, 0). \end{cases}$$

Form (19) and (20) it follows that the initial condition in the latter problem satisfies for any $k > 0$ the estimate $\mathbf{E}\|V^\varepsilon(\cdot, 0)\|_{C^k(\mathbb{R}^d)}^k \leq C_k \varepsilon^{k\delta/2}$. If we multiply equation (29) by V_2^ε and integrate the resulting relation over $\mathbb{R}^d \times (0, T)$, then integrating by parts and combining estimates (25), (26) and the estimates for Φ^j , we obtain

$$(30) \quad \mathbf{E}\|V_2^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C\varepsilon^\delta.$$

Denote

$$\begin{aligned} \langle a \rangle^0(s) &= \int_{\mathbb{T}^d} (\widehat{a}^0(z, s) - a^{\text{eff}}) dz \\ \langle a \rangle^k(s) &= \int_{\mathbb{T}^d} (\widehat{a}^k(z, s) - a^{k, \text{eff}}) dz, \quad k = 1, 2, \dots \end{aligned}$$

It follows from the definition of \widehat{a}^k that for any $k \geq 0$ and $l > 0$ there is a constant $C_{l,k}$ such that $\mathbf{E}\|(\widehat{a}^k - \langle a \rangle^k)\|_{C^k(\mathbb{T}^d \times [0, T])}^N \leq C_{N,k}$. Since for each $s \in \mathbb{R}$ the mean value of $(\widehat{a}^k(\cdot, s) - \langle a \rangle^k(s))$ is equal to zero, the problem

$$\Delta_z \zeta^{k, im}(z, s) = (\widehat{a}^k(z, s) - \langle a \rangle^k(s))^{im}$$

has for each i and m a unique up to an additive constant periodic solution. Letting $\Theta^{k, im}(z, s) = \nabla \zeta^{k, im}(z, s)$, we obtain a stationary in s vector functions $\Theta^{k, im}$ such that

$$\text{div } \Theta^{k, im}(z, s) = (\widehat{a}^k(z, s) - \langle a \rangle^k(s))^{im}, \quad \mathbf{E}\|\Theta^{k, im}\|_{C^k(\mathbb{T}^d \times [0, T])}^N \leq C_{N,k}.$$

It is then straightforward to check that for the functions

$$\begin{aligned} F^\varepsilon(x, t) &= \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left[\widehat{a}^j\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) - \langle a \rangle^j\left(\frac{t}{\varepsilon^\alpha}\right) \right]^{im} \frac{\partial^2}{\partial x_i \partial x_m} u^k(x, t) \\ &= \varepsilon^{1-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left\{ \text{div} \left[\Theta^{j, im}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) \frac{\partial^2}{\partial x_i \partial x_m} u^k(x, t) \right] \right. \\ &\quad \left. - \Theta^{j, im}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) \nabla \left(\frac{\partial^2}{\partial x_i \partial x_m} u^k(x, t) \right) \right\} \end{aligned}$$

the following estimate is fulfilled:

$$(31) \quad \mathbf{E}\|F^\varepsilon\|_{L^2(0, T; H^{-1}(\mathbb{R}^d))}^2 \leq C\varepsilon^\delta.$$

Therefore, a solution to the problem

$$(32) \quad \begin{cases} \partial_t V_{1,2}^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_{1,2}^\varepsilon \right] = F^\varepsilon(x, t), \\ V_{1,2}^\varepsilon(x, 0) = 0. \end{cases}$$

admits the estimate

$$(33) \quad \mathbf{E} \|V_{1,2}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^d))}^2 \leq C\varepsilon^\delta.$$

Splitting $V_1^\varepsilon = V_{1,1}^\varepsilon + V_{1,2}^\varepsilon$, we conclude that $V_{1,1}^\varepsilon$ solves the following problem

$$(34) \quad \begin{cases} \partial_t V_{1,1}^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_{1,1}^\varepsilon \right] = \\ \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left[\langle a \rangle^j \left(\frac{t}{\varepsilon^\alpha} \right) - a^{j,\text{eff}} \right]^{im} \frac{\partial^2 u^k}{\partial x_i \partial x_m}, \\ V_{1,1}^\varepsilon(x, 0) = 0, \end{cases}$$

By construction the strong mixing coefficient of \widehat{a}^k remains unchanged and is equal to $\gamma(\cdot)$. Denote by $V_{1,1}^{0,\varepsilon}$ the solution of the following problem

$$(35) \quad \begin{cases} \partial_t V_{1,1}^{0,\varepsilon} - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_{1,1}^{0,\varepsilon} \right] = \varepsilon^{-\frac{\alpha}{2}} \left[\langle a \rangle^0 \left(\frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right]^{im} \frac{\partial^2 u^0}{\partial x_i \partial x_m} \\ V_{1,1}^{0,\varepsilon}(x, 0) = 0, \end{cases}$$

Lemma 4.1 *The solution of problem (35) converges in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ equipped with strong topology, to the solution of (12).*

Proof. We consider an auxiliary problem

$$(36) \quad \begin{cases} \partial_t V_{\text{aux}}^\varepsilon - \operatorname{div} \left[a^{\text{eff}} \nabla V_{\text{aux}}^\varepsilon \right] = \varepsilon^{-\alpha/2} \left[\langle a \rangle^0 \left(\frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right]^{ij} \frac{\partial^2 u^0}{\partial x^i \partial x^j} \\ V_{\text{aux}}^\varepsilon(x, 0) = 0, \end{cases}$$

and notice that this problem admits an explicit solution

$$V_{\text{aux}}^\varepsilon = \varepsilon^{\alpha/2} \zeta^{ij} \left(\frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2 u^0}{\partial x^i \partial x^j} \quad \text{with} \quad \zeta(s) = \int_0^s \left[\langle a \rangle^0(r) - a^{\text{eff}} \right] dr.$$

Due to [12, Lemma VIII.3.102], [12, Theorem VIII.3.97] and Assumption **c5**, the invariance principle holds for the process $\varepsilon^{\alpha/2} \zeta^{ij} \left(\frac{t}{\varepsilon^\alpha} \right)$, that is $\varepsilon^{\alpha/2} \zeta^{ij} \left(\frac{t}{\varepsilon^\alpha} \right)$,

converges in law, as $\varepsilon \rightarrow 0$, in $C([0, T])^{d^2}$ to a d^2 -dimensional Brownian motion with the covariance matrix Λ . Since u^0 satisfies estimates (8), the last convergence implies that $V_{\text{aux}}^\varepsilon$ converges in law in $C((0, T); L^2(\mathbb{R}^d))$ to the solution of problem (12).

Next, we represent $V_{1,1}^{0,\varepsilon}$ as $V_{1,1}^{0,\varepsilon}(x, t) = \mathcal{Z}^\varepsilon(x, t) + V_{\text{aux}}^\varepsilon(x, t)$. Then \mathcal{Z}^ε solves the problem

$$(37) \quad \begin{cases} \partial_t \mathcal{Z}^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \mathcal{Z}^\varepsilon \right] = \operatorname{div} \left(\left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right] \nabla V_{\text{aux}}^\varepsilon(x, t) \right) \\ \mathcal{Z}^\varepsilon(x, 0) = 0, \end{cases}$$

and our goal is to show that \mathcal{Z}^ε goes to zero in probability in $L^2((0, T) \times \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$. To this end we consider one more auxiliary problem that reads

$$(38) \quad \begin{cases} \partial_t \mathcal{Y}^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \mathcal{Y}^\varepsilon \right] = \operatorname{div} \left(\left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right] \Xi(x, t) \right) \\ \mathcal{Y}^\varepsilon(x, 0) = 0. \end{cases}$$

If the vector function $\Xi \in L^2((0, T) \times \mathbb{R}^d)$, then this problem has a unique solution, and, by the standard energy estimate,

$$\|\mathcal{Y}^\varepsilon\|_{L^2(0, T; H^1(\mathbb{R}^d))} + \|\partial_t \mathcal{Y}^\varepsilon\|_{L^2(0, T; H^{-1}(\mathbb{R}^d))} \leq C \|\Xi\|_{L^2((0, T) \times \mathbb{R}^d)}.$$

According to [20, Lemma 1.5.2] the family $\{\mathcal{Y}^\varepsilon\}$ is locally compact in $L^2((0, T) \times \mathbb{R}^d)$. Combining this with Aronson's estimate (see [2]) we conclude that the family $\{\mathcal{Y}^\varepsilon\}$ is compact in $L^2((0, T) \times \mathbb{R}^d)$.

Assume for a while that Ξ is smooth and satisfies estimates (8). Multiplying equation (38) by a test function of the form $\varphi(x, t) + \varepsilon \chi^0 \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \varphi(x, t)$ with $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^d)$ and integrating the resulting relation yields

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^d} \mathcal{Y}^\varepsilon \left(\partial_t \varphi + \varepsilon^{1-\alpha} (\partial_t \chi^0) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \varphi + \varepsilon \chi^0 \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_t \nabla \varphi(x, t) \right) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \partial_{x_m} \mathcal{Y}^\varepsilon a^{im} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \left[\partial_{x_i} \varphi + (\partial_{x_i} \chi^{0,j}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_j} \varphi + \varepsilon \chi^{0,j} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_i} \partial_{x_j} \varphi \right] dx dt \\ & = \int_0^T \int_{\mathbb{R}^d} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right]^{im} \Xi^m \left[\partial_{x_i} \varphi + (\partial_{x_i} \chi^{0,j}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_j} \varphi + \varepsilon \chi^{0,j} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right] dx dt \end{aligned}$$

Since $\int_{\mathbb{T}^d} \chi^0(z, s) dz = 0$, we have $\|(\partial_t \chi^0) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \varphi\|_{L^2(0, T; H^{-1}(\mathbb{R}^d))} \leq C\varepsilon$. Therefore, $\int_0^T \int_{\mathbb{R}^d} \mathcal{Y}^\varepsilon \varepsilon^{1-\alpha} (\partial_t \chi^0) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla \varphi dx dt$ tends to zero, as $\varepsilon \rightarrow 0$. Con-

sidering (4) and (6) we obtain

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \partial_{x_m} \mathcal{Y}^\varepsilon a^{im} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \left[\partial_{x_i} \varphi + (\partial_{x_i} \chi^{0,j}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_j} \varphi \right] dx dt \\ &= - \int_0^T \int_{\mathbb{R}^d} \mathcal{Y}^\varepsilon \left\{ a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) [\mathbf{I} + (\nabla \chi^0) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right)] \right\}^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} dx dt \end{aligned}$$

and

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right]^{im} \Xi^m \left[\partial_{x_i} \varphi + (\partial_{x_i} \chi^{0,j}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_j} \varphi + \varepsilon \chi^{0,j} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right] dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} \left\{ a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) [\mathbf{I} + (\nabla \chi^0) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right)] - a^{\text{eff}} \right\}^{im} \Xi^m \partial_{x_i} \varphi dx dt \\ &\quad - \int_0^T \int_{\mathbb{R}^d} \left\{ a^{\text{eff}} \right\}^{im} \Xi^m (\partial_{x_i} \chi^{0,j}) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \partial_{x_j} \varphi dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right]^{im} \Xi^m \varepsilon \chi^{0,j} \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} dx dt \rightarrow 0, \end{aligned}$$

as $\varepsilon \rightarrow 0$. Denoting by \mathcal{Y}^0 the limit of \mathcal{Y}^ε for a subsequence, we conclude that

$$\int_0^T \int_{\mathbb{R}^d} \mathcal{Y}^0 \left(-\partial_t \varphi - (a^{\text{eff}})^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right) dx dt = 0.$$

Therefore, $\mathcal{Y}^0 = 0$, and the whole family \mathcal{Y}^ε a.s. converges to 0 in $L^2((0, T) \times \mathbb{R}^d)$. By the density argument this convergence also holds for any $\Xi \in L^2((0, T) \times \mathbb{R}^d)$. Since $V_{\text{aux}}^\varepsilon$ converges in law in $C((0, T); L^2(\mathbb{R}^d))$, the solution of problem (37) converges to zero in probability in $L^2((0, T) \times \mathbb{R}^d)$, and the statement of the lemma follows. \square

From the last lemma it follows that the solution of problem (34) converges in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ equipped with strong topology, to the solution of (12). Combining this convergence with (30) and (33) we conclude that V^ε converges in law in the same space to the solution of (12). This completes the proof of Theorem 1.

5 Proof of Theorem 2

The beginning is the same as in Section 4. We consider the following expression:

$$V^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x, t) - \sum_{k=0}^{J^0} \varepsilon^{k\delta} \left(u^k(x, t) + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) \right) \right\},$$

where $\chi^j(z, y)$ and $u^k(x, t)$ are defined in (13) and (14), respectively. We substitute V^ε for u^ε in (1) using Itô's formula:

$$(39) \quad \begin{aligned} dV^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V^\varepsilon \right] dt &= -\varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta} \left[\partial_t u^k \right. \\ &+ \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)} (\mathcal{L}_y \chi^j) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \partial_t \nabla u^k \left. \right] dt \\ &+ \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(1-\alpha+(k+j)\delta)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) dB_t \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \varepsilon^{k\delta-1} \left[(\operatorname{div} a) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta} (\operatorname{div}(a \nabla \chi^j)) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \right] \nabla u^k \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \widehat{a}^{j,im} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \frac{\partial^2}{\partial x_i \partial x_m} u^k dt \\ &+ \varepsilon^{-\frac{\alpha}{2}} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta+1} (a^{im} \chi^{j,l}) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \frac{\partial^3}{\partial x_i \partial x_m \partial x_l} u^k dt. \end{aligned}$$

Here the $n \times n$ matrix $\sigma(y)$ is such that $\sigma(y)\sigma^*(y) = 2q(y)$, B_t is a standard n -dimensional Brownian motion. Due to (7) and (13)

$$\begin{aligned} &- \sum_{k=0}^{J^0} \varepsilon^{k\delta} \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha)} (\mathcal{L}_y \chi^j) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k \\ &+ \sum_{k=0}^{J^0} \varepsilon^{k\delta-1} \left[(\operatorname{div} a) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) + \sum_{j=0}^{J^0-k} \varepsilon^{j\delta} (\operatorname{div}(a \nabla \chi^j)) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \right] \nabla u^k \\ &= -\varepsilon^{(J^0+1)\delta-1} \sum_{k=0}^{J^0} (\mathcal{L}_y \chi^{J^0-k}) \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k. \end{aligned}$$

Considering equations (14) and the definitions of $a^{k,\text{eff}}$ and $\widehat{a}^k(z, y)$, we obtain an expression similar to that in (24)

$$\begin{aligned}
(40) \quad & dV^\varepsilon(x, t) - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V^\varepsilon \right] dt \\
&= \left(\varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=0}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{a}^k \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - a^{k,\text{eff}} \right]^{im} \frac{\partial^2 u^j}{\partial x_i \partial x_m} \right) dt \\
&+ \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(1-\alpha+(k+j)\delta)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) dB_t \\
&+ \mathcal{R}^\varepsilon(x, t) dt,
\end{aligned}$$

with $a^{0,\text{eff}} = a^{\text{eff}}$ and the initial condition

$$V^\varepsilon(x, 0) = \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha/2)} \chi^j \left(\frac{x}{\varepsilon}, \xi_0 \right) \nabla u^k(x, 0);$$

and

$$(41) \quad \mathcal{R}^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{N^0} \varepsilon^{1+j\delta} \vartheta^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \Phi^j(x, t)$$

with periodic in z smooth functions $\vartheta^j = \vartheta^j(z, y)$ of at most polynomial growth in y , and Φ^j satisfying (27).

We represent V^ε as the sum $V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon + V_3^\varepsilon$ where V_1^ε and V_2^ε solve problems equivalent to (28) and (29):

$$(42) \quad \begin{cases} \partial_t V_1^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_1^\varepsilon \right] \\ = \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left[\widehat{a}^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{j,\text{eff}} \right]^{im} \frac{\partial^2 u^k}{\partial x_i \partial x_m}, \\ V_1^\varepsilon(x, 0) = 0, \end{cases}$$

and

$$(43) \quad \begin{cases} \partial_t V_2^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_2^\varepsilon \right] = \mathcal{R}^\varepsilon(x, t), \\ V_2^\varepsilon(x, 0) = V^\varepsilon(x, 0). \end{cases}$$

The last term V_3^ε satisfies the SPDE:

$$(44) \quad \begin{aligned} & dV_3^\varepsilon(x, t) - \operatorname{div} \left[\mathbf{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_3^\varepsilon \right] dt \\ &= \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(1-\alpha+(k+j)\delta)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) dB_t \end{aligned}$$

with initial condition $V_3^\varepsilon(x, 0) = 0$.

We have

$$\begin{aligned} \mathbf{E} \|\mathcal{R}^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 &\leq C \varepsilon^{1-\alpha/2} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} (1 + |y|)^{N_1} (1 + |x|)^{-2n} \rho(y) dy dx dt \\ &\leq C \varepsilon^{1-\alpha/2}. \end{aligned}$$

Similarly, $\mathbf{E} \|V_2^\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^d)}^2 \leq C \varepsilon^{1-\alpha/2}$. Therefore, V_2^ε still satisfies (30) and thus does not contribute in the limit.

We turn to V_1^ε . The statement similar to that of Lemma 4.1 still holds. Indeed the equivalent of F^ε

$$H^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=0}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{\mathbf{a}}^k \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \langle \mathbf{a} \rangle^k \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \right]^{im} \frac{\partial^2 u^j}{\partial x_i \partial x_m}$$

admits the estimate (31):

$$(45) \quad \mathbf{E} \|H^\varepsilon\|_{L^2(0, T; H^{-1}(\mathbb{R}^d))}^2 \leq C \varepsilon^{2-\alpha}.$$

We split $V_1^\varepsilon = V_{1,1}^\varepsilon + V_{1,2}^\varepsilon$, where

- $V_{1,2}^\varepsilon$ solves (32) with H^ε on the right-hand side instead of F^ε , it admits estimate (33);
- $V_{1,1}^\varepsilon$ solves (34).

According to [28, Theorem 3] the processes

$$A^k(t) = \int_0^t (\langle \mathbf{a} \rangle^k(\xi_s) - a^{k, \text{eff}}) ds$$

satisfy the functional central limit theorem (invariance principle), that is the process

$$A^{\varepsilon, k}(t) = \varepsilon^{\frac{\alpha}{2}} \int_0^{\varepsilon^{-\alpha} t} (\langle \mathbf{a} \rangle^k(\xi_s) - a^{k, \text{eff}}) ds$$

converges in law in $C([0, T]; \mathbb{R}^{d^2})$ to a d^2 -dimensional Brownian motion with covariance matrix

$$(\Lambda_k) = \{(\Lambda_k)^{ijml}\} = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial y_{r_1}} (Q^k)^{ij}(y) \right] q^{r_1 r_2}(y) \left[\frac{\partial}{\partial y_{r_2}} (Q^k)^{ml}(y) \right] \rho(y) dy.$$

with matrix-function Q^0 defined in (15) and Q^k given by

$$(46) \quad \mathcal{L}Q^k(y) = \langle \mathbf{a} \rangle^k(y), \quad k = 1, \dots$$

By the same arguments as those in the proof of Theorem 1 (see also [16, Lemma 5.1]), we obtain the same conclusions as in Lemma 4.1.

To finish the proof of Theorem 2, we need to control V_3^ε , solution of problem (44). The following crucial statement will be proved in the next section, in this section it is taken for granted.

Proposition 5.1 *For a solution of problem (44) the following estimate holds:*

$$(47) \quad \mathbf{E} \|V_3^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha} |\log \varepsilon|.$$

Combining (30) and (47), together with Lemma 4.1, completes the proof of Theorem 2. Let us again emphasize that the diffusive case cannot be deduced from our first case because of the presence of V_3^ε .

6 Proof of Proposition 5.1

In order to prove Proposition 5.1 we consider first the following problem:

$$(48) \quad \begin{aligned} dV_B^\varepsilon(x, t) - \operatorname{div} \left[\mathbf{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_B^\varepsilon \right] dt \\ = \varepsilon^{(1-\alpha)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^0 \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^0(x, t) dB_t \end{aligned}$$

with initial condition $V_B^\varepsilon(x, 0) = 0$. In fact we keep only the smallest power of ε on the right-hand side of (44) (since $\delta = 2 - \alpha > 0$). Our goal is to prove the following estimate:

$$(49) \quad \mathbf{E} \|V_B^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha} |\log \varepsilon|.$$

This inequality (49) implies the desired statement of Proposition 5.1 on V_3^ε because the other terms in (44) have larger powers of ε and thus V_B^ε is the largest term in V_3^ε when ε goes to zero.

6.1 Construction of a mild solution

Our aim is to prove that the solution V_B^ε of (48) is given by:

$$(50) \quad V_B^\varepsilon(x, t) = \int_0^t \left[\int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, y, s) G\left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, y, s\right) dy \right] dB_s,$$

where Γ^ε is the fundamental solution of the following parabolic equation:

$$(51) \quad \frac{\partial u^\varepsilon}{\partial t}(x, t) = \operatorname{div} \left[a\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^\varepsilon \right]$$

and

$$(52) \quad G^\varepsilon(y, t) = G\left(\frac{y}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}, y, t\right) = \varepsilon^{1-\alpha} \sigma(\xi_{t/\varepsilon^\alpha}) \nabla \chi^0\left(\frac{y}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^0(y, t).$$

Note that the latter function is bounded by $\varepsilon^{1-\alpha} K_G$. The stochastic integral in (50) has to be defined properly since $\Gamma^\varepsilon(x, t, y, s)$ is measurable w.r.t. the σ -field $\mathcal{F}_{t/\varepsilon^\alpha}$ generated by the random variables B_u with $u \leq t/\varepsilon^\alpha$. The correct definition can be found in [23] and is based on Malliavin's calculus.

The stationary process ξ satisfies the following SDE:

$$(53) \quad d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dB_t.$$

It is well known that under the assumption **c3**, ξ satisfies for any $T \geq 0$ and any $p \geq 1$

$$\mathbf{E} \left(\sup_{t \in [0, T]} |\xi_t|^p \right) \leq C$$

where C is a positive constant depending on p , T , and the constants in condition **c3**. Moreover from [6, Proposition 2.6], under **(C)**, we have for any $\eta > 0$ and $p \geq 1$:

$$(54) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^\eta \mathbf{E} \left(\sup_{t \in [0, T]} \left| \xi_{\frac{t}{\varepsilon^\alpha}} \right|^p \right) = 0.$$

In what follows we borrow some notations from Nualart [22]. Recall that B is a d -dimensional Brownian motion. Let f be in $C_p^\infty(\mathbb{R}^{dn})$ (set of all infinitely continuously differentiable functions such that the function and all of its partial derivatives have polynomial growth) with

$$f(x) = f(x_1^1, \dots, x_1^d; \dots; x_n^1, \dots, x_n^d).$$

We define a smooth random variable F by:

$$F = f(B(t_1), \dots, B(t_n))$$

for $0 \leq t_1 < t_2 < \dots < t_n \leq T$. The class of smooth random variables is denoted by \mathcal{S} . Then the Malliavin derivative $D_t F$ is given by

$$D_t^j(F) = \sum_{i=1}^d \frac{\partial f}{\partial x_i^j}(B(t_1), \dots, B(t_n)) \mathbf{1}_{[0, t_i]}(t)$$

(see Definition 1.2.1 in [22]). $D_t(F)$ is the d -dimensional vector $D_t(F) = (D_t^j(F), j = 1, \dots, d)$. Moreover, this derivative $D_t(F)$ is a random variable with values in the Hilbert space $L^2([0, T]; \mathbb{R}^d)$. The space $\mathbb{D}^{1,p}$, $p \geq 1$, is the closure of the class of smooth random variables with respect to the norm

$$\|F\|_{1,p} = \left[\mathbf{E}(|F|^p) + \mathbf{E} \left(\|DF\|_{L^2([0,T]; \mathbb{R}^d)}^p \right) \right]^{1/p}.$$

For $p = 2$, $\mathbb{D}^{1,2}$ is a Hilbert space. Then by induction we can define $\mathbb{D}^{k,p}$ the space of k -times differentiable random variables where the k derivatives are in $L^p(\Omega)$. Finally

$$\mathbb{D}^{k,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{k,p}, \quad \mathbb{D}^\infty = \bigcap_{k \in \mathbb{N}} \mathbb{D}^{k,\infty}.$$

The next result can be found in [22], Theorems 2.2.1 and 2.2.2.

Lemma 6.1 *Under **c3**, the coordinate ξ_t^i belongs to $\mathbb{D}^{1,\infty}$ for any $t \in [0, T]$ and $i = 1, \dots, d$. Moreover for any $j = 1, \dots, d$ and any $p \geq 1$*

$$(55) \quad \sup_{0 \leq r \leq T} \mathbf{E} \left(\sup_{r \leq t \leq T} |D_r^j \xi_t^i|^p \right) < +\infty.$$

The derivative $D_r^j \xi_t^i$ satisfies the following linear equation:

$$D_r^j \xi_t^i = \sigma_{i,j}(\xi_r) + \sum_{1 \leq k, l \leq d} \int_r^t \tilde{\sigma}_{i,k}^l(s) D_r^j(\xi_s^k) dB_s^l + \sum_{k=1}^d \int_r^t \tilde{b}_{i,k}(s) D_r^j(\xi_s^k) ds$$

for $r \leq t$ a.e. and $D_r^j \xi_t^i = 0$ for $r > t$ a.e., where σ^j is the column number j of the matrix σ and where for $1 \leq i, j \leq d$ and $1 \leq l \leq d$, $\tilde{b}_{i,j}(s)$ and $\tilde{\sigma}_{i,j}^l(s)$ are given by (17):

$$\tilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \quad \tilde{\sigma}_{i,j}^l(s) = (\partial_{x_j} \sigma_{i,l})(\xi_s).$$

The Malliavin derivative of G^ε defined by (52) can be computed by a chain rule argument:

$$D_r G^\varepsilon(y, s) = \varepsilon^{1-\alpha} \nabla u^0(y, s) D_r \xi_{s/\varepsilon^\alpha} \left[\tilde{\sigma}(s) \nabla \chi^0 \left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}} \right) + \sigma(\xi_{s/\varepsilon^\alpha}) \nabla_z \nabla \chi^0 \left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}} \right) \right].$$

Hence

$$\|D_r G^\varepsilon(y, s)\| \leq \varepsilon^{1-\alpha} K_G \|D_r \xi_{s/\varepsilon^\alpha}\| \leq \varepsilon^{1-\alpha} K_G \psi^\varepsilon(r)$$

where

$$(56) \quad \psi^\varepsilon(r) = \sup_{t \in [0, T]} \|D_r \xi_{t/\varepsilon^\alpha}\| = \sup_{\tau \in [r, T/\varepsilon^\alpha]} \|D_r \xi_\tau\|.$$

Equation (50) is well defined if we can control the Malliavin derivative of the fundamental solution Γ^ε . In the rest of this paper, for two positive constants c and C , the function $g_{c,C}(x, t)$ is defined as follows:

$$g_{c,C}(x, t) = \frac{c}{t^{\frac{d}{2}}} \exp\left(-\frac{C|x|^2}{t}\right), \quad t > 0, \quad x \in \mathbb{R}^d.$$

It is well known (see among other [7], Chapter 9, [2] or [29]) that there exist two constants ς and ϖ depending only on the constant λ in Assumption **a1** and the dimension d , such that

$$(57) \quad \Gamma^\varepsilon(x, t, y, s) \leq g_{\varsigma, \varpi}(x - y, t - s).$$

This inequality is called the **Aronson estimate** (recall that Γ^ε is non negative).

For the next result let us assume that the matrix a satisfies **a1** and the following regularity conditions:

(R) For any $1 \leq j, k \leq d$

$$|\nabla_z a(z, y)| + |\nabla_y a(z, y)| + \left| \frac{\partial^2}{\partial z_j \partial y_k} a(z, y) \right| \leq K_a$$

Note that **(R)** is weaker than **c1**.

Proposition 6.1 *Under **a1** and **(R)**, the fundamental solution Γ^ε of (51) and its spatial derivatives belong to $\mathbb{D}^{1,\infty}$ for every $(t, s) \in [0, T]^2$, $s < t$ and $(x, y) \in (\mathbb{R}^n)^2$. Moreover it satisfies the following inequalities: there exist two constants ϱ and ϖ that only depend on the uniform ellipticity constant λ and on K_a , such that*

$$(58) \quad |\nabla_x \Gamma^\varepsilon(x, t, y, s)| \leq \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s),$$

$$(59) \quad |D_r \Gamma^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) g_{\varrho, \varpi}(x-y, t-s),$$

and

$$(60) \quad |D_r \nabla_x \Gamma^\varepsilon(x, t, y, s)| \leq \frac{\psi^\varepsilon(r)}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s).$$

The constants ϱ and ϖ do not depend on ε . The quantity ψ^ε is defined by (56).

Just remark that Estimate (58) holds under the weaker assumption than **(R)**, namely it is sufficient to assume that $|\nabla_z a(z, y)| \leq K_a$. The other derivatives of a in **(R)** are used to control the Malliavin derivatives. The proof of this result is quite involved and based on the construction of Γ^ε by the parametrix method. For the reader's convenience we postpone it to the Appendix.

As a consequence of [1, Theorem 5.10] and [1, Theorem 5.11] one can easily deduce that the right-hand side of Equation (50) is well defined and is the unique classical solution of (48).

6.2 Intermediate result

We begin this section by proving the following result.

Lemma 6.2 *The following estimate holds: for any $p > 1$ and any η such that $4 - 2\alpha - \eta > 0$, there exists a constant C , independent of ε , such that*

$$(61) \quad \mathbf{E} \|V_B^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha-\eta} |\log \varepsilon| \left[1 + \left[\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right].$$

Proof. We know that

$$\begin{aligned}
V_B^\varepsilon(x, t) &= \int_0^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\
&= \int_{t-\varepsilon^2}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\
&\quad + \int_0^{t-\varepsilon^2} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\
(62) \qquad &= J_1^\varepsilon(x, t) + J_2^\varepsilon(x, t).
\end{aligned}$$

Denote $t_\varepsilon = t - \varepsilon^2$. We first rearrange the term J_1^ε :

$$\begin{aligned}
J_1^\varepsilon(x, t) &= \int_{t_\varepsilon}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\
&= \varepsilon^{1-\alpha} \int_{t_\varepsilon}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\
&= \varepsilon^{1-\alpha} \int_{t_\varepsilon}^t j^\varepsilon(x, t, s) dB_s.
\end{aligned}$$

Using isometric property of the anticipating Itô integral (see Eq. (3.5) in [23]) we get

$$\begin{aligned}
(63) \qquad \mathbf{E}((J_1^\varepsilon(x, t))^2) &= \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t |j^\varepsilon(x, t, s)|^2 ds \\
&\quad + \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t |D_r j^\varepsilon(x, t, s)|^2 ds dr.
\end{aligned}$$

By the Aronson estimate (57), $\Gamma^\varepsilon(x, t, x', s) \leq g_{\varsigma, \varpi}(x - x', t - s)$. Moreover for any $\mathbf{k} = (k_0, k_1, \dots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k}, N}$ such that

$$(64) \qquad |\partial^{\mathbf{k}} u^0| \leq C_{\mathbf{k}, N} (1 + |x|)^{-N},$$

where $\partial^{\mathbf{k}} u^0(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \dots \frac{\partial^{k_d}}{\partial x_d^{k_d}} u^0(x, t)$. The matrix σ is bounded and χ^0 is at most of polynomial growth w.r.t. y . This yields

$$\begin{aligned}
|j^\varepsilon(x, t, s)| &= \left| \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' \right| \\
&\leq C_N \int_{\mathbb{R}^d} g_{\varsigma, \varpi}(x - x', t - s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\
&\leq C_N (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}
\end{aligned}$$

for all t, s such that $0 \leq s < t \leq T$ and for all $N > 0$. Therefore from (54)

$$\begin{aligned} \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t |j^\varepsilon(x, t, s)|^2 ds &\leq C_N \varepsilon^{2-2\alpha-\eta} \int_{t_\varepsilon}^t (1 + |x|)^{-2N} \varepsilon^\eta \mathbf{E} (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} ds \\ &\leq C_N \varepsilon^{4-2\alpha-\eta} (1 + |x|)^{-2N}. \end{aligned}$$

The Malliavin derivative of j^ε is obtained by a chain rule argument, and the estimates (57) and (59) lead to

$$\begin{aligned} |D_r j^\varepsilon(x, t, s)| &\leq \int_{\mathbf{R}^d} \left| D_r \Gamma^\varepsilon(x, t, x', s) \sigma(\xi_{\frac{s}{\varepsilon^\alpha}}) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) \right| dx' \\ &\quad + \int_{\mathbf{R}^d} \Gamma^\varepsilon(x, t, x', s) \left| D_r \left[\sigma(\xi_{\frac{s}{\varepsilon^\alpha}}) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \right] \right| |\nabla u^0(x', s)| dx' \\ &\leq \psi^\varepsilon(r) \int_{\mathbf{R}^d} g_{\varsigma, \varpi}(x - x', t - s) \theta\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) |\nabla u^0(x', s)| dx' \end{aligned}$$

with

$$\theta(z, y) = \left| \sigma(y) \nabla_y \chi^0(z, y) \right| + \left| \nabla_y \sigma(y) \nabla_y \chi^0(z, y) + \sigma(y) \partial_y^2 \chi^0(z, y) \right|.$$

Hence

$$\begin{aligned} |D_r j^\varepsilon(x, t, s)| &\leq C \psi^\varepsilon(r) \int_{\mathbf{R}^d} g_{\varsigma, \varpi}(x - x', t - s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\ &\leq C \psi^\varepsilon(r) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}. \end{aligned}$$

Thereby for any $p > 1$, denoting by q the Hölder conjugate of p and using Hölder's inequality for the expectation and Jensen's inequality for the time integrals, we get

$$\begin{aligned} &\mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t |D_r j^\varepsilon(x, t, s)|^2 dr ds \\ &\leq C^2 \mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^2 (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} (1 + |x|)^{-2N} ds dr \\ &\leq C^2 \left[\mathbf{E} \left(\int_{t_\varepsilon}^t (\psi^\varepsilon(r))^2 dr \right)^p \right]^{1/p} \left[\mathbf{E} \left(\int_{t_\varepsilon}^t (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} ds \right)^q \right]^{1/q} (1 + |x|)^{-2N} \\ &\leq \frac{C^2 (t - t_\varepsilon)}{(1 + |x|)^{2N}} \left[\mathbf{E} \left(\int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right) \right]^{1/p} \left[\mathbf{E} \left(\int_{t_\varepsilon}^t (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2Kq} ds \right) \right]^{1/q} \\ &\leq C^2 \varepsilon^{2+2/q-\eta} \left[\mathbf{E} \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} (1 + |x|)^{-2N}; \end{aligned}$$

the last inequality here is a consequence of (54). Coming back to (63) we obtain

$$\mathbf{E}((J_1^\varepsilon(x, t))^2) \leq C\varepsilon^{4-2\alpha-\eta} \left[1 + \varepsilon^{2/q} \left[\mathbf{E} \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right] (1 + |x|)^{-2N}$$

and finally

$$(65) \quad \mathbf{E} \left(\|J_1^\varepsilon\|_{L^2(\mathbb{R}^d \times [0, T])}^2 \right) \leq C\varepsilon^{4-2\alpha-\eta} \left[1 + \left[\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right].$$

Recalling (52), since the mean value of $\nabla_y \chi^0(z, y)$ in z is equal to zero, there exists a periodic in z function $X^0 = X^0(z, y)$ such that $\operatorname{div}_z X^0(z, y) = \sigma(y) \nabla_y \chi^0(z, y)$. Moreover, X^0 is smooth and has at most polynomial growth in y .

For estimating J_2^ε , we use an integration by parts formula:

$$\begin{aligned} J_2^\varepsilon(x, t) &= \varepsilon^{1-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\ &= \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \frac{\partial^2}{\partial x^2} u^0(x', s) dx' dB_s \\ &+ \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \nabla_y \Gamma^\varepsilon(x, t, x', s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\ &= \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} w_1^\varepsilon(x, t, s) dB_s + \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} w_2^\varepsilon(x, t, s) dB_s \\ (66) \quad &= I_1^\varepsilon(x, t) + I_2^\varepsilon(x, t). \end{aligned}$$

Using again isometric property of the anticipating Itô integral we have for $j = 1$ or 2

$$\mathbf{E}((I_j^\varepsilon)^2) = \varepsilon^{4-2\alpha} \mathbf{E} \int_0^{t_\varepsilon} |w_j^\varepsilon(x, t, s)|^2 ds + \varepsilon^{4-2\alpha} \mathbf{E} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} |D_r w_j^\varepsilon(x, t, s)|^2 ds dr.$$

Using (57), (59) and (64), by the same arguments as in the proof of (65) we have

$$(67) \quad \mathbf{E} \|I_1^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T])}^2 \leq C\varepsilon^{4-2\alpha-\eta} \left[1 + \left(\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right)^{1/p} \right].$$

In order to obtain an upper bound for I_2^ε we use Aronson's estimate for the derivative of Γ^ε :

$$\begin{aligned}
|w_2^\varepsilon(x, t, s)| &= \left| \int_{\mathbb{R}^d} \nabla_{x'} \Gamma^\varepsilon(x, x', t, s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' \right| \\
&\leq \int_{\mathbb{R}^d} \frac{1}{|t-s|^{1/2}} g_{\varsigma, \varpi}(x-x', t-s) \left| X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) \right| dx' \\
&\leq \frac{C}{|t-s|^{1/2}} \int_{\mathbb{R}^d} g_{\varsigma, \varpi}(x-x', t-s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\
&\leq \frac{C}{|t-s|^{1/2}} (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}.
\end{aligned}$$

Hence

$$\mathbf{E} \int_0^{t_\varepsilon} |w_2^\varepsilon(x, t, s)|^2 ds \leq C |\log(\varepsilon)| \varepsilon^{-\eta} (1 + |x|)^{-2N}.$$

For the Malliavin derivative we proceed as before with (60):

$$|D_r w_2^\varepsilon(x, t, s)| \leq \psi^\varepsilon(r) \int_{\mathbb{R}^d} \frac{1}{|t-s|^{1/2}} g_{\varsigma, \varpi}(x-x', t-s) \theta\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) |\nabla u^0(x', s)| dx'$$

with

$$\theta(z, y) = \left| X^0(z, y) \right| + \left| \nabla_y X^0(z, y) \right|.$$

Again

$$\begin{aligned}
&\mathbf{E} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} |D_r w_2^\varepsilon(x, t, s)|^2 dr ds \\
&\leq C (1 + |x|)^{-2N} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} \frac{1}{|t-s|} \mathbf{E} \left[(\psi^\varepsilon(r))^2 (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} \right] ds dr \\
&\leq C (1 + |x|)^{-2N} \varepsilon^{-\eta} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} \frac{1}{|t-s|} \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} ds dr \\
&\leq C |\log(\varepsilon)| \varepsilon^{-\eta} (1 + |x|)^{-2N} \int_0^T \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} dr.
\end{aligned}$$

From the last two inequalities we deduce that

$$\mathbf{E} \|I_2^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha-\eta} |\log(\varepsilon)| \left[1 + \int_0^T \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} dr \right].$$

The last estimate combined with (65) and (67) yields the desired inequality (61). This completes the proof. \square

6.3 Uniform estimates of the Malliavin derivative

Now to obtain the estimate (49), from Lemma 6.2, we only need to control

$$\psi^\varepsilon(r) = \sup_{t \in [0, T]} \|D_r \xi_{t/\varepsilon^\alpha}\| = \sup_{\tau \in [r, T/\varepsilon^\alpha]} \|D_r \xi_\tau\|$$

and show that under our standing conditions it admits uniform in ε estimates. Let us also emphasize that Condition **(S)** has not been used until now.

Lemma 6.3 *Under Conditions **c2**, **c3** and **(S)**, there exists a constant C_p such that for any T and $\varepsilon > 0$*

$$\mathbf{E}(|\psi^\varepsilon(r)|^p) \leq C_p.$$

Proof. Recall that $Z(t) = D_r \xi_t$ is the matrix-valued process defined by:

$$Z(t) = \sigma(\xi_r) + \int_r^t \tilde{b}(s)Z(s)ds + \sum_{1 \leq l \leq d} \int_r^t \tilde{\sigma}^l(s)Z(s)dB_s^l$$

and σ is bounded as specified in condition **c2**. Each column Z^j of Z satisfies the linear d -dimensional SDE

$$Z_t^j = \sigma_j(\xi_r) + \int_r^t \tilde{b}(s)Z^j(s)ds + \sum_{1 \leq l \leq d} \int_r^t \tilde{\sigma}^l(s)Z^j(s)dB_s^l$$

where σ_j is the j -th column of σ . We apply the results contained in Appendix B (see also Section 6.7) of [14], more precisely Equation (B.11). The process $X_t = |Z_t^j|$ satisfies the scalar linear equation:

$$\begin{aligned} dX_t &= \left(\mathcal{Q}(t, \Theta_t) + \frac{1}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t) \Theta_t, \Theta_t)^2 \right) X_t dt \\ &+ X_t \sum_{l=1}^d (\tilde{\sigma}^l(t) \Theta_t, \Theta_t) dB_t^l \\ &= X_t (\mathbf{b}(t) dt + \mathbf{s}(t) dB_t) \end{aligned}$$

where \mathcal{Q} is defined by (18) and $\Theta_t = Z_t/|Z_t|$ belongs to \mathbb{S}^{d-1} , \mathbf{b} is the one dimensional process

$$\mathbf{b}(t) = \mathcal{Q}(t, \Theta_t) + \frac{1}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t) \Theta_t, \Theta_t)^2$$

and \mathfrak{s} is the d -dimensional process

$$\mathfrak{s}(t) = ((\tilde{\sigma}^l(t)\Theta_t, \Theta_t), 1 \leq l \leq d).$$

From condition **c3**, the process \mathfrak{s} is bounded: there exists a constant C_1 such that $\|\mathfrak{s}(u)\|^2 \leq (C_1)^2$. Hence

$$\begin{aligned} (X_t)^\mathfrak{p} &= (X_r)^\mathfrak{p} + \int_r^t \mathfrak{p}(X_u)^\mathfrak{p} (\mathfrak{b}(u) du + \mathfrak{s}(u) dB_u) \\ &\quad + \frac{\mathfrak{p}(\mathfrak{p} - 1)}{2} \int_r^t (X_u)^\mathfrak{p} \|\mathfrak{s}(u)\|^2 du \\ &= (X_r)^\mathfrak{p} + \int_r^t \mathfrak{p}(X_u)^\mathfrak{p} \left(\mathfrak{b}(u) + \frac{\mathfrak{p} - 1}{2} \|\mathfrak{s}(u)\|^2 \right) du + M_t \end{aligned}$$

where M is the martingale:

$$M_t = \frac{\mathfrak{p}(\mathfrak{p} - 1)}{2} \int_r^t (X_u)^\mathfrak{p} \mathfrak{s}(u) dB_u.$$

Condition **(S)** leads to

$$(68) \quad (X_t)^\mathfrak{p} + c\mathfrak{p} \int_r^t (X_u)^\mathfrak{p} du \leq (X_r)^\mathfrak{p} + M_t.$$

In particular from condition **c2** for any $T \geq r$

$$\sup_{t \in [0, T]} \mathbf{E} [(X_t)^\mathfrak{p}] + c\mathfrak{p} \mathbf{E} \int_r^T (X_u)^\mathfrak{p} du \leq \mathbf{E} (X_r)^\mathfrak{p} \leq \frac{1}{\lambda^\mathfrak{p}}.$$

Now coming back to (68) we have

$$(69) \quad \sup_{t \in [r, T]} (X_t)^\mathfrak{p} \leq (X_r)^\mathfrak{p} + \sup_{t \in [r, T]} M_t.$$

From Davis inequality (see [21], Chapter 1, Theorem 6), together with **c2**, we obtain

$$\begin{aligned} \mathbf{E} \left[\sup_{t \in [r, T]} (X_t)^\mathfrak{p} \right] &\leq \frac{1}{\lambda^\mathfrak{p}} + \mathbf{E} \left[\sup_{t \in [r, T]} |M_t| \right] \\ &\leq \frac{1}{\lambda^\mathfrak{p}} + \kappa_\mathfrak{p} \mathbf{E} \left[\left(\int_r^T (X_u)^{2\mathfrak{p}} \|\mathfrak{s}(u)\|^2 du \right)^{1/2} \right]. \end{aligned}$$

with $\kappa_{\mathbf{p}} = 10(\mathbf{p}(\mathbf{p} - 1))^2$. By Young's inequality

$$\begin{aligned} & \mathbf{E} \left[\left(\int_r^T (X_u)^{2\mathbf{p}} \|\mathfrak{s}(u)\|^2 du \right)^{1/2} \right] \\ & \leq \mathbf{E} \left[\left(\sup_{t \in [r, T]} (X_t)^{\mathbf{p}} \right)^{1/2} \left(\int_r^T (X_u)^{\mathbf{p}} \|\mathfrak{s}(u)\|^2 du \right)^{1/2} \right] \\ & \leq \frac{1}{2\kappa_{\mathbf{p}}} \mathbf{E} \left[\sup_{t \in [r, T]} (X_t)^{\mathbf{p}} \right] + \frac{\kappa_{\mathbf{p}}}{2} \mathbf{E} \left[\left(\int_r^T (X_u)^{\mathbf{p}} \|\mathfrak{s}(u)\|^2 du \right) \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2} \mathbf{E} \left[\sup_{t \in [r, T]} (X_t)^{\mathbf{p}} \right] & \leq \frac{1}{\lambda^{\mathbf{p}}} + \frac{(\kappa_{\mathbf{p}})^2}{2} \mathbf{E} \left[\int_r^T (X_u)^{\mathbf{p}} \|\mathfrak{s}(u)\|^2 du \right] \\ & \leq \frac{1}{\lambda^{\mathbf{p}}} + \frac{(\kappa_{\mathbf{p}})^2 (C_1)^2}{2} \mathbf{E} \left[\int_r^T (X_u)^{\mathbf{p}} du \right] \\ & \leq \frac{1}{\lambda^{\mathbf{p}}} \left[1 + \frac{(\kappa_{\mathbf{p}})^2 (C_1)^2}{2\mathbf{p}\mathbf{c}} \right]. \end{aligned}$$

This achieves the proof. \square

Note that in dimension one ($d = 1$), Condition **(S)** follows from the inequality

$$(70) \quad \partial_x b(\xi_t) + \frac{\mathbf{p} - 1}{2} (\partial_x \sigma(\xi_t))^2 \leq -\mathbf{c}.$$

Indeed, since $\zeta_t = D_r \xi_t$ satisfies the linear one-dimensional SDE:

$$\zeta_t = \sigma(\xi_r) + \int_r^t \partial_x \sigma(\xi_u) \zeta_u dB_u + \int_r^t \partial_x b(\xi_u) \zeta_u du,$$

an explicit formula for $|\zeta_t|^{\mathbf{p}}$ reads

$$\begin{aligned} |\zeta_t|^{\mathbf{p}} & = |\sigma(\xi_r)|^{\mathbf{p}} \exp \left[\int_r^t \mathbf{p} \partial_x \sigma(\xi_u) dB_u - \frac{1}{2} \int_r^t \mathbf{p}^2 (\partial_x \sigma(\xi_u))^2 du \right] \\ & \quad \exp \left[\mathbf{p} \int_r^t \left(\partial_x b(\xi_u) + \frac{\mathbf{p} - 1}{2} (\partial_x \sigma(\xi_u))^2 \right) du \right]. \end{aligned}$$

Under (70) we have

$$|\zeta_t|^p \leq \frac{1}{\lambda^p} e^{-\mathfrak{c}p(t-r)} \exp \left[\int_r^t \mathfrak{p} \partial_x \sigma(\xi_u) dB_u - \frac{1}{2} \int_r^t \mathfrak{p}^2 (\partial_x \sigma(\xi_u))^2 du \right]$$

and taking the expectation, we get

$$\sup_{t \geq r} \mathbf{E} |\zeta_t|^p \leq \frac{1}{\lambda^p}.$$

This last inequality holds even if \mathfrak{c} is equal to zero. However, if $\mathfrak{c} > 0$, we can change the order of the expectation and the supremum.

Appendix: construction and properties of the fundamental solution

Here we prove Proposition 6.1. In other words we want to prove that the fundamental solution Γ^ε of a parabolic equation

$$\frac{\partial u}{\partial t}(x, t) = \operatorname{div} \left[\mathfrak{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u \right]$$

is in $\mathbb{D}^{1,2}$ (Malliavin differentiability) and that the Aronson estimates for the Malliavin derivative (Inequalities (59) and (60)) hold.

One construction of the fundamental solution Γ^ε can be found in [19], Chapter IV, sections 11 to 13 or [7], Chapter I. It is based on the property that for some $\hbar \in (0, 1)$, there exists a non negative random variable $\kappa(\omega)$ such that a.s.

$$\left| \mathfrak{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \mathfrak{a} \left(\frac{x'}{\varepsilon}, \xi_{\frac{t'}{\varepsilon^\alpha}} \right) \right| \leq \frac{K_a}{\varepsilon} |x - x'| + \kappa(\omega) \left| \frac{t - t'}{\varepsilon^\alpha} \right|^{\hbar/2}$$

(see for example [5] for details). Hence the existence of Γ^ε is guaranteed. But the constants in (57) or (58) may depend on ε if we follow this construction.

For the Malliavin differentiability property of Γ^ε , we use the approach developed in Alòs et al. [1]. For a fixed ε the solution V_B^ε defined by (48) is well defined and satisfies all required properties. But $\varepsilon > 0$ is a parameter of the equation and thus it may appear in (59) and (60) on Γ^ε , and therefore on V_B^ε or v^ε . Indeed, following the proof in [1] we might have extra negative powers of ε in the estimates of the Malliavin derivative of Γ^ε . In other words for the initial homogenization problem, we need more accurate estimates on Γ^ε as in Proposition 6.1.

Preliminary remarks

Let T be a fixed positive constant. The time variable belongs to the interval $[0, T]$. Recall that we have defined ψ^ε by (56):

$$\psi^\varepsilon(r) = \sup_{t \in [0, T]} \|D_r \xi_{t/\varepsilon^\alpha}\| = \sup_{\tau \in [r, T/\varepsilon^\alpha]} \|D_r \xi_\tau\|.$$

We denote by ξ^ε the process $\xi_t^\varepsilon = \xi_{\frac{t}{\varepsilon^{\alpha-2}}}$, and for any $r \in [0, T/\varepsilon^2]$ we have

$$\sup_{t \in [0, T/\varepsilon^2]} \|D_r \xi_t^\varepsilon\| = \sup_{\tau \in [0, T/\varepsilon^\alpha]} \|D_r \xi_\tau\| = \psi^\varepsilon(r).$$

Now if $\tilde{\Gamma}^\varepsilon$ is the fundamental solution for:

$$(71) \quad \frac{\partial u}{\partial t}(x, t) = \operatorname{div} \left[a \left(x, \xi_t^\varepsilon \right) \nabla u(x, t) \right],$$

then:

$$(72) \quad \Gamma^\varepsilon(x, t, y, s) = \frac{1}{\varepsilon^d} \tilde{\Gamma}^\varepsilon \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \frac{y}{\varepsilon}, \frac{s}{\varepsilon^2} \right).$$

Note that if Γ^ε is defined for $0 \leq s \leq t \leq T$, then $\tilde{\Gamma}^\varepsilon$ should be defined on $0 \leq s \leq t \leq T/\varepsilon^2$. Remark that if $\tilde{\Gamma}^\varepsilon$ satisfies Estimates (57)–(60), with some constants ς , ϖ and ϱ , uniformly w.r.t to ε , then Γ^ε verifies the same inequalities with the same constants.

From now on and in the rest of the appendix, we denote by $a^\varepsilon(x, t)$ the matrix:

$$a^\varepsilon(x, t) = a \left(x, \xi_t^\varepsilon \right) = a \left(x, \xi_{\frac{t}{\varepsilon^{\alpha-2}}} \right)$$

and we consider only $\tilde{\Gamma}^\varepsilon$ the fundamental solution of (71). Moreover the uniform ellipticity condition **a1** and the regularity condition **(R)** hold.

Let us emphasize again that the pathwise existence of $\tilde{\Gamma}^\varepsilon$ is justified in [19], Chapter IV. Hence we concentrate ourselves mainly on the uniform estimates. Note that the Aronson inequality (57) holds for $\tilde{\Gamma}^\varepsilon$ with constants independent of ε (see [2] and [29]). Our goal is to derive estimates (58), (59) and (60) for $\tilde{\Gamma}^\varepsilon$, that is to show that for any $(x, y) \in (\mathbb{R}^d)^2$ and any $0 \leq s < t$ we have

$$|\nabla_x \tilde{\Gamma}^\varepsilon(x, t, y, s)| \leq \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s),$$

$$|D_r \tilde{\Gamma}^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) g_{\varrho, \varpi}(x - y, t - s),$$

and

$$|D_r \nabla_x \tilde{\Gamma}^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x - y, t - s).$$

Let us emphasize that these estimates hold for any $s < t$ in $(0, +\infty)$. In order to obtain this result, we use and adapt the construction of the fundamental solution developed in [7], Chapter 9 (see also [30] for the parametrix). The scheme is the following: consider first the case where a^ε just depends on the time variable and derive the desired estimates; then extend the result to the general case by the parametrix method. Here only the space variable is frozen in the first stage. Thus we avoid the problem due to the lack of regularity w.r.t. t .

Parametrix method and the estimate on the gradient

First assume that a^ε just depends on t , that is $a = a(y)$. In this case the fundamental solution $\tilde{\Gamma}^\varepsilon$ is denoted by Z^ε and is given by the formula: for any $s < t$ and $(x, y) \in (\mathbb{R}^d)^2$

$$(73) \quad Z^\varepsilon(x - y, t, s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, s, \zeta) d\zeta,$$

where V^ε is the following function:

$$V^\varepsilon(t, s, \zeta) = \exp\left(-\left\langle \int_s^t a^\varepsilon(u) du, \zeta \right\rangle\right).$$

From Condition **a1**, a^ε verifies a.s. the estimates

$$\lambda^{-1}(t-s)|\zeta|^2 \leq \left\langle \int_s^t a^\varepsilon(u) du, \zeta \right\rangle \leq \lambda(t-s)|\zeta|^2.$$

From the above expression for Z^ε we deduce that for any $k \geq 1$ and $1 \leq j_\ell \leq d$ with $1 \leq \ell \leq k$

$$\partial_{x_{j_1} \dots x_{j_k}}^k Z^\varepsilon(x - y, t, s) = \frac{(i)^k}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, s, \zeta) (\zeta_{j_1} \dots \zeta_{j_k}) d\zeta.$$

As in [7], Chapter 9, Theorem 1, we obtain that:

$$(74) \quad |\partial_{x_{j_1} \dots x_{j_k}}^k Z^\varepsilon(x - y, t, s)| \leq \frac{1}{(t-s)^{k/2}} g_{\zeta, \varpi}(x - y, t - s).$$

In particular the Aronson estimates (57) and (58) can be derived.

Now we define the parametrix, also denoted by Z^ε , as the fundamental solution of (71) for $a^\varepsilon(\mathfrak{z}, t)$ where $\mathfrak{z} \in \mathbb{R}^d$ is a fixed parameter:

$$\frac{\partial u}{\partial t}(x, t) = \operatorname{div} \left[a^\varepsilon(\mathfrak{z}, t) \nabla u(x, t) \right].$$

We have again the representation

$$(75) \quad \forall s \leq t, \quad Z^\varepsilon(x - y, t, s, \mathfrak{z}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, s, \zeta, \mathfrak{z}) d\zeta,$$

with

$$V^\varepsilon(t, s, \zeta, \mathfrak{z}) = \exp \left(- \left\langle \int_s^t a^\varepsilon(\mathfrak{z}, u) du, \zeta, \zeta \right\rangle \right).$$

The above arguments give Estimates (57) to (60). The next result is equivalent to Lemma 5 in [7], Chapter 9, Section 3.

Lemma 1 *Suppose that f is a measurable function on $\mathbb{R}^d \times [0, +\infty)$ and satisfies*

$$|f(x, t)| \leq k \exp(a|x|^2)$$

for some constants k and $a < \varpi/T$. Then the integral

$$F(x, t) = \int_0^t \left(\int_{\mathbb{R}^d} Z^\varepsilon(x - \zeta, t, s, \zeta) f(\zeta, s) d\zeta \right) ds$$

is well defined for $0 \leq t \leq T$ and the derivative $\nabla_x F$ exists for $0 < t \leq T$ and

$$\nabla_x F(x, t) = \int_0^t \left(\int_{\mathbb{R}^d} \nabla_x Z^\varepsilon(x - \zeta, t, s, \zeta) f(\zeta, s) d\zeta \right) ds.$$

Proof. We skip the proof because of the fact that the arguments are the same as the proof of Lemma IX.5 in [7] (see also [7], Chapter 1, Section 3 for more details). Since we consider only the gradient, we only need regularity of the function f (see Theorem I.2 in [7]). \square

The parametrix method suggests to construct $\tilde{\Gamma}^\varepsilon$ in the form

$$(76) \quad \begin{aligned} \tilde{\Gamma}^\varepsilon(x, t, y, s) &= Z^\varepsilon(x - y, t, s, y) \\ &+ \int_s^t \int_{\mathbb{R}^d} Z^\varepsilon(x - \zeta, t, r, \zeta) \Phi^\varepsilon(\zeta, r, y, s) d\zeta dr. \end{aligned}$$

If the function Φ^ε is measurable with suitable growth condition, we can apply Lemma 1 and thus $\tilde{\Gamma}^\varepsilon$ is the fundamental solution if and only if

$$\Phi^\varepsilon(x, t, y, s) = \mathcal{K}^\varepsilon(x, t, y, s) + \int_s^t \int_{\mathbb{R}^d} \mathcal{K}^\varepsilon(x, t, \zeta, r) \Phi^\varepsilon(\zeta, r, y, s) d\zeta dr$$

where

$$\mathcal{K}^\varepsilon(x, t, y, s) = \operatorname{div} [(a^\varepsilon(x, t) - a^\varepsilon(y, t)) \nabla_x Z^\varepsilon(x - y, t, s, y)].$$

Notice that in the expression $a^\varepsilon(x, t) - a^\varepsilon(y, t)$, the matrix is evaluated two times at the same point ξ_t^ε . Hence formally the function Φ^ε is the sum of iterated kernels

$$(77) \quad \Phi^\varepsilon(x, t, y, s) = \sum_{m=1}^{\infty} \mathcal{K}_m^\varepsilon(x, t, y, s)$$

where $\mathcal{K}_m^\varepsilon$ is the kernel:

$$\mathcal{K}_m^\varepsilon(x, t, y, s) = \int_s^t \int_{\mathbb{R}^d} \mathcal{K}^\varepsilon(x, t, \zeta, r) \mathcal{K}_{m-1}^\varepsilon(\zeta, r, y, s) d\zeta dr.$$

Let us follow the scheme of [7] to obtain (58), (59) and (60).

We first estimate the space derivative (58). From Condition **a1** and **(R)**, the matrix a^ε satisfies the following properties.

p1. a^ε is uniformly elliptic: for any $(t, x, \zeta) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$

$$\lambda^{-1}|\zeta|^2 \leq a^\varepsilon(x, t)\zeta \cdot \zeta \leq \lambda|\zeta|^2.$$

p2. a^ε is continuous on $\mathbb{R}^d \times \mathbb{R}_+$ and satisfies the Lipschitz condition w.r.t. x , uniformly w.r.t. t :

$$(78) \quad |a^\varepsilon(x, t) - a^\varepsilon(x', t)| \leq K_a |x - x'|$$

for some constant K_a independent of ε and t .

Remark that continuity of a^ε w.r.t. t is not used after. We will use the following notations: a_i is the i -th column of \mathbf{a} , γ is the vector-function defined by:

$$\gamma_i(z, y) = \operatorname{div}(a_i(z, y)) = \sum_{j=1}^n \frac{\partial a_{ji}}{\partial z_j}(z, y)$$

and $\gamma^\varepsilon(x, t) = \gamma(x, \xi_t^\varepsilon)$. The kernel \mathcal{K}^ε satisfies:

$$(79) \quad \begin{aligned} \mathcal{K}^\varepsilon(x, t, y, s) &= \sum_{i,j=1}^n (a_{ij}^\varepsilon(x, t) - a_{ij}^\varepsilon(y, t)) \frac{\partial^2 Z^\varepsilon}{\partial x_i \partial x_j}(x - y, t, s, y) \\ &+ \sum_{i=1}^n \gamma_i^\varepsilon(x, t) \frac{\partial Z^\varepsilon}{\partial x_i}(x - y, t, s, y). \end{aligned}$$

Lemma 2 *In (77), the series converge and the sum Φ^ε is measurable and verifies:*

$$(80) \quad |\Phi^\varepsilon(x, t, y, s)| \leq \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x - y, t - s).$$

The constants ϱ and ϖ do not depend on ε , but on λ and d , whereas ϱ also depends on the Lipschitz constant K_a of a w.r.t. z .

Proof. Hence from the previous estimate (74) on the derivatives of Z^ε and the Lipschitz property of a , we obtain

$$\begin{aligned} |\mathcal{K}^\varepsilon(x, t, y, s)| &\leq K_a |x - y| \frac{1}{t-s} g_{\varsigma, \varpi}(x - y, t - s) \\ &+ K_a \frac{1}{\sqrt{t-s}} g_{\varsigma, \varpi}(x - y, t - s) \\ &\leq \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x - y, t - s). \end{aligned}$$

Again ς , ϖ or ϱ may differ from line to line but they never depend on ε . Thus \mathcal{K}^ε satisfies exactly Inequality (4.6) of [7], Chapter 9, Section 4. Then the convergence of the series in (77) can be proved by the same arguments. Indeed for any $0 < \eta < 1$

$$\begin{aligned} |\mathcal{K}_2^\varepsilon(x, t, y, s)| &\leq \int_s^t \int_{\mathbb{R}^d} |\mathcal{K}^\varepsilon(x, t, \zeta, r)| |\mathcal{K}^\varepsilon(\zeta, r, y, s)| d\zeta dr \\ &\leq \int_s^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{(t-r)(r-s)}} g_{\varrho, \varpi}(x - \zeta, t - r) g_{\varrho, \varpi}(\zeta - y, r - s) d\zeta dr \\ &\leq \int_s^t \frac{M(\eta, \varpi) \varrho^2}{\sqrt{(t-r)(r-s)}} \frac{1}{(t-s)^{\frac{d}{2}}} \exp\left(-\varpi(1-\eta) \frac{|x-y|^2}{t-s}\right) dr \end{aligned}$$

using Lemma IX.7 in [7]. $M(\eta, \varpi)$ is a constant depending on η and ϖ (and also on d). And by direct computations (see Lemma I.2 in [7])

$$\int_s^t \frac{1}{\sqrt{(t-r)(r-s)}} dr = \pi.$$

Thereby we get two constants ϱ and ϖ such that

$$|\mathcal{K}_2^\varepsilon(x, t, y, s)| \leq g_{\varrho, \varpi}(x - y, t - s).$$

Iterating this computation we obtain by induction for $m \geq 2$:

$$|\mathcal{K}_m^\varepsilon(x, t, y, s)| \leq \frac{M^m}{(1 + m/2)!} (t - s)^{m/2-1} g_{\varrho, \varpi}(x - y, t - s)$$

where M is a constant depending on ϱ and ϖ and $(\cdot)!$ stands for the gamma function (see the proof of Theorem IX.2 in [7] for the details). The convergence of the series and the estimate (80) can be then deduced. \square

Using Lemma 1, we deduce that $\tilde{\Gamma}^\varepsilon$ is well-defined and the property (58) is obtained with the formula

$$\begin{aligned} \partial_{x_j} \tilde{\Gamma}^\varepsilon(x, t, y, s) &= \partial_{x_j} Z^\varepsilon(x - y, t, s, y) \\ (81) \quad &+ \int_s^t \int_{\mathbb{R}^d} \partial_{x_j} Z^\varepsilon(x - \zeta, t, r, \zeta) \Phi^\varepsilon(\zeta, r, y, s) d\zeta dr, \end{aligned}$$

together with estimate (74) on Z^ε and (80) on Φ^ε . We underline that only the properties **p1** and **p2** of a^ε are required to obtain (58). Let us formulate precisely our result.

Lemma 3 *Let $a = a(x, t)$ be a measurable matrix function defined on $\mathbb{R}^d \times \mathbb{R}^+$, and assume that the uniform ellipticity condition **a1.** holds. Assume moreover that there exists $K_a > 0$ such that $|a(x, t) - a(x', t)| \leq K_a |x - x'|$ for almost all $t \in \mathbb{R}^+$. Then the fundamental solution of*

$$\frac{\partial u}{\partial t} = \operatorname{div} \left[a(x, t) \nabla u \right],$$

satisfies estimate (58) where the constant ϖ only depends on the uniform ellipticity constant λ and the dimension d , and the constant ϱ only depends on λ , d and K_a .

The Malliavin derivative of $\widetilde{\Gamma}^\varepsilon$

Now we turn to the estimates (59) and (60) for $\widetilde{\Gamma}^\varepsilon$. Recall that we assume Conditions **a1** and **(R)** to hold. Let us summarize some properties of the matrix a^ε .

Lemma 4 *The matrix a^ε satisfies the following properties.*

p3. For each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $a^\varepsilon(x, t)$ is $\mathcal{F}_{t/\varepsilon^{\alpha-2}}$ -measurable.

p4. For each (t, x) the random variable $a^\varepsilon(x, t)$ belongs to $\mathbb{D}^{1,2}$.

p5. For every (t, x) , (t', x') and r , such that t, t' and r are in $[0, T/\varepsilon^2]$

$$\begin{aligned} |D_r a^\varepsilon(x, t)| &\leq K_a \psi^\varepsilon(r) \\ |D_r(a^\varepsilon(x, t) - a^\varepsilon(x', t))| &\leq K_a \psi^\varepsilon(r) |x - x'|. \end{aligned}$$

Proof. The assertion **p3** follows immediately from the definition of a^ε and ξ^ε . Since $\xi \in \mathbb{D}^{1,2}$, and since a is smooth (at least of class C^1), a^ε also belongs to $\mathbb{D}^{1,2}$ (classical chain rule, see Proposition 1.2.3 in [24]). Hence **p4** is proved.

Now we want to obtain suitable estimates on the Malliavin derivative of a^ε . Once again $\xi \in \mathbb{D}^{1,2}$ and a is smooth. Thus for any fixed x , we have for any $k = 1, \dots, d$

$$D_r^k a_{ij}^\varepsilon(x, t) = \sum_{\ell=1}^d \frac{\partial a_{ij}}{\partial y_\ell}(x, \xi_t^\varepsilon) D_r^k(\xi_t^{\varepsilon, \ell}).$$

Thus $D_r a^\varepsilon(x, t) = 0$ if $r > t/\varepsilon^{\alpha-2}$ and for $r \leq t/\varepsilon^{\alpha-2}$:

$$(82) \quad |D_r^k a_{ij}^\varepsilon(x, t)| \leq \left| \frac{\partial a_{ij}}{\partial y_\ell} \right| |D_r(\xi_t^\varepsilon)| \leq K_a \psi^\varepsilon(r).$$

The same computation shows that

$$\begin{aligned} &D_r^k(a_{ij}^\varepsilon(x, t) - a_{ij}^\varepsilon(x', t)) \\ &= \sum_{l=1}^n \left[\frac{\partial a_{ij}}{\partial y_l}(x, \xi_t^\varepsilon) - \frac{\partial a_{ij}}{\partial y_l}(x', \xi_t^\varepsilon) \right] D_r^k(\xi_t^{\varepsilon, l}). \end{aligned}$$

Hence

$$|D_r^k(a_{ij}^\varepsilon(x, t) - a_{ij}^\varepsilon(x', t))| \leq K_a \psi^\varepsilon(r) |x - x'|.$$

This completes the proof. \square

Let us remark that the above construction of $\tilde{\Gamma}^\varepsilon$ has been made pathwise, ω by ω . We want to prove now that $\tilde{\Gamma}^\varepsilon$ is also Malliavin differentiable. A straightforward consequence of the previous definitions is that for any $s < t$, the random variables $Z^\varepsilon(x - y, t, s)$, $\Phi^\varepsilon(x, t, y, s)$ and $\tilde{\Gamma}^\varepsilon$ are $\mathcal{F}_{t/\varepsilon^{\alpha-2}}$ -measurable.

Let us first consider the Malliavin derivative of Z^ε . From the representation (73), this derivative can be computed explicitly: for $j = 1, \dots, d$

$$\begin{aligned} D_r^j Z^\varepsilon(x - y, t, s) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} D_r^j V^\varepsilon(t, s, \zeta) d\zeta \\ &= -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, s, \zeta) \left\langle \int_s^t D_r^j a^\varepsilon(u) du, \zeta, \zeta \right\rangle d\zeta. \end{aligned}$$

Thus

$$D_r^j Z^\varepsilon(x - y, t, s) = \text{Trace} \left[\left(\int_s^t D_r^j a^\varepsilon(u) du \right) \partial_x^2 Z^\varepsilon(x - y, t, s) \right].$$

Thereby

$$|D_r^j Z^\varepsilon(x - y, t, s)| \leq \left| \int_s^t \frac{\partial a}{\partial y_k}(\xi_u^\varepsilon) D_r^j \xi_u^{k, \varepsilon} du \right| \frac{1}{t-s} g_{\varsigma, \varpi}(x - y, t - s).$$

Since we assume that the derivative of a is bounded (Condition **(R)**), we obtain:

$$|D_r Z^\varepsilon(x - y, t, s)| \leq K_a \psi^\varepsilon(r) g_{\varsigma, \varpi}(x - y, t - s).$$

Therefore we deduce (59). Similar computations give:

$$D_r \partial_{x_j} Z^\varepsilon(x - y, t, s) = -i \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, s, \zeta) \left\langle \int_s^t D_r^j a^\varepsilon(u) du, \zeta, \zeta \right\rangle \zeta_j d\zeta,$$

and using the estimate on the third derivative of Z^ε w.r.t. x , we obtain (60):

$$(83) \quad |D_r \partial_{x_i} Z^\varepsilon(x - y, t, s)| \leq K_a \psi^\varepsilon(r) \frac{1}{(t-s)^{1/2}} g_{\varsigma, \varpi}(x - y, t - s).$$

In other words if a^ε does not depend on x , that is if a does not depend on z , the estimates (57), (58), (59) and (60) hold for Z^ε . In (59) and (60), the

implied constants depend also on the Lipschitz constant of the matrix a w.r.t. the second variable y . Similar computations and arguments also show:

$$|D_r \partial_{x_i x_j}^2 Z^\varepsilon(x - y, t, s)| \leq K_a \psi^\varepsilon(r) \frac{1}{(t - s)} g_{\varsigma, \varpi}(x - y, t - s).$$

Lemma 5 (Malliavin differentiability of Φ^ε) *The function Φ^ε belongs to $\mathbb{D}^{1, \infty}$ for every $(t, s) \in [0, T]^2$, $s < t$ and $(x, y) \in (\mathbb{R}^d)^2$. Moreover there exists two constants $\varrho > 0$ and $\varpi > 0$ such that*

$$(84) \quad |D_r \Phi^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) \frac{1}{\sqrt{t - s}} g_{\varrho, \varpi}(x - y, t - s).$$

Proof.

Recall that

$$\gamma_i(z, y) = \operatorname{div}(a_i(z, y)) = \sum_{j=1}^d \frac{\partial a_{ji}}{\partial z_j}(z, y)$$

and $\gamma^\varepsilon(x, t) = \gamma(x, \xi_t^\varepsilon)$. Note that from Condition **R**, the process γ^ε belongs also to $\mathbb{D}^{1, \infty}$. Indeed

$$D_r^k \gamma_i^\varepsilon(x, t) = \sum_{\ell=1}^d \frac{\partial \gamma_i}{\partial y_\ell}(x, \xi_t^\varepsilon) D_r^k(\xi_t^{\varepsilon, \ell}) = \sum_{\ell=1}^d \sum_{j=1}^d \frac{\partial^2 a_{ij}}{\partial z_j \partial y_\ell}(x, \xi_t^\varepsilon) D_r^k(\xi_t^{\varepsilon, \ell}).$$

From (79) the Malliavin derivative of \mathcal{K}^ε is given by:

$$\begin{aligned} D_r \mathcal{K}^\varepsilon(x, t, y, s) &= \sum_{i, j=1}^n [D_r a_{ij}^\varepsilon(x, t) - D_r a_{ij}^\varepsilon(y, t)] \frac{\partial^2 Z^\varepsilon}{\partial x_i \partial x_j}(x - y, t, s, y) \\ &+ \sum_{i=1}^n D_r \gamma_i^\varepsilon(x, t) \frac{\partial Z^\varepsilon}{\partial x_i}(x - y, t, s, y) \\ &+ \sum_{i, j=1}^n (a_{ij}^\varepsilon(x, t) - a_{ij}^\varepsilon(y, t)) D_r \frac{\partial^2 Z^\varepsilon}{\partial x_i \partial x_j}(x - y, t, s, y) \\ &+ \sum_{i=1}^n \gamma_i^\varepsilon(x, t) D_r \frac{\partial Z^\varepsilon}{\partial x_i}(x - y, t, s, y). \end{aligned}$$

From our previous assumptions and properties we deduce that

$$\begin{aligned}
|D_r \mathcal{K}^\varepsilon(x, t, y, s)| &\leq K_a \psi^\varepsilon(r) \frac{|x-y|}{(t-s)} g_{\varrho, \varpi}(x-y, t-s) \\
&+ K_a \psi^\varepsilon(r) \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s) \\
&+ K_a \frac{|x-y|}{(t-s)} \psi^\varepsilon(r) g_{\varrho, \varpi}(x-y, t-s) \\
&+ K_a \psi^\varepsilon(r) \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s) \\
&\leq \psi^\varepsilon(r) \frac{1}{\sqrt{t-s}} g_{\varrho, \varpi}(x-y, t-s).
\end{aligned}$$

By induction (same technics as the proof of Lemma 2), we obtain for $m \geq 2$

$$|D_r \mathcal{K}_m^\varepsilon(x, t, y, s)| \leq \frac{M^m}{(1+m/2)!} \psi^\varepsilon(r) (t-s)^{m/2-1} g_{\varrho, \varpi}(x-y, t-s)$$

for some constant $M > 0$ depending on ϱ and ϖ . Indeed for $m = 2$

$$\begin{aligned}
|D_r \mathcal{K}_2^\varepsilon(x, t, y, s)| &\leq \int_s^t \int_{\mathbb{R}^d} |D_r \mathcal{K}^\varepsilon(x, t, \zeta, \tau)| |\mathcal{K}^\varepsilon(\zeta, \tau, y, s)| d\zeta d\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} |\mathcal{K}^\varepsilon(x, t, \zeta, \tau)| |D_r \mathcal{K}^\varepsilon(\zeta, \tau, y, s)| d\zeta d\tau \\
&\leq 2\psi^\varepsilon(r) \int_s^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{(t-\tau)(\tau-s)}} g_{\varrho, \varpi}(x-\zeta, t-\tau) g_{\varrho, \varpi}(\zeta-y, \tau-s) d\zeta d\tau
\end{aligned}$$

and we conclude by classical arguments to have an estimate on the integral. By the closability of the operator D we conclude that

$$(85) \quad D_r \Phi^\varepsilon(x, t, y, s) = \sum_{m=1}^{\infty} D_r \mathcal{K}_m^\varepsilon(x, t, y, s)$$

and that Estimate (84) holds. Since $\psi^\varepsilon(r)$ belongs to $L^p(\Omega)$ for any p , $\Phi \in \mathbb{D}^{1, \infty}$. This achieves the proof of the Lemma. \square

In the next lemma we prove that $\tilde{\Gamma}^\varepsilon \in \mathbb{D}^{1, \infty}$ and that the same Gaussian estimates hold for the Malliavin derivative.

Lemma 6 (Malliavin differentiability of Γ) *The fundamental solution $\tilde{\Gamma}^\varepsilon$ and its spatial derivative belong to $\mathbb{D}^{1,\infty}$ for every $(t, s) \in [0, T]^2$, $s < t$ and $(x, y) \in (\mathbb{R}^d)^2$ and (59) and (60) hold:*

$$\begin{aligned} |D_r \tilde{\Gamma}^\varepsilon(x, t, y, s)| &\leq \psi^\varepsilon(r) g_{\varrho, \varpi}(x - y, t - s), \\ |D_r \frac{\partial}{\partial x_i} \tilde{\Gamma}^\varepsilon(x, t, y, s)| &\leq \psi^\varepsilon(r) \frac{1}{\sqrt{t - s}} g_{\varrho, \varpi}(x - y, t - s). \end{aligned}$$

Proof. From the definition of $\tilde{\Gamma}^\varepsilon$ (Equation (76)), and the two previous lemmas, and the properties of the Malliavin derivative D we obtain that:

$$\begin{aligned} (86) \quad D_\tau \tilde{\Gamma}^\varepsilon(x, t, y, s) &= D_\tau Z^\varepsilon(x - y, t, s, y) \\ &+ \int_s^t \int_{\mathbb{R}^d} D_\tau Z^\varepsilon(x - \zeta, t, \tau, \zeta) \Phi^\varepsilon(\zeta, \tau, y, s) d\zeta d\tau \\ &+ \int_s^t \int_{\mathbb{R}^d} Z^\varepsilon(x - \zeta, t, \tau, \zeta) D_\tau \Phi^\varepsilon(\zeta, \tau, y, s) d\zeta d\tau. \end{aligned}$$

Moreover the inequalities (83) and (84) imply that

$$|D_r \tilde{\Gamma}^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) g_{\varrho, \varpi}(x - y, t - s).$$

See Lemma I.4.3 in [7]. From Equation (81), we can compute the Malliavin derivative of $\frac{\partial}{\partial x_i} \tilde{\Gamma}^\varepsilon(x, t, y, s)$ and we have:

$$\begin{aligned} D_r \frac{\partial}{\partial x_i} \tilde{\Gamma}^\varepsilon(x, t, y, s) &= D_r \frac{\partial Z^\varepsilon}{\partial x_i}(x - y, t, s, y) \\ &+ \int_s^t \int_{\mathbb{R}^d} D_r \frac{\partial Z^\varepsilon}{\partial x_i}(x - \zeta, t, \tau, \zeta) \Phi^\varepsilon(\zeta, \tau, y, s) d\zeta d\tau \\ &+ \int_s^t \int_{\mathbb{R}^d} \frac{\partial Z^\varepsilon}{\partial x_i}(x - \zeta, t, \tau, \zeta) D_r \Phi^\varepsilon(\zeta, \tau, y, s) d\zeta d\tau. \end{aligned}$$

Again with Lemma I.4.3 in [7], the estimates (83) and (84) imply (60). This completes the proof. \square

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