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Diffusion approximation for random parabolic operators with oscillating coefficients

Marina Kleptsyna[‡] Andrey Piatnitski[§] and Alexandre Popier[‡]

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Abstract

We consider Cauchy problem for a divergence form second order parabolic operator with rapidly oscillating coefficients that are periodic in spatial variables and random stationary ergodic in time. As was proved in [24] and [12] in this case the homogenized operator is deterministic. The paper focuses on non-diffusive scaling, when the oscillation in spatial variables is faster than that in temporal variable. Our goal is to study the asymptotic behaviour of the normalized difference between solutions of the original and the homogenized problems.

1 Introduction

In this work we consider the asymptotic behaviour of solutions to the following Cauchy problem

$$(1) \quad \begin{aligned} \frac{\partial}{\partial t} u^\varepsilon &= \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^\varepsilon \right] \\ u^\varepsilon(x, 0) &= \varphi(x). \end{aligned}$$

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Here ε is a small positive parameter that tends to zero, α satisfies the inequality $0 < \alpha < 2$, $a(z, s)$ is a positive definite matrix whose entries are periodic in z variable and random stationary ergodic in s .

It is known (see [24, 12]) that this problem admits homogenization and that the homogenized operator is deterministic and has constant coefficients. The homogenized Cauchy problem takes the form

$$(2) \quad \begin{aligned} \frac{\partial}{\partial t} u^0 &= \operatorname{div}(a^{\text{eff}} \nabla u^0) \\ u^0(x, 0) &= \varphi(x). \end{aligned}$$

The formula for the effective matrix a^{eff} can be found in [12].

The goal of this paper is to study the limit behaviour of the difference $u^\varepsilon - u^0$, as $\varepsilon \rightarrow 0$.

In the existing literature there is a number of works devoted to homogenization of random parabolic problems. The results obtained in [15] and [19] for random divergence form elliptic operators also apply to the parabolic case. In the presence of large lower order terms the limit dynamics might remain random and show diffusion or even more complicated behaviour. The papers [5], [20], [14] focus on the case of time dependent parabolic operators with periodic in spatial variables and random in time coefficients. The fully random case has been studied in [21], [3], [4], [10].

One of the important aspects of homogenization theory is estimating the rate of convergence. For random operators the first estimates have been obtained in [22]. Further important progress in this direction was achieved in the recent works [9], [8].

Problem (1) in the case of diffusive scaling $\alpha = 2$ was studied in our previous work [13]. It was shown that, under proper mixing conditions, the difference $u^\varepsilon - u^0$ is of order ε , and that the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$ after subtracting an appropriate corrector, converges in law to a solution of some limit SPDE.

In the present paper we consider the case $0 < \alpha < 2$. In other words, bearing in mind the diffusive scaling, we assume that the oscillation in spatial variables is faster than that in time. In this case the principal part of the asymptotics of $u^\varepsilon - u^0$ consists of a finite number of correctors, the oscillating part of each of them being a solution of an elliptic PDE with periodic in spatial variable coefficients. The number of correctors increases as α approaches 2. After subtracting these correctors, the resulting expression divided by $\varepsilon^{\alpha/2}$ converges in law to a solution of the limit SPDE.

The case $\alpha > 2$ will be considered elsewhere.

The paper is organized as follows. In Section 2 we introduce the studied problem and provide all the assumptions. Then we formulate the main results of the paper.

In Section 3 we consider a number of auxiliary problems and define the higher order terms of the asymptotics of solution.

Section 4 focuses on the proof of Theorem 1.

In Section 5 we consider the special case of diffusive dependence on time. Namely, we assume that $a(z, s) = a(z, \xi_s)$ where ξ is a stationary diffusion process in \mathbb{R}^n and $a(z, y)$ is a periodic in z smooth deterministic function. It should be emphasized that in the said diffusive case Theorem 1 does not apply because the coefficients a^{ij} do not possess the required regularity in time. That is why in the diffusive case we provide another proof of convergence which is based on the Itô calculus and an estimate based on anticipating stochastic integral.

In Section 6 we justify this estimate using anticipating calculus and the properties of the fundamental solution of a stochastic parabolic equation with random coefficients. The construction and the required properties of the fundamental solution are postponed in the Appendix.

2 Problem setup and main results

In this section we provide all the assumptions on the data of problem (1), introduce some notations and formulate the main results.

The studied Cauchy problem reads

$$(3) \quad \begin{aligned} \frac{\partial}{\partial t} u^\varepsilon &= \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^\varepsilon \right] && \text{in } \mathbb{R}^d \times (0, T] \\ u^\varepsilon(x, 0) &= \varphi(x); \end{aligned}$$

here ε is a small positive parameter. Our aim is to study the behaviour of a solution u^ε as $\varepsilon \rightarrow 0$.

We assume that the following conditions hold true:

- c1.** the matrix $a(z, s) = \{a^{ij}(z, s)\}_{i,j=1}^d$ is symmetric and satisfies uniform ellipticity conditions

$$\lambda |\zeta|^2 \leq a(z, s) \zeta \cdot \zeta \leq \lambda^{-1} |\zeta|^2, \quad \zeta \in \mathbb{R}^d, \quad \lambda > 0;$$

- c2.** the coefficients $a^{ij}(z, s)$ are periodic in z with the period $[0, 1]^d$ and random stationary ergodic in s . Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with an ergodic dynamical system τ_s , we assume that $a^{ij}(z, s, \omega) = \mathbf{a}^{ij}(z, \tau_s \omega)$, where $\{\mathbf{a}^{ij}(z, \omega)\}_{i,j=1}^d$ is a collection of random periodic in z functions that satisfy the above uniform ellipticity conditions.
- c3.** The realizations $a^{ij}(z, s)$ are smooth. For any $N \geq 1$ and $k \geq 2$ there exist $C_{N,k}$ such that

$$\mathbf{E} \|a^{ij}\|_{C^N(\mathbb{T}^d \times [0,1])}^k \leq C_{N,k};$$

here and in what follows we identify periodic functions with functions on the torus \mathbb{T}^d , \mathbf{E} stands for the expectation.

- c4.** $\varphi \in C_0^\infty(\mathbb{R}^d)$. In fact, this condition can be essentially relaxed, see Remark 1.
- c5.** Mixing condition. The strong mixing coefficient $\gamma(r)$ of $a(z, \cdot)$ satisfies the inequality

$$\int_0^\infty (\gamma(r))^{1/2} dr < \infty.$$

For the reader's convenience we recall here the definition of strong mixing coefficient. Let $\mathcal{F}_{\leq s}$ and $\mathcal{F}_{\geq s}$ be the σ -algebras generated by $\{a(z, t) : z \in \mathbb{T}^d, t \leq s\}$ and $\{a(z, t) : z \in \mathbb{T}^d, t \geq s\}$, respectively. We set

$$\gamma(r) = \sup |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|,$$

where the supremum is taken over all $A \in \mathcal{F}_{\leq 0}$ and $B \in \mathcal{F}_{\geq r}$.

According to [12] the sequence u^ε converges in probability, as $\varepsilon \rightarrow 0$, to a solution u^0 of problem (2).

In order to formulate the main results we also need a number of auxiliary functions and quantities. The first auxiliary problem reads

$$(4) \quad \operatorname{div}(a(z, s)\nabla\chi^0(z, s)) = -\operatorname{div} a(z, s), \quad z \in \mathbb{T}^d;$$

here s and ω are parameters. It has a unique up to an additive constant periodic solution. This constant is chosen in such a way that

$$(5) \quad \int_{\mathbb{T}^d} \chi^0(z, s) dz = 0 \quad \text{for all } s \text{ and } \omega.$$

Then, for $j = 1, 2, \dots, J^0$ with $J^0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$, the higher order correctors are introduced as periodic solutions to the equations

$$(6) \quad \operatorname{div}(a(z, s)\nabla\chi^j(z, s)) = \partial_s\chi^{j-1}(z, s),$$

where $\lfloor \cdot \rfloor$ stands for the integer part. Due to (5) for $j = 1$ this equation is solvable in the space of periodic in z functions. A solution χ^1 is uniquely defined up to an additive constant. Choosing the constant in a proper way yields

$$\int_{\mathbb{T}^d} \chi^1(z, s) dz = 0 \quad \text{for all } s \text{ and } \omega$$

and thus the solvability of the equation for χ^2 . Iterating this procedure, we define all χ^j , $j = 1, 2, \dots, J^0$.

Next, we introduce the functions $u^j = u^j(x, t)$, $j = 1, \dots, J^0$. Recall that u^0 has been defined in (2). Functions u^j solve the following problems:

$$(7) \quad \begin{aligned} \frac{\partial}{\partial t}u^j &= \operatorname{div}(a^{\text{eff}}\nabla u^j) + \sum_{k=1}^j a^{k,\text{eff}} \frac{\partial^2}{\partial x^2}u^{j-k} \\ u^j(x, 0) &= 0 \end{aligned}$$

with

$$a^{k,\text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} a(z, s)\nabla\chi^k(z, s) dz.$$

To characterize the diffusive term in the limit equation we introduce the matrix

$$\Xi(s) = \int_{\mathbb{T}^d} \left[(a(z, s) + a(z, s)\nabla\chi^0(z, s)) - \mathbf{E}(a(z, s) + a(z, s)\nabla\chi^0(z, s)) \right] dz.$$

By construction the function Ξ is stationary and satisfies condition **c5**. Denote

$$\Lambda = \frac{1}{2} \int_0^\infty \mathbf{E} \left(\Xi(s) \otimes \Xi(0) + \Xi(0) \otimes \Xi(s) \right) ds,$$

where $(\Xi(s) \otimes \Xi(0))^{ijkl} = \Xi^{ij}(s)\Xi^{kl}(0)$. Under condition **c5**. the integral on the right-hand side converges.

The first main result of this paper is

Theorem 1 *Let conditions **c1**–**c5** be fulfilled, and assume that $\alpha < 2$. Then the functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left(u^\varepsilon - u^0 - \sum_{j=0}^{J_0} \varepsilon^{j(2-\alpha)} u^j \right)$$

converge in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ to a solution of the following SPDE

$$(8) \quad \begin{aligned} dv^0 &= \operatorname{div}(a^{\text{eff}} \nabla v^0) dt + \Lambda \frac{\partial^2}{\partial x^2} u^0 dW_t \\ v^0(x, 0) &= 0; \end{aligned}$$

where W is the standard d^2 -dimensional Brownian motion.

3 Auxiliary problems

We begin by considering problem (4). This equation has a unique up to an additive constant periodic solution. Since $\chi^0(\cdot, s)$ only depends on $a(\cdot, s)$, the solution with zero average is stationary, and the strong mixing coefficient of the pair $(a(\cdot, s), \chi(\cdot, s))$ is the same as that for $a(\cdot, s)$. By the classical elliptic estimates (see [7]), under our standing assumptions we have

$$(9) \quad \|\chi^0\|_{L^\infty(\mathbb{T}^d \times (0, \infty))} \leq C, \quad \mathbf{E} \|\chi^0\|_{C^k(\mathbb{T}^d \times [0, 1])}^N \leq C_{k, N}.$$

By the similar arguments, solutions χ^j of equations (6) are stationary, satisfy strong mixing condition with the same coefficient $\gamma(r)$, and the following estimates hold:

$$(10) \quad \mathbf{E} \|\chi^j\|_{C^k(\mathbb{T}^d \times [0, 1])}^N \leq C_{k, N}.$$

Solutions u^0 and u^j of problems (2) and (7) are smooth functions. Moreover, for any $\mathbf{k} = (k_0, k_1, \dots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k}, N}$ such that

$$(11) \quad |D^{\mathbf{k}} u^j| \leq C_{\mathbf{k}, N} (1 + |x|)^{-N},$$

where $D^{\mathbf{k}} f(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \dots \frac{\partial^{k_d}}{\partial x_d^{k_d}} f(x, t)$.

Next, we write down the following ansatz

$$V^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x, t) - \sum_{k=0}^{J^0} \varepsilon^{k\delta} \left(u^k(x, t) + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla u^k(x, t) \right) \right\},$$

here and in what follows the symbol δ stands for $2 - \alpha$.

4 Proof of Theorem 1

In this section we prove Theorem 1. Denote

$$\widehat{a}^0(z, s) = a(z, s) + a(z, s) \nabla \chi^0(z, s) + \nabla (a(z, s) \chi^0(z, s)),$$

$$\widehat{a}^k(z, s) = a(z, s) \nabla \chi^k(z, s) + \nabla (a(z, s) \chi^k(z, s)), \quad k = 1, 2, \dots,$$

and

$$\widehat{a}^{k, \text{eff}} = \mathbf{E} \int_{\mathbb{T}^d} [\widehat{a}^k(z, s)] dz, \quad k = 1, 2, \dots$$

Substituting V^ε for u^ε in (3) and taking into account (6), (2) and (7) yields

$$\begin{aligned} \partial_t V^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V^\varepsilon \right] &= \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \varepsilon^{j\delta} \left[\widehat{a}^0 \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j \\ (12) \quad &+ \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{a}^k \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{k, \text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j + \mathcal{R}^\varepsilon(x, t), \\ V^\varepsilon(x, 0) &= \sum_{j=0}^{J^0} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, 0 \right) \nabla u^0(x, 0) \end{aligned}$$

with

$$(13) \quad \mathcal{R}^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{N_0} \varepsilon^{1+j\delta} \theta^j \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \Phi^j(x, t),$$

where $\theta^j(z, s)$ are periodic in z , stationary in s and satisfy the estimates

$$(14) \quad \mathbf{E} (\|\theta^j\|_{C(\mathbb{T}^d \times [0,1])}^k) \leq C_k;$$

Φ^j are smooth functions such that

$$(15) \quad |D^k \Phi^j| \leq C_{\mathbf{k}, N} (1 + |x|)^{-N},$$

and N_0 is a finite number; we do not specify these quantities explicitly because we do not need this. We represent V^ε as the sum $V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon$, where V_1^ε and V_2^ε solve the following problems:

$$(16) \quad \left\{ \begin{array}{l} \partial_t V_1^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_1^\varepsilon \right] = \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \varepsilon^{j\delta} \left[\widehat{a}^0 \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{\text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j \\ \quad + \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{a}^k \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) - a^{k, \text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j, \\ V_1^\varepsilon(x, 0) = 0, \end{array} \right.$$

and

$$(17) \quad \left\{ \begin{array}{l} \partial_t V_2^\varepsilon - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha} \right) \nabla V_2^\varepsilon \right] = \mathcal{R}^\varepsilon(x, t), \\ V_2^\varepsilon(x, 0) = V^\varepsilon(x, 0). \end{array} \right.$$

From (9) and (10) it follows that the initial condition in the latter problem satisfies for any $k > 0$ the estimate $\mathbf{E} \|V^\varepsilon(\cdot, 0)\|_{C(\mathbb{T}^d)}^k \leq C_k \varepsilon^{k\delta/2}$. Combining this estimate with (13), (14) and the estimates for Φ^j , by means of the standard parabolic estimates we obtain

$$(18) \quad \mathbf{E} \|V_2^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^\delta.$$

Denote

$$\begin{aligned} \langle a \rangle^0(s) &= \int_{\mathbb{T}^d} (\widehat{a}^0(z, s) - a^{\text{eff}}) dz, \\ \langle a \rangle^k(s) &= \int_{\mathbb{T}^d} (\widehat{a}^k(z, s) - a^{k, \text{eff}}) dz, \quad k = 1, 2, \dots \end{aligned}$$

It follows from the definition of \widehat{a}^k that for any $k \geq 0$ and $l > 0$ there is a constant $C_{l, k}$ such that $\mathbf{E} \|(\widehat{a}^k - \langle a \rangle^k)\|_{C^k(\mathbb{T}^d \times [0, 1])}^l \leq C_{l, k}$. Since for each $s \in \mathbb{R}$ the mean value of $(\widehat{a}^k(\cdot, s) - \langle a \rangle^k(s))$ is equal to zero, the problem

$$\Delta_z \zeta^k(z, s) = (\widehat{a}^k(z, s) - \langle a \rangle^k(s))$$

has a unique up to an additive constant periodic solution. Letting $\Theta^k(z, s) = \nabla \zeta^k(z, s)$, we obtain a stationary in s vector functions Θ^k such that

$$\operatorname{div} \Theta^k(z, s) = (\widehat{a}^k(z, s) - \langle a \rangle^k(s)), \quad \mathbf{E} \|\Theta^k\|_{C^k(\mathbb{T}^d \times [0,1])}^l \leq C.$$

It is then straightforward to check that for the functions

$$F^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(k+j)\delta} \left[\widehat{a}^k\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) - \langle a \rangle^k\left(\frac{t}{\varepsilon^\alpha}\right) \right] \frac{\partial^2}{\partial x^2} u^j(x, t)$$

the following estimate is fulfilled:

$$\mathbf{E} \|F^\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))}^2 \leq C\varepsilon^\delta.$$

Therefore, a solution to the problem

$$(19) \quad \begin{cases} \partial_t V_{1,2}^\varepsilon - \operatorname{div} \left[a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) \nabla V_{1,2}^\varepsilon \right] = F^\varepsilon(x, t), \\ V_{1,2}^\varepsilon(x, 0) = 0. \end{cases}$$

admits the estimate

$$(20) \quad \mathbf{E} \|V_{1,2}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^d))}^2 \leq C\varepsilon^\delta.$$

Splitting $V_1^\varepsilon = V_{1,1}^\varepsilon + V_{1,2}^\varepsilon$, we conclude that $V_{1,1}^\varepsilon$ solves the following problem

$$(21) \quad \begin{cases} \partial_t V_{1,1}^\varepsilon - \operatorname{div} \left[a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) \nabla V_{1,1}^\varepsilon \right] = \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \varepsilon^{j\delta} \left[\langle a \rangle^0\left(\frac{t}{\varepsilon^\alpha}\right) - a^{\text{eff}} \right] \frac{\partial^2 u^j}{\partial x^2} \\ \quad + \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\langle a \rangle^k\left(\frac{t}{\varepsilon^\alpha}\right) - a^{k,\text{eff}} \right] \frac{\partial^2 u^j}{\partial x^2}, \\ V_{1,1}^\varepsilon(x, 0) = 0, \end{cases}$$

By construction the strong mixing coefficient of \widehat{a}^k remains unchanged and is equal to $\gamma(\cdot)$. In exactly the same way as in the proof of [13, Lemma 5.1] one can show that the solution of problem (21) converges in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ equipped with strong topology to a solution of (8). Combining this convergence with (18) and (20) we conclude that V^ε converges in law in the same space to a solution of (8). This completes the proof of Theorem 1.

Remark 1 The regularity assumption on φ given in condition **c4.** can be weakened. Namely, the statement of Theorem 1 holds if φ is $J^0 + 1$ times continuously differentiable and the corresponding partial derivatives decay at infinity sufficiently fast.

5 Diffusive case

In this section we consider the special case of problem (3). Namely, we assume that the matrix $a(z, s)$ has the form

$$a(z, s) = a(z, \xi_s),$$

where ξ_s is a stationary diffusion process in \mathbb{R}^n with a generator

$$\mathcal{L} = q(y) \frac{\partial^2}{\partial y^2} + b(y) \frac{\partial}{\partial y}.$$

We still assume that Conditions **c1** and **c4** hold, that is:

- c1.** the matrix $a(z, s) = \{a^{ij}(z, s)\}_{i,j=1}^d$ is symmetric and satisfies uniform ellipticity conditions

$$\lambda |\zeta|^2 \leq a(z, s) \zeta \cdot \zeta \leq \lambda^{-1} |\zeta|^2, \quad \zeta \in \mathbb{R}^d, \quad \lambda > 0;$$

- c4.** $\varphi \in C_0^\infty(\mathbb{R}^d)$ (with Remark 1).

Moreover we suppose that the matrix-functions $a(z, y)$, $q(y)$ and vector-function $b(y)$ possess the following properties:

- a1.** $a = a(z, y)$ is periodic in z and smooth in both variables z and y . Moreover, for each $N > 0$ there exists $C_N > 0$ such that

$$\|a\|_{C^N(\mathbb{T}^d \times \mathbb{R}^n)} \leq C_N.$$

- a2.** The matrix $q = q(y)$ satisfies the uniform ellipticity conditions: there exist $\lambda > 0$ such that

$$\lambda^{-1} |\zeta|^2 \leq q(y) \zeta \cdot \zeta \leq \lambda |\zeta|^2, \quad y, \zeta \in \mathbb{R}^n.$$

Moreover there exists a matrix $\sigma = \sigma(y)$ such that $q(y) = \sigma^*(y) \sigma(y)$.

- a3.** The matrix function σ and vector-function b are smooth, for each $N > 0$ there exists $C_N > 0$ such that

$$\|\sigma\|_{C^N(\mathbb{R}^n)} \leq C_N, \quad \|b\|_{C^N(\mathbb{R}^n)} \leq C_N.$$

a4. The following inequality holds for some $R > 0$ and $C_0 > 0$ and $p > -1$:

$$b(y) \cdot y \leq -C_0|y|^p \quad \text{for all } y \in \{y \in \mathbb{R}^n : |y| \geq R\}.$$

According to [21] under conditions **a2.** and **a4.** a diffusion process ξ with the generator \mathcal{L} has an invariant measure in \mathbb{R}^n that has a smooth density $\rho = \rho(y)$. For any $N > 0$ it holds

$$(1 + |y|)^N \rho(y) \leq C_N$$

with some constant C_N . The function ρ is the unique up to a multiplicative constant bounded solution of the equation $\mathcal{L}^* \rho = 0$; here $*$ denotes the formally adjoint operator. We assume that the process ξ_t is stationary and distributed with the density ρ .

Now we introduce the last condition which looks a little bit peculiar. Indeed this assumption will not directly appear in our next Theorem. We define

$$(22) \quad \tilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \quad \tilde{\sigma}_{i,j}^l(s) = (\partial_{x_j} \sigma_{i,l})(\xi_s).$$

These coefficients will be implied in the dynamic of the Malliavin derivative of ξ (see Lemma 6.1). The notation \mathbb{S}^{d-1} stands for the unit sphere of \mathbb{R}^d .

a5. For some $p > 2$, a.s. for any $t \geq 0$ and any $\theta \in \mathbb{S}^{d-1}$

$$Q(t, \theta) + \frac{p}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \theta)^2 \leq 0$$

where

$$Q(t, \theta) = (\tilde{b}(t)\theta, \theta) + \frac{1}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \tilde{\sigma}^l(t)\theta) - \sum_{l=1}^d (\tilde{\sigma}^l(t)\theta, \theta)^2.$$

This hypothesis is used to obtain upper bounds for the Malliavin derivative of ξ (see Lemma 6.2).

We introduce a number of corrector. The first one $\chi^0 = \chi^0(z, y)$ is a periodic solution of the equation

$$(23) \quad \operatorname{div}_z (a(z, y) \nabla_z \chi^0(z, y)) = -\operatorname{div}_z a(z, y);$$

here $y \in \mathbb{R}^n$ is a parameter. We choose an additive constant in such a way that $\int_{\mathbb{T}^d} \chi^0(z, y) dz = 0$. Higher order correctors are defined as periodic solutions of the equations

$$(24) \quad \operatorname{div}_z (\mathbf{a}(z, y) \nabla_z \chi^j(z, y)) = -\mathcal{L}_y \chi^{j-1}(z, y), \quad j = 1, 2, \dots, J^0.$$

Notice that $\int_{\mathbb{T}^d} \chi^{j-1}(z, y) dz = 0$ for all $j = 1, 2, \dots, J^0$, thus the compatibility condition is satisfied and the equations are solvable. Moreover, by the classical Schauder estimates, for any $N > 0$ there exists C_N such that

$$\|\chi^j\|_{C^N(\mathbb{T}^d \times \mathbb{R}^n)} \leq C_N.$$

Remark 2 It is interesting to compare the correctors defined in (23) and (24) with the correctors used in the proof of Theorem 1. It follows from (4) and (23) that the zero order correctors coincide. Observe however that the higher order correctors need not coincide.

The effective matrix is given by

$$\mathbf{a}^{\text{eff}} = \int_{\mathbb{R}^n} \int_{\mathbb{T}^d} \left(\mathbf{a}(z, y) + \mathbf{a}(z, y) \frac{\partial}{\partial z} \chi^0(z, y) \right) \rho(y) dz dy.$$

We also introduce the matrices

$$\mathbf{a}^{k, \text{eff}} = \int_{\mathbb{R}^n} \int_{\mathbb{T}^d} [\mathbf{a}(z, y) \nabla_z \chi^k(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^k(z, y))] \rho(y) dz dy, \quad k = 1, 2, \dots,$$

and matrix valued functions

$$\begin{aligned} \widehat{\mathbf{a}}^0(z, y) &= \mathbf{a}(z, y) + \mathbf{a}(z, y) \nabla_z \chi^0(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^0(z, y)), \\ \widehat{\mathbf{a}}^k(z, y) &= \mathbf{a}(z, y) \nabla_z \chi^k(z, y) + \nabla_z (\mathbf{a}(z, y) \chi^k(z, y)), \quad k = 1, 2, \dots, \\ \langle \mathbf{a} \rangle^0(y) &= \int_{\mathbb{T}^d} (\widehat{\mathbf{a}}^0(z, y) - \mathbf{a}^{\text{eff}}) dz, \\ \langle \mathbf{a} \rangle^k(y) &= \int_{\mathbb{T}^d} (\widehat{\mathbf{a}}^k(z, y) - \mathbf{a}^{k, \text{eff}}) dz, \quad k = 1, 2, \dots \end{aligned}$$

The functions $u^j = u^j(x, t)$ are defined as solutions of problems

$$(25) \quad \begin{aligned} \frac{\partial}{\partial t} u^j &= \operatorname{div}(\mathbf{a}^{\text{eff}} \nabla u^j) + \sum_{k=1}^j \mathbf{a}^{k, \text{eff}} \frac{\partial^2}{\partial x^2} u^{j-k} \\ u^j(x, 0) &= 0 \end{aligned}$$

Since for each $j = 1, 2, \dots$ problem (7) has a unique solution, the functions u^j are uniquely defined.

Finally, we consider the equations

$$(26) \quad \mathcal{L}Q^j(y) = \langle a \rangle^j(y), \quad j = 0, 1, \dots$$

According to [21], for each j this equation has a unique up to an additive constant solution of at most polynomial growth. Denote

$$(27) \quad \Lambda^2 = \{(\Lambda^2)^{ijml}\} = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial y^{r_1}} (Q^0)^{ij}(y) \right] q^{r_1 r_2}(y) \left[\frac{\partial}{\partial y^{r_2}} (Q^0)^{ml}(y) \right] \rho(y) dy.$$

The matrix Λ^2 is non-negative. Consequently its square root Λ is well defined.

In the diffusive case the following result holds:

Theorem 2 *Under conditions c1., c4. and a1. – a5. the normalized functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left(u^\varepsilon - u^0 - \sum_{j=0}^{J_0} \varepsilon^{j(2-\alpha)} u^j \right)$$

converge in law, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d \times (0, T))$ to a solution of (8) with the standard d^2 -dimensional Brownian motion W . and Λ defined in (27)

Proof. Consider the following expression:

$$V^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x, t) - \sum_{k=0}^{J_0} \varepsilon^{k\delta} \left(u^k(x, t) + \sum_{j=0}^{J_0-k} \varepsilon^{(j\delta+1)} \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) \right) \right\},$$

where $\chi(z, y)$ and $u^k(x, t)$ are defined in (24) and (25), respectively. Considering equations (23)–(25) and the definitions of $a^{k, \text{eff}}$ and $\widehat{a}^k(z, y)$, and applying

Ito's formula to V^ε , after straightforward rearrangements we obtain

$$\begin{aligned}
(28) \quad & dV^\varepsilon(x, t) - \operatorname{div} \left[\mathbf{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V^\varepsilon \right] dt \\
&= \varepsilon^{-\alpha/2} \left(\sum_{j=0}^{J^0} \varepsilon^{j\delta} \left[\widehat{\mathbf{a}}^0 \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \mathbf{a}^{\text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j \right) dt \\
&+ \left(\varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{\mathbf{a}}^k \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \mathbf{a}^{k, \text{eff}} \right] \frac{\partial^2}{\partial x^2} u^j \right) dt \\
&+ \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(1-\alpha+(k+j)\delta)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) dB_t \\
&+ \mathcal{R}^\varepsilon(x, t) dt,
\end{aligned}$$

with the initial condition

$$V^\varepsilon(x, 0) = \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1-\alpha/2)} \chi^j \left(\frac{x}{\varepsilon}, \xi_0 \right) \nabla u^k(x, 0);$$

here the $n \times n$ matrix $\sigma(y)$ is such that $\sigma(y)\sigma^*(y) = 2q(y)$, B is a standard n -dimensional Brownian motion, and

$$(29) \quad \mathcal{R}^\varepsilon(x, t) = \varepsilon^{-\alpha/2} \sum_{j=0}^{N^0} \varepsilon^{1+j\delta} \vartheta^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \Phi^j(x, t)$$

with periodic in z smooth functions $\vartheta^j = \vartheta^j(z, y)$ of at most polynomial growth in y , and Φ^j satisfying (15).

We have

$$\begin{aligned}
\mathbf{E} \|\mathcal{R}^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 &\leq C \varepsilon^{1-\alpha/2} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} (1 + |y|)^{N_1} (1 + |x|)^{-2n} \rho(y) dy dx dt \\
&\leq C \varepsilon^{1-\alpha/2}.
\end{aligned}$$

Similarly, $\mathbf{E} \|V^\varepsilon(\cdot, 0)\|_{L^2(\mathbb{R}^d)}^2 \leq C \varepsilon^{1-\alpha/2}$. Therefore, the function \mathcal{R}^ε on the right-hand side of (28) and the initial condition do not contribute to the limit equation.

In order to estimate the contribution of the stochastic term on the right-hand side of (28), let V_F^ε be a solution of

$$(30) \quad \begin{aligned} & dV_F^\varepsilon(x, t) - \operatorname{div} \left[\mathbf{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_F^\varepsilon \right] dt \\ &= \sum_{k=0}^{J^0} \sum_{j=0}^{J^0-k} \varepsilon^{(1-\alpha+(k+j)\delta)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^j \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^k(x, t) dB_t \end{aligned}$$

with initial condition $V_F^\varepsilon(x, 0) = 0$. The following crucial statement is admitted for the moment and will be proved in the next section.

Lemma 5.1 *For a solution of problem (30) the following estimate holds:*

$$(31) \quad \mathbf{E} \|V_F^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha} |\log \varepsilon|.$$

It is straightforward to check that the function

$$\begin{aligned} H^\varepsilon(x, t) &= \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \varepsilon^{j\delta} \left[\widehat{\mathbf{a}}^0 \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \mathbf{a}^{\text{eff}} - \langle \mathbf{a} \rangle^0 \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \right] \frac{\partial^2}{\partial x^2} u^j \\ &+ \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \left[\widehat{\mathbf{a}}^k \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) - \mathbf{a}^{k, \text{eff}} - \langle \mathbf{a} \rangle^k \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \right] \frac{\partial^2}{\partial x^2} u^j \end{aligned}$$

admits the estimate

$$(32) \quad \mathbf{E} \|H^\varepsilon\|_{L^2(0, T; H^{-1}(\mathbb{R}^d))}^2 \leq C \varepsilon^{2-\alpha}.$$

Therefore, it is sufficient to characterize the limit behaviour of a solution of the following equation

$$(33) \quad \begin{aligned} & \partial_t V_3^\varepsilon(x, t) - \operatorname{div} \left[\mathbf{a} \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_3^\varepsilon \right] \\ &= \varepsilon^{-\alpha/2} \left(\sum_{j=0}^{J^0} \varepsilon^{j\delta} \langle \mathbf{a} \rangle^0 \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \frac{\partial^2}{\partial x^2} u^j \right) \\ &+ \varepsilon^{-\alpha/2} \sum_{j=0}^{J^0} \sum_{k=1}^{J^0-j} \varepsilon^{(k+j)\delta} \langle \mathbf{a} \rangle^k \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \frac{\partial^2}{\partial x^2} u^j \end{aligned}$$

with initial condition $V_3^\varepsilon(x, 0) = 0$.

According to [21] the processes

$$A^k(t) = \int_0^t \langle a \rangle^k(\xi_s) ds$$

satisfy the functional central limit theorem (invariance principle), that is the process

$$A^{\varepsilon,k}(t) = \varepsilon^{\frac{\alpha}{2}} \int_0^{\varepsilon^{-\alpha} t} \langle a \rangle^k(\xi_s) ds$$

converges in law in $(C[0, T])^{d^2}$ to a d^2 -dimensional Brownian motion with covariance matrix

$$(\Lambda_k)^2 = \{(\Lambda_k^2)^{ijml}\} = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial y^{r_1}} (Q^k)^{ij}(y) \right] q^{r_1 r_2}(y) \left[\frac{\partial}{\partial y^{r_2}} (Q^k)^{ml}(y) \right] \rho(y) dy.$$

with matrix-function Q^k defined in (26). It was proven in [13, Lemma 5.1] that, given the fact that $A^{\varepsilon,k}(\cdot)$ satisfies the invariance principle, a solution V_3^ε of problem (33) converges in law in $L^2(\mathbb{R}^d \times (0, T))$ towards a solution of problem (8) with $\Lambda = \Lambda_0$ defined in (27). Combining this convergence results with estimate (32) and the statement of Lemma 5.1 completes the proof of Theorem 2. \square

6 Proof of Lemma 5.1

In order to prove Lemma 5.1 we consider first the following problem:

$$(34) \quad \begin{aligned} dV_B^\varepsilon(x, t) - \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V_B^\varepsilon \right] dt \\ = \varepsilon^{(1-\alpha)} \sigma \left(\xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla_y \chi^0 \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^0(x, t) dB_t \end{aligned}$$

with initial condition $V_B^\varepsilon(x, 0) = 0$. In fact we keep only the smallest power of ε in the right-hand side of (30) (since $\delta = 2 - \alpha > 0$). Our goal is to prove the following estimate:

$$(35) \quad \mathbf{E} \|V_B^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha} |\log \varepsilon|.$$

Then the statement of Lemma 5.1 can be proven in exactly the same way as the previous estimate.

The stationary process ξ satisfies the following SDE:

$$(36) \quad d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dB_t.$$

It is well known that under the assumption **a3**, ξ satisfies for any $T \geq 0$ and any $p \geq 2$

$$\mathbf{E} \left(\sup_{t \in [0, T]} |\xi_t|^p \right) \leq C$$

where C is a positive constant depending on $p, T, K_{b, \sigma}$.

For the Malliavin calculus we borrow some notations from Nualart [18] (see the details inside). D is the Malliavin derivative defined on smooth random variables F (see Definition 1.2.1 in [18]). Since B is a d -dimensional Brownian motion, $D_t^j(F)$, $t \in [0, T]$, $j = 1, \dots, d$, is the derivative of a random variable F as an element of $L^2([0, T] \times \Omega; \mathbb{R}^d)$. Hence $D_t(F)$ is a d -dimensional vector. The space $\mathbb{D}^{1, p}$, $p \geq 1$, is the closure of the class of smooth random variables with respect to the norm

$$\|F\|_{1, p} = [\mathbf{E}(|F|^p) + \mathbf{E}(\|DF\|_{L^2}^p)]^{1/p}.$$

For $p = 2$, $\mathbb{D}^{1, 2}$ is a Hilbert space. Then by induction we can define $\mathbb{D}^{k, p}$ the space of k -times differentiable random variables where the k derivatives are in $L^p(\Omega)$. Finally

$$\mathbb{D}^{k, \infty} = \bigcap_{p \geq 1} \mathbb{D}^{k, p}, \quad \mathbb{D}^\infty = \bigcap_{k \in \mathbb{N}} \mathbb{D}^{k, \infty}.$$

The next result can be found in [18], Theorems 2.2.1 and 2.2.2.

Lemma 6.1 *Under **a3**, the coordinate ξ_t^i belongs to $\mathbb{D}^{1, \infty}$ for any $t \in [0, T]$ and $i = 1, \dots, d$. Moreover*

$$(37) \quad \sup_{0 \leq r \leq T} \mathbf{E} \left(\sup_{r \leq t \leq T} |D_r^j \xi_t^i|^p \right) < +\infty.$$

The derivative $D_r^j \xi_t^i$ satisfies the following linear equation:

$$D_r^j \xi_t^i = \sigma_{i, j}(\xi_r) + \sum_{1 \leq k, l \leq d} \int_r^t \tilde{\sigma}_{i, k}^l(s) D_r^j(\xi_s^k) dB_s^l + \sum_{k=1}^d \int_r^t \tilde{b}_{i, k}(s) D_r^j(\xi_s^k) ds$$

for $r \leq t$ a.e. and $D_r^j \xi_t = 0$ for $r > t$ a.e., where σ^j is the column number j of the matrix σ and where for $1 \leq i, j \leq d$ and $1 \leq l \leq d$, $\tilde{b}_{i,j}(s)$ and $\tilde{\sigma}_{i,j}^l(s)$ are given by (22):

$$\tilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \quad \tilde{\sigma}_{i,j}^l(s) = (\partial_{x_j} \sigma_{i,l})(\xi_s).$$

In the rest of this section for two positive constants c and C , the function $g_{c,C}(x, t)$ is defined for $t > 0$ and $x \in \mathbb{R}^d$ as follows:

$$g_{c,C}(x, t) = \frac{c}{t^{\frac{d}{2}}} \exp\left(-\frac{C|x|^2}{t}\right).$$

We also define

$$(38) \quad \psi^\varepsilon(r) = \sup_{t \in [0, T]} \|D_r \xi_{t/\varepsilon^\alpha}\|$$

and

$$(39) \quad G^\varepsilon(y, t) = G\left(\frac{y}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}, y, t\right) = \varepsilon^{1-\alpha} \sigma(\xi_{t/\varepsilon^\alpha}) \nabla \chi^0\left(\frac{y}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^0(y, t).$$

Note that the latter function is bounded by $\varepsilon^{1-\alpha} K_G$. The Malliavin derivative of G^ε can be computed by a chain rule argument:

$$D_r G^\varepsilon(y, s) = \varepsilon^{1-\alpha} \nabla u^0(y, s) D_r \xi_{s/\varepsilon^\alpha} \left[\tilde{\sigma}(s) \nabla \chi^0\left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) + \sigma(\xi_{s/\varepsilon^\alpha}) \nabla_z \nabla \chi^0\left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \right].$$

Hence

$$\|D_r G^\varepsilon(y, s)\| \leq \varepsilon^{1-\alpha} K_G \|D_r \xi_{s/\varepsilon^\alpha}\| \leq \varepsilon^{1-\alpha} K_G \psi^\varepsilon(r).$$

6.1 Uniform estimates of the Malliavin derivative

In this section we consider the quantity

$$\psi^\varepsilon(r) = \sup_{t \in [0, T]} \|D_r \xi_{t/\varepsilon^\alpha}\| = \sup_{\tau \in [r, T/\varepsilon^\alpha]} \|D_r \xi_\tau\|$$

and show that under our standing conditions it admits uniform in ε estimates.

Lemma 6.2 *Under Condition a5., there exists a constant C_p such that for any T and $\varepsilon > 0$*

$$\mathbf{E}(|\psi^\varepsilon(r)|^p) \leq C_p.$$

Proof. Recall that if $Z(t) = D_r \xi_t$ is the matrix-valued process defined by:

$$Z(t) = \sigma(\xi_r) + \int_r^t \tilde{b}(s) Z(s) ds + \sum_{1 \leq l \leq d} \int_r^t \tilde{\sigma}^l(s) Z(s) dB_s^l$$

and σ is bounded as specified in Condition **a2**. Each column Z^j of Z satisfies the linear d -dimensional SDE

$$Z_t^j = \sigma_j(\xi_r) + \int_r^t \tilde{b}(s) Z^j(s) ds + \sum_{1 \leq l \leq d} \int_r^t \tilde{\sigma}^l(s) Z^j(s) dB_s^l$$

where σ_j is the j -th column of σ . We apply the results contained in Appendix B (see also Section 6.7) of [11], more precisely Equation (B.11). For any $p' < p$, the process $X_t = |Z_t^j|^{p'}$ satisfies the scalar linear equation:

$$\begin{aligned} dX_t &= \left(p' Q(t, \Theta_t) + \frac{(p')^2}{2} \sum_{l=1}^d (\tilde{\sigma}^l(t) \Theta_t, \Theta_t)^2 \right) X_t dt \\ &+ p' X_t \sum_{l=1}^d (\tilde{\sigma}^l(t) \Theta_t, \Theta_t) dB_t^l \end{aligned}$$

where $\Theta_t = Z_t/|Z_t|$ belongs to \mathbb{S}^{d-1} . Hence

$$\begin{aligned} X_t &= |\sigma_j(\xi_r)| \exp \left[\int_r^t \left(p' Q(s, \Theta_s) + \frac{(p')^2}{2} \sum_{l=1}^d (\tilde{\sigma}^l(s) \Theta_s, \Theta_s)^2 \right) ds \right. \\ &\quad \left. + p' \int_r^t \sum_{l=1}^d (\tilde{\sigma}^l(s) \Theta_s, \Theta_s) dB_s^l - \frac{(p')^2}{2} \int_r^t \sum_{l=1}^d (\tilde{\sigma}^l(s) \Theta_s, \Theta_s)^2 ds \right]. \end{aligned}$$

From our assumption **a5.**, $X_t \leq M_t$ where M is the martingale:

$$M_t = \lambda^{-1} \exp \left[p' \int_r^t \sum_{l=1}^d (\tilde{\sigma}^l(s) \Theta_s, \Theta_s) dB_s^l - \frac{(p')^2}{2} \int_r^t \sum_{l=1}^d (\tilde{\sigma}^l(s) \Theta_s, \Theta_s)^2 ds \right].$$

Since $\tilde{\sigma}^l(s)$ is bounded (condition **a2.**), M is a true martingale. By Doob's inequality,

$$\mathbf{E} \left(\sup_{t \geq r} |Z(t)|^p \right) = \mathbf{E} \left(\sup_{t \geq r} (X_t)^{p/(p')} \right) \leq \left(\frac{p}{p-p'} \right)^{p'/p} = C_p.$$

This achieves the proof. □

Note that in dimension one ($d = 1$), Condition **a5**. becomes

$$\partial_x b(\xi_t) + \frac{p-1}{2}(\partial_x \sigma(\xi_t))^2 \leq 0.$$

This assumption can be easily deduced since $\zeta_t = D_r \xi_t$ satisfies the linear SDE:

$$\zeta_t = \sigma(\xi_r) + \int_r^t \partial_x \sigma(\xi_u) \zeta_u dB_u + \int_r^t \partial_x b(\xi_u) \zeta_u du,$$

which can be solved explicitly. Hence

$$\begin{aligned} |\zeta_t|^p &= |\sigma(\xi_r)|^p \exp \left[\int_r^t p \partial_x \sigma(\xi_u) dB_u - \frac{1}{2} \int_r^t p^2 (\partial_x \sigma(\xi_u))^2 du \right] \\ &\quad \exp \left[p \int_r^t \left(\partial_x b(\xi_u) + \frac{p-1}{2} (\partial_x \sigma(\xi_u))^2 \right) du \right]. \end{aligned}$$

The condition **a5**. is here also necessary.

As an example, let us consider the multidimensional Ornstein-Uhlenbeck process:

$$d\xi_t = B\xi_t dt + \Sigma dB_t,$$

where B and Σ are two $d \times d$ matrices. Here $\tilde{b}(t) = B$ and $\tilde{\sigma}^l(t) = 0$. Therefore, condition **a5**. is reduced to $(B\theta, \theta) \leq 0$.

6.2 Construction of a mild solution

Our aim is to prove that the solution V_B^ε of (34) is given by:

$$(40) \quad V_B^\varepsilon(x, t) = \int_0^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, y, s) G\left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, y, s\right) dy dB_s,$$

where Γ^ε is the fundamental solution of the PDE:

$$(41) \quad \frac{\partial u^\varepsilon}{\partial t}(x, t) = \operatorname{div} \left[a\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^\varepsilon \right].$$

The stochastic integral in (40) has to be defined properly since $\Gamma^\varepsilon(x, t, y, s)$ is measurable w.r.t. the σ -field $\mathcal{F}_{t/\varepsilon^\alpha}$ generated by the random variables B_u with $u \leq t/\varepsilon^\alpha$.

It is well known (see among other [6], Chapter 9, [2] or [23]) that there exist two constants ς and ϖ such that if $2i + j \leq 2$ and $t > s$ then

$$(42) \quad |\partial_t^i \partial_x^j \Gamma^\varepsilon(x, t, y, s)| \leq \frac{1}{(t-s)^{\frac{2i+j}{2}}} g_{\varsigma, \varpi}(x-y, t-s).$$

These inequalities are called **Aronson's estimates**. These constants ς and ϖ depend in fact only on the constant λ of **c1**. The key result of the Malliavin derivative of Γ^ε is the following.

Proposition 6.1 *The fundamental solution Γ^ε of (41) and its spatial derivatives belong to $\mathbb{D}^{1, \infty}$ for every $(t, s) \in [0, T]^2$, $s < t$ and $(x, y) \in (\mathbb{R}^n)^2$. Moreover it satisfies the following inequalities:*

$$(43) \quad |D_r \Gamma^\varepsilon(x, t, y, s)| \leq \psi^\varepsilon(r) g_{\varsigma, \varpi}(x-y, t-s),$$

$$(44) \quad |D_r \partial_x \Gamma^\varepsilon(x, t, y, s)| \leq \frac{\psi^\varepsilon(r)}{(t-s)^{1/2}} g_{\varsigma, \varpi}(x-y, t-s),$$

and

$$(45) \quad |D_r \partial_x^2 \Gamma^\varepsilon(x, t, y, s)| \leq \frac{\psi^\varepsilon(r)}{(t-s)} g_{\varsigma, \varpi}(x-y, t-s).$$

Here the constants ς and ϖ just depend on the uniform ellipticity constant λ and ψ^ε depends only on ξ and is defined by (38).

The proof of this result is quite long and is based on the construction of Γ . For a more pleasant reading, we postpone it in the Appendix.

As a consequence of [1, Theorem 5.10] and [1, Theorem 5.11] one can easily deduce that the right-hand side of Equation (40) is well defined and is the unique classical solution of (34).

6.3 Proof of Lemma 5.1

Recalling (39), since the mean value of $\nabla_y \chi^0(z, y)$ in z is equal to zero, there exists a periodic in z function $X^0 = X^0(z, y)$ such that $\operatorname{div}_z X^0(z, y) = \sigma(y) \nabla_y \chi^0(z, y)$. Moreover, X^0 is smooth and has at most polynomial growth in y .

The statement of Lemma 5.1 is now an immediate consequence of Lemma 6.2 (with $p > 2$) and the next result.

Lemma 6.3 *The following estimate holds: for any $p > 1$*

$$(46) \quad \mathbf{E} \|V_B^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C \varepsilon^{4-2\alpha} |\log \varepsilon| \left[1 + \left[\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right].$$

Proof. We know that

$$(47) \quad \begin{aligned} V_B^\varepsilon(x, t) &= \int_0^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\ &= \int_{t-\varepsilon^2}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\ &\quad + \int_0^{t-\varepsilon^2} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\ &= J_1^\varepsilon(x, t) + J_2^\varepsilon(x, t). \end{aligned}$$

We will denote by t_ε the time $t - \varepsilon^2$ and we begin with J_1^ε

$$\begin{aligned} J_1^\varepsilon(x, t) &= \int_{t_\varepsilon}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) G^\varepsilon(x', s) dx' dB_s \\ &= \varepsilon^{1-\alpha} \int_{t_\varepsilon}^t \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\ &= \varepsilon^{1-\alpha} \int_{t_\varepsilon}^t j^\varepsilon(x, t, s) dB_s. \end{aligned}$$

Using isometric property of the anticipating Ito integral (see Eq. (3.5) in [17]) we get

$$(48) \quad \begin{aligned} \mathbf{E}((J_1^\varepsilon(x, t))^2) &= \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t |j^\varepsilon(x, t, s)|^2 ds \\ &\quad + \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t |D_r j^\varepsilon(x, t, s)|^2 ds dr. \end{aligned}$$

By the Aronson estimate (42), $\Gamma^\varepsilon(x, t, x', s) \leq g_{\varsigma, \varpi}(x - x', t - s)$. Moreover for any $\mathbf{k} = (k_0, k_1, \dots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k}, N}$ such that

$$(49) \quad |\partial^{\mathbf{k}} u^0| \leq C_{\mathbf{k}, N} (1 + |x|)^{-N},$$

where $\partial^{\mathbf{k}} u^0(x, t) = \frac{\partial^{k_0}}{\partial t^{k_0}} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \dots \frac{\partial^{k_d}}{\partial x_d^{k_d}} u^0(x, t)$. The matrix σ is bounded and χ^0 is at most of polynomial growth w.r.t. y . This yields

$$\begin{aligned} |j^\varepsilon(x, t, s)| &= \left| \int_{\mathbf{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' \right| \\ &\leq C_N \int_{\mathbf{R}^d} g_{\varsigma, \varpi}(x - x', t - s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\ &\leq C_N (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}. \end{aligned}$$

for all t, s such that $0 \leq s < t \leq T$ and for all $N > 0$. Therefore,

$$\begin{aligned} \varepsilon^{2-2\alpha} \mathbf{E} \int_{t_\varepsilon}^t |j^\varepsilon(x, t, s)|^2 ds &\leq C_N \varepsilon^{2-2\alpha} \int_{t_\varepsilon}^t (1 + |x|)^{-2N} \mathbf{E} (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} ds \\ &\leq C_N \varepsilon^{4-2\alpha} (1 + |x|)^{-2N}. \end{aligned}$$

The Malliavin derivative of j^ε is given by a chain rule argument and the estimates (42) and (43) leads to

$$\begin{aligned} |D_r j^\varepsilon(x, t, s)| &\leq \int_{\mathbf{R}^d} \left| D_r \Gamma^\varepsilon(x, t, x', s) \sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) \right| dx' \\ &\quad + \int_{\mathbf{R}^d} \Gamma^\varepsilon(x, t, x', s) \left| D_r \left[\sigma\left(\xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \right] \right| |\nabla u^0(x', s)| dx' \\ &\leq \psi^\varepsilon(r) \int_{\mathbf{R}^d} g_{\varsigma, \varpi}(x - x', t - s) \theta\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) |\nabla u^0(x', s)| dx' \end{aligned}$$

with

$$\theta(z, y) = \left| \sigma(y) \nabla_y \chi^0(z, y) \right| + \left| \nabla_y \sigma(y) \nabla_y \chi^0(z, y) + \sigma(y) \partial_y^2 \chi^0(z, y) \right|.$$

Hence

$$\begin{aligned} |D_r j^\varepsilon(x, t, s)| &\leq C \psi^\varepsilon(r) \int_{\mathbf{R}^d} g_{\varsigma, \varpi}(x - x', t - s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\ &\leq C \psi^\varepsilon(r) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}. \end{aligned}$$

Thereby for any $p > 1$ and for q the Hölder conjugate of p

$$\begin{aligned}
& \mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t |D_r j^\varepsilon(x, t, s)|^2 dr ds \\
& \leq C^2 \mathbf{E} \int_{t_\varepsilon}^t \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^2 (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} (1 + |x|)^{-2N} ds dr \\
& \leq C^2 \left[\mathbf{E} \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \left[\mathbf{E} \int_{t_\varepsilon}^t (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2qK} ds \right]^{1/q} (1 + |x|)^{-2N} \\
& \leq C^2 \varepsilon \left[\mathbf{E} \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} (1 + |x|)^{-2N}.
\end{aligned}$$

Coming back to (48) we obtain

$$\mathbf{E}((J_1^\varepsilon(x, t))^2) \leq C \varepsilon^{3-2\alpha} \left[\varepsilon + \left[\mathbf{E} \int_{t_\varepsilon}^t (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right] (1 + |x|)^{-2N}$$

and finally

$$(50) \quad \mathbf{E} \left(\|J_1^\varepsilon\|_{L^2(\mathbb{R}^d \times [0, T])}^2 \right) \leq C \varepsilon^{4-2\alpha} \left[1 + \left[\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right]^{1/p} \right].$$

For J_2^ε , we use an integration by parts:

$$\begin{aligned}
J_2^\varepsilon(x, t) &= \varepsilon^{1-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) \sigma(\xi_{\frac{s}{\varepsilon^\alpha}}) \nabla_y \chi^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\
&= \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \Gamma^\varepsilon(x, t, x', s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \frac{\partial^2}{\partial x^2} u^0(x', s) dx' dB_s \\
&+ \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} \int_{\mathbb{R}^d} \nabla_y \Gamma^\varepsilon(x, t, x', s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' dB_s \\
&= \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} w_1^\varepsilon(x, t, s) dB_s + \varepsilon^{2-\alpha} \int_0^{t_\varepsilon} w_2^\varepsilon(x, t, s) dB_s \\
(51) \quad &= I_1^\varepsilon(x, t) + I_2^\varepsilon(x, t).
\end{aligned}$$

Using again isometric property of the anticipating Ito integral we have for $j = 1$ or 2

$$\mathbf{E}((I_j^\varepsilon)^2) = \varepsilon^{4-2\alpha} \mathbf{E} \int_0^{t_\varepsilon} |w_j^\varepsilon(x, t, s)|^2 ds + \varepsilon^{4-2\alpha} \mathbf{E} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} |D_r w_j^\varepsilon(x, t, s)|^2 ds dr.$$

Using (42), (43) and (49), by the same arguments as in the proof of (50) we have

$$(52) \quad \mathbf{E} \|I_1^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C\varepsilon^{4-2\alpha} \left[1 + \left(\mathbf{E} \int_0^T (\psi^\varepsilon(r))^{2p} dr \right)^{1/p} \right].$$

In order to obtain an upper bound for I_2^ε we use Aronson's estimate for the derivative of Γ^ε :

$$\begin{aligned} |w_2^\varepsilon(x, t, s)| &= \left| \int_{\mathbb{R}^d} \nabla_{x'} \Gamma^\varepsilon(x, x', t, s) X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) dx' \right| \\ &\leq \int_{\mathbb{R}^d} \frac{1}{|t-s|^{1/2}} g_{s, \varpi}(x-x', t-s) \left| X^0\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) \nabla u^0(x', s) \right| dx' \\ &\leq \frac{C}{|t-s|^{1/2}} \int_{\mathbb{R}^d} g_{s, \varpi}(x-x', t-s) (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x'|)^{-N} dx' \\ &\leq \frac{C}{|t-s|^{1/2}} (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^K (1 + |x|)^{-N}. \end{aligned}$$

Hence

$$\mathbf{E} \int_0^{t_\varepsilon} |w_2^\varepsilon(x, t, s)|^2 ds \leq C |\log(\varepsilon)| (1 + |x|)^{-2N}.$$

For the Malliavin derivative we proceed as before with (44):

$$|D_r w_2^\varepsilon(x, t, s)| \leq \psi^\varepsilon(r) \int_{\mathbb{R}^d} \frac{1}{|t-s|^{1/2}} g_{s, \varpi}(x-x', t-s) \theta\left(\frac{x'}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}\right) |\nabla u^0(x', s)| dx'$$

with

$$\theta(z, y) = \left| X^0(z, y) \right| + \left| \nabla_y X^0(z, y) \right|.$$

Again

$$\begin{aligned} &\mathbf{E} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} |D_r w_2^\varepsilon(x, t, s)|^2 dr ds \\ &\leq C(1 + |x|)^{-2N} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} \frac{1}{|t-s|} \mathbf{E} \left[(\psi^\varepsilon(r))^2 (1 + |\xi_{\frac{s}{\varepsilon^\alpha}}|)^{2K} \right] ds dr \\ &\leq C(1 + |x|)^{-2N} \int_0^{t_\varepsilon} \int_0^{t_\varepsilon} \frac{1}{|t-s|} \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} ds dr \\ &\leq C |\log(\varepsilon)| (1 + |x|)^{-2N} \int_0^T \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} dr. \end{aligned}$$

From the last two inequalities we deduce that

$$\mathbf{E} \|I_2^\varepsilon\|_{L^2(\mathbb{R}^d \times (0, T))}^2 \leq C\varepsilon^{4-2\alpha} |\log(\varepsilon)| \left[1 + \int_0^T \mathbf{E} [(\psi^\varepsilon(r))^{2p}]^{1/p} dr \right].$$

The last estimate combined with (50) and (52) yields the desired inequality (46). This completes the proof. \square

Appendix: construction and properties of the fundamental solution

Here we develop the required arguments in order to prove Proposition 6.1. In other words we want to prove that the fundamental solution Γ^ε of the PDE (41)

$$\frac{\partial u^\varepsilon}{\partial t}(x, t) = \operatorname{div} \left[a \left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla u^\varepsilon \right].$$

is in $\mathbb{D}^{1,2}$ (Malliavin differentiability) and that the Aronson estimates for the Malliavin derivative (Inequalities (43), (44) and (45)) hold.

To simplify the notations, we assume that $\varepsilon = 1$ and thus Γ is the fundamental solution of the PDE:

$$(53) \quad \frac{\partial u}{\partial t}(x, t) = \operatorname{div} \left[a \left(x, \xi_t \right) \nabla u \right].$$

Let T be a fixed positive constant. We assume that the time variables belong to the interval $[0, T]$. For any $r \in [0, T]$ we define

$$(54) \quad \psi(r) = \sup_{t \in [r, T]} |D_r \xi_t|.$$

From Lemma 6.1 we have for any $p \geq 2$

$$\sup_{r \in [0, T]} \mathbf{E} (\psi(r)^p) < +\infty.$$

We denote by \mathcal{L} and $a(x, t)$ the operator and the matrix:

$$\mathcal{L} = \operatorname{div} \left[a \left(x, \xi_t \right) \nabla \right], \quad a(x, t) = a \left(x, \xi_t \right).$$

The fundamental solution can be defined by the parametrix method:

$$(55) \quad \Gamma(x, t, y, s) = Z(x, t, y, s) + \int_s^t \int_{\mathbb{R}^d} Z(x, t, \zeta, r) \Phi(\zeta, r, y, s) d\zeta dr.$$

One construction of the fundamental solution Γ can be found in [16], Chapter IV, sections 11 to 13 (or [6], Chapter I). For the Malliavin differentiability property of Γ , we use the approach developed in Alòs et al. [1].

For a fixed ε the solution V_B^ε is well defined and satisfies all required properties. But $\varepsilon > 0$ is a parameter of the equation and thus it will appear in all estimates (43), (44) and (45) on Γ^ε , and therefore on v^ε . Indeed, following the proof in [1] we might have extra negative powers of ε in the estimates of the Malliavin derivative of Γ^ε . Even the constants ς and ϖ may depend on ε and may go to ∞ as ε goes to zero. In other words for the initial homogenization problem, we need more accurate estimates on Γ^ε as in Proposition 6.1.

In order to obtain this property and thus to prove Proposition 6.1 completely, another construction of the fundamental solution Γ^ε can be done ([6], Chapter 9, or [23]). We denote by $a^\varepsilon(x, t)$ the matrix:

$$a^\varepsilon(x, t) = a\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right).$$

First assume that a^ε just depends on t . Hence the fundamental solution Γ^ε is given by the formula:

$$(56) \quad \Gamma^\varepsilon(x, t, y, s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, \zeta, s) d\zeta,$$

where $V^\varepsilon(t, \zeta, s)$ is the following function:

$$V^\varepsilon(t, \zeta, s) = \exp\left(-\int_s^t \langle \bar{a}^\varepsilon(u) \zeta, \zeta \rangle du\right).$$

From this expression of Γ^ε , the Aronson estimates (42) can be derived (see Theorem 1, Chapter 9 in [6] for the details). Hence the Malliavin derivative of Γ^ε is:

$$\begin{aligned} D_r \Gamma^\varepsilon(x, t, y, s) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} D_r V^\varepsilon(t, \zeta, s) d\zeta \\ &= -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, \zeta, s) \left(\int_s^t \left\langle \frac{\partial a}{\partial y_k}(\xi_{u/\varepsilon^\alpha}) \zeta, \zeta \right\rangle (D_r \xi_{u/\varepsilon^\alpha}^k) du \right) d\zeta. \end{aligned}$$

Since we assume that the derivative of a is bounded (condition **a2**), we obtain:

$$|D_r \Gamma(x, t, y, s)| \leq K_a \psi^\varepsilon(r) (t-s) \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, \zeta, s) |\zeta|^2 d\zeta.$$

Once again using Theorem 1, Chapter 9 in [6] to control

$$\left| \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V^\varepsilon(t, \zeta, s) |\zeta|^2 d\zeta \right| \leq \frac{1}{(t-s)} g_{\varsigma, \varpi}(x-y, t-s),$$

we deduce (43). The same inequalities (44) and (45) hold for the spatial derivatives of Γ .

For the general case, that is, if a depends also on x , let us begin by the following remark. If a just depends on x , then the fundamental solution Γ^ε for $a^\varepsilon(x) = a(x/\varepsilon)$ satisfies:

$$(57) \quad \Gamma^\varepsilon(x, t, y, s) = \frac{1}{\varepsilon^d} \Gamma^1(x/\varepsilon, t/\varepsilon^2, y/\varepsilon, s/\varepsilon^2).$$

Therefore if Γ^1 satisfies Estimate (42) with some constants ς and ϖ , then Γ^ε satisfies the same inequality with the same constants. Similar property holds for a^ε : if Γ^1 (resp. Γ^ε) is the fundamental solution for:

$$(58) \quad \frac{\partial u}{\partial t}(x, t) = \operatorname{div} \left[a \left(x, \xi_{\frac{t}{\varepsilon^{\alpha-2}}} \right) \nabla u \right].$$

(resp. for (41)), then (57) holds. Hence if we can have uniform estimates w.r.t. ε for Γ^1 , then the same estimates hold for Γ^ε . Let us recall how to construct the solution Γ^1 :

$$\Gamma^1(x, t, y, s) = Z(x, t, y, s) + \int_s^t \int_{\mathbb{R}^d} Z(x, t, \zeta, r) \Phi(\zeta, r, y, s) d\zeta dr.$$

where for a fixed y , Z is given by (56) with:

$$V(t, \zeta, s) = \exp \left(- \int_s^t \langle a(y, \xi_{u/\varepsilon^{\alpha-2}}) \zeta, \zeta \rangle du \right).$$

The function Φ is again the sum of iterated kernels

$$\Phi(x, t, y, s) = \sum_{m=1}^{\infty} \mathcal{K}_m(x, t, y, s)$$

where \mathcal{K}_m is an iterated kernel:

$$\mathcal{K}_m(x, t, y, s) = \int_s^t \int_{\mathbb{R}^n} \mathcal{K}(x, t, \zeta, r) \mathcal{K}_{m-1}(\zeta, r, y, s) d\zeta dr,$$

with

$$\begin{aligned} \mathcal{K}(x, t, y, s) &= \left[a\left(x, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right) - a\left(y, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right) \right] \partial_x^2 Z(x, t, y, s) \\ &+ \partial_x a\left(x, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right) \nabla_x Z(x, t, y, s). \end{aligned}$$

Notice that in the expression $a\left(x, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right) - a\left(y, \xi_{\frac{t}{\varepsilon^{\alpha-2}}}\right)$ the matrix a is evaluated two times at the same point $\xi_{\frac{t}{\varepsilon^{\alpha-2}}}$. Then the properties of \mathcal{K} , Φ and thus Γ^1 can be obtained similarly to [1]. The Malliavin derivatives of \mathcal{K} , Φ and thus Γ^1 only involve the Malliavin derivative of $\xi_{t/\varepsilon^{\alpha-2}}$, the Aronson estimates of Z , and the Lipschitz constant of a w.r.t. x . In other words all bounds (42), (43), (44) and (45) hold. The constants ς and ϖ do not depend on ε , and the random variable $\psi^\varepsilon(r)$ depends only on the Malliavin derivative of $\xi_{t/\varepsilon^\alpha}$.

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