On the Oseen vortices in dimension N = 2 for the inhomogeneous Navier-Stokes equations for radial initial density

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Abstract: This paper is dedicated to the proof of the existence of Lamb Oseen vortex for the inhomogeneous Navier-Stokes equation when $N = 2$. We restrict our study to the case of radial initial data ρ_0 which belongs to $C^\alpha(\mathbb{R}^2)$ with $0 < \alpha < 1$. To do this we recall the construction of fundamental solution for reaction diffusion equations. We point out that when $\rho_0 \neq 1$, our Lamb Oseen solution are not self similar. We prove also that the vorticity of inhomogeneous Navier-Stokes equations (when curlu_0 is radial and in $L^1(\mathbb{R}^2)$) converges asymptotically in time to the Oseen solution of Navier Stokes equation provided that the fluctuation of the initial density is sufficiently small.

1 Introduction

In this paper, we are concerned with the following model of incompressible viscous fluid with variable density:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = 0, \\ \mathrm{div} u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density, $Du = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor. We denote by μ the viscosity coefficients of the fluid, which is assumed to satisfy $\mu > 0$. The term $\nabla \Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\mathrm{div} u = 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathbb{T}_a^N$ with period a_i, in the i-th direction. We restrict ourselves to the case $N = 2$. The equations (1.1) are invariant under the sealing transformation:

$$\rho(t,x) \to \rho(\lambda^2 t, \lambda x);\ u(t,x) \to \lambda u(\lambda^2 t, \lambda x) \ \text{ and } \ \mathrm{curl} u(t,x) \to \lambda^2 u(\lambda^2 t, \lambda x), \ \ \lambda > 0. \tag{1.2}$$

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We observe that the vorticity equation can be rewritten as follows:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t \mathrm{curl} u + u \cdot \nabla \mathrm{curl} u - \mu \mathrm{div}(\frac{1}{\rho} \nabla \mathrm{curl} u) + \nabla^\perp(\frac{1}{\rho}) \cdot \nabla \Pi = 0, \\ \mathrm{div} u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0), \end{cases} \tag{1.3}$$

with $\nabla^\perp = (-\partial_2, \partial_1)$. Let us recall now some classical results for the Navier-Stokes equations in dimension $N = 2$.

1.1 Classical results of global strong solution for the Navier-Stokes equations

For the Navier Stokes equation, it is well known that there exists global self similar solution when $N = 2$. In order to obtain such solution, the main difficulty consists in proving the existence of global strong solution for initial vorticity in a functional space X which is invariant for the norm by the scaling of the equation (1.2). In addition X must be sufficiently large such that he admits homogeneous initial velocity of order -1. A natural candidate is the space $\mathcal{M}(\mathbb{R}^2)$ of all finite measures on $\mathbb{R}^2$, equipped with the total variation norm. The existence of global solution with initial data was first proved by Cottet [7] and independently by Giga, Miyakawa and Osada [20]. In [20], the authors proved also the uniqueness when the atomic part of the initial vorticity is sufficiently small. Let us mention that the Navier-Stokes equations are well-posed for any large initial vorticity when $\mathrm{curl} u_0 \in L^1(\mathbb{R}^2)$ (see [4]). All the previous results use in a crucial way the bound of the L^1 norm of the vorticity for $t > 0$, this is a consequence in some sense of the maximum principle.
In the particular case where $\mu = \alpha \delta_0$ for $\alpha \in \mathbb{R}$ and ($\rho = 1$ the Navier Stokes equation) there exists a particular self similar solution, the so called Lamb-Oseen vortex given by:

$$\mathrm{curl} u(t,x) = \frac{\alpha}{t} G(\frac{x}{\sqrt{t}}),\ u(t,x) = \frac{\alpha}{\sqrt{t}} v_G(\frac{x}{\sqrt{t}}),\ x \in \mathbb{R}^2,\ t > 0, \tag{1.4}$$

where:

$$G(\xi) = \frac{1}{4\pi} e^{-\frac{|\xi|^2}{4}},\ v_G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} (1 - e^{-\frac{|\xi|^2}{4}}),\ \xi \in \mathbb{R}^2.$$

In [16], Gallay and Wayne showed the uniqueness of the Lamb-Oseen solution by rewriting the system in self similar variables and using Lyapounov functions for the Fokker-Planck equations combined with the so called log Sobolev inequalities and Csiszár-Kullback inequalities (see [26]). We refer also to [15] for an another proof using symetrization techniques for parabolic equations The uniqueness for any $\mathrm{curl} u_0 \in \mathcal{M}(\mathbb{R}^2)$ is proved in [14], it allows to obtain the existence and the uniqueness of global self similar solution when $N = 2$.

1.2 Results of global strong solution for the non homogenous Navier-Stokes equations

Compared with the homogeneous Navier-Stokes equation ($\rho = 1$), to obtain self similar solutions seems more tricky. Indeed it is not clear due to the term $\nabla^\perp(\frac{1}{\rho}) \cdot \nabla \Pi$ in (1.3)

that the L^1 norm of the vorticity curl$u$ remains bound in L^1 norm all along the time. Similarly if we try to show the existence of global self similar solution with $u_0 \in B^{\frac{N}{p}-1}_{p,\infty}$ small enough (which corresponds to the results obtained in [6] for the Navier-Stokes equation) and $\rho_0 - 1 \in B^{\frac{N}{p}}_{p,\infty} \cap L^\infty$, then it seems difficult to propagate the regularity $B^{\frac{N}{p}}_{p,\infty}$ on the density via the mass equation. Indeed the velocity u will be a priori log Lipschitz and it will involve a loss of derivative on the density (see [21] for more development on this question, where we prove the existence of global strong solution for initial data small enough in $(B^{\frac{N}{p}+\varepsilon}_{p,\infty} \cap L^\infty) \times B^{\frac{N}{p}-1}_{p,\infty}$).
Danchin in [9] proved the local wellposedness of the system (1.3) if the initial data $(\rho_0 - 1, u_0)$ are in the critical Besov space $B^{\frac{N}{p}}_{p,1} \times B^{\frac{N}{p}-1}_{p,1}$ for $1 \leq p \leq N$. Furthermore the solution is global if the initial data are sufficiently small in $B^{\frac{N}{p}}_{p,1} \times B^{\frac{N}{p}-1}_{p,1}$. The smallness assumption for the initial density in [9] the smallness assumption on the fluctuation of the initial density was removed in [1, 2], and in [10] the authors extends the result of [9] to the case $1 \leq p < 2N$. In [21], we generalize the result of [6] to the case of the inhomogeneous Navier-Stokes equation by allowing to the initial velocity to belong to $B^{\frac{N}{p}-1}_{p,r}$ with $1 \leq r \leq +\infty$ and p well chosen. By opposite we need to require to be slightly subcritical in terms of regularity for the fluctuateion of the density which must be in $B^{\frac{N}{p}+\varepsilon}_{p,r} \cap L^\infty$ with $\varepsilon > 0$. Danchin and Mucha in [10] introduced the Lagrangian approach and proved the existence global strong solution for small initial data with $(\rho_0 - 1, u_0) \in \mathcal{M}(B^{\frac{N}{p}-1}_{p,1}) \times B^{\frac{N}{p}}_{p,1}$ ($\mathcal{M}(B^{\frac{N}{p}-1}_{p,1})$ corresponds here to the multiplier set of $B^{\frac{N}{p}-1}_{p,1}$). This result is very interesting since it allows to the authors to deal with initial density which are not necessary continuous. We refer also to [11, 18, 8] for some extensions of this previous results.
To finish let us recall that (1.1) is globally well-posed in dimension $N = 2$ (see [3], see also more recently [8]), the question remains open for general viscosity coefficients depending on the density.

1.3 Fundamental solution for parabolic system and Parametrix

In the present paper we wish to exhibit Lamb Oseen solution for the non homogeneous Navier-Stokes equation when $N = 2$. It means solutions such that the initial vorticity corresponds to $\alpha\delta_0$ with $\alpha \in \mathbb{R}$. As we explained previously the term $\nabla^\perp(\frac{1}{\rho}) \cdot \nabla\Pi$ in (1.3) prevents any L^1 control of the voracity all along the time. It is then natural to search solution such that $\nabla^\perp(\frac{1}{\rho}) \cdot \nabla\Pi = 0$. To do this, we can assume geometrical condition on the velocity and the density, more precisely we are going to consider rotationally invariant initial data of the form:

$$\rho_0(x) = \rho_0(|x|) \text{ and } u_0(x) = \nabla^\perp\psi(|x|), \tag{1.5}$$

with $\frac{1}{\rho_0} \in L^\infty(\mathbb{R}^2)$. It is important to observe that this choice of initial density does not allow to obtain the existence of global self similar solution since ρ_0 should be homogeneous

of degree 0. Let us search now solution of (1.1) verifying the following property:

$$\rho(t,x)=\rho(t,|x|) \text{ and } u(t,x)=\nabla^{\perp}\psi_1(t,|x|)=\frac{x^{\perp}}{|x|}\psi_1'(t,|x|), \tag{1.6}$$

with ψ_1 a radial function. We observe then at least heuristically that via the mass equation, the density is stationary and verifies for all $t>0$:

$$\rho(t,|x|)=\rho_0(|x|). \tag{1.7}$$

Furthermore the velocity and the vorticity u and curl$u$ verify:

$$\begin{cases}\rho_0\partial_t u-\mu\Delta u=0,\\ \mathrm{div}u=0,\\ u(0,\cdot)=u_0.\end{cases}\qquad\begin{cases}\partial_t\mathrm{curl}u-\mu\mathrm{div}(\frac{1}{\rho_0}\nabla\mathrm{curl}u)=0\\ \mathrm{div}u=0,\\ \mathrm{curl}u(0,\cdot)=\mathrm{curl}u_0.\end{cases} \tag{1.8}$$

We observe in particular that for such initial data curl$u$ verifies a linear heat equation and conserve the mass L^1 at least if curlu_0 is positive (it suffices to apply the maximum principle). More generally we have for all $t>0$:

$$\|\mathrm{curl}u(t,\cdot)\|_{L^1(\mathbb{R}^2)}\leq\|\mathrm{curl}u_0\|_{L^1}. \tag{1.9}$$

Now we are interested in constructing Lamb Oseen solution which corresponds to the case where curl$u_0=\alpha\delta_0$ with δ_0 the Dirac measure.
To do this, we are interested in constructing the fundamental solution associated to the equations (1.8). We assume now that ρ_0 verifies the following conditions:

$$\begin{cases}\rho_0(x)=\rho_0(|x|)\ \ \forall x\in\mathbb{R}^2,\\ \frac{1}{\rho_0}\in L^{\infty}(\mathbb{R}^2),\rho_0\in L^{\infty},\\ \exists C>0,\ \forall(x,y)\in\mathbb{R}^2\ \ |\frac{1}{\rho_0(x)}-\frac{1}{\rho_0(y)}|\leq C|x-y|^{\alpha}\ \text{ with } 0<\alpha<1,\\ \text{for }\ i\in\{1,2\}\ \ \exists C_i>0,\ \forall(x,y)\in\mathbb{R}^2\ \ |\frac{x_i\rho_0'(|x|)}{\rho_0^2(x)}-\frac{y_i\rho_0'(|y|)}{\rho_0^2(y)}|\leq C_i|x-y|^{\alpha}\\ \text{with } 0<\alpha<1,\end{cases} \tag{1.10}$$

We are going now to recall some results due to A. Friedman and Ladyzenskaya et al (see [12, 24]) on the construction of the fundamental solution for linear parabolic equations. More precisely we wish to solve the following equation:

$$\partial_t Z(x,\xi,t,\tau)-\mu\mathrm{div}(\frac{1}{\rho_0}\nabla Z)(x,\xi,t,\tau)=\delta_{(x-\xi)}\delta_{(t-\tau)}. \tag{1.11}$$

with Z the fundamental solution of (1.8). We have in particular for any continuous function f with compact support:

$$\lim_{t\to\tau}\int_{\mathbb{R}^2}Z(x,\xi,t,\tau)f(\xi)d\xi=0. \tag{1.12}$$

This implies that the initial data associated to $Z(x,\xi,t,0)$ is δ_0.
We define now a parametric (this method is due to E. E. Levi (see [24])):

$$Z_0(x-\xi,\xi,t,\tau)=\frac{\rho_0(|\xi|)}{4\mu\pi(t-\tau)}\exp\big(-\frac{\rho_0(|\xi|)}{4\mu(t-\tau)}|x-\xi|^2\big), \tag{1.13}$$

with $Z_0(x-\xi,\xi,t,\tau)=0$ when $t<\tau$. We know in particular that:

$$\partial_t[Z_0(x-\xi,\xi,t,\tau)]-\frac{\mu}{\rho_0(|\xi|)}\Delta_x Z_0(x-\xi,\xi,t,\tau)=\delta(x-\xi)\delta(t-\tau). \tag{1.14}$$

We define now the following kernel:

$$K(x,\xi,t,\tau)=\mu(\frac{1}{\rho_0(|\xi|)}-\frac{1}{\rho_0(|x|)})\Delta_x Z_0(x-\xi,\xi,t,\tau)+\mathcal{L}_1(x,t,\frac{\partial}{\partial x})Z_0(x-\xi,\xi,t,\tau), \tag{1.15}$$

with:

$$\mathcal{L}_1(x,t,\frac{\partial}{\partial x})Z_0(x-\xi,\xi,t,\tau)=-\mu\nabla(\frac{1}{\rho_0})(x,t)\cdot\nabla_x Z_0(x-\xi,\xi,t,\tau). \tag{1.16}$$

We deduce that:

$$\partial_t[Z_0(x-\xi,\xi,t,\tau)]-\mu\mathrm{div}_x(\frac{1}{\rho_0}\nabla_x[Z_0(x-\xi,\xi,t,\tau)])=\delta(x-\xi)\delta(t-\tau)+K(x,\xi,t,\tau). \tag{1.17}$$

We search now a solution $Z(x,\xi,t,\tau)$ of (1.11) under the following form:

$$Z(x,\xi,t,\tau)=Z_0(x-\xi,\xi,t,\tau)+\int_\tau^t d\lambda\int_{\mathbb{R}^2}Z_0(x-y,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy. \tag{1.18}$$

Assume that Q is a function satisfying a Hölder condition in (y,λ) then we have when we set $Z'(x,\xi,t,\tau)=\int_\tau^t d\lambda\int_{\mathbb{R}^2}Z_0(x-y,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy$:

$$\partial_t[Z'(x,\xi,t,\tau)]-\mu\mathrm{div}_x(\frac{1}{\rho_0}\nabla_x Z'(x,\xi,t,\tau))=Q(x,\xi,t,\tau)+\int_\tau^t d\lambda\int_{\mathbb{R}^2}K(x,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy. \tag{1.19}$$

From (1.17), (1.18), (1.19) we deduce that (1.11) is verified if Q is solution of the following Volterra equation:

$$Q(x,\xi,t,\tau)+\int_\tau^t d\lambda\int_{\mathbb{R}^2}K(x,y,t,\lambda)Q(y,\xi,\lambda,\tau)dy+K(x,\xi,t,\tau)=0. \tag{1.20}$$

Proposition 1.1 *There exists a solution Q of (1.20) with:*

$$Q(x,\xi,t,\tau)=\sum_{m=1}^{+\infty}(-1)^m K_m(x,\xi,t,\tau), \tag{1.21}$$

where K_m is defined as follows:

$$K_m(x,\xi,t,\tau)=\int_\tau^t d\lambda\int_{\mathbb{R}^2}K(x,y,t,\lambda)K_{m-1}(y,\xi,\lambda,\tau)dy. \tag{1.22}$$

In addition there exists $c,C>0$ such that for any (x,ξ,t,τ):

$$|Q(x,\xi,t,\tau)|\le c(t-\tau)^{-\frac{4-\alpha}{2}}\exp(-C\frac{|x-\xi|^2}{t-\tau}). \tag{1.23}$$

proof: We refer to [24], where the authors prove that there exists $c, C > 0$ such that for any $m \geq 1$:

$$|K_m(x,\xi,t,\tau)| \leq c^m (\frac{\pi}{C})^{m-1} \frac{\Gamma^m(\frac{\alpha}{2})}{\Gamma(\frac{m\alpha}{2})} (t-\tau)^{\frac{m\alpha-4}{2}} \exp(-C\frac{|x-\xi|^2}{t-\tau}). \tag{1.24}$$

The series in (1.21) is then uniformly convergent and we obtain (1.1). ■

Let us now consider the Oseen solution u_{α,ρ_0} of (1.8) which verifies the heat equation:

$$\begin{cases} \partial_t \mathrm{curl} u_{\alpha,\rho_0} - \mu \mathrm{div}(\frac{1}{\rho_0} \nabla \mathrm{curl} u_{\alpha,\rho_0}) = 0 \\ \mathrm{curl} u(0,\cdot) = \alpha \delta_0. \end{cases} \tag{1.25}$$

This solution corresponds to the Oseen tourbillon for the non homogeneous Navier Stokes equation with initial density $\rho_0(|x|)$ and initial tourbillon $\alpha\delta_0$. We recover u by using the Biot Savart law.
We observe with the previous notation that for any $t > 0$:

$$\mathrm{curl} u_{\alpha,\rho_0}(t,\cdot) = Z(x,0,t,0). \tag{1.26}$$

We have the following property (see [24]).

Proposition 1.2 *We have:*

- $$|D_t^r D_x^s \mathrm{curl} u_{\alpha,\rho_0}(t,x)| \leq C t^{-\frac{N+2r+s}{2}} \exp(-C_1 \frac{|x|^2}{t}) \tag{1.27}$$
  where $2r + s \leq 2$, $t > \tau$.

- $$\begin{aligned} &|D_t^r D_x^s \mathrm{curl} u_{\alpha,\rho_0}(t,x) - D_t^r D_{x'}^s \mathrm{curl} u_{\alpha,\rho_0}(x',t)| \leq \\ &C\big[|x-x'|^\gamma t^{-\frac{N+2+\gamma}{2}} + |x-x'|^\beta t^{-\frac{N+2-\alpha+\beta}{2}}\big] \exp(-C_1 \frac{|x|^2}{t}), \end{aligned} \tag{1.28}$$
  where $2r + s = 2$ *(i.e* $r = 0$, $s = 2$ *and* $r = 1$, $s = 0$*),* $0 \leq \gamma$, $0 \leq \beta \leq \alpha$, $t > 0$.

- $$\begin{aligned} &|D_t^r D_x^s \mathrm{curl} u_{\alpha,\rho_0}(t,x) - D_{t'}^r D_x^s \mathrm{curl} u_{\alpha,\rho_0}(t',x)| \leq \\ &C\big[(t-t')(t')^{-\frac{N+2r+s+2}{2}} + (t-t')^{\frac{2-2r-s+\alpha}{2}} (t')^{-\frac{N+2}{2}}\big] \exp(-C_1 \frac{|x|^2}{t}), \end{aligned} \tag{1.29}$$
  where $2r + s = 1, 2$ *and* $t > t' > 0$.

To simplify the notation we assume now that $\mu = 1$. Let us state our main result.

Theorem 1.1 *Let* $N = 2$*, assume that* ρ_0 *verifies the assumptions (1.10) and* $u_0(x) = \nabla^\perp \psi(|x|)$ *with* $\mathrm{curl} u_0 \in L^1(\mathbb{R}^2)$*. Then there exists a unique global strong solution* curl$u$ *for the system (1.8) and furthermore for any* $p \in [1, +\infty]$ *there exists* $C_p > 0$ *such that for all* $t > 0$*:*

$$\|\mathrm{curl} u(t,\cdot)\|_{L^p} \leq \frac{C_p}{t^{1-\frac{1}{p}}} \|\mathrm{curl} u_0\|_{L^1} \quad \forall t > 0. \tag{1.30}$$

Furthermore $(\rho_0, \mathrm{curl}u(t,\cdot))$ *is the unique solution for the non homogeneous Navier Stokes equation (1.3) in the sense of the mild solution.*

Let us assume now in addition that $\frac{1}{\rho_0} - 1 \in L^2(\mathbb{R}^2)$ *and that there exists* $\varepsilon > 0$ *sufficiently small and* $C > 0$ *such that for* $t_0 > 0$*:*

$$\int_{\mathbb{R}^2} |\mathrm{curl}u_0(x)|^2 \exp(\frac{|x-x_0|^2}{4t_0})dx \leq C. \tag{1.31}$$

and:

$$\|\frac{1}{\rho_0} - 1\|_{L^\infty} \leq \varepsilon. \tag{1.32}$$

Then for any $\alpha \in \mathbb{R}$ *If* $\int_{\mathbb{R}^2} \mathrm{curl}u_o(x)dx = \alpha$ *then we have for a constant* γ *with* $0 < \gamma < \frac{1}{2}$ *and some constant* C_p *depending only on* $p \in [1, +\infty]$*:*

$$(t+t_0)^{1-\frac{1}{p}} \|\mathrm{curl}u(t) - \frac{\alpha}{t+t_0} G(\frac{x-x_0}{\sqrt{t+t_0}})\|_{L^p} \leq C_p \frac{1}{(1+\frac{t}{t_0})^\gamma}. \tag{1.33}$$

In addition there exists Lamb Oseen solution of (1.3) for initial data $(\rho(0,\cdot), \mathrm{curl}u(0,\cdot)) = (\rho_0, \alpha\delta_0)$ *with* $\alpha \in \mathbb{R}$ *and* $(\rho(t,\cdot), \mathrm{curl}u(t,\cdot)) = (\rho_0, Z(\cdot,0,t,0))$ *with* Z *defined by (1.26).*

Remark 1 *The uniqueness (for the equation (1.1) of the Lamb Oseen solution is true if* α *is small enough and if* $(\rho_0 - 1) \in B^{2+\alpha}_{1,\infty}$*, this is provided from [21]. However the uniqueness of the Lamb Oseen solution for large* α *remains open.*
Let us also mention that in [21], the existence of strong solution in finite time is proved for initial data $u_0 \in \overline{C_c^\infty \cap B^1_{1,\infty}}^{B^1_{1,\infty}}$*, it does not cover the case of vorticity* $\mathrm{curl}u_0$ *which are finite measure. In this sense the existence of Oseen vortices extend the results of [21]. The uniqueness (for (1.1 with* $(\rho_0 - 1) \in B^{2+\alpha}_{1,\infty}$*) of the solution for* $\mathrm{curl}u_0 \in L^1(\mathbb{R}^2)$ *with the condition of the previous theorem provides of [21] since* $\overline{C_c^\infty \cap B^0_{\infty,1}}^{B^0_{1,\infty}}$ *is embedded in* $L^1(\mathbb{R}^2)$ *and the the uniqueness is a local problem.*

Remark 2 *The problem concerning the existence or not of self similar solution with* $\rho_0 \neq 1$ *remains actually open.*

Remark 3 *We would like to point out that a vorticity in* $L^1(\mathbb{R}^2)$ *does not imply a velocity field in* $\mathbb{R}^2$*. Indeed if* $u_0 \in L^2(\mathbb{R}^2)$ *and* $\mathrm{curl}u_0 \in L^1(\mathbb{R}^2)$ *then we can verify that* $\int_{\mathbb{R}^2} \mathrm{curl}u_0(x)dx = 0$*. Since the integral of* $\mathrm{curl}u$ *is conserved, it implies that if* $\int_{\mathbb{R}^2} \mathrm{curl}u_0(x)dx \neq 0$ *then* $\mathrm{curl}u_0$ *will be never in* $L^2(\mathbb{R}^2)$*.*

Remark 4 *In [16, 17] the authors estimate the time asymptotic rate of convergence to* 0 *of* $\|\mathrm{curl}u(t) - \frac{\alpha}{t} G(\frac{\cdot}{\sqrt{t}})\|_{L^1}$ *when* $\int_{\mathbb{R}^2} w_0(x)dx = \alpha$ *and* $w_0 \in L^1(\mathbb{R}^2)$ *for the Navier-Stokes equations. We extend this result to the case of non homogeneous Navier-Stokes equation, indeed it is easy to observe that (1.33) implies that* $\|\mathrm{curl}u(t) - \frac{\alpha}{t} G(\frac{\cdot}{\sqrt{t}})\|_{L^1}$ *converges to* 0 *when* t *goes to* $+\infty$*. In particular we generalize the results of [25] since we do not assume any smallness assumption on* $\|\mathrm{curl}u(1,\cdot) - \alpha G\|_{L^1}$*.*

We wish now to weaken the conditions (1.10) on the initial density ρ_0. Indeed in the previous theorem, we need to assume that $\nabla\frac{1}{\rho_0}$ belongs to $C^\alpha(\mathbb{R}^2)$ for $0<\alpha<1$. This is essentially due to the fact that we consider the solution of the equation (1.3) on curl$u$. Let us deal now with the equation (1.1), we have then the following result.

Theorem 1.2 *Let $N=2$, assume that ρ_0 is radial and $\frac{1}{\rho_0}$ belongs to $C^\alpha(\mathbb{R}^2)$ with $0<\alpha<1$. In addition we have $0<c<\rho_0\leq M<+\infty$ and $u_0(x)=\nabla^\perp\psi(|x|)$ with $u_0\in L^{2,\infty}(\mathbb{R}^2)$. Then there exists a unique global strong solution u for the system (1.8). Furthermore (ρ_0,u) is solution for the system (1.1) in the sense of the mild solution.*

Remark 5 *The uniqueness of the solution u for the equation (1.1) is true if u_0 is small enough in $B^{\frac{N}{p}-1}_{p,\infty}$ with suitable p such that $1\leq p\leq 2$, this is provided from [21] (if in addition $(\rho_0-1)\in B^{\frac{N}{p}}_{p,\infty}$). In particular we can choose $u_0=\alpha\frac{x^\perp}{|x|^2}$ which is in $B^1_{\infty,\infty}\cap L^{2,\infty}$ and such that* curl$u_0=\alpha\delta$. *Indeed we can observe that $u_0(x)=\nabla^\perp\ln(|x|)$ and we have (since the Fourier transform of $\ln(|x|)$ is $-\frac{C}{|x|^2}$ with $C>0$ a universal constant:*

$$\mathcal{F}\Delta_l u(0)(\xi)=\frac{1}{|\xi|^2}\varphi(\frac{\xi}{2^l}),$$

we refer to [21] for the definition of Littlewood-Paley theory with $\varphi\in C^\infty(\mathbb{R}^2)$ and supp$\varphi\in C(\frac{1}{4},\frac{3}{4})$. We deduce than using the inverse Fourier transform that:

$$\Delta_l u_0(x)=\mathcal{F}\psi(-2^l x)\ \ with\ \ \psi(z)=\frac{\varphi(z)}{|z|^2}.$$

We conclude that $\ln(|x|)\in B^2_{1,\infty}$.

Let us give an other direct consequence of the form that take the solution u of (1.8) which is simply an heat equation.

Corollary 1 *Let $N=2$, assume that ρ_0 is radial and $0<c<\rho_0\leq M<+\infty$. Furthermore let us consider $u_0(x)=\nabla^\perp\psi(|x|)$ with $u_0\in B^{\frac{N}{p}-1}_{p,\infty}$ with $1\leq p<+\infty$. Assume that there exists $\varepsilon>0$ sufficiently small such that:*

$$\|\frac{1}{\rho_0}-1\|_{\mathcal{M}(B^{\frac{N}{p}-1}_{p,\infty})}\leq\varepsilon,$$

then there exists a unique solution u of (1.8). (ρ_0,u) is solution of (1.1).

Remark 6 *The proof follows the same idea than in [10] and in your case is a simple fixed pointed for the solution of (1.8). It is a priori not clear to know if (ρ_0,u) is the unique solution of (1.1) even for small initial velocity. Indeed the initial data is completely critical for the initial density and the velocity.*

2 Proof of the theorem 1.1 and 1.2

The existence part of the theorem is a direct consequence of the construction of a fundamental solution for the equation (1.8). Indeed the solution reads as follows:

$$\mathrm{curl}u(t,x) = \int_{\mathbb{R}^2} Z(x,t,\xi,0)\mathrm{curl}u_0(\xi)d\xi. \tag{2.34}$$

We observe now that since $\mathrm{curl}u_0$ is radial $\mathrm{curl}u(t,\cdot)$ is also radial for any $t>0$. Indeed it provides from the fact that $\mathrm{curl}u(t,\cdot)$ is a convolution between $\mathrm{curl}u_0$ with a kernel $K(x-\xi,t) = Z(x,\xi,t,0)$ which is radial. The convolution preserves the radial property. Using Biot Savart formula we deduce that u is solution of the non homogeneous Navier-Stokes equation since $\rho_0 u\cdot\nabla u$ is a gradient of a radial function. Indeed we have for $t>0$:

$$u(t,x) = \frac{1}{2\pi}\nabla_x^{\perp}\int_{\mathbb{R}^2}\ln(|x-y|)\mathrm{curl}u(t,y)dy,$$

and we have that the initial data is verified in the sense that for any $\varphi\in C_c^{\infty}(\mathbb{R}^2)$

It implies that there exists a radial function F such that $\rho_0 u\cdot\nabla u = \nabla F(t,|x|)$. It shows in particular that u is solution of the equation (1.3). The proof of (1.30) is a direct consequence of (2.34) and (1.27).

Let us now prove the asymptotic time decay estimate (1.33). Following [16, 17], we define the rescaled vorticity $w_1(\tau,\xi) = \mathrm{curl}u_1(\tau,\xi)$ and the density $\rho_1(\tau,\xi)$ by:

$$\begin{aligned} w(t,x) = \frac{1}{(t+t_0)}w_1(\frac{x-x_0}{\sqrt{t+t_0}},\log(1+\frac{t}{t_0}))\ \text{ and }\ \rho(t,x) = \rho_1(\frac{x-x_0}{\sqrt{t+t_0}},\log(1+\frac{t}{t_0})),\\ \text{with }\ \xi = \frac{x-x_0}{\sqrt{t+t_0}}\ \text{ and }\ \tau = \log(1+\frac{t}{t_0})\in[0,+\infty[. \end{aligned} \tag{2.35}$$

Here we have defined w as follows $w = \mathrm{curl}u$. We have in particular:

$$w_1(\tau,\xi) = t_0 e^{\tau}w(t_0(e^{\tau}-1),\sqrt{t_0}e^{\frac{\tau}{2}}\xi+x_0)\ \text{ and }\ \rho_1(\tau,\xi) = \rho(t_0(e^{\tau}-1),\sqrt{t_0}e^{\frac{\tau}{2}}\xi+x_0) \tag{2.36}$$

The equation verified by $w_1(\tau,\xi)$ reads:

$$\partial_\tau w_1 - \mathrm{div}(\frac{1}{\rho_1}\nabla w_1) - \frac{1}{2}\xi\cdot\nabla w_1 - w_1 = 0. \tag{2.37}$$

We now define $\mathcal{L}$ as follows:

$$\mathcal{L}f = \Delta f + \frac{1}{2}\xi\cdot\nabla f + f,$$

which is the so called Fokker-Planck operator. We observe that $\mathrm{Ker}\mathcal{L} = \mathrm{Vect}\ G$ with G defined as follows:

$$G(\xi) = \frac{1}{4\pi}e^{\frac{-|\xi|^2}{4}}.$$

G in kinetic theory is called the Maxwellian. We are going now to estimate $\widetilde{w} = w_1 - \alpha G$ which satisfies:

$$\partial_\tau \widetilde{w} - \Delta \widetilde{w} - \frac{1}{2}\xi \cdot \nabla \widetilde{w} - \widetilde{w} = \mathrm{div}((\frac{1}{\rho_1} - 1)\nabla w_1). \tag{2.38}$$

The idea is classical now for the Fokker Planck equations, we are going to estimate the time asymptotic convergence of a solution w_1 of (2.37) to the Maxwellian G. To do this, the idea consists (see also [25]) in multiplying the equation (2.38) by $G^{-1}\widetilde{w}$ and estimate the norm $|\widetilde{w}|_{G,2}$ with:

$$|\widetilde{w}|_{G,2} = \|G^{-\frac{1}{2}}\widetilde{w}\|_{L^2(\mathbb{R}^2)}. \tag{2.39}$$

The idea is to observe that if $L = G^{-\frac{1}{2}}\mathcal{L}G^{\frac{1}{2}}$ then $L = -\Delta + \frac{|\xi|^2}{16} - \frac{1}{2}$ is a harmonic oscillator with spectrum $\{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$ (see [25]). Moreover 0 is a simple eigenvalue with eigenvector $G^{\frac{1}{2}}$. In particular if f is in the domain of L with $\int_{\mathbb{R}^2} G^{\frac{1}{2}} f = \langle f, G^{\frac{1}{2}}\rangle = 0$, then:

$$\int_{\mathbb{R}^2} fLf dx \geq \frac{1}{2}\|f\|^2_{L^2}. \tag{2.40}$$

Coming back to $\mathcal{L}$, we obtain if $G^{-\frac{1}{2}}\widetilde{w}$ belongs to the domain of L with $\int_{\mathbb{R}^2} \widetilde{w} dx = 0$, then for any $0 < \gamma < \frac{1}{2}$ taking $\widetilde{w} = G^{\frac{1}{2}} f$ we have:

$$\begin{aligned}\int_{\mathbb{R}^2} G^{-1}\widetilde{w}\mathcal{L}\widetilde{w} &= -(1-\gamma)\int_{\mathbb{R}^2} fLf + \gamma \int_{\mathbb{R}^2} G^{-1}\widetilde{w}\mathcal{L}\widetilde{w} \\ &\leq -\frac{1}{2}(1-\gamma)|\widetilde{w}|^2_{G,2} + \gamma \int_{\mathbb{R}^2} G^{-1}\widetilde{w}\mathcal{L}\widetilde{w}\end{aligned}$$

Now using integration by parts on the formula of L, we have:

$$\int_{\mathbb{R}^2} G^{-1}\widetilde{w}\mathcal{L}\widetilde{w} \leq -\frac{1}{2}(1-2\gamma)|\widetilde{w}|^2_{G,2} - \gamma(\|\nabla(G^{-\frac{1}{2}}\widetilde{w})\|^2_{L^2} + |\frac{|\xi|}{4}\widetilde{w}|^2_{G,2}).$$

From Young inequality we obtain:

$$\int_{\mathbb{R}^2} G^{-1}\widetilde{w}\mathcal{L}\widetilde{w} \leq -\frac{1}{2}(1-2\gamma)|\widetilde{w}|^2_{G,2} - \gamma(\frac{1}{3}|\nabla\widetilde{w}|^2_{G,2} + \frac{1}{2}|\frac{|\xi|}{4}\widetilde{w}|^2_{G,2}).$$

Next we have:

$$\begin{aligned}&\int_{\mathbb{R}^2} \mathrm{div}((\frac{1}{\rho_1} - 1)\nabla w_1)\widetilde{w}\, G^{-1} d\xi = \int_{\mathbb{R}^2} \mathrm{div}((\frac{1}{\rho_1} - 1)\nabla\widetilde{w})\widetilde{w}\, G^{-1} d\xi + \alpha\int_{\mathbb{R}^2} \mathrm{div}((\frac{1}{\rho_1} - 1)\nabla G)\widetilde{w}\, G^{-1} d\xi \\ &= -\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)|\nabla\widetilde{w}|^2 G^{-1} d\xi - \frac{1}{2}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\nabla\widetilde{w}\cdot\xi G^{-1}\widetilde{w} d\xi \\ &\qquad + \frac{\alpha}{2}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\xi\cdot\nabla\widetilde{w} d\xi + \frac{\alpha}{4}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\widetilde{w}|\xi|^2 d\xi\end{aligned}$$

Multiplying the momentum equation (2.38) by $G^{-1}\widetilde{w}$ and combining all the previous estimates, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\widetilde{w}|^2_{G,2} + \frac{1}{2}(1-2\gamma)|\widetilde{w}|^2_{G,2} + \gamma(\frac{1}{3}|\nabla\widetilde{w}|^2_{G,2} + \frac{1}{2}|\frac{|\xi|}{4}\widetilde{w}|^2_{G,2}) \leq -\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)|\nabla\widetilde{w}|^2 G^{-1} d\xi \\ &- \frac{1}{2}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\nabla\widetilde{w}\cdot\xi G^{-1}\widetilde{w} d\xi + \frac{\alpha}{2}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\xi\cdot\nabla\widetilde{w} d\xi + \frac{\alpha}{4}\int_{\mathbb{R}^2} (\frac{1}{\rho_1} - 1)\widetilde{w}|\xi|^2 d\xi.\end{aligned} \tag{2.41}$$

Using Young inequality we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\widetilde{w}|^2_{G,2}+\frac{1}{2}(1-2\gamma)|\widetilde{w}|^2_{G,2}+\gamma(\frac{1}{3}|\nabla\widetilde{w}|^2_{G,2}+\frac{1}{2}|\frac{|\xi|}{4}\widetilde{w}|^2_{G,2})\\
&\leq\|\frac{1}{\rho_1}-1\|_{L^\infty}|\nabla\widetilde{w}|^2_{G,2}+\frac{\varepsilon}{4}\|\frac{1}{\rho_1}-1\|_{L^\infty}|\nabla\widetilde{w}|^2_{G,2}+\frac{1}{4\varepsilon}\|\frac{1}{\rho_1}-1\|_{L^\infty}||\xi|^2\widetilde{w}|^2_{G,2} \qquad (2.42)\\
&+\frac{|\alpha|}{2}\|(\frac{1}{\rho_1}-1)G^{\frac{1}{2}}\|_{L^2}|\nabla\widetilde{w}|_{G,2}+\frac{|\alpha|}{4}\|(\frac{1}{\rho_1}-1)G^{\frac{1}{2}}\|_{L^2}||\xi|\widetilde{w}|_{G,2}.
\end{aligned}$$

We observe now that:

$$\|(\frac{1}{\rho_1}(\tau,\cdot)-1)G^{\frac{1}{2}}\|_{L^2}\leq\frac{1}{2\sqrt{\pi t_0}}e^{-\frac{\tau}{2}}\|\frac{1}{\rho_0}-1\|_{L^2}. \qquad (2.43)$$

From (2.42) and (2.43), we deduce from Young inequality that for $\varepsilon',\varepsilon_1>0$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}|\widetilde{w}(\tau)|^2_{G,2}+\frac{1}{2}(1-2\gamma)|\widetilde{w}(\tau)|^2_{G,2}+\gamma(\frac{1}{3}|\nabla\widetilde{w}(\tau)|^2_{G,2}+\frac{1}{2}|\frac{|\xi|}{4}\widetilde{w}(\tau)|^2_{G,2})\\
&\leq|\nabla\widetilde{w}(\tau)|^2_{G,2}(\|\frac{1}{\rho_1(\tau)}-1\|_{L^\infty}+\frac{\varepsilon}{4}\|\frac{1}{\rho_1(\tau)}-1\|_{L^\infty}+\frac{|\alpha|\varepsilon'}{8\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2})+\frac{|\alpha|}{8\varepsilon'\sqrt{\pi t_0}}e^{-\frac{\tau}{2}}\|\frac{1}{\rho_0}-1\|_{L^2}e^{-\tau}\\
&\quad+||\xi|^2\widetilde{w}(\tau)|^2_{G,2}(\frac{1}{4\varepsilon}\|\frac{1}{\rho_1}-1\|_{L^\infty}+\frac{\varepsilon_1|\alpha|}{16\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2})+\frac{|\alpha|}{16\varepsilon_1\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2}e^{-\tau}\\
&\leq|\nabla\widetilde{w}(\tau)|^2_{G,2}(\|\frac{1}{\rho_0}-1\|_{L^\infty}+\frac{\varepsilon}{4}\|\frac{1}{\rho_0}-1\|_{L^\infty}+\frac{|\alpha|\varepsilon'}{8\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2})+\frac{|\alpha|}{8\varepsilon'\sqrt{\pi t_0}}e^{-\frac{\tau}{2}}\|\frac{1}{\rho_0}-1\|_{L^2}e^{-\tau}\\
&\quad+||\xi|^2\widetilde{w}(\tau)|^2_{G,2}(\frac{1}{4\varepsilon}\|\frac{1}{\rho_0}-1\|_{L^\infty}+\frac{\varepsilon_1|\alpha|}{16\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2})+\frac{|\alpha|}{16\varepsilon_1\sqrt{\pi t_0}}\|\frac{1}{\rho_0}-1\|_{L^2}e^{-\tau}
\end{aligned} \qquad (2.44)$$

Since $\|\frac{1}{\rho_0}-1\|_{L^\infty}$ is sufficiently small, we deduce by bootstrap that there exists $0<C<\frac{1}{2}$ and $C_1>0$ such that:

$$\frac{1}{2}\frac{d}{d\tau}|\widetilde{w}(\tau)|^2_{G,2}+C|\widetilde{w}(\tau)|^2_{G,2}\leq C_1e^{-\tau}. \qquad (2.45)$$

From Grönwall lemma, we have for any $\tau>0$:

$$|\widetilde{w}(\tau)|^2_{G,2}\leq e^{-2C\tau}|\widetilde{w}(0)|^2_{G,2}+2C_1e^{-\tau}. \qquad (2.46)$$

Now since we have

$$D=\int_{\mathbb{R}^2}|w_0(x)|^2\exp(\frac{|x-x_0|^2}{4t_0})dx=\int_{\mathbb{R}^2}|w_1(0,\xi)|^2\exp(\frac{|\xi|^2}{4})d\xi\leq C<+\infty,$$

we deduce that there exists $C_2>0$ depending on D such that:

$$|\widetilde{w}(\tau)|_{G,2}\leq C_2e^{-C\tau}, \qquad (2.47)$$

withe $0 < C < \frac{1}{2}$. It implies in particular that setting:

$$\int_{\mathbb{R}^2} |t_0 e^\tau w(t_0(e^\tau-1), \sqrt{t_0} e^{\frac{\tau}{2}}\xi + x_0) - \alpha G(\xi)|^2 G^{-1}(\xi) d\xi \leq C_2^2 e^{-2C\tau}$$

$$\int_{\mathbb{R}^2} |t_0 e^\tau w(t_0(e^\tau-1), z) - \alpha G(\frac{z-x_0}{\sqrt{t_0} e^{\frac{\tau}{2}}})|^2 G^{-1}(\frac{z-x_0}{\sqrt{t_0} e^{\frac{\tau}{2}}}) d\xi \leq C_2^2 e^{-2C\tau}$$

$$\int_{\mathbb{R}^2} |(t_0+t) w(t,z) - \alpha G(\frac{z-x_0}{\sqrt{t_0+t}})|^2 d\xi \leq C_2^2 \frac{1}{(1+\frac{t}{t_0})^{2C}}$$

$$\sqrt{t_0+t}\, \|\big(w(t,\cdot) - \frac{\alpha}{t+t_0} G(\frac{\cdot - x_0}{\sqrt{t_0+t}})\big) G^{-1}(\frac{z-x_0}{\sqrt{t_0} e^{\frac{\tau}{2}}})\|_{L^2} \leq C_2 \frac{1}{(1+\frac{t}{t_0})^C}$$

From Höder inequality we deduce that for $1 \leq p \leq 2$ since by definition $w = \mathrm{curl} u$:

$$(t_0+t)^{1-\frac{1}{p}}\, \|\mathrm{curl} u(t,\cdot) - \frac{\alpha}{t+t_0} G(\frac{\cdot - x_0}{\sqrt{t_0+t}}\|_{L^2} \leq \frac{C_{2,p}}{(1+\frac{t}{t_0})^C}.$$

By duality we prove the same estimate for $2 \leq p \leq +\infty$. It concludes the proof of the theorem 1.1. ■

2.1 Proof of the theorem 1.2

It suffices to construct as previously a fundamental solution of the equation:

$$\rho_0 \partial_t \Delta u - \Delta u = 0.$$

We have then an explicit solution which is also solution of (1.1).

Acknowledgements

The author has been partially funded by the ANR project INFAMIE ANR-15-CE40-0011.

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