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Numerical simulation of a viscous compressible fluid based on a splitting method.

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Abstract

In this paper, a new approximation to solve the equations of a stationary, viscous and slightly compressible fluid is proposed. This method is based on a splitting of the equation into an elliptic and a hyperbolic part.

1 Introduction

In several recent papers, it appears that great progress has been made on the understanding of the compressible Navier-Stokes equations. For example, for the system

$$-\mu\Delta v - (\lambda + \mu)\nabla\operatorname{div} v + \nabla(K\rho^\gamma) = -\operatorname{div}(\rho v \otimes v) + \rho g \quad (1)$$

$$\operatorname{div}(\rho v) = 0 \quad \text{in } \Omega \quad (2)$$

$$v = 0 \quad \text{on } \partial\Omega \quad (3)$$

which corresponds to a 2D isentropic flow with speed v and density ρ , A. Novotný in [11] was able to prove the existence of a solution in $W^{1,2}(\Omega) \times L^\gamma(\Omega)$ when the density force g is in L^∞ and $\gamma > 1$. One of the main tool in his proof is a decomposition of the operator. The same idea is used in an another framework by Novotný and M. Padula in [1]. In this approach, the first step is to express the speed with the help of the Helmholtz decomposition, which reads:

$$\begin{aligned} v &= u + \nabla\varphi \\ \operatorname{div} u &= 0 \end{aligned}$$

The system (3) is then decomposed into three parts:

- A Stokes like system for the divergence-free part of the speed u and the **effective pressure** $P = \rho^\gamma - (\lambda + 2\mu)\operatorname{div}v$
- A Neumann problem for the potential part of the speed φ
- A non-linear transport equation for the density ρ

A crucial point in the proof is that the effective pressure is more regular than ρ^γ and $\operatorname{div}v$ separately.

Recently, M. Padula has discovered a more natural decomposition of the equations. This new splitting permitted her to sketch a simplified existence and uniqueness result for the isothermal flow (which corresponds to $\gamma = 1$ or to a linearization of the isentropic pressure law around $\rho = \rho_0$), under the assumption of the smallness of the data. Her work is presented in [14]. Roughly speaking, the system (3) is now split into two parts:

- A Stokes like system for the speed v and the **effective pressure** $P = \rho - (\lambda + 2\mu)\operatorname{div}v$
- A linear transport equation for the density ρ

An important difficulty in studying the compressible Navier-Stokes system (3) is that it is neither elliptic nor hyperbolic. With this splitting, elliptic and hyperbolic features are now clearly separated, and the theoretical study is simplified.

In this paper, we intend to show that this splitting can also be very fruitful numerically speaking. Indeed, the Stokes and the incompressible Navier-Stokes systems have been extensively studied. Building numerical approximations of these systems is not easy because of the possibility of spurious numerical solutions, but robust methods now exist (for a description of the theory, see Brezzi-Fortin [7], Bernardi-Maday [4]...). On the other hand, many schemes have been developed for the numerical treatment of the transport equations.

When numerically solving the compressible Navier-Stokes equations two **independent** dimensionless numbers play a crucial role: the Mach number M , which measures the compressibility effects, and the Reynolds number R , which measures the viscosity effects. Even with a small Reynolds number (big viscosity), in some physical configurations, the Mach number may be big. In that case the transport phenomena can not be damped by the viscosity, because there is no second derivative of ρ in the system (3)). In the numerical treatment, oscillations may appear without a carefully chosen approximation. The reverse situation is analog: if the Mach number is small, we are almost in the case of incompressible fluid. Therefore, the approximate density may present the same pathologies than the pressure in the Stokes problem.

In that context, the splitting permitted us to apply a very natural numerical scheme to solve the compressible Navier-Stokes equations, considered as a **coupled problem of two systems of partial differential equations**. We shall also present an iterative algorithm to numerically solve this coupled problem.

For the approximation we decided to use two very classical and robust finite element methods:

- The quadratic Crouzeix-Raviart element (also known as the P2-bubble + P1 element) for the approximation of the Stokes-like problem.
- The Streamline Upwind Petrov Galerkin (SUPG) method of Hughes for the transport problem.

Of course many other approximation techniques may have been used (like spectral elements for the Stokes problem or high order finite volume for the transport problem) with the same splitting of the equations.

The paper is organized as follow:

- We first rewrite the equations under a dimensionless form. Thus, the assumption of smallness of g is equivalent to an assumption of smallness of some dimensionless quantities. We are then able to show how the splitting can be used to derive an existence and uniqueness result when g is small enough and $\gamma = 1$ (isothermal fluid). This result is sketched in the paper of Padula [14], but we found it interesting to provide a detailed exposition. Indeed, some ideas of the proof are new, and we are convinced that they will also help the numerical analysis of compressible flows. It must be pointed out here that the global existence of a solution is still an open question in the case of isothermal flow.
- The technique of the previous proof is then adapted to construct an iterative algorithm to solve (3). It is shown to be convergent when g is small enough.
- Then, an approximation scheme, adapted to compressible flows, and numerical results are presented as an application of the previous theory. We did not perform error estimate or convergence proof for our approximation. Anyway, we present convincing numerical experiments which prove that our approximation scheme has good precision properties.

2 Dimensionless form of the equations

2.1 Equations

We present a model of viscous compressible fluid, based on an isothermal assumption. This allows us to present our method in a simple way. But it can be extended to more realistic fluids and boundary conditions.

We consider a stationary, viscous, compressible and isothermal fluid, in a bounded open set Ω of R^2 or R^3 . The speed v and the density ρ satisfy the Navier-Stokes equations (see [9]):

$$-\mu\Delta v - (\lambda + \mu)\nabla\operatorname{div} v + \nabla(K\rho^\gamma) = -\operatorname{div}(\rho v \otimes v) + \rho g \quad (4)$$

$$\operatorname{div}(\rho v) = 0 \quad \text{in } \Omega \quad (5)$$

$$v = 0 \quad \text{on } \partial\Omega \quad (6)$$

K , λ and μ are constant, > 0 , and g is a vector field. The pressure law is of the form $p = K\rho^\gamma$ and here, $\gamma = 1$ because the fluid is isothermal.

2.2 Dimensionless equations

Our goal is to prove an existence and uniqueness result on (6) based on an assumption of smallness of g . But in the proof, it is equivalent and more elegant to use an assumption of smallness of some dimensionless constants such as the Reynolds number or the Mach number. Therefore we shall first write (6) in dimensionless form.

We consider the following physical dimensions

- $[L]$: length unit
- $[T]$: time unit
- $[M]$: mass unit

The dimensions of the physical parameters units (6) are:

x	spatial position	$[L]$
μ	dynamic viscosity	$[M][L]^{-1}[T]^{-1}$
λ	second viscosity	$[M][L]^{-1}[T]^{-1}$
K	pressure law coefficient	$[L]^2[T]^{-2}$
v	speed	$[L][T]^{-1}$
ρ	mass density	$[M][L]^{-3}$
g	force density	$[L][T]^{-2}$

Let L be a characteristic length of our problem (i.e., the diameter of Ω), G a characteristic force density (i.e. the mean value of $|g|$ on Ω , or any other norm of g), and ρ_0 , the mean value of the density on Ω . We then set $x = Lx'$:

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x'}$$

And by taking $V = \sqrt{GL}$, which is a characteristic speed of the problem, with $v' = v/V$, we have

$$-\mu' \Delta' v' - (\lambda' + \mu') \nabla' \operatorname{div}' v' + \nabla' (K' \sigma') = -\operatorname{div}' (\rho' v' \otimes v') + (1 + \sigma') g' \quad (7)$$

$$\operatorname{div}' (\rho' v') = 0 \quad \text{in } \Omega' \quad (8)$$

$$v' = 0 \quad \text{on } \partial\Omega' \quad (9)$$

where

- $\mu' = \frac{\mu}{\rho_0 V L}$
- $\rho' = \frac{\rho}{\rho_0}$

- $\lambda' = \frac{\lambda}{\rho_0 V L}$
- $K' = \frac{K}{V^2}$
- $g' = g/G$
- Ω' is the image of Ω in the transformation $x' = x/L$.

The Reynolds number is here

$$R = \frac{\rho_0 V L}{\mu}$$

and the Mach number is

$$M = \frac{V}{\sqrt{K}}$$

When all the parameters are fixed, it is clear that the smallness of g is equivalent to the smallness of R and M .

In order to simplify the notations we will omit the primes in the sequel.

Setting $\alpha = \lambda/\mu$, we obtain

$$-\frac{1}{R}(\Delta v + (1 + \alpha)\nabla \operatorname{div} v) + \frac{1}{M^2}\nabla \rho = -\operatorname{div}(\rho v \otimes v) + \rho g \quad (10)$$

$$\operatorname{div}(\rho v) = 0 \quad \text{in } \Omega \quad (11)$$

$$v = 0 \quad \text{on } \partial\Omega \quad (12)$$

3 Theoretical study

3.1 Preliminary

3.1.1 Notations

Ω is an open set of R^d ($d = 2$ or 3). We denote by $\partial\Omega$ its boundary, which is supposed to be of regularity C^3 . For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the usual Banach space of functions u defined on Ω such that $|u|^p$ is summable. $\|u\|_{0,p} = \left(\int_{\Omega} |u|^p\right)^{\frac{1}{p}}$

For $k > 0$, $W^{k,p}$ is the usual Sobolev space of functions in $L^p(\Omega)$ such that all the derivatives up to order k are in $L^p(\Omega)$. If k is an integer, the norm in $W^{k,p}$ is:

$$\|u\|_{k,p} = \left(\sum_{i=0}^k \sum_{|\alpha|=i} \int_{\Omega} |D^{\alpha} u|^p \right)^{\frac{1}{p}}$$

Here, $\alpha = (\alpha_1 \cdots \alpha_d) \in N^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and the notation $D^{\alpha} u$ means

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

For $k \geq 1$ and $p \geq 2$, the trace on $\partial\Omega$ of elements of $W^{k,p}$ is well defined and we can set

$$W_0^{k,p} = \{u \in W^{k,p}, u = 0 \text{ on } \partial\Omega\}$$

we also define

$$\overline{W}^{k,p} = \left\{ u \in W^{k,p}, \int_{\Omega} u = 0 \right\}$$

If u is a vector valued function and if each component of u is in a space E , we shall write $u \in E$, rather than $u \in E^d$. The Sobolev injection theorems will be frequently applied (see, for example, Brézis [3], Adams [2]...).

C will be a generic constant, > 0 , which depends only on Ω .

Here, our study will be restricted to rather strong solutions of (6). More precisely, we shall assume that g is in the Sobolev space $W^{1,2}$. We look for a solution such that

- the speed v is in $W_0^{3,2}$ (thus, it is in $C^1(\overline{\Omega})$ thanks to the Sobolev injection theorem and because the space dimension $d \leq 3$)
- the density ρ is in $\overline{W}^{2,2}$ (thus $\sigma \in C(\overline{\Omega})$)

Our approach can be extended to less regular solutions, i.e. with v in $W_0^{2,p}$, $p > 2$ (see [1]).

3.1.2 Some well known results

Let's recall first some regularity properties that will be used below.

Stokes problem The Stokes problem reads

$$-\Delta u + \nabla P = F \tag{13}$$

$$\operatorname{div} u = g \tag{14}$$

$$u = 0 \text{ on } \partial\Omega \tag{15}$$

it is well set for F and g in $L^2(\Omega)$, and (u, P) in $W^{1,2} \times L^2$. Moreover, thanks to the ellipticity of (15) and the regularity of Ω , the following estimates hold (they are proved for example in Galdi [8]):

$$\|u\|_{3,2} + \|P\|_{2,2} \leq C \left(\|F\|_{1,2} + \|g\|_{2,2} \right)$$

$$\|u\|_{2,2} + \|P\|_{1,2} \leq C \left(\|F\|_{0,2} + \|g\|_{1,2} \right)$$

Using the relation

$$\nabla \wedge \nabla \wedge = -\Delta + \nabla \nabla \cdot$$

this estimate is still true for the boundary value problem

$$\nabla \wedge \nabla \wedge u + \nabla P = F \quad (16)$$

$$\operatorname{div} u = g \quad (17)$$

$$\int_{\Omega} P = \int_{\Omega} g = 0 \quad (18)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (19)$$

Transport equation Let v be a regular vector field on $\bar{\Omega}$ (at least in $W^{3,2}$). We suppose that $v \cdot n = 0$ on the boundary and that v is small in $W^{3,2}$ norm. The transport problem

$$\sigma + \operatorname{div}(\sigma v) = P \quad (20)$$

is well set for P in $L^2(\Omega)$ and σ in $L^2(\Omega)$ (see Lax-Philips [13], Rauch [16]). The following regularity estimates are proved by Novotný in [12] (for Ω regular)

$$\begin{aligned} \|\sigma\|_{k,2} &\leq C \|P\|_{k,2} \\ \|\operatorname{div}(\sigma v)\|_{k,2} &\leq C \|P\|_{k,2} \quad 0 \leq k \leq 2 \end{aligned}$$

Taking the derivatives of (20), these inequalities are easy to prove for regular solutions (but the construction of regular sequences converging to the weak solution of (20) is quite technical).

3.2 The splitting method

3.2.1 splitting of the density

Because it is easier to work with functions of zero mean value, we first set

$$\rho = 1 + \sigma$$

then, $\int_{\Omega} \sigma = 0$. σ will be called in the sequel the perturbation of density.

3.2.2 splitting into a linear and a non linear problems

Let

$$F(\sigma, v) = -\operatorname{div}((1 + \sigma)v \otimes v) + (1 + \sigma)g \quad (21)$$

we then have

$$\begin{aligned} -\frac{1}{R}(\Delta v + (1 + \alpha)\nabla \operatorname{div} v) + \frac{1}{M^2}\nabla \sigma &= F(\sigma, v) \\ \operatorname{div}(v) + \operatorname{div}(\sigma v) &= 0 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

It is then natural to study, for **fixed** w and G , the linear problem

$$-\frac{1}{R}(\Delta v + (1 + \alpha)\nabla \operatorname{div} v) + \frac{1}{M^2}\nabla \sigma = G \quad (22)$$

$$\operatorname{div}(v) + \operatorname{div}(\sigma w) = 0 \quad \text{in } \Omega \quad (23)$$

$$v = 0 \quad \text{on } \partial\Omega \quad (24)$$

The resolution of this problem defines a linear mapping $\Lambda_w : G \mapsto (\sigma, v)$. The solutions of (12) are then the fixed points of the non linear mapping $(\tau, w) \mapsto G = F(\tau, w) \mapsto \Lambda_w(G) = (\sigma, v)$.

3.2.3 Study of the linear problem

splitting We use here the approach initiated by Padula in [14]. The difficulty is that the linear problem (24) is neither elliptic nor hyperbolic. We shall split it into two easier problems: a standard Stokes problem on v and a transport problem on σ . The coupling of these two problems will be achieved through a new unknown $P = \frac{R}{M^2}\sigma - (2 + \alpha)\text{div}v$.

We want to study

$$\begin{aligned} -\frac{1}{R}(\Delta v + (1 + \alpha)\nabla\text{div}v) + \frac{1}{M^2}\nabla\sigma &= G \\ \text{div}(v) + \text{div}(\sigma w) &= 0 \quad \text{in}\Omega \\ v &= 0 \quad \text{on}\partial\Omega \end{aligned}$$

w is a vector field, such that $w = 0$ on the boundary of Ω . Moreover, if v is a solution of (6), it is easy to check that $\text{div}v = 0$ on $\partial\Omega$. Thus, we suppose that it is also true for w , because at the end of the fixed point procedure, we must have $w = v$.

Using the relation $\nabla \wedge \nabla \wedge = -\Delta + \nabla\nabla\cdot$, we can write

$$\begin{aligned} \frac{1}{R}(\nabla \wedge \nabla \wedge v + \nabla P) &= G \\ \text{div}v &= -\text{div}(\sigma w) \quad \text{in}\Omega \\ v &= 0 \quad \text{on}\partial\Omega \\ \int_{\Omega} P &= 0 \end{aligned}$$

Our second equation is then

$$\sigma + \text{div}\left(\sigma\frac{M^2}{R}(2 + \alpha)w\right) = \frac{M^2}{R}P \quad (25)$$

We have seen that it is natural to set $w = 0$ and $\text{div}w = 0$ on $\partial\Omega$. Therefore, we define the following Banach space

$$W_{0,0}^{3,2} = \left\{w \in W_0^{3,2}, w = 0 \text{ et } \text{div}w = 0 \text{ on } \partial\Omega\right\}$$

For w in $W_{0,0}^{3,2}$, we also introduce

$$\overline{W}_w^{2,2} = \left\{\sigma \in \overline{W}^{2,2}, \text{div}(\sigma w) \in \overline{W}^{2,2}\right\}$$

It is a Banach space, with the norm

$$\|\sigma\|_{2,2,w} = \|\sigma\|_{2,2} + \|\text{div}\sigma w\|_{2,2}$$

or with the equivalent norm (because $\operatorname{div} \sigma w = 0$ on $\partial\Omega$)

$$\|\sigma\|_{2,2,w} = \|\sigma\|_{2,2} + \|\Delta \operatorname{div} \sigma w\|_{0,2} \quad (26)$$

The following result can then be stated

Theorem 1 *there exists a constant $\beta > 0$, which depends only on Ω , such that if $M^2/R < \beta$ then problem (24) admits one and only one solution for G in $W^{1,2}(\Omega)$ and w in $W_{0,0}^{3,2}$, $\|w\|_{3,2} < 1$. Moreover the following estimates hold*

$$\begin{aligned} \|v\|_{3,2} &\leq CR \|G\|_{1,2} \\ \|\sigma\|_{2,2} &\leq CR \|G\|_{1,2} \end{aligned}$$

Proof:

Define the map $L : \sigma \rightarrow \sigma'$, by

- v and P are solutions of

$$\frac{1}{R} (\nabla \wedge \nabla \wedge v + \nabla P) = G \quad (27)$$

$$\operatorname{div} v = -\operatorname{div}(\sigma w) \quad \text{in } \Omega \quad (28)$$

$$v = 0 \quad \text{on } \partial\Omega \quad (29)$$

$$\int_{\Omega} P = 0 \quad (30)$$

- σ' is solution of

$$\sigma' + \operatorname{div} \left(\sigma' \frac{M^2}{R} (2 + \alpha) w \right) = \frac{M^2}{R} P \quad (31)$$

Obviously, if (v, σ) is a solution of (24), then it also verifies (30) and (31). Conversely, if σ is a fixed point of L , and if v is given by (30), then (v, σ) is solution of (24).

Then we have to find the fixed points of L . More precisely, we are going to prove that if $\frac{M^2}{R} w$ is small enough, L is actually a contracting mapping from $\overline{W}_w^{2,2}$ into itself. This will prove the existence and uniqueness of σ in that space.

- L is from $\overline{W}_w^{2,2}$ into $\overline{W}_w^{2,2}$:

Indeed, if σ is in $\overline{W}_w^{2,2}$, then $\operatorname{div}(\sigma w)$ is in $\overline{W}^{2,2}$. Moreover, G belonging to $W^{1,2}$, after resolution of the Stokes problem, v is in $W^{3,2}$ and P in $\overline{W}^{2,2}$. Then, σ' is in $\overline{W}_w^{2,2}$.

- L is a contraction

$\|L(\sigma_1) - L(\sigma_2)\|_{2,2,w}$ should be evaluated as a function of $\|\sigma_1 - \sigma_2\|_{2,2,w}$, where σ_1 and σ_2 are two elements of $\overline{W}_w^{2,2}$. v_i , P_i , and σ'_i , are the notations for the quantities associated to σ_i ($i = 1, 2$) after resolution of (30) and (31).

From (30), we get

$$\|v_1 - v_2\|_{3,2} + \|P_1 - P_2\|_{2,2} \leq C \|\sigma_1 - \sigma_2\|_{2,2,w}$$

and

$$\Delta(P_1 - P_2) = 0$$

The resolution of the transport problem (31) gives

$$\begin{aligned} \|\sigma'_1 - \sigma'_2\|_{2,2} &\leq C \frac{M^2}{R} \|P_1 - P_2\|_{2,2} \\ \|\operatorname{div}(\sigma'_1 - \sigma'_2)w\|_{2,2} &\leq C \|P_1 - P_2\|_{2,2} \end{aligned}$$

Those two inequalities hold under the condition that $\frac{M^2}{R}w$ is small enough.

However, they are not sufficient to prove the contraction. Actually they lead to

$$\|\sigma'_1 - \sigma'_2\|_{2,2,w} \leq C \left(1 + \frac{M^2}{R}\right) \|\sigma_1 - \sigma_2\|_{2,2,w}$$

and we cannot assume that the constant C , which depends on Ω , is small enough. To overcome this difficulty, let us take the laplacian of the transport equation

$$\Delta(\sigma'_1 - \sigma'_2) + \Delta \operatorname{div} \left((\sigma'_1 - \sigma'_2) \frac{M^2}{R} (2 + \alpha)w \right) = \frac{M^2}{R} \Delta(P_1 - P_2) = 0$$

According to the Leibniz formula, we have

$$\Delta(\sigma'_1 - \sigma'_2) + \operatorname{div} \left(\Delta(\sigma'_1 - \sigma'_2) \frac{M^2}{R} (2 + \alpha)w \right) = \frac{M^2}{R} r$$

where r is a sum of products of derivatives of $(\sigma'_1 - \sigma'_2)$ up to order 2 and derivatives of w up to order 3. It reads as a transport equation for $\Delta(\sigma'_1 - \sigma'_2)$. Using the regularity estimates, and the Sobolev inequalities, we get

$$\|\operatorname{div} \Delta(\sigma'_1 - \sigma'_2)w\|_{0,2} \leq C \|r\|_{0,2} \leq C \|\sigma'_1 - \sigma'_2\|_{2,2} \|w\|_{3,2}$$

On the other hand, $\operatorname{div} \Delta(\sigma'_1 - \sigma'_2)w = \Delta \operatorname{div}(\sigma'_1 - \sigma'_2)w + r'$, where, as before, r' is a sum of products of derivatives of $(\sigma'_1 - \sigma'_2)$ up to order 2 and derivatives of w up to order 3. Hence, we obtain

$$\|\sigma'_1 - \sigma'_2\|_{2,2} + \|\Delta \operatorname{div}(\sigma'_1 - \sigma'_2)w\|_{0,2} \leq C \frac{M^2}{R} \left(1 + \|w\|_{3,2}\right) \|\sigma_1 - \sigma_2\|_{2,2,w}$$

and, by the norm equivalence (26),

$$\|L(\sigma_1) - L(\sigma_2)\|_{2,2,w} \leq C \frac{M^2}{R} (1 + \|w\|_{3,2}) \|\sigma_1 - \sigma_2\|_{2,2,w}$$

L is then a contraction in $\overline{W}_w^{2,2}$, for $\frac{M^2}{R} (1 + \|w\|_{3,2})$ small enough, for instance if $\|w\|_{3,2} \leq 1$ and $\frac{M^2}{R} \leq \beta = \frac{1}{2C}$

- *A posteriori estimate*

Let σ be the unique fixed point of L . We have

$$\|\sigma\|_{2,2,w} - \|L(0)\|_{2,2,w} \leq \|\sigma - L(0)\|_{2,2,w} = \|L(\sigma) - L(0)\|_{2,2,w} \leq C \frac{M^2}{R} \|\sigma\|_{2,2,w}$$

moreover, $\sigma_0 = L(0)$ is solution of

$$\sigma_0 + \operatorname{div} \left(\sigma_0 \frac{M^2}{R} (2 + \alpha) w \right) = \frac{M^2}{R} P_0$$

then

$$\|\sigma_0\|_{2,2,w} \leq C \|P_0\|_{2,2}$$

and P_0 is solution of (30) with $\sigma = 0$, thus

$$\|P_0\|_{2,2} \leq CR \|G\|_{1,2}$$

and for $\frac{M^2}{R}$ small enough, we get

$$\|\sigma\|_{2,2,w} \leq CR \|G\|_{1,2}$$

Finally, we deduce, thanks to (30)

$$\|v\|_{3,2} \leq CR \|G\|_{1,2}$$

and the proof is completed. \square

3.2.4 Study of the global problem

We are now in a position to give our main result for the problem (6).

Theorem 2 *Let B be the ball*

$$B = \left\{ (\tau, w) \in \overline{W}^{2,2} \times W_{0,0}^{3,2}, \|\tau\|_{2,2} + \|w\|_{3,2} \leq 1 \right\}$$

there exists a constant β depending only on Ω such that if $R + \frac{M^2}{R} \leq \beta$, then problem (6) admits one and only one solution in B .

Proof :

We consider the mapping $H : (\tau, w) \rightarrow (\sigma, v)$, defined as follow :

- Following definition (21), we set

$$G = F(\tau, w)$$

- then (σ, v) is the solution of the previous linear problem (24).

Each solution (σ, v) of (6) is a fixed point of H . And each fixed point of H is a solution of (6).

- **Let us show that $H(B) \subset B$**

If $\frac{M^2}{R}$ is small enough, we know that H is well defined, and that

$$\|\sigma\|_{2,2} + \|v\|_{3,2} \leq CR \|F(\tau, w)\|_{1,2}$$

Sobolev inequalities give

$$\|F(\tau, w)\|_{1,2} \leq C(1 + \|\tau\|_{2,2})(\|w\|_{3,2}^2 + \|g\|_{1,2})$$

thus, for R small enough, $\|\tau\|_{2,2} + \|w\|_{3,2} \leq 1 \implies \|\sigma\|_{2,2} + \|v\|_{3,2} \leq 1$

- **Let us show that H is a contraction from B to B**

With obvious notations, we have

$$\|\sigma_1 - \sigma_2\|_{2,2} + \|v_1 - v_2\|_{3,2} \leq CR \|F(\tau_1, w_1) - F(\tau_2, w_2)\|_{1,2}$$

moreover, if (τ_1, w_1) and (τ_2, w_2) are in B , then the Sobolev injections give

$$\|F(\tau_1, w_1) - F(\tau_2, w_2)\|_{1,2} \leq C \left((1 + \|g\|_{1,2}) \|\tau_1 - \tau_2\|_{2,2} + \|w_1 - w_2\|_{3,2} \right)$$

for R small enough, H is thus a contraction from B to B . \square

3.3 Iterative algorithm

With the aid of the previous theory, an algorithm of resolution of (6) will now be given. The construction we have described to prove theorem (2) is actually made up of two interacting iterative procedures. For practical reasons, we have to modify it in order to build a simpler algorithm.

3.3.1 Presentation

Let E be the set

$$E = \left\{ (v, \sigma) \in W_{0,0}^{3,2} \times \overline{W}^{2,2}, \operatorname{div}(\sigma v) \in \overline{W}^{2,2} \right\} \quad (32)$$

(E is not a linear space)

We consider the following mapping

$$\Gamma : \begin{array}{ccc} E & \longrightarrow & E \\ (v, \sigma) & \longrightarrow & (v', \sigma') \end{array} \quad (33)$$

defined as follow

- $(v', P') \in W_{0,0}^{3,2} \times \overline{W}^{2,2}$ is the solution of

$$\frac{1}{R} (\nabla \wedge \nabla \wedge v' + \nabla P') = F(\sigma, v) \quad (34)$$

$$\operatorname{div} v' = -\operatorname{div}(\sigma v) \quad \text{in } \Omega \quad (35)$$

$$v' = 0 \quad \text{on } \partial\Omega \quad (36)$$

$$\int_{\Omega} P' = 0 \quad (37)$$

- $\sigma' \in \overline{W}^{2,2}$ is the solution of

$$\sigma' + \operatorname{div} \left(\sigma' \frac{M^2}{R} (2 + \alpha) v' \right) = \frac{M^2}{R} P' \quad (38)$$

A fixed point of Γ is a solution of (6) and every solution of (6) is a fixed point of Γ .

3.3.2 Remarks

The advantages of the algorithm (37), (38) are:

- (37) and (38) are linear problems
- the theory of each problem is well known and a great amount of numerical methods have been developed to solve it.
- because it is decomposed into several problems, (6) can be solved, even if it is too huge to stand alone in memory.
- only the transport linear system has to be factorized at each iteration

The drawbacks are:

- the convergence may be slow and we cannot expect more than linear convergence.
- the algorithm is divergent if the Mach number or the Reynolds number are too big.

This algorithm should thus be considered as a first step to a faster and more efficient resolution algorithm. Actually, we decided to adopt it because it permitted us very easily to reuse already written finite element software.

3.3.3 Stability

We show here the stability of our iterative scheme

Theorem 3 *Let $\eta > 0$, be small enough. Let E be defined as in (32). Suppose that*

$$\|g\|_{1,2} \leq \eta$$

Let $(v, \sigma) \in E$, such that

$$\begin{aligned} \|v\|_{3,2} + \|\sigma\|_{2,2} &\leq \eta \\ \|\operatorname{div}(\sigma v)\|_{2,2} &\leq \eta \end{aligned}$$

If we set $(v', \sigma') = \Gamma(v, \sigma)$ (c.f. definition (33)), then there exists $\beta > 0$, such that for $R + \frac{M^2}{R} < \beta$, we also have

$$\begin{aligned} \|v'\|_{3,2} + \|\sigma'\|_{2,2} &\leq \eta \\ \|\operatorname{div}(\sigma' v')\|_{2,2} &\leq \eta \end{aligned}$$

and the constants β and η depend only on Ω .

proof

this result is the consequence of the following estimates (already proved before)

- $\|F(\sigma, v)\|_{1,2} \leq C(1 + \|\sigma\|_{2,2})(\|v\|_{3,2}^2 + \|g\|_{1,2})$
- $\|v'\|_{3,2} + \|P'\|_{2,2} \leq C \left(R \|F(\sigma, v)\|_{1,2} + \|\operatorname{div}(\sigma v)\|_{2,2} \right)$
- $\|\Delta P'\|_{0,2} \leq CR \|F(\sigma, v)\|_{1,2}$
- $\|\sigma'\|_{2,2} \leq C \frac{M^2}{R} \|P'\|_{2,2}$
- $\|\Delta \operatorname{div}(\sigma' v')\|_{0,2} \leq C \left(\|\Delta P'\|_{0,2} + \|\sigma'\|_{2,2} \|v'\|_{3,2} \right). \square$

3.3.4 Convergence

Theorem 4 *Let $\eta > 0$, be small enough. Let E be defined as in (32). Suppose that*

$$\|g\|_{1,2} \leq \eta$$

Let $(v_0, \sigma_0) \in E$, such that

$$\begin{aligned} \|v_0\|_{3,2} + \|\sigma_0\|_{2,2} &\leq \eta \\ \|\operatorname{div}(\sigma_0 v_0)\|_{2,2} &\leq \eta \end{aligned}$$

If we set $(v_{n+1}, \sigma_{n+1}) = \Gamma(v_n, \sigma_n)$ (c.f. definition (33)), then there exists $\beta > 0$, such that for $R + \frac{M^2}{R} < \beta$, the sequence (v_n, σ_n) converges in $W_{0,0}^{2,2} \times \overline{W}^{1,2}$ towards the unique solution in E (noted (v, σ)) of (6).

proof

Denote by (v, σ) the unique solution of (6) in E . Existence and uniqueness of that solution is given by theorem 2. $P = \frac{R}{M^2} \sigma + \operatorname{div}(\sigma(2 + \alpha)v)$ is the effective pressure related to that solution.

We then have

$$\begin{aligned} \frac{1}{R} (\nabla \wedge \nabla \wedge (v_{n+1} - v) + \nabla(P_{n+1} - P)) &= F(\sigma_n, v_n) - F(\sigma, v) \\ \operatorname{div}(v_{n+1} - v) &= -\operatorname{div}(\sigma_n v_n) + \operatorname{div}(\sigma v) \quad \text{in } \Omega \\ v_{n+1} - v &= 0 \quad \text{on } \partial\Omega \\ \int_{\Omega} P_{n+1} - P &= 0 \end{aligned}$$

therefore, using the previous stability result and the regularity of the Stokes problem, we deduce

$$\begin{aligned} \|v_{n+1} - v\|_{2,2} + \|P_{n+1} - P\|_{1,2} &\leq CR \left(\|\sigma_n - \sigma\|_{1,2} + \|v_n - v\|_{2,2} \right) \\ &\quad + C\eta \left(\|\sigma_n - \sigma\|_{1,2} + \|v_n - v\|_{2,2} \right) \end{aligned}$$

It is worthy to notice here that because of the term $-\operatorname{div}(\sigma_n v_n) + \operatorname{div}(\sigma v)$ we lost an order of derivative in the estimate. A stability result in a stronger norm gives us a convergence result in a coarser norm.

We then consider the transport equation

$$\begin{aligned} \sigma - \sigma_{n+1} + \operatorname{div} \left((\sigma - \sigma_{n+1}) \frac{M^2}{R} (2 + \alpha) v \right) &= \frac{M^2}{R} (P_{n+1} - P) \\ &+ \operatorname{div} \left(\sigma_{n+1} \frac{M^2}{R} (2 + \alpha) (v_{n+1} - v) \right) \end{aligned}$$

thus

$$\|\sigma_{n+1} - \sigma\|_{1,2} \leq C \frac{M^2}{R} \left(\|P_{n+1} - P\|_{1,2} + \|\sigma_{n+1}\|_{2,2} \|v_{n+1} - v\|_{2,2} \right)$$

and for $R + \frac{M^2}{R} + \eta$ small enough, the algorithm is convergent. \square

4 Numerical study

4.1 Choice of the approximations

We shall now describe the finite element methods used to solve (38) and (37). The speed and the pressure of the Stokes problem and the density of the transport problem will be computed on the same mesh (with the same numbering of the elements but not necessarily of the vertices). In this way, the quantities of a problem are easily accessed by the other problem.

4.1.1 Stokes problem

The classical quadratic finite element of Crouzeix-Raviart (described in Cuvelier [5], Crouzeix-Raviart [10], Pironneau [15]...), and the following weak formulation are used:

To solve¹ (for $u \in W^{1,2}, P \in L^2$)

$$\begin{aligned} -\Delta u + \nabla P &= F \\ \operatorname{div} u &= h \quad \text{in } \Omega \\ u &= u_b \quad \text{on } \partial\Omega \\ \int_{\Omega} P &= 0 \end{aligned}$$

we start with the following bilinear and linear forms

$$\begin{aligned} A(u, w) &= \int_{\Omega} \nabla u \cdot \nabla w \\ B(w, P) &= - \int_{\Omega} P \operatorname{div} w \end{aligned}$$

¹Of course, the following compatibility relation is supposed to hold (ν is the outward normal to Ω on $\partial\Omega$): $\int_{\partial\Omega} u_b \cdot \nu = \int_{\Omega} h$

and the following weak formulation

$$\begin{aligned} A(u, w) + B(w, P) &= \int_{\Omega} Fw \quad \forall w \in W_0^{1,2} \\ B(u, Q) &= \int_{\Omega} hQ \quad \forall Q \in L^2 \\ u &= u_b \quad \text{on } \partial\Omega \end{aligned}$$

which is restricted to the finite dimension space spanned by the finite element basis of Crouzeix-Raviart.

It must be remembered that in the case of a triangular quadrangular element, the unknowns are the two components of the speed at the vertices and the centroid of the element, and the pressure and its gradient at the centroid. The approximation is thus quadratic and continuous for the speed, and linear and discontinuous for the pressure. The precision is then quadratic for the speed (in the $W^{1,2}$ norm) and for the pressure (in the L^2 norm), when the solution is regular enough. Stability and convergence of this approximation are studied for example in Cuvelier [5], Pironneau [15], Crouzeix-Raviart [10], Brezzi-Fortin [7], Bernardi-Maday [4]...

The size of the associated linear system can be reduced because the unknowns at the centroids of the elements can be eliminated (2 speed components and 3 pressure unknowns) without loss of precision. Those unknowns can be recovered *a posteriori*.

Moreover, this finite element method can obviously be adapted to the non-linear incompressible Navier-Stokes system. A non-linear system has to be solved (for example with the Newton method).

4.1.2 Transport equation

A weak formulation of Hughes [17], the Streamline Upwind Petrov Galerkin (or SUPG in short), is used to solve the transport problem, with quadratic quadrangular finite elements. For $\sigma \in W^{1,2}, P \in W^{1,2}$, the general transport equation reads:

$$\begin{aligned} \sigma + \text{div}(\sigma w) &= P \\ \sigma &= \sigma_0 \quad \text{on } \Gamma^- \\ \Gamma^- &= \{x \in \partial\Omega, n(x) \cdot w(x) < 0\} \end{aligned}$$

Let B be the bilinear form on $W^{1,2} \times W^{1,2}$

$$B(\sigma, \tau) = \int_{\Omega} (\sigma + \text{div}(\sigma w)) (\tau + \varepsilon \nabla \tau \cdot w) - \int_{\Gamma^-} \sigma \tau (w \cdot n)$$

the formulation is then

$$B(\sigma, \tau) = \int_{\Omega} P (\tau + \varepsilon \nabla \tau \cdot w) - \int_{\Gamma^-} \sigma_0 \tau (w \cdot n) \quad \forall \tau \in W^{1,2}$$

ε is a small parameter which stabilizes the approximation. It should be of order h , where h is the maximal diameter of the finite elements (for a theoretical study, see Johnson & Al. [6]). For a quadratic finite element basis, the error can be proved to be of order $h^{5/2}$ in a mesh-dependant norm, intermediate between the L^2 and the H^1 norm. Numerically, the error behaves like h^3 in the L^2 norm, and like h^2 in the H^1 norm.

4.2 Numerical results

4.2.1 Numerical convergence study

Our first test is devoted to a numerical precision study. We shall try to guess the convergence orders for the speed and the density.

Consider the two disks

$$\begin{aligned} D_1 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\} \\ D_2 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 4\} \end{aligned}$$

and set $\Omega = D_2 \setminus \overline{D_1}$. The following speed vector field, and density are then considered

$$v = \frac{2}{3}\left(r - \frac{1}{r}\right) e_\theta \quad (39)$$

$$\rho = 1 + \sigma = Ar^{-2a^2/K} \exp\left(\frac{a^2}{2K}\left(r^2 - \frac{1}{r^2}\right)\right) \quad (40)$$

where

$$\begin{aligned} r^2 &= x^2 + y^2 \\ e_r &= \frac{1}{r} \begin{bmatrix} x \\ y \end{bmatrix} \\ e_\theta &= \frac{1}{r} \begin{bmatrix} -y \\ x \end{bmatrix} \\ a &= 2/3 \\ K &= 10 = 1/M^2 \end{aligned}$$

The constant A is given by the relation

$$\int_{\Omega} \rho = 1$$

We found

$$A \simeq 0.992426\dots$$

it is then easy to check that (σ, v) is a solution to

$$\Delta v + \nabla \operatorname{div} v + \frac{1}{M^2} \nabla \sigma = -\operatorname{div}(\rho v \otimes v) \quad (41)$$

$$\operatorname{div}((1 + \sigma)v) = 0 \quad \text{in } \Omega \quad (42)$$

$$\int_{\Omega} \sigma = 0 \quad (43)$$

$$v = e_{\theta} \quad \text{on } \partial D_2 \quad (44)$$

$$v = 0 \quad \text{on } \partial D_1 \quad (45)$$

A numerical resolution of (45) as been performed for $1/M^2 = 10$ and for several mesh stepsizes h . The following table (see also figure 4.2.3) gives the H^1 and the L^2 norm of the error for the speed and the perturbation of density $\sigma = \rho - 1$.

h	H^1 speed error	L^2 density error	H^1 density error
1	0,00789573	0,000397334	0,012118
1/2	0,0020248	0,0000979495	0,0060295
1/4	0,000490715	0,0000243944	0,00306567

The error curves in a logarithmic scale are almost lines, and their slopes gives the approximation order

H^1 speed order	L^2 density order	H^1 density order
2,04482213558421	2,00548812857888	0,975835962359618

It thus appears that the precision of our approximation is the same as for the incompressible flow. There is no loss of precision due to the transport phenomena. It is also an indication that we could have use linear finite element to solve the transport problem, with the same precision.

4.2.2 Numerical stability study

The stability of the iterative algorithm is now studied on the previous problem (45).

The algorithm (37), (38) has been tested for different values of M . In order to obtain faster convergence, the described algorithm must be slightly modified. We implicit, at each iteration, the right hand side of (37), and actually solve

$$\frac{1}{R} (\nabla \wedge \nabla \wedge v' + \nabla P') = F(\sigma, v') \quad (\text{rather than } F(\sigma, v)) \quad (46)$$

$$\operatorname{div} v' = -\operatorname{div}(\sigma v) \quad \text{in } \Omega \quad (47)$$

$$v' = e_{\theta} \quad \text{on } \partial D_2 \quad (48)$$

$$v' = 0 \quad \text{on } \partial D_1 \quad (49)$$

$$\int_{\Omega} P' = 0 \quad (50)$$

by the Newton method.

This modification allows us to achieve convergence with fewer iterations. Of course, each iteration requires more CPU time because the Jacobian matrix in the Newton method has to be factorized. This approach is used in our test, but keeping a factorization for several iterations is certainly more efficient.

The results are presented on figure 4.2.3 (where, as above, $K = 1/M^2$). We have plotted the discrete quadratic norm of $\sigma_{n+1} - \sigma_n$ divided by the discrete quadratic norm of $\sigma_1 - \sigma_0$ during the iterations.

We notice that the algorithm is divergent for $M^2 \geq 1/20$

The convergence (or the divergence) is obviously linear and the logarithm of the contracting constants is given by the slope of the linear part of the curves. Actually, if we suppose that

$$\|\sigma_{n+1} - \sigma_n\| = (C + \varepsilon_n) \|\sigma_n - \sigma_{n-1}\|$$

where ε_n is a sequence which tends rapidly to zero, it is easy to prove that, asymptotically

$$\log \frac{\|\sigma_{n+1} - \sigma_n\|}{\|\sigma_1 - \sigma_0\|} = n \log C + C_0$$

thus, we are able to plot $\log C$ against the constant K . We obtain the plot of figure 4.2.3. The points are very close to a line of slope -1 . It is a numerical proof that the contracting constant is of the form $C = AM^2$ (when the Reynolds number is fixed). This result is in accordance with the previous theory where we have seen the importance of the ratio $\frac{M^2}{R}$. It is also an indication that theorem 4 could probably be improved. In that theorem we were only able to prove that the contracting constant is of the form $C(R + \frac{M^2}{R} + \eta)$.

On the other hand, a very simple relaxation method permits us to make the iterative algorithm converge up to $M^2 = 2$: instead of Γ , we consider the following mapping

$$\Gamma' : \begin{array}{ccc} E & \longrightarrow & E \\ (v, \sigma) & \longrightarrow & (v', \sigma'') \end{array}$$

defined as follow

- v' and P' are solutions of

$$\begin{aligned} \frac{1}{R} (\nabla \wedge \nabla \wedge v' + \nabla P') &= F(\sigma, v') \\ \operatorname{div} v' &= -\operatorname{div}(\sigma v) \quad \text{in } \Omega \end{aligned}$$

and subject to the usual boundary conditions.

- σ' is solution of

$$\sigma' + \operatorname{div} \left(\sigma' \frac{M^2}{R} (2 + \alpha) v' \right) = \frac{M^2}{R} P'$$

- Finally, we set $\sigma'' = (1 - \omega)\sigma' + \omega\sigma$, with $0 < \omega < 1$ and ω close to 1 when the Mach number increases.

We obtain the results of figure 4.2.3 (with $\omega = 0.9$).

4.2.3 An another example

In our last example, we consider

$$\begin{aligned} E_1 &= \{(x, y) \in \mathbb{R}^2, x^2 + 16y^2 \leq 1\} \\ D_2 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 4\} \end{aligned}$$

and the problem

$$-\frac{1}{R}(\Delta v + (1 + \alpha)\nabla \operatorname{div} v) + \frac{1}{M^2}\nabla \rho = -\operatorname{div}(\rho v \otimes v) + \rho g \quad (51)$$

$$\operatorname{div}(\rho v) = 0 \quad \text{in } \Omega \quad (52)$$

$$v = e_\theta \quad \text{on } \partial D_2 \quad (53)$$

$$v = 0 \quad \text{on } \partial E_1 \quad (54)$$

which corresponds to a rotating elliptic cylinder in a 2D compressible fluid limited by a circular cylinder. The equations are written in the referential linked to the ellipse. It explains the presence of the term $g = \frac{1}{4}r$ which is the centrifugal acceleration.

We fix $M^2 = 1/20$ and we obtain the results of figures 4.2.3, 4.2.3, 4.2.3 and 4.2.3 for different Reynolds numbers. For $R = 1$, we find $\sup \sigma \simeq 0.3$, and $\inf \sigma \simeq -0.3$. To plot, we chose a scale between -0.06 and 0.06 . Indeed, the compressibility effects are localized near the extremities of the ellipse. With a larger scale, only the variations close to these extremities would have been visible. For the other values of R , our scales respect the variations of σ (which are of order 0.05). We explain our results by the fact that the shear forces due to the viscosity diminish when the Reynolds number increases. When $R \rightarrow \infty$, the ellipse behaves as a circle of radius 1.

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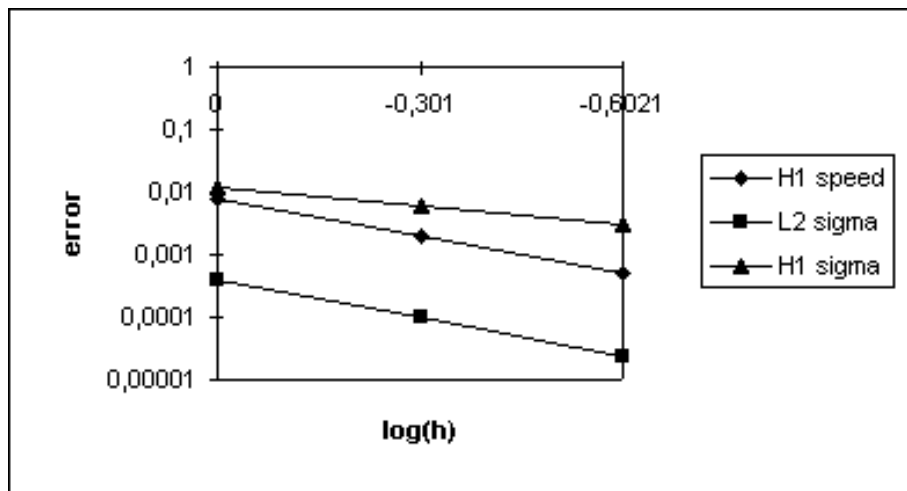


Figure 1: Approximations errors

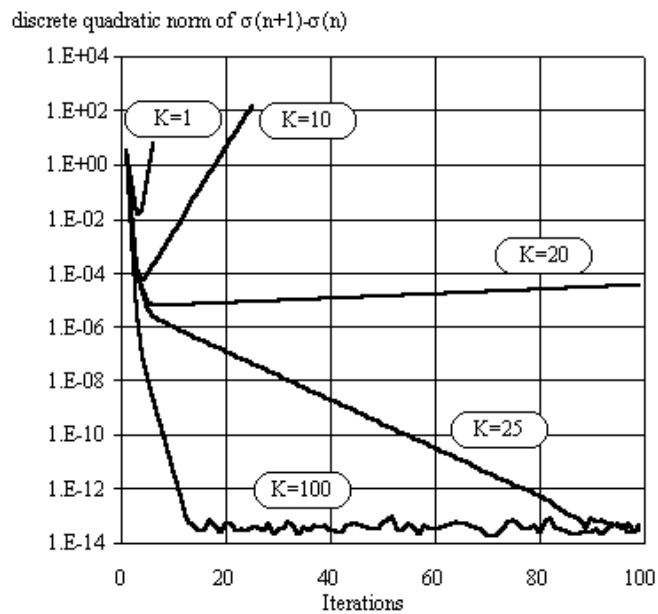


Figure 2: Convergence of the density

normalized discrete quadratic norm of $\sigma(n+1)-\sigma(n)$

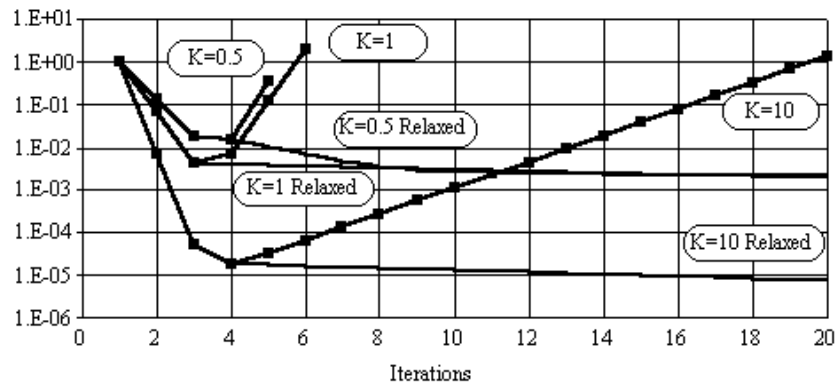


Figure 3: Convergence with relaxation

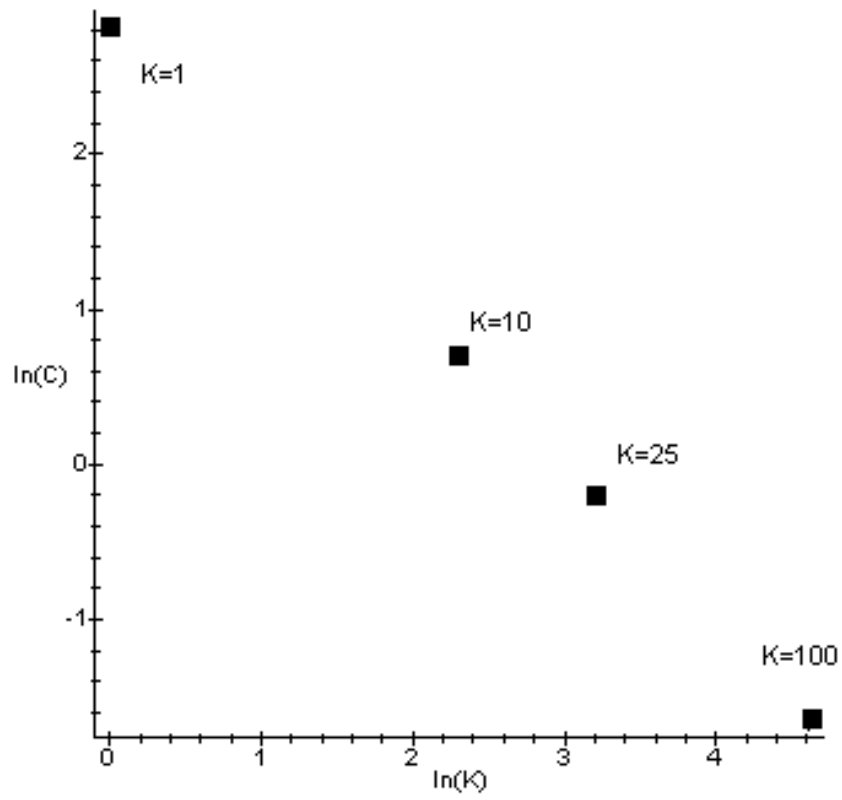


Figure 4: Contracting constant depending on the Mach number

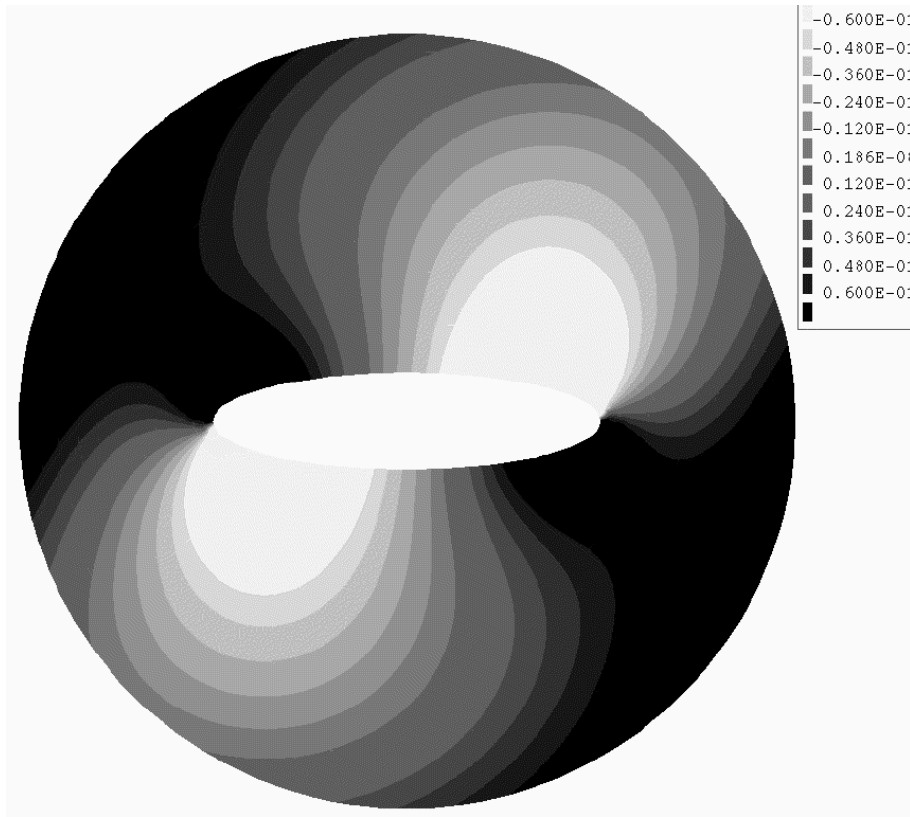


Figure 5: Rotating elliptic cylinder, $R = 1$

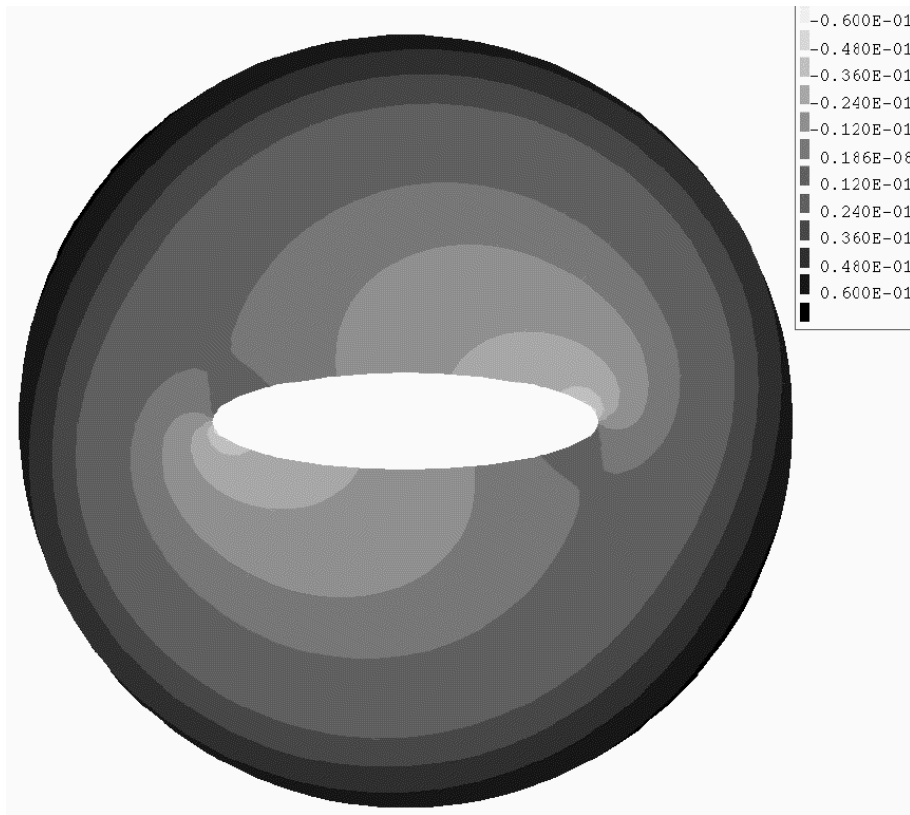


Figure 6: Rotating elliptic cylinder, $R = 10$

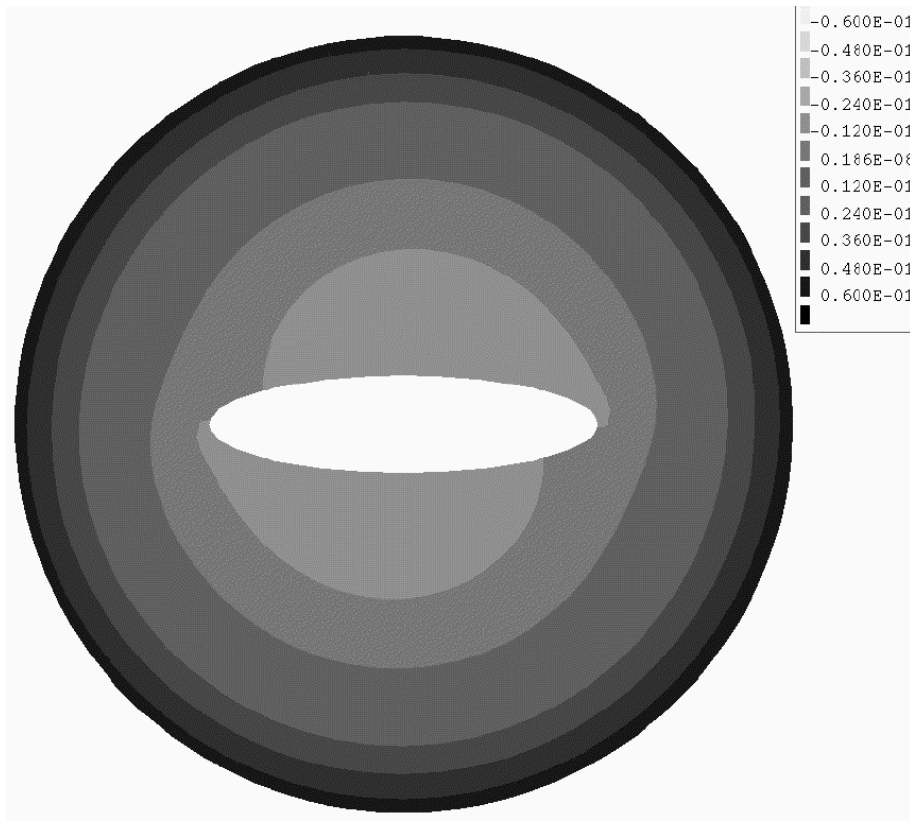


Figure 7: Rotating elliptic cylinder, $R = 100$

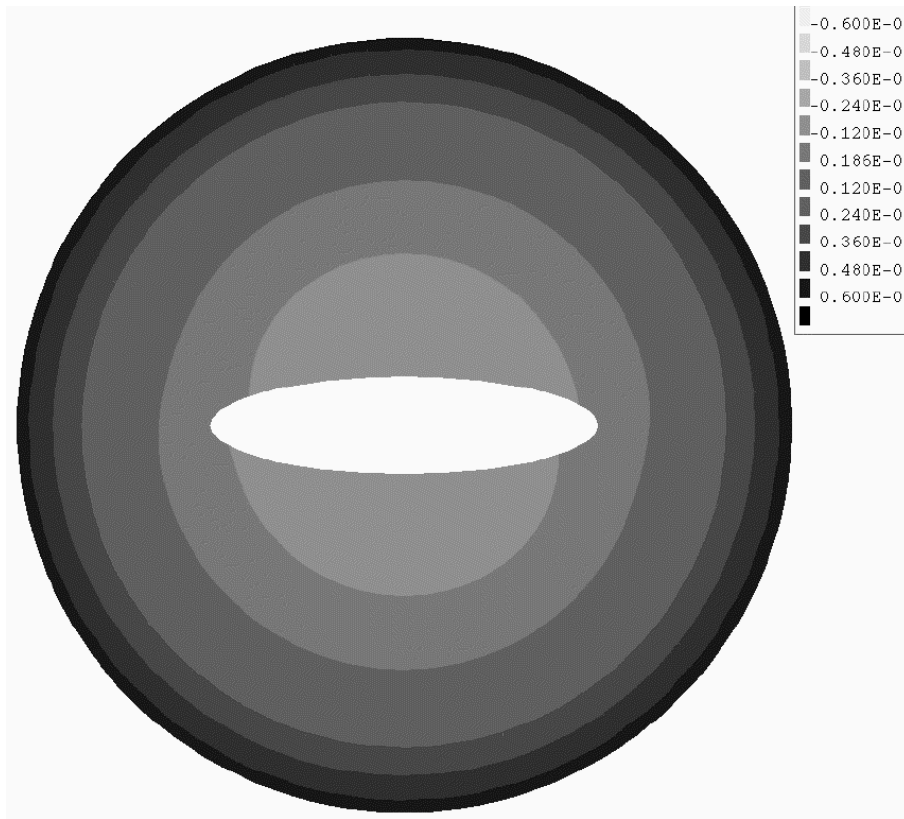


Figure 8: Rotating elliptic cylinder, $R = 1$