

# Solutions in $H^1$ of the steady transport equation in a bounded polygon with a fully non-homogeneous velocity.

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## Abstract

This article studies the solutions in  $H^1$  of a steady transport equation with a divergence-free driving velocity that is  $W^{1,\infty}$ , in a two-dimensional bounded polygon. Since the velocity is assumed fully non-homogeneous on the boundary, existence and uniqueness of the solution require a boundary condition on the open part  $\Gamma^-$  where the normal component of  $\mathbf{u}$  is strictly negative. In a previous article, we studied the solutions in  $L^2$  of this steady transport equation. The methods, developed in this article, can be extended to prove existence and uniqueness of a solution in  $H^1$  with Dirichlet boundary condition on  $\Gamma^-$  only in the case where the normal component of  $\mathbf{u}$  does not vanish at the boundary of  $\Gamma^-$ . In the case where the normal component of  $\mathbf{u}$  vanishes at the boundary of  $\Gamma^-$ , under appropriate assumptions, we construct local  $H^1$  solutions in the neighborhood of the end-points of  $\Gamma^-$ , which allow us to establish existence and uniqueness of the solution in  $H^1$  for the transport equation with a Dirichlet boundary condition on  $\Gamma^-$ .

## Résumé

Cet article étudie les solutions dans  $H^1$  d'une équation de transport stationnaire avec une vitesse de régularité  $W^{1,\infty}$  à divergence nulle, dans un polygone borné. La vitesse étant supposée non nulle sur la frontière, l'existence et l'unicité de la solution requièrent une condition sur la partie de la frontière où la composante normale de la vitesse est strictement négative. Dans un précédent article, nous avons étudié les solutions dans  $L^2$  de cette équation de transport stationnaire. Les méthodes, développées dans cet article, peuvent être étendues pour prouver l'existence et l'unicité d'une solution dans  $H^1$  avec une condition de Dirichlet sur  $\Gamma^-$  seulement dans le cas où la composante normale de  $\mathbf{u}$  ne s'annule pas à la frontière de  $\Gamma^-$ . Dans le cas où la composante normale de  $\mathbf{u}$  s'annule à la frontière de  $\Gamma^-$ , sous des hypothèses appropriées, nous construisons des solutions locales au voisinage des points frontières de  $\Gamma^-$  de régularité  $H^1$ , qui nous permettent d'établir l'existence et l'unicité de la solution dans  $H^1$  de l'équation de transport avec une condition de Dirichlet sur  $\Gamma^-$ .

**Key words.** transport equation, nonstandard boundary condition, localization methods

**AMS subject classifications.** 35A05, 35D05, 35D10, 76A05

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## 0 Introduction.

Transport equations are studied in many frameworks. In [2,7,12] the stress  $z$ , i.e, the transported quantity, is not assumed regular, while they impose strong conditions on the fluid velocity  $\mathbf{u}$ , which indicates the direction of the transport. Contrary to this, in [1,4], the velocity has only bounded variation with its divergence integrable, but the stress is assumed bounded or continuous. In fact, we have to choose the regularity of  $z$  and  $u$  for the product  $\mathbf{u} \cdot \nabla z$  to be well defined in some distributional sense. Thus, in [7], V. Girault and L.R. Scott, for defining  $\mathbf{u} \cdot \nabla z$  with the weaker assumptions, studied the transport equation with the stress  $z$  in  $L^2(\Omega)$ , the velocity  $\mathbf{u}$  given in  $H^1(\Omega)^d$ , with  $\operatorname{div} \mathbf{u} = 0$ , and the right hand side given in  $L^2(\Omega)$ , where  $\Omega$  is a Lipschitz-continuous domain. These Authors established existence and uniqueness of the solution for the transport equation by using the essential technique of Puel and Roptin [11] and the renormalizing argument of DiPerna and Lions [5]. In a following article [8], they extended their results from  $L^2$  to  $H^1$  for the transport equation. By another technique, in particular a Yosida approximation, V. Girault and L. Tartar [9] studied the solutions in  $L^p$ ,  $p \geq 2$ , of the transport equation, when the right hand side is in  $L^p$ .

However, all these approaches of transport equations assume that the normal component of the fluid velocity  $\mathbf{u}$  vanishes on the boundary of the domain. Indeed, in the contrary case, the problem is no longer well-posed and the unicity requires a boundary condition. However, it is not possible to define the trace on the boundary of the stress  $z$  when it is not regular but only square-integrable. Nevertheless, such transport equation with  $\mathbf{u} \cdot \mathbf{n} \neq 0$ , where  $\mathbf{n}$  denotes the unit exterior normal to the boundary, arises in the problem of fully nonhomogeneous second grade fluid [7]: multiple solutions imply that additional boundary conditions should be imposed.

In a previous article [3], we established existence and uniqueness of the solution, in the space where  $z$  and  $\mathbf{u} \cdot \nabla z$  are  $L^2$ , for the transport equation, with a boundary condition on the open part of the boundary where the normal component of  $\mathbf{u}$  is strictly negative, where  $\Omega$  is a Lipschitz-continuous domain of  $\mathbb{R}^d$ ,  $\mathbf{u}$  is given in  $H^1(\Omega)^d$  such that  $\operatorname{div} \mathbf{u} = 0$ , the right hand side is given in  $L^2(\Omega)$ , and  $\mathcal{W}$  is a given real parameter different from 0. We showed that it is possible to define the normal component of  $z\mathbf{u}$  on the boundary and, hence, to prove that the problem is well-posed by requiring a condition for the normal component of  $z\mathbf{u}$  on the part of the boundary where  $\mathbf{u} \cdot \mathbf{n} < 0$ .

The present article studies the steady transport problem : find  $z \in H^1(\Omega)$  such that

$$\begin{cases} z + \mathcal{W}\mathbf{u} \cdot \nabla z = l & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma^- \end{cases} \quad (0.1)$$

where  $\Omega$  is a bounded polygon of  $\mathbb{R}^2$ ,  $\mathbf{u}$  is given in  $W^{1,\infty}(\Omega)^2$  such that  $\operatorname{div} \mathbf{u} = 0$ ,  $\Gamma^-$  is the open part of the boundary of  $\Omega$  such that  $\mathbf{u} \cdot \mathbf{n} < 0$ ,  $l$  is given in  $H^1(\Omega)$ , and  $\mathcal{W}$  is a given real parameter different from 0. But, in a such framework, if we look for a solution in  $H^1$ , a difficulty arises when  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the boundary of  $\Gamma^-$ , as we shall see in examples given below. Indeed, the fact that the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes at a boundary point of  $\Gamma^-$  leads to a discontinuity for the partial derivatives of the solution  $z$  at this point and the solution  $z$  has not always the regularity  $H^1$ , see examples 4, 5 and 7. As we will see in the following examples, the regularity of the solution seems to depend on the multiplicity of the root of the equation  $\mathbf{u} \cdot \mathbf{n} = 0$  at the boundary of  $\Gamma^-$  and on the sign of  $\mathbf{u} \cdot \boldsymbol{\tau}_-$ , where  $\boldsymbol{\tau}_-$  is the unit tangent vector to the boundary at the point  $m$ , directed towards  $\Gamma^-$  : in

these examples, the solution of the transport equation is  $H^1$  if the multiplicity of the root is 0 or 1 and if the sign of  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  is negative in  $\mathbf{m}$ . On the contrary, the solution is not  $H^1$  if the multiplicity of the root is strictly greater than 1 or if the sign of  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  in  $\mathbf{m}$  is positive, which is consistent with the assumptions (3.5) of the Theorem 3.1.

When  $\mathbf{u} \cdot \mathbf{n}$  does not vanish at the boundary of  $\Gamma^-$ , by using results and tools of [3], we can prove existence and uniqueness of the solution  $H^1$  for the steady transport equation with the boundary condition on  $\Gamma^-$ . In contrast, when  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the boundary of  $\Gamma^-$ , the previous method does not work anymore. In this case, we split the right-hand side of the transport equation, which gives us a set of localized problems, and the solution  $H^1$  of the transport problem is the sum of the solutions  $H^1$  of the localized problems. For solving the problems localized in the neighborhoods of the points where  $\mathbf{u} \cdot \mathbf{n}$  vanishes such as simple roots of the equation  $\mathbf{u} \cdot \mathbf{n} = 0$ , in the case where  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  is negative at each of these points, we use a change of variables, which allows us to explain the local solution  $H^1$  of the transport equation in integral form. Next, we extend this local solution to the whole domain  $\Omega$  and we obtain the  $H^1$  solutions of the transport problems localized around these roots. Instead, to solve the problems localized far enough of these roots, the methods of [3] yield the  $H^1$  solutions.

After this introduction, this article is organized as follows. In section 1, we study several examples of transport problems, which show the link between the regularity of the solution ( $L^2$  or  $H^1$ ) and both the multiplicity of the roots of the equation  $\mathbf{u} \cdot \mathbf{n} = 0$  at the end points of  $\Gamma^-$  and the sign of  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  at these points. Section 2 is devoted to the solution in  $H^1$  of the transport problem when the normal component of the velocity does not vanish on  $\bar{\Gamma}^-$ . In section 3, we deal with the solutions in  $H^1$  of the transport problem in the case where the normal component of the velocity vanishes on  $\bar{\Gamma}^-$ .

We end this introduction by recalling some basic results of [3] that we shall use throughout this article. Let  $\Gamma'$  be an open part of the boundary  $\partial\Omega$  of class  $C^{0,1}$  and, for  $r > 2$ ,  $T_{1,r}^{\Gamma'}$  the mapping  $v \mapsto v|_{\Gamma'}$  defined on  $W^{1,r}(\Omega)$ . We denote by  $W^{1-\frac{1}{r},r}(\Gamma')$  (see [10]) the space  $T_{1,r}^{\Gamma'}(W^{1,r}(\Omega))$  which is equipped with the norm:

$$\|\varphi\|_{W^{1-\frac{1}{r},r}(\Gamma')} = \inf\{\|v\|_{W^{1,r}(\Omega)}, v \in W^{1,r}(\Omega) \text{ and } v|_{\Gamma'} = \varphi\}. \quad (0.2)$$

For fixed  $\mathbf{u}$  in  $H^1(\Omega)^2$ , let us introduce the space

$$X_{\mathbf{u}}(\Omega) = \{z \in L^2(\Omega), \mathbf{u} \cdot \nabla z \in L^2(\Omega)\}, \quad (0.3)$$

which is a Hilbert space equipped with the norm

$$\|z\|_{\mathbf{u}} = (\|z\|_{L^2(\Omega)}^2 + \|\mathbf{u} \cdot \nabla z\|_{L^2(\Omega)}^2)^{1/2}. \quad (0.4)$$

In the same way we define

$$Y_{\mathbf{u}}(\Omega) = \{z \in L^2(\Omega), \mathbf{u} \cdot \nabla z \in L^1(\Omega)\}.$$

We recall a theorem ( see [3]) concerning the normal component of boundary values of  $(z\mathbf{u})$  where  $z$  belongs to  $Y_{\mathbf{u}}(\Omega)$ .

**Theorem 0.1** *Let  $\Omega$  be a Lipschitz-continuous domain of  $\mathbb{R}^d$ , let  $\mathbf{u}$  belong to  $H^1(\Omega)^d$  with  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and let  $r > d$  be a real number. We denote by  $r'$  the real number defined*

by:  $\frac{1}{r} + \frac{1}{r'} = 1$ . The mapping  $\gamma'_{\mathbf{n}} : z \mapsto (z\mathbf{u}) \cdot \mathbf{n}|_{\partial\Omega}$  defined on  $\mathcal{D}(\overline{\Omega})^d$  can be extended by continuity to a linear and continuous mapping, still denoted by  $\gamma'_{\mathbf{n}}$ , from  $Y_{\mathbf{u}}(\Omega)$  into  $W^{-1/r', r'}(\partial\Omega)$ .

From this theorem and with a density argument, we derive the following Green's formula: let  $r > d$  be a real number and let  $\mathbf{u}$  be in  $H^1(\Omega)^d$  with  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ ,

$$\forall z \in Y_{\mathbf{u}}(\Omega), \forall \varphi \in W^{1,r}(\Omega), \int_{\Omega} z(\mathbf{u} \cdot \nabla \varphi) \, d\mathbf{x} + \int_{\Omega} \varphi(\mathbf{u} \cdot \nabla z) \, d\mathbf{x} = \langle (z\mathbf{u}) \cdot \mathbf{n}, \varphi \rangle_{\partial\Omega}. \quad (0.5)$$

Let  $\Gamma_0$  and  $\Gamma_1$  be two non empty open parts of  $\partial\Omega$  that have a finite number of connected components and verify

$$\Gamma^0 \cap \Gamma^1 = \emptyset, \quad \partial\Omega = \overline{\Gamma^0} \cup \overline{\Gamma^1},$$

such that  $\overline{\Gamma^0} \cap \overline{\Gamma^1}$  has a finite number of connected components.

We introduce the space  $W^{-1/r', r'}(\Gamma_0) = (W_0^{1-1/r, r}(\Gamma_0))'$ , where

$$W_0^{1-1/r, r}(\Gamma_0) = \{v|_{\Gamma_0}, v \in W^{1,r}(\Omega), v|_{\Gamma_1} = 0\}, \quad (0.6)$$

and we denote  $\langle \cdot, \cdot \rangle_{\Gamma_0}$  the duality pairing between these two spaces. Note that if  $z \in Y_{\mathbf{u}}(\Omega)$ , then  $(z\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_0} \in W^{-1/r', r'}(\Gamma_0)$  and, in the same way as previously, we have the Green's formula :  $\forall z \in Y_{\mathbf{u}}(\Omega), \forall \varphi \in W^{1,r}(\Omega)$ , with  $\varphi|_{\Gamma_1} = 0, \forall \mathbf{u} \in H^1(\Omega)^d$  with  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ ,

$$\int_{\Omega} z(\mathbf{u} \cdot \nabla \varphi) \, d\mathbf{x} + \int_{\Omega} \varphi(\mathbf{u} \cdot \nabla z) \, d\mathbf{x} = \langle (z\mathbf{u}) \cdot \mathbf{n}, \varphi \rangle_{\Gamma_0}. \quad (0.7)$$

Then, we can define the following space :

$$X_{\mathbf{u}}(\Gamma_0) = \{z \in X_{\mathbf{u}}, (z\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_0} = 0\}. \quad (0.8)$$

From now on, we suppose that  $d = 2$  and  $\Omega \subset \mathbb{R}^2$ . Let us denote by  $\Gamma^-$  and  $\Gamma^{0,+}$  the following open portions of  $\partial\Omega$

$$\Gamma^- = \bigcup_{i \in I} \omega_i, \quad (0.9)$$

where the sequence  $(\omega_i)_{i \in I}$  represents the set of the open sets  $\omega_i$  of  $\partial\Omega$  such that  $\mathcal{W} \mathbf{u} \cdot \mathbf{n} < 0$  almost everywhere in  $\omega_i$ . In the same way,

$$\Gamma^{0,+} = \bigcup_{j \in J} \omega'_j, \quad (0.10)$$

where the open sets  $\omega'_j$  of  $\partial\Omega$  are such that  $\mathcal{W} \mathbf{u} \cdot \mathbf{n} \geq 0$  almost everywhere in  $\omega'_j$ . Let us note that these definitions imply

$$\Gamma^- \cap \Gamma^{0,+} = \emptyset.$$

We assume that  $\Gamma^-$  and  $\Gamma^{0,+}$  have a finite number of connected components and verify

$$\partial\Omega = \overline{\Gamma^-} \cup \overline{\Gamma^{0,+}}, \quad \overline{\Gamma^-} \cap \overline{\Gamma^{0,+}} = \{\mathbf{m}_1, \dots, \mathbf{m}_q\}, \quad (0.11)$$

where  $\mathbf{m}_k, 1 \leq k \leq q$ , denote points of the boundary  $\partial\Omega$ .

Let us define the space  $U$  by

$$U = \{\mathbf{v} \in H^1(\Omega)^2; \operatorname{div} \mathbf{v} = 0\}. \quad (0.12)$$

Finally, we recall basic results of [3] that we apply in the particular case where  $d = 2$ .

**Proposition 0.2** *Let  $\Omega$  be a Lipschitz-continuous domain of  $\mathbb{R}^2$ , let  $\mathbf{u}$  be given in  $U$ , defined by (0.12), and let  $\Gamma^-$  and  $\Gamma^{0,+}$  be defined by (0.9) and (0.10), verifying (0.11). Let  $z$  belong to  $X_{\mathbf{u}}(\Gamma^-)$  and  $w$  to  $X_{\mathbf{u}}(\Gamma^{0,+})$ . Then,  $z$  and  $w$  verify the following inequalities*

$$\int_{\Omega} (\mathcal{W}\mathbf{u} \cdot \nabla z) z \, d\mathbf{x} \geq 0, \quad \int_{\Omega} (\mathcal{W}\mathbf{u} \cdot \nabla w) w \, d\mathbf{x} \leq 0. \quad (0.13)$$

Considering the problem: for  $\mathbf{u}$  in  $U$ ,  $l$  in  $L^2(\Omega)$  and  $\mathcal{W}$  in  $\mathbb{R}^*$ , find  $z$  in  $L^2(\Omega)$  such that:

$$\begin{cases} z + \mathcal{W}\mathbf{u} \cdot \nabla z = l & \text{in } \Omega, \\ (z\mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Gamma^- \end{cases} \quad (0.14)$$

In [3], we prove the following result of existence and uniqueness in  $L^2$ .

**Theorem 0.3** *Let  $\Omega$  be a lipschitz-continuous domain of  $\mathbb{R}^2$  and let  $\Gamma^-$  and  $\Gamma^{0,+}$  be defined by (0.9) and (0.10), verifying (0.11). For all  $\mathbf{u}$  in  $U$ , defined by (0.12), all  $l$  in  $L^2(\Omega)$  and all real numbers  $\mathcal{W}$  in  $\mathbb{R}^*$ , the transport problem (0.14) has a unique solution  $z$  in  $L^2(\Omega)$ .*

## 1 Examples of transport problems.

In this section, we study different examples of transport problems (0.1), obtained with different choices of velocities  $\mathbf{u}$ , functions  $l$  as right hand side and domains  $\Omega \subset \mathbb{R} \times ]0, +\infty[$ .

First, we choose  $\mathbf{u}(x, y) = (x, -y)$ , that verifies  $\operatorname{div} \mathbf{u} = 0$  everywhere, which corresponds to the following transport problem : find  $z \in H^1(\Omega)$  satisfying

$$\begin{cases} z + x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = l & \text{in } \Omega, \\ z|_{\Gamma^-} = 0 \end{cases} \quad (1.1)$$

This problem was introduced in [3] in the particular case where  $\Omega = ]0, 1[ \times ]1, 2[$ . In the examples 1, 2 and 3, we study the problem (1.1) for different examples of functions  $l$  and domains  $\Omega$ . We can set

$$\begin{cases} X = xy \\ Y = \ln y \end{cases} \quad .$$

Setting  $\Omega_* = \{(X, Y) \in \mathbb{R}^2, (Xe^{-Y}, e^Y) \in \Omega\}$ ,  $\Gamma_*^- = \{(X, Y) \in \mathbb{R}^2, (Xe^{-Y}, e^Y) \in \Gamma^-\}$  and  $Z(X, Y) = z(x, y)$ , we derive the following equivalent problem : Find  $Z \in H^1(\Omega_*)$  satisfying

$$\begin{cases} Z - \frac{\partial Z}{\partial Y} = l(Xe^{-Y}, e^Y) \\ Z|_{\Gamma_*^-} = 0 \end{cases} \quad .$$

Hence, if  $a \in [\min_{(x,y) \in \bar{\Omega}}(y), \max_{(x,y) \in \bar{\Omega}}(y)]$ , we find the general solution of the first equation of (1.1):

$$z(x, y) = y \int_{\ln y}^{\ln a} e^{-t} l(xye^{-t}, e^t) dt + yC(xy), \quad (1.2)$$

where  $C$  if any function in  $L^2$ . Thus, we have an infinity of solutions. In order to obtain a well-posed problem (see [3]), it is necessary to require a boundary condition on  $\Gamma^-$ , which allows us to compute the function  $C$ .

Second, another choice of velocity  $\mathbf{u}$  is:  $\mathbf{u}(x, y) = (xy^2 + y, -y^2)$ , as in the example 4 (or  $\mathbf{u}(x, y) = (-xy^2 - y, y^2)$ , as in the example 5). Setting  $X = xy^2 + y$ ,  $Y = \frac{1}{y}$  and  $Z(X, Y) = z(x, y)$ , we have the equivalence

$$z + \mathbf{u} \cdot \nabla z = l \iff Z + \frac{\partial Z}{\partial Y} = \tilde{l},$$

where  $\tilde{l}(X, Y) = l(XY^2 - Y, \frac{1}{Y})$ , which implies

$$\forall (x, y) \in \Omega, z(x, y) = e^{-\frac{1}{y}} \int_{\frac{1}{3}}^{\frac{1}{y}} e^{tl} ((xy^2 + y)t^2 - t, \frac{1}{t}) dt + e^{-\frac{1}{y}} C(xy^2 + y),$$

where  $C$  is any function in  $L^2$ . With the choice of  $l = 1$ , we obtain

$$z(x, y) = 1 - e^{\frac{1}{3} - \frac{1}{y}} + e^{-\frac{1}{y}} C(xy^2 + y). \quad (1.3)$$

For a given velocity  $\mathbf{u}$ , we introduce the following notations :

$$\Gamma^0 \text{ is the interior of the set } \{\mathbf{x} \in \partial\Omega, (\mathbf{u} \cdot \mathbf{n})(\mathbf{x}) = 0\}, \quad (1.4)$$

$$\Gamma^+ \text{ is the interior of the set } \{\mathbf{x} \in \partial\Omega, (\mathbf{u} \cdot \mathbf{n})(\mathbf{x}) > 0\}. \quad (1.5)$$

In the first two examples, the function  $\mathbf{u} \cdot \mathbf{n}$  does not vanish on  $\overline{\Gamma^-}$  and we verify that the regularity  $H^1$  of the solution  $z$  depends on the regularity  $H^1$  of  $l$ .

### 1.1 Example 1 : $\Omega = ]0, 1[ \times ]1, 2[$ , $l(x, y) = x^{\frac{2}{3}}$ , $\mathbf{u}(x, y) = (x, -y)$ .

Let  $\Omega$  be the square  $]0, 1[ \times ]1, 2[ \subset \mathbb{R}^2$  (see figure 1.1). We can verify that

$$\lim_{x \rightarrow 0^+} l'_x(x, y) = +\infty, \text{ but } l \in H^1(\Omega).$$

Moreover  $\Gamma^- = \Gamma_3$  (in red),  $\Gamma^0 = \Gamma_4$  (in green),  $\Gamma^+ = \Gamma_1 \cup \Gamma_2$  (in blue). From (1.2) and  $a = 2$ , in view of the boundary condition  $z|_{\Gamma^-} = 0$ , we derive  $C = 0$  and

$$\forall (x, y) \in [0, 1] \times [1, 2], z(x, y) = \frac{3}{5} x^{\frac{2}{3}} \left(1 - \frac{y^{\frac{5}{3}}}{2^{\frac{5}{3}}}\right).$$

We can verify that  $z \in H^1(\Omega)$ . Thus, we have taken  $l \in H^1(\Omega)$  and we have obtained  $z \in H^1(\Omega)$  and the problem (1.1) is well-posed.

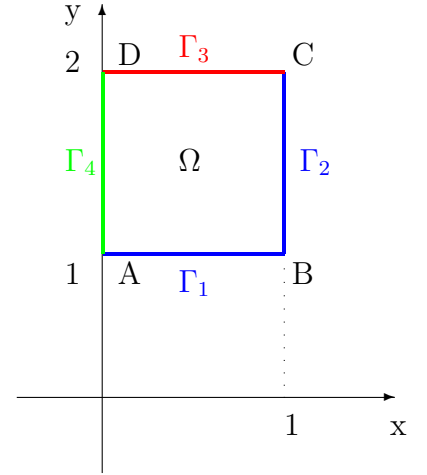


figure 1.1

### 1.2 Example 2 : $\Omega = ]0, 1[ \times ]1, 2[$ , $l(x, y) = \sqrt{x}$ , $\mathbf{u}(x, y) = (x, -y)$ .

Here, we have the same domain  $\Omega$  as previously, therefore the sets  $\Gamma^-$ ,  $\Gamma^0$  and  $\Gamma^+$  are the same, but we have now a function  $l$  that not belongs to  $H^1(\Omega)$ . As previously, from (1.2) and  $a = 2$ , in view of the boundary condition  $z|_{\Gamma^-} = 0$ , we derive  $C = 0$  and

$$\forall (x, y) \in [0, 1] \times [1, 2], z(x, y) = \frac{\sqrt{x}}{6} (4 - y\sqrt{2y}).$$

We can verify that  $z \in L^2(\Omega)$ , but  $z \notin H^1(\Omega)$ . Clearly, the reason why is that  $l \notin H^1(\Omega)$  and the problem (1.1) has no  $H^1$  solution.

In the following three examples, the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes on  $\overline{\Gamma^-}$ . In Example 3, the two assumptions of (3.5) are verified and the solution  $z$  is  $H^1$ , as expected by Theorem 3.1. On the contrary, in Examples 4 and 5, one of the two assumptions of (3.5) is not verified (the first in Example 4 and the second in Example 5) and the solution  $z$  is not  $H^1$ , thus proving the necessity of the two hypotheses (3.5) in Theorem 3.1.

**1.3 Example 3 :**  $\Omega = \text{triangle}(A(-\frac{1}{2}, \frac{1}{2}), B(\frac{1}{2}, \frac{1}{2}), C(\frac{1}{2}, \frac{3}{2}))$ ,  
 $l(x, y) = 1$ ,  $\mathbf{u}(x, y) = (x, -y)$ .

In this example, the set  $\Gamma^-$  is the line  $]A, C[$  and the function  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma^-}$  vanishes at the endpoint  $A$ .

We can verify that  $\Gamma^0 = \emptyset$  and  $\Gamma^+ = ]A, B] \cup ]B, C[$ . From (1.2),  $a = \frac{1}{2}$  and  $l = 1$  we derive

$$\forall (x, y) \in \Omega, z(x, y) = 1 - 2y + y C(xy).$$

Considering the boundary condition, we have

$$(z|_{\Gamma^-} = 0) \iff (\forall (x, y) \in \Gamma^-, C(xy) = 2 - \frac{1}{y}).$$

Setting  $X = xy$ , we have the following equivalence:

$$\forall (x, y) \in \Gamma^-, \begin{cases} y = x + 1 \\ -0.5 < x < 0.5 \end{cases} \iff \begin{cases} y = \frac{1 + \sqrt{1 + 4X}}{2} \\ -0.25 < X < 0.75 \end{cases}$$

Finally, we obtain the unique solution

$$\forall (x, y) \in \Omega, z(x, y) = 1 - \frac{2y}{1 + \sqrt{1 + 4xy}}.$$

Indeed,  $z \in L^2(\Omega)$  but we must verify that  $z \in H^1(\Omega)$ . We have

$$z'_x(x, y) = \frac{4y^2}{(1 + \sqrt{1 + 4xy})^2} \frac{1}{\sqrt{1 + 4xy}}.$$

For computing  $\int \int_{\Omega} (z'_x)^2 dx dy$ , we make the substitution  $\begin{cases} X = xy \\ y = y \end{cases}$ , the jacobian of which is  $\frac{1}{y}$ . We obtain

$$\int \int_{\Omega} (z'_x)^2 dx dy \leq \int_{0.5}^{1.5} 16y^3 dy \left( \int_{y^2-y}^{0.5y} \frac{1}{1 + 4X} dX \right) = \int_{0.5}^{1.5} 4y^3 (\ln(1 + 2y) - 2 \ln(2y - 1)) dy.$$

This last integral converges because  $\int_{0.5}^{1.5} \ln(2y - 1) dy$  is convergent in the neighbourhood of 0.5. We can compute  $\int \int_{\Omega} (z'_y)^2 dx dy$  in the same way. Thus, we obtain that the solution  $z$  belongs to  $H^1(\Omega)$  and, therefore, the problem (1.1) is well-posed.

Note that, the fact that the function  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma^-}$  vanishes at the point  $A$  leads to a discontinuity for the partial derivatives of the solution  $z$  in  $A$ . However, the function  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma^-}$  has a simple root in  $A$  and, moreover,  $\mathbf{u} \cdot \boldsymbol{\tau}_-(A) = -\frac{\sqrt{2}}{2} < 0$  in  $A$ , where  $\boldsymbol{\tau}_-(A)$  is the

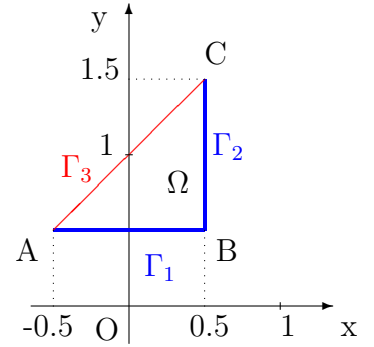


figure 1.2

unit tangent vector oriented towards  $\Gamma^-$ . Thus, the assumptions (3.5) of Theorem 3.1 are verified, which explains that the solution  $z$  is still in  $H^1$ .

In the two following examples, we change the function  $\mathbf{u}$ . In the Example 4, the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the end point of  $\Gamma^-$  with an order two and the solution  $z$  is not  $H^1$ , which is consistent with the Theorem 3.1, since the assumption (3.5) is not verified. In the Example 5, the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the end point  $A$  of  $\Gamma^-$  with an order one (simple root), but the assumption (3.5) is no longer verified, since the function  $\mathbf{u} \cdot \boldsymbol{\tau}_-(A)$  is positive in  $A$ , and again, but for another reason, the solution  $z$  is not  $H^1$ .

**1.4 Example 4 :**  $\Omega = \text{triangle}(A(-2, \frac{1}{3}), B(0, \frac{1}{3}), C(0, 1))$ ,  $l(x, y) = 1$ ,  $\mathbf{u}(x, y) = (2xy + 1, -y^2)$ .

In this example, we change the function  $\mathbf{u}$ . We shall see that  $\Gamma^-$  is the line  $]A, C[$  and, as in the example 3, the function  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma^-}$  vanishes at the endpoint  $A$ . However, contrary to the example 3, the solution  $z$  does not belong to  $H^1(\Omega)$ . The reason why is that, contrary to the previous example, the root in  $A$  is a double root, as we will show below. We can verify that  $\Gamma^0 = \emptyset$  and  $\Gamma^+ = ]A, B] \cup ]B, C[$ .

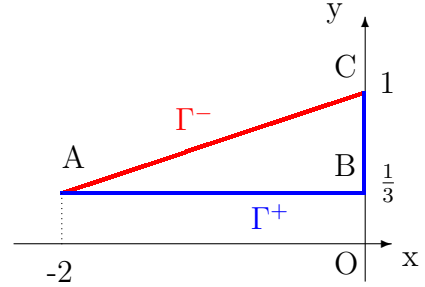


figure 1.3

We consider the following transport problem : find  $z \in H^1(\Omega)$  satisfying

$$\begin{cases} z + \mathbf{u} \cdot \nabla z = l & \text{in } \Omega, \\ z|_{\Gamma^-} = 0 \end{cases} \quad (1.6)$$

Since, for all  $(x, y) \in ]A, C[$ ,  $\mathbf{u} \cdot \mathbf{n} = -\frac{(x+2)^2}{\sqrt{10}}$ , we obtain  $\Gamma^- = ]A, C[$ . As we saw previously, the solution  $z$  is expressed by (1.3). Next, in view of  $X = xy^2 + y$ , we have the following equivalence

$$\begin{cases} y = \frac{x}{3} + 1 \\ -2 \leq x \leq 0 \end{cases} \iff \begin{cases} y = \left(\frac{X-\frac{1}{9}}{3}\right)^{\frac{1}{3}} + \frac{1}{3} \\ \frac{1}{9} \leq X \leq 1 \end{cases}$$

Then we derive

$$\forall (x, y) \in \Gamma^-, z(x, y) = 0 \iff C(X) = e^{\frac{1}{3}} - e^{\frac{1}{y}} = e^{\frac{1}{3}} - e^{\frac{1}{\left(\frac{X-\frac{1}{9}}{3}\right)^{\frac{1}{3}} + \frac{1}{3}}},$$

which allows us to compute the unique solution  $z$  of Problem (1.6) :

$$\forall (x, y) \in \Omega, z(x, y) = 1 - e^{\frac{1}{\alpha(x, y)} - \frac{1}{y}}, \quad (1.7)$$

with the function  $\alpha$  defined in  $\Omega$  by

$$\forall (x, y) \in \Omega, \alpha(x, y) = \left(\frac{xy^2 + y - \frac{1}{9}}{3}\right)^{\frac{1}{3}} + \frac{1}{3}.$$



Since the domain  $\Omega$  is below the segment  $[AC]$  and since the branch of hyperbola  $\begin{cases} y = -\frac{1}{x} \\ x < 0 \end{cases}$  is above the segment  $[AC]$ , we derive that, for all  $(x, y) \in \bar{\Omega}$ ,  $0 < xy + 1 \leq 1$ . The function  $(x, y) \mapsto xy + 1$  is continuous on the compact  $\bar{\Omega}$ , therefore there exists  $m_0 > 0$  such that  $\forall (x, y) \in \Omega$ ,  $xy + 1 \geq m_0$ . Note that  $m_0 \leq (-2)^{\frac{1}{3}} + 1 = \frac{1}{3}$ , which gives

$$\forall (x, y) \in \Omega, \quad m_0 \leq xy + 1 \leq 1, \quad \text{with } 0 < m_0 \leq \frac{1}{3}.$$

Hence, we obtain

$$\forall (x, y) \in \Omega, \quad \frac{m_0}{3} \leq \frac{1}{3}(1 - (1 - 3m_0)^{\frac{1}{3}}) \leq \alpha(x, y) \leq 1. \quad (1.8)$$

Let us show that  $z'_x$  does not belong to  $L^2(\Omega)$ . Considering (1.7), we compute

$$\forall (x, y) \in \Omega, \quad z'_x(x, y) = \frac{\alpha'_x(x, y)}{(\alpha(x, y))^2} e^{\frac{1}{\alpha(x, y)} - \frac{1}{y}},$$

with

$$\alpha'_x(x, y) = \frac{y^2}{9\left(\frac{xy^2 + y - \frac{1}{9}}{3}\right)^{\frac{2}{3}}}.$$

From (1.8), we derive

$$\forall (x, y) \in \Omega, \quad |z'_x(x, y)| \geq \frac{1}{3^4 e^2} \frac{1}{\left(\frac{xy^2 + y - \frac{1}{9}}{3}\right)^{\frac{2}{3}}}.$$

Using this estimation yields

$$\int \int_{\Omega} (z'_x(x, y))^2 dx dy \geq \frac{1}{3^8 e^4} \int_{\frac{1}{3}}^1 dy \left( \int_{3(y-1)}^0 \frac{1}{\left(\frac{xy^2 + y - \frac{1}{9}}{3}\right)^{\frac{4}{3}}} dx \right).$$

Making the substitution  $\begin{cases} X = xy^2 + y \\ Y = \frac{1}{y} \end{cases}$ , the jacobian of which is -1, we obtain

$$\int \int_{\Omega} (z'_x(x, y))^2 dx dy \geq \frac{1}{3^8 e^4} \int_1^3 dY \left( \int_{3\left(\frac{1}{Y} - \frac{1}{3}\right)^3 + \frac{1}{9}}^{\frac{1}{Y}} \frac{1}{\left(\frac{X - \frac{1}{9}}{3}\right)^{\frac{4}{3}}} dX \right) \geq \frac{1}{3^5 e^4} \int_1^3 \frac{1}{3 - Y} dY - \frac{2^{\frac{2}{3}}}{3^5 e^4}.$$

Since  $\int_1^3 \frac{1}{3 - Y} dY = +\infty$ , we obtain  $\int \int_{\Omega} (z'_x(x, y))^2 dx dy = +\infty$ .

Finally, the solution  $z$  of the example 4, contrary to the previous example, does not belong to  $H^1(\Omega)$  and, therefore, the problem (1.6) is not well-posed.

1.5 **Example 5** :  $\Omega = \text{triangle}(A(-\frac{4}{3}, \frac{2}{3}), B(-\frac{11}{6}, \frac{1}{6}), C(-\frac{4}{3}, \frac{1}{2}))$ ,  
 $l(x, y) = 1$ ,  $\mathbf{u}(x, y) = (-2xy - 1, y^2)$ .

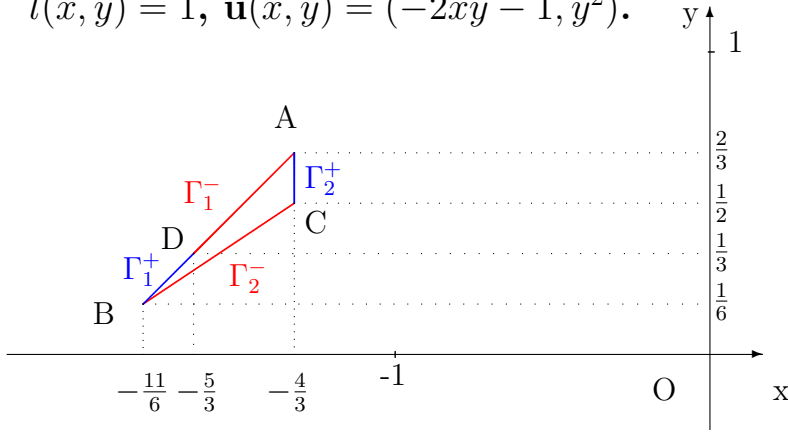


figure 1.4

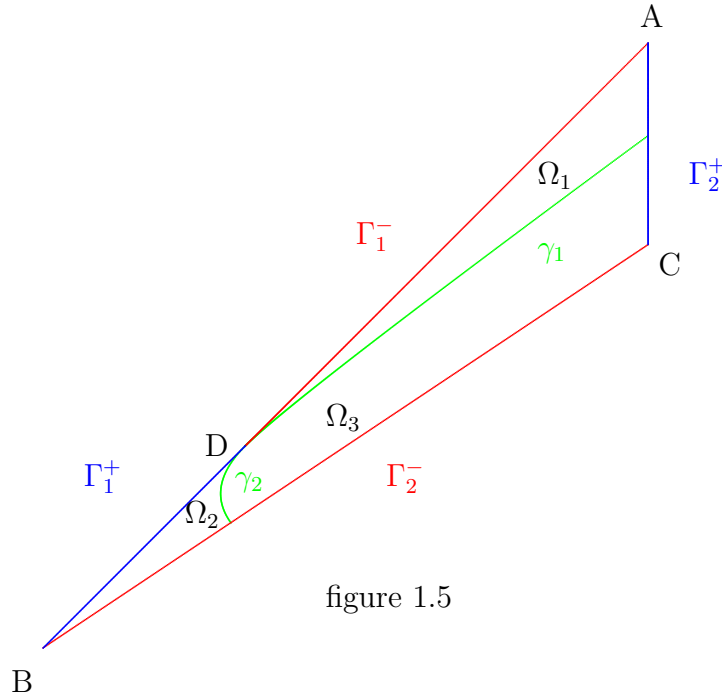


figure 1.5

We can verify:  $(x, y) \in (AB) \Leftrightarrow x - y + 2 = 0$  and  $(x, y) \in (BC) \Leftrightarrow 2x - 3y + \frac{25}{6} = 0$ .  
 The set  $\Gamma^-$  is composed of two parts :  $\Gamma_1^- \begin{cases} x - y + 2 = 0 \\ \frac{1}{3} < y < \frac{2}{3} \end{cases}$  and  $\Gamma_2^- \begin{cases} 2x - 3y + \frac{25}{6} = 0 \\ \frac{1}{6} < y < \frac{1}{2} \end{cases}$ .  
 The set  $\Gamma^+$  is composed of two parts :  $\Gamma_1^+ \begin{cases} x - y + 2 = 0 \\ \frac{1}{6} < y < \frac{1}{3} \end{cases}$  and  $\Gamma_2^+ \begin{cases} x = -\frac{4}{3} \\ \frac{1}{2} < y < \frac{2}{3} \end{cases}$ .

For all  $\mathbf{m} = (x, y) \in \Gamma_1^-$ ,  $\mathbf{u} \cdot \mathbf{n}(\mathbf{m}) = (3x + 5)(x + 1)$  and for all  $\mathbf{m} = (x, y) \in \Gamma_2^-$ ,  
 $\mathbf{u} \cdot \mathbf{n}(\mathbf{m}) = -9y^2 + \frac{25}{3}y - 2$ . Therefore, the function  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma}$  vanishes at the unique point

$D(-\frac{5}{3}, \frac{1}{3})$ , with an order one with respect to the parameter of the line  $(AB)$ . Thus, we have

$$\mathbf{u} \cdot \mathbf{n}_{|\Gamma}(D) = 0, \quad \frac{\partial \mathbf{u}}{\partial \boldsymbol{\tau}_-} \cdot \mathbf{n}(D) \neq 0 \text{ and } (\mathbf{u} \cdot \boldsymbol{\tau}_-)|_{\Gamma}(D) > 0, \quad (1.9)$$

where  $\boldsymbol{\tau}_-(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$  is unit tangent vector, oriented towards  $\Gamma^-$ .

Setting  $\begin{cases} X = -xy^2 - y \\ Y = \frac{1}{y} \end{cases}$ , by technics analogous to the previous examples, we obtain the solutions of the equation  $z + \mathbf{u} \cdot \nabla z = l$

$$Z(X, Y) = 1 + e^Y(C(X) - 1) \Leftrightarrow z(x, y) = 1 + e^{\frac{1}{y}}(C(-xy^2 - y) - 1),$$

where  $C$  is a function to be determined by the boundary conditions.

Setting  $\alpha(y) = X(y - 2, y) = -y^3 + 2y^2 - y$ , we can verify

$$z \cdot \mathbf{n}_{|\Gamma_1^-} = 0 \Leftrightarrow \forall y \in ]\frac{1}{3}, \frac{2}{3}[, C(\alpha(y)) = 1 - e^{\frac{1}{y}} \Leftrightarrow \forall X \in ]-\frac{4}{27}, -\frac{2}{27}[, C(X) = 1 - e^{\frac{1}{\alpha^{-1}(X)}}.$$

In the same way, setting  $\beta(y) = X(\frac{3}{2}y - \frac{25}{12}, y) = -\frac{3}{2}y^3 + \frac{25}{12}y^2 - y$ , we can verify

$$z \cdot \mathbf{n}_{|\Gamma_2^-} = 0 \Leftrightarrow \forall y \in ]\frac{1}{6}, \frac{1}{2}[, C(\beta(y)) = 1 - e^{\frac{1}{y}} \Leftrightarrow \forall X \in ]-\frac{1}{6}, -\frac{25}{216}[, C(X) = 1 - e^{\frac{1}{\beta^{-1}(X)}}.$$

Taking into account these boundary conditions, setting  $y_1 = \beta^{-1}(-\frac{4}{27})$  and using a function  $\alpha_1$ , which is a restriction of the function  $\alpha$ , and functions  $\alpha_2$  and  $\alpha_3$ , which are restrictions of the function  $\beta$ , we express the solution  $z$  by splitting the domain  $\Omega$  into three sub domains  $\Omega_i$ ,  $i = 1, 2, 3$ :

$$z_{|\Omega_i} = 1 - e^{\frac{1}{y} - \frac{1}{\alpha_i^{-1}(-xy^2 - y)}}, \quad (1.10)$$

where  $\Omega_1$  is defined by

$$\Omega_1 = \{(x, y) \in \Omega, y > \frac{1}{3}, -xy^2 - y > -\frac{4}{27}\} \text{ with } \alpha_1 : \begin{array}{l} [\frac{1}{3}, \frac{2}{3}] \rightarrow [-\frac{4}{27}, -\frac{2}{27}] \\ y \mapsto -y^3 + 2y^2 - y \end{array},$$

where  $\Omega_2$  is defined by

$$\Omega_2 = \{(x, y) \in \Omega, y < \frac{1}{3}, -xy^2 - y > -\frac{4}{27}\} \text{ with } \alpha_2 : \begin{array}{l} [\frac{1}{6}, y_1] \rightarrow [-\frac{4}{27}, -\frac{25}{216}] \\ y \mapsto -\frac{3}{2}y^3 + \frac{25}{12}y^2 - y \end{array}$$

and where  $\Omega_3$  is defined by

$$\Omega_3 = \{(x, y) \in \Omega, -xy^2 - y < -\frac{4}{27}\} \text{ with } \alpha_3 : \begin{array}{l} [y_1, \frac{1}{2}] \rightarrow [-\frac{1}{6}, -\frac{4}{27}] \\ y \mapsto -\frac{3}{2}y^3 + \frac{25}{12}y^2 - y \end{array}.$$

Note that the domains  $\Omega_1$  and  $\Omega_3$  are adjacent and are separated by the curve  $\gamma_1$  and the domains  $\Omega_2$  and  $\Omega_3$  are adjacent and are separated by the curve  $\gamma_2$  (see the figure 1.5), where  $\gamma_1$  and  $\gamma_2$  are defined by

$$\gamma_1 \begin{cases} -xy^2 - y = -\frac{4}{27} \\ \frac{1}{3} < y < \frac{3+\sqrt{3}}{8} \end{cases}, \quad \gamma_2 \begin{cases} -xy^2 - y = -\frac{4}{27} \\ y_1 < y < \frac{1}{3} \end{cases}.$$

Considering the expressions of the solution given by (1.10), we obtain

$$\forall y \in ]\frac{1}{3}, \frac{3 + \sqrt{3}}{8}[, (z|_{\Omega_1} - z|_{\Omega_3})|_{\gamma_1}(y) = (e^{-\frac{1}{y^1}} - e^{-3})e^{\frac{1}{y}},$$

which implies that the solution  $z$  is discontinuous on the curve  $\gamma_1$ . Computing the gradient of the solution  $z$  yields

$$\nabla z = \widetilde{\nabla z|_{\Omega_1}} + \widetilde{\nabla z|_{\Omega_2}} + \widetilde{\nabla z|_{\Omega_3}} - \delta_{\gamma_1},$$

where the wide tildes denote the extensions by zero and where the distribution  $\delta_{\gamma_1}$  is defined by

$$\forall \varphi \in (\mathcal{D}(\Omega))^2, \langle \delta_{\gamma_1}, \varphi \rangle = \int_{\gamma_1} (z|_{\Omega_1} - z|_{\Omega_3}) \varphi \cdot \mathbf{n}_1 ds,$$

where  $\mathbf{n}_1$  is the unit exterior normal vector to the boundary of the domain  $\Omega_1$ . Finally, since the distribution  $\delta_{\gamma_1}$  does not belong to  $L^2(\Omega)$ , we obtain that the solution  $z$  of the example 5 does not belong to  $H^1(\Omega)$  and, therefore, the problem (1.6) is not well-posed.

For explaining further in details, in view of (1.9), the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the boundary point  $D$  of  $\Gamma_1^-$  with an order one with respect to the parameter of the line  $(AB)$ , but, since we have  $(\mathbf{u} \cdot \boldsymbol{\tau}_-)|_{\Gamma}(D) > 0$ , the solution  $z$  in the neighborhood of  $D$  on the  $\Omega_2$  side depends of the boundary condition on  $\Gamma_2^-$ , which is far from  $D$ . This means that we cannot localize the transport problem in a neighborhood of the boundary point  $D$  and, therefore, we cannot apply the technics of the proof of Theorem 3.1. Thus, the assumption  $\mathbf{u}(\mathbf{m}) \cdot \boldsymbol{\tau}_-(\mathbf{m}) < 0$  of Theorem 3.1 is not only a technical assumption, but a basic assumption as well as the other assumption  $\frac{\partial \mathbf{u}}{\partial \boldsymbol{\tau}_-}(\mathbf{m}) \cdot \mathbf{n}_-(\mathbf{m}) \neq 0$  of (3.5).

## 2 Transport equations in $H^1$ when $\mathbf{u} \cdot \mathbf{n}$ does not vanish on $\overline{\Gamma^-}$

Let us recall the following problem studied in [3]. Let  $\Omega$  be a bounded domain of  $\mathbb{R}^2$  and  $\Gamma^-$  be defined by (0.9), verifying (0.11): for  $\mathbf{u}$  in  $H^1(\Omega)^d$ , with  $\operatorname{div} \mathbf{u} = 0$ ,  $l$  in  $L^2(\Omega)$  and  $\mathcal{W}$  in  $\mathbb{R}^*$ , find  $z$  in  $L^2(\Omega)$  such that

$$\begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = l & \text{in } \Omega \\ (z \mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Gamma^-. \end{cases} \quad (2.1)$$

The main result is given by Theorem 3.3 in [3], which gives the existence and the uniqueness of solution in  $L^2(\Omega)$  in the case where  $\Omega$  is a Lipschitz-continuous domain of  $\mathbb{R}^d$ . Now, we are interested by  $H^1$  solutions in the two dimensions case. In order to find  $H^1$  solutions, we assume that  $\Omega$  is a bounded polygon, we suppose that  $\mathbf{u}$  belongs to  $W^{1,\infty}(\Omega)^2$  and we shall impose another boundary condition.

Thus, we are led to study the following problem: let  $\Omega$  be a bounded polygon, for  $\mathbf{u}$  in  $U \cap W^{1,\infty}(\Omega)^2$ , where  $U$  is defined by (0.12),  $l$  in  $H^1(\Omega)$  and  $\mathcal{W}$  in  $\mathbb{R}^*$ , find  $z$  in  $H^1(\Omega)$  such that

$$\begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = l & \text{in } \Omega \\ z = 0 & \text{on } \Gamma^-. \end{cases} \quad (2.2)$$

Let  $\Omega$  be a bounded polygon. We begin to establish a result of existence and uniqueness in the particular where  $l$  vanishes on  $\Gamma^-$ .

**Theorem 2.1** *Let  $\Omega$  be a bounded polygon,  $\Gamma^-$  be defined by (0.9), verifying (0.11) and  $U$  be defined by (0.12). For all  $\mathbf{u}$  in  $U \cap W^{1,\infty}(\Omega)^2$  such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq \frac{1}{2|\mathcal{W}|}, \quad (2.3)$$

*all  $l$  in  $H^1(\Omega)$  such that  $l|_{\Gamma^-} = 0$  and all real number  $\mathcal{W}$  in  $\mathbb{R}^*$ , the transport problem (2.2) has a unique solution  $z$  in  $H^1(\Omega)$ .*

**Proof.** Formally,  $\nabla z$  satisfies

$$\nabla z + \mathcal{W} \mathbf{u} \cdot \nabla (\nabla z) = \nabla l - \mathcal{W} \nabla \mathbf{u} \cdot \nabla z.$$

Let us define a sequence  $(\mathbf{F}_n)$  of functions  $\mathbf{F}_n \in X_{\mathbf{u}}(\Gamma^-)^2$ ,  $n \in \mathbb{N}$ , by recurrence, where  $X_{\mathbf{u}}(\Gamma^-)$  is defined by (0.8). We set  $\mathbf{F}_0 = \mathbf{0}$  and assume that the function  $\mathbf{F}_n \in X_{\mathbf{u}}(\Gamma^-)^2$  is given for  $n \in \mathbb{N}$ . Then, applying Theorem 0.3, we define each component  $F_{n+1,1}$  and  $F_{n+1,2}$  of  $\mathbf{F}_{n+1}$  as the unique solution of a transport equation from the type (2.1), of such so that we define  $\mathbf{F}_{n+1} \in X_{\mathbf{u}}(\Gamma^-)^2$  as the unique solution of the transport equation

$$\begin{cases} \mathbf{F}_{n+1} + \mathcal{W} \mathbf{u} \cdot \nabla \mathbf{F}_{n+1} = \nabla l - \mathcal{W} \nabla \mathbf{u} \cdot \mathbf{F}_n & \text{in } \Omega \\ (\mathbf{F}_{n+1} \mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Gamma^-. \end{cases} \quad (2.4)$$

Since  $\mathbf{F}_{n+1}$  belongs to  $X_{\mathbf{u}}(\Gamma^-)^2$ , the basic result of Proposition (0.2) implies

$$\int_{\Omega} (\mathcal{W} \mathbf{u} \cdot \nabla F_{n+1,i}) F_{n+1,i} d\mathbf{x} \geq 0, \text{ for } i = 1, 2.$$

Then, taking the scalar product of both sides of the first equation of (2.4) with  $\mathbf{F}_{n+1}$  yields

$$\|\mathbf{F}_{n+1}\|_{L^2(\Omega)}^2 \leq (\nabla l, \mathbf{F}_{n+1}) - \mathcal{W} (\nabla \mathbf{u} \cdot \mathbf{F}_n, \mathbf{F}_{n+1}).$$

Hence, we derive

$$\|\mathbf{F}_{n+1}\|_{L^2(\Omega)} \leq \|\nabla l\|_{L^2(\Omega)} + |\mathcal{W}| \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\mathbf{F}_n\|_{L^2(\Omega)}.$$

In view of the bound (2.3), we obtain

$$\|\mathbf{F}_{n+1}\|_{L^2(\Omega)} \leq \|\nabla l\|_{L^2(\Omega)} + \frac{1}{2} \|\mathbf{F}_n\|_{L^2(\Omega)},$$

which implies, by a recurrence argument, that  $\mathbf{F}_n$  is uniformly bounded in  $L^2(\Omega)$  and  $\forall n \in \mathbb{N}$ ,

$$\|\mathbf{F}_n\|_{L^2(\Omega)} \leq 2\|\nabla l\|_{L^2(\Omega)}. \quad (2.5)$$

Owing to (2.5),  $\mathbf{u} \cdot \nabla \mathbf{F}_{n+1}$  is also uniformly bounded in  $L^2(\Omega)$ . Therefore we can pass to the limit in the first equation of (2.4) and there exists a function  $\mathbf{F} \in L^2(\Omega)^2$  such that

$$\mathbf{F} + \mathcal{W} (\mathbf{u} \cdot \nabla \mathbf{F} + \nabla \mathbf{u} \cdot \mathbf{F}) = \nabla l. \quad (2.6)$$

Let us set  $z = l - \mathcal{W} \mathbf{u} \cdot \mathbf{F}$ . From the previous equation, we derive  $\mathbf{F} = \nabla z$  and we obtain  $z = l - \mathcal{W} \mathbf{u} \cdot \nabla z$ , which gives that  $z$  is solution of the first equation of (2.2).

Next, from Green's formula (0.7) and  $(\mathbf{F}_{n+1}\mathbf{u}) \cdot \mathbf{n}|_{\Gamma^-} = \mathbf{0}$ , we derive  $\forall \varphi \in W^{1,r}(\Omega)^2$ , with  $\varphi|_{\Gamma^{0,+}} = \mathbf{0}$ ,

$$(\mathbf{F}_{n+1}\mathbf{u}, \nabla\varphi) + (\varphi\mathbf{u}, \nabla\mathbf{F}_{n+1}) = \langle (\mathbf{F}_{n+1}\mathbf{u}) \cdot \mathbf{n}, \varphi \rangle_{\Gamma^-} = 0.$$

Using the above convergence, we can pass to the limit and we obtain

$$\forall \varphi \in W^{1,r}(\Omega)^2, \text{ with } \varphi|_{\Gamma^{0,+}} = 0, \quad (\mathbf{F}\mathbf{u}, \nabla\varphi) + (\varphi\mathbf{u}, \nabla\mathbf{F}) = 0,$$

which implies, with again the Green's formula (0.7),  $\langle (\mathbf{F}\mathbf{u}) \cdot \mathbf{n}, \varphi \rangle_{\Gamma^-} = 0$ . Thus, we obtain  $(\mathbf{F}\mathbf{u}) \cdot \mathbf{n}|_{\Gamma^-} = 0$ , that is to say,

$$(\nabla z \mathbf{u}) \cdot \mathbf{n}|_{\Gamma^-} = 0. \quad (2.7)$$

Hence, we can use a density result of [3](Corollary 2.11, page 1012): since, for  $i = 1, 2$ ,  $\frac{\partial z}{\partial x_i}$  belongs to  $X_{\mathbf{u}}(\Gamma^-)$ , there exist two sequences  $(\varphi_{1,n})$  et  $(\varphi_{2,n})$  such that, for  $i = 1, 2$ ,  $\varphi_{i,n} \in \mathcal{D}(\overline{\Omega}, \Gamma^-)$  and

$$\lim_{n \rightarrow +\infty} \varphi_{i,n} = \frac{\partial z}{\partial x_i} \quad \text{strongly in } X_{\mathbf{u}}(\Gamma^-),$$

where  $X_{\mathbf{u}}(\Gamma^-)$  is defined in (0.8). Setting  $\varphi_n = \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix}$ , from the above convergence and the regularity of  $\mathbf{u}$  we derive

$$\lim_{n \rightarrow +\infty} \mathbf{u} \cdot \varphi_n = \mathbf{u} \cdot \nabla z \quad \text{strongly in } L^2(\Omega).$$

Noting that

$$\nabla(\mathbf{u} \cdot \nabla z) = \begin{pmatrix} \frac{\partial \mathbf{u}}{\partial x_1} \cdot \nabla z + \mathbf{u} \cdot \nabla \left( \frac{\partial z}{\partial x_1} \right) \\ \frac{\partial \mathbf{u}}{\partial x_2} \cdot \nabla z + \mathbf{u} \cdot \nabla \left( \frac{\partial z}{\partial x_2} \right) \end{pmatrix},$$

the convergences in  $X_{\mathbf{u}}(\Gamma^-)$  give, for  $i = 1, 2$ ,

$$\lim_{n \rightarrow +\infty} (\mathbf{u} \cdot \nabla \varphi_{i,n}) = \mathbf{u} \cdot \nabla \left( \frac{\partial z}{\partial x_i} \right) \quad \text{strongly in } L^2(\Omega),$$

$$\lim_{n \rightarrow +\infty} \left( \frac{\partial \mathbf{u}}{\partial x_i} \cdot \varphi_n \right) = \frac{\partial \mathbf{u}}{\partial x_i} \cdot \nabla z \quad \text{strongly in } L^2(\Omega).$$

These convergences imply

$$\lim_{n \rightarrow +\infty} \nabla(\mathbf{u} \cdot \varphi_n) = \nabla(\mathbf{u} \cdot \nabla z).$$

Thus, we obtain that

$$\lim_{n \rightarrow +\infty} (\mathbf{u} \cdot \varphi_n) = \mathbf{u} \cdot \nabla z \quad \text{strongly in } H^1(\Omega).$$

In view of  $\varphi_n|_{\Gamma^-} = \mathbf{0}$ , we obtain

$$(\mathbf{u} \cdot \nabla z)|_{\Gamma^-} = 0.$$

Considering that  $z + \mathcal{W} \mathbf{u} \cdot \nabla z = l$  and  $l|_{\Gamma^-} = 0$ , we obtain

$$z|_{\Gamma^-} = 0.$$

Thus, we have proven the existence of solution for the transport problem (2.2).

Concerning the uniqueness, let us consider  $z \in H^1(\Omega)$  solution of the problem

$$\begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = 0 & \text{in } \Omega \\ z = 0 & \text{on } \Gamma^-. \end{cases} \quad (2.8)$$

For proving the uniqueness of solution of Problem (2.2), we must show that necessarily  $z = 0$ . Taking the scalar product in  $L^2(\Omega)$  of the previous equation by  $z$  yields

$$\|z\|_{L^2(\Omega)} + \mathcal{W}(\mathbf{u} \cdot \nabla z, z) = 0.$$

Since  $z$  belongs to  $X_{\mathbf{u}}(\Gamma^-)$ , Proposition (0.2) implies  $\mathcal{W}(\mathbf{u} \cdot \nabla z, z) \geq 0$  and we derive

$$\|z\|_{L^2(\Omega)} \leq 0.$$

This gives  $z = 0$ , which gives the uniqueness of solution of Problem (2.2).  $\diamond$ .

Now, we do not assume that  $l$  vanishes on  $\Gamma^-$ . If  $\mathbf{m}$  belongs to  $\overline{\Gamma^-}$  and does not belong to  $\overline{\Gamma^{+,0}}$ , we denote by

$$\mathbf{n}_-(\mathbf{m}) \text{ the unit exterior normal vector to } \overline{\Gamma^-} \text{ in } \mathbf{m} \quad (2.9)$$

(one or other of the two unit exterior normal vectors if  $\mathbf{m}$  is a vertex of the polygon). If  $\mathbf{m}$  belongs to  $\overline{\Gamma^-} \cap \overline{\Gamma^{+,0}}$ , then  $\mathbf{m}$  is the common endpoint of two adjacent straight segments  $\gamma_+$  and  $\gamma_-$  such that  $\gamma_+ \subset \overline{\Gamma^{+,0}}$  and  $\gamma_- = [\mathbf{m}, \mathbf{m}_-] \subset \overline{\Gamma^-}$  with  $\mathbf{m} \neq \mathbf{m}_-$ . We denote by

$$\mathbf{n}_-(\mathbf{m}) \text{ the unit exterior normal vector to } \gamma_-, \quad (2.10)$$

and by

$$\boldsymbol{\tau}_-(\mathbf{m}) \text{ the unit tangent vector } \frac{1}{\|\mathbf{m}\mathbf{m}_-\|} \mathbf{m}\mathbf{m}_-. \quad (2.11)$$

First, we assume that the normal component of the velocity does not vanish on  $\overline{\Gamma^-}$ . Since  $\mathbf{u} \cdot \mathbf{n}$  is continuous on the sides of the polygon  $\Omega$ , this implies that the end points of  $\Gamma^-$  are vertices of the polygon. The following theorem gives assumptions implying existence and uniqueness for problem (2.2).

**Theorem 2.2** *Let  $\Omega$  be a bounded polygon,  $\Gamma^-$  be defined by (0.9), verifying (0.11) and  $U$  be defined by (0.12). For all  $\mathbf{u}$  in  $U \cap W^{1,\infty}(\Omega)^2$  such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq \frac{1}{2|\mathcal{W}|} \quad (2.12)$$

and such that

$$\forall \mathbf{m} \in \overline{\Gamma^-}, \mathbf{u}(\mathbf{m}) \cdot \mathbf{n}_-(\mathbf{m}) \neq 0, \quad (2.13)$$

where  $\mathbf{n}_-(\mathbf{m})$  is defined by (2.9) or by (2.10), all  $l$  in  $H^1(\Omega)$  and all real number  $\mathcal{W}$  in  $\mathbb{R}^*$ , the transport problem (2.2) has a unique solution  $z$  in  $H^1(\Omega)$ .

**Proof.** Since the end points of  $\Gamma^-$  are vertices, we have

$$\overline{\Gamma^-} = \bigcup_{j \in J} \Gamma_j, \quad (2.14)$$

where the sets  $\Gamma_j$  are sides of the polygon  $\Omega$ . Since  $\mathbf{u}$  is continuous on  $\partial\Omega$ , for all  $j \in J$ , we denote  $\eta_j = \min_{m \in \Gamma_j} (|\mathbf{u}(\mathbf{m}) \cdot \mathbf{n}_j|)$ . From (2.13), we derive that, for all  $j \in J$ ,  $\eta_j > 0$ , which implies that  $(\frac{l}{\mathbf{u} \cdot \mathbf{n}_j})|_{\Gamma_j}$  belongs to  $H^{\frac{1}{2}}(\Gamma_j)$ . So, there exists  $z_0$  in  $H^2(\Omega)$  verifying, for all  $j \in J$ ,

$$\begin{cases} (\frac{\partial z_0}{\partial n})|_{\Gamma_j} = (\frac{l}{\mathcal{W} \mathbf{u} \cdot \mathbf{n}_j})|_{\Gamma_j} \\ z_0|_{\Gamma_j} = 0. \end{cases} \quad (2.15)$$

Hence, we derive that

$$(z_0 + \mathcal{W} \mathbf{u} \cdot \nabla z_0)|_{\Gamma^-} = l|_{\Gamma^-}. \quad (2.16)$$

Next, applying Theorem 2.1, let  $z^* \in H^1(\Omega)$  be the unique solution of the problem

$$\begin{cases} z^* + \mathcal{W} \mathbf{u} \cdot \nabla z^* = l - z_0 - \mathcal{W} \mathbf{u} \cdot \nabla z_0 & \text{in } \Omega \\ z^* = 0 & \text{on } \Gamma^- \end{cases}.$$

Then,  $z = z^* + z_0$  verifies  $z + \mathcal{W} \mathbf{u} \cdot \nabla z = l$  and  $z|_{\Gamma^-} = 0$ . Thus, we have proven the existence of solution for the transport problem (2.2). We prove the uniqueness in the same way as in the previous theorem.  $\diamond$

### 3 Transport equations in $H^1$ when $\mathbf{u} \cdot \mathbf{n}$ vanishes on $\overline{\Gamma^-}$

We assume that  $\Omega$  is a bounded convex polygon, but the fact that the normal component of the velocity can vanish on the boundary introduces a singularity at the end points of  $\Gamma^-$  and we will be forced to make assumptions at the end points of  $\Gamma^-$ , as we could expect from the examples of the Section 2. We denote by

$$S \text{ the set of the vertices of the polygon } \Omega \quad (3.1)$$

and let the set  $E$  be defined by

$$E = \{\mathbf{m} \in \overline{\Gamma^-} \cap \overline{\Gamma^{+,0}}, \mathbf{u}(\mathbf{m}) \cdot \mathbf{n}_-(\mathbf{m}) = 0\}, \quad (3.2)$$

where  $\mathbf{n}_-(\mathbf{m})$  is defined by (2.10). Note that, in view of the assumption (0.11), the set  $E$  is finite. In addition, we make the assumption that the velocity  $\mathbf{u}$  is such that

$$\{\mathbf{m} \in \overline{\Gamma^-}, \mathbf{u}(\mathbf{m}) \cdot \mathbf{n}_-(\mathbf{m}) = 0\} \subset E, \quad (3.3)$$

which means that  $\mathbf{u} \cdot \mathbf{n}$  does not vanish in a point located in the interior of  $\overline{\Gamma^-}$ .

The next theorem, which is the main result of the paper, gives assumptions implying existence and uniqueness for problem (2.2), in the case where the normal component of the velocity vanishes on the boundary. Note that, the first assumption of (3.5) means that the function  $\mathbf{u} \cdot \mathbf{n}$  must have only simple roots at the end points of  $\Gamma^-$ , which seems consistent with the previously studied examples. At first glance, the second assumption of (3.5) seems to be a technical assumption, related to the method used in the proof of Theorem 3.1. Indeed, we need this assumption, in the proof of the theorem, probably because, in



the case where  $\mathbf{u}(\mathbf{m}) \cdot \boldsymbol{\tau}_-(\mathbf{m}) > 0$ , it does not seem possible to localize the problem around the points of the set  $E$ : on either side of the point where  $\mathbf{u} \cdot \mathbf{n}$  vanishes, the expressions of the solution  $z$  are determined by boundary conditions located in two different places of the boundary, which leads to a discontinuity of the solution  $z$ , see Example 5. In fact, as it appears in Example 5, this second assumption seems necessary to obtain a solution  $z$  in  $H^1$ .

**Theorem 3.1** *Let  $\Omega$  be a bounded convex polygon,  $\Gamma^-$  be defined by (0.9), verifying (0.11) and  $U$  be defined by (0.12). For all  $\mathbf{u}$  in  $U \cap W^{1,\infty}(\Omega)^2$ , verifying (3.3), such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq \frac{1}{2|\mathcal{W}|} \quad (3.4)$$

and such that

$$\forall \mathbf{m} \in E, \quad \frac{\partial \mathbf{u}}{\partial \boldsymbol{\tau}_-}(\mathbf{m}) \cdot \mathbf{n}_-(\mathbf{m}) \neq 0 \text{ and } \mathbf{u}(\mathbf{m}) \cdot \boldsymbol{\tau}_-(\mathbf{m}) < 0, \quad (3.5)$$

where  $\mathbf{n}_-(\mathbf{m})$  (respectively  $\boldsymbol{\tau}_-(\mathbf{m})$ ,  $E$ ) is defined by (2.10) (respectively (2.11), (3.2)), all  $l$  in  $H^1(\Omega)$  and all real number  $\mathcal{W}$  in  $\mathbb{R}^*$ , the transport equation (2.2) has a unique solution  $z$  in  $H^1(\Omega)$ .

**Proof.** Let us split up  $\overline{\Gamma^-}$  into straight segments as

$$\begin{aligned} \overline{\Gamma^-} &= \bigcup_{j=1}^q \gamma_j, \quad \gamma_j \cap \gamma_k = \emptyset \text{ if } k \notin \{j-1, j, j+1\}, \\ \gamma_j \cap \gamma_k &= \emptyset \text{ or } \gamma_j \cap \gamma_k \in S \text{ if } k \in \{j-1, j+1\}, \quad 1 \leq j \leq q, \quad 0 \leq k \leq q+1 \end{aligned} \quad (3.6)$$

with the convention  $\gamma_{q+l} = \gamma_l$  for  $l = 0, 1$ , and let  $\mu_0 > 0$  be defined by

$$\mu_0 = \min_{\substack{\gamma_j \cap \gamma_k = \emptyset \\ 1 \leq j, k \leq q}} d(\gamma_j, \gamma_k), \quad (3.7)$$

where  $d(., .)$  is the euclidian distance in  $\mathbb{R}^2$ . Then, for  $0 < \mu \leq \frac{1}{2}\mu_0$ , in order to localize around the sets  $\gamma_j$ , let us define the functions  $(\theta_{j,\mu})_{1 \leq j \leq q} \in \mathcal{D}(\mathbb{R}^2)$  by

$$\forall \mathbf{x} \in \mathbb{R}^2, \quad \theta_{j,\mu}(\mathbf{x}) = \begin{cases} 1 & \text{if } d(\mathbf{x}, \gamma_j) \leq \frac{1}{2}\mu \\ 0 & \text{if } d(\mathbf{x}, \gamma_j) \geq \mu. \end{cases} \quad (3.8)$$

and,  $\forall \mathbf{x} \in \mathbb{R}^2$ ,  $\theta_{q+1,\mu}(\mathbf{x}) = 0$ . Setting, for  $1 \leq j \leq q$  and  $0 < \mu \leq \frac{1}{2}\mu_0$ ,

$$l_{j,\mu} = \theta_{j,\mu}(1 - \theta_{j+1,\mu})l \quad (3.9)$$

and

$$l_\mu = \left(1 - \sum_{j=1}^q \theta_{j,\mu}(1 - \theta_{j+1,\mu})\right)l, \quad (3.10)$$

where  $l$  is the right hand side of the transport equation, we obtain

$$l = l_\mu + \sum_{j=1}^q l_{j,\mu} \quad (3.11)$$

and we can verify that

$$\forall \mathbf{x} \in \Gamma^-, l_\mu(\mathbf{x}) = 0. \quad (3.12)$$

From the development of  $l$  given by (3.11), we derive  $q + 1$  problems, constructed from (2.2) by substituting  $l_\mu, l_{j,\mu}, 1 \leq j \leq q$ , to  $l$ . First, the problem  $(P_\mu)$  : find  $z$  in  $H^1(\Omega)$  such that

$$(P_\mu) \begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = l_\mu & \text{in } \Omega \\ z = 0 & \text{on } \Gamma^- \end{cases} \quad (3.13)$$

and, second, the problems  $(P_{j,\mu})_{1 \leq j \leq q}$  : find  $z$  in  $H^1(\Omega)$  such that

$$(P_{j,\mu}) \begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = l_{j,\mu} & \text{in } \Omega \\ z = 0 & \text{on } \Gamma^-. \end{cases} \quad (3.14)$$

Note that, because of the linearity, the solution of the problem (2.2) will be the sum of the solution of the problem  $(P_\mu)$  and the solutions of problems  $(P_{j,\mu})_{1 \leq j \leq q}$ .

In view of (3.12), applying Theorem 2.1, we derive that

$$\text{the problem } (P_\mu) \text{ has a unique solution } z_\mu \in H^1(\Omega). \quad (3.15)$$

Next, we have to solve the problems  $(P_{j,\mu})_{1 \leq j \leq q}$ . We denote by  $\mathbf{n}_j$  the exterior unit normal vector of the side of the polygon which contains  $\gamma_j$  and, for  $i = -1, 1$ , by  $\mathbf{S}_j^i$  the end points of  $\gamma_j$ , with the convention that, if  $\gamma_j \cap \gamma_{j+i} \neq \emptyset$  for  $i = -1$  or  $i = 1$ , then  $\gamma_j \cap \gamma_{j+i} = \{\mathbf{S}_j^i\}$ . Note that, for each point  $\mathbf{S}_j^i, i = -1, 1, 1 \leq j \leq q$ , we have four possibilities :  $\mathbf{S}_j^i \in \gamma_{j+i}, \mathbf{S}_j^i \notin \gamma_{j+i}$  with  $\mathbf{S}_j^i \notin E, \mathbf{S}_j^i \notin \gamma_{j+i}$  with  $\mathbf{S}_j^i \in (E \cap S), \mathbf{S}_j^i \notin \gamma_{j+i}$  with  $\mathbf{S}_j^i \in (E \cap S^c)$ , where  $S^c$  is the complementary set of  $S$  in  $\mathbb{R}^2$ . We shall not consider all the cases, because there are similar cases, but we shall study some cases, which will be models for the other cases. Note that, for  $i = 1, 2$ , if  $\mathbf{S}_j^i$  is not a vertex of the polygon, then  $\mathbf{u} \cdot \mathbf{n}_j(\mathbf{S}_j^i) = 0$ , that is to say  $\mathbf{S}_j^i \in E$ .

1) **First case:**  $\mathbf{S}_j^i \in \gamma_{j+i}, i = -1, 1$ .

Note that, in view of (3.6) and (3.3),  $\mathbf{S}_j^i \in S$  and  $\mathbf{u}(\mathbf{S}_j^i) \cdot \mathbf{n}_{j+i} \neq 0, \mathbf{u}(\mathbf{S}_j^i) \cdot \mathbf{n}_j \neq 0$ , for  $i = -1, 1$ . Moreover,  $l_{j,\mu} = 0$  on  $\gamma_k$ , for  $k \notin \{j-1, j\}$ . Since  $\mathbf{u}(\mathbf{S}_j^{-1}) \cdot \mathbf{n}_{j-1} \neq 0$ , there exist a real number  $\mu_1 > 0$  such that, for all  $\mathbf{x}$  verifying  $d(\mathbf{S}_j^{-1}, \mathbf{x}) \leq \mu_1$ , we have  $\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_{j-1} \neq 0$ . Then, with the notation

$$\gamma_{j-1,1} = \{\mathbf{x} \in \gamma_{j-1}, d(\mathbf{S}_j^{-1}, \mathbf{x}) \leq \mu_1\}, \quad \gamma_{j-1,2} = \{\mathbf{x} \in \gamma_{j-1}, d(\mathbf{S}_j^{-1}, \mathbf{x}) > \mu_1\},$$

taking

$$0 < \mu \leq \min\left(\frac{1}{2}\mu_0, \mu_1\right), \quad (3.16)$$

in the same way as in the proof of Theorem 2.2, there exists  $z_{0,j,\mu}$  in  $H^2(\Omega)$  verifying,

$$\begin{cases} \left(\frac{\partial z_{0,j,\mu}}{\partial n}\right)_{|\gamma_j} = \left(\frac{l_{j,\mu}}{\mathcal{W} \mathbf{u} \cdot \mathbf{n}_j}\right)_{|\gamma_j}, \\ z_{0,j,\mu}|_{\gamma_j} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \left(\frac{\partial z_{0,j,\mu}}{\partial n}\right)_{|\gamma_{j-1}} = \begin{cases} \left(\frac{l_{j,\mu}}{\mathcal{W} \mathbf{u} \cdot \mathbf{n}_{j-1}}\right) & \text{on } \gamma_{j-1,1} \\ 0 & \text{on } \gamma_{j-1,2} \end{cases}, \\ z_{0,j,\mu}|_{\gamma_{j-1}} = 0 \end{cases} \quad (3.17)$$

and, for  $1 \leq k \leq q$ ,  $k \neq j$ ,  $k \neq j - 1$ ,

$$\begin{cases} (\frac{\partial z_{0,j,\mu}}{\partial n})|_{\gamma_k} = 0 \\ z_{0,j,\mu}|_{\gamma_k} = 0 \end{cases}.$$

Next, applying Theorem 2.1, let  $z_{j,\mu}^* \in H^1(\Omega)$  be the unique solution of the problem

$$\begin{cases} z_{j,\mu}^* + \mathcal{W} \mathbf{u} \cdot \nabla z_{j,\mu}^* = l_{j,\mu} - z_{0,j,\mu} - \mathcal{W} \mathbf{u} \cdot \nabla z_{0,j,\mu} & \text{in } \Omega \\ z_{j,\mu}^* = 0 & \text{on } \Gamma^-, \end{cases}$$

Then,  $z_{j,\mu} = z_{j,\mu}^* + z_{0,j,\mu}$  verifies  $z_{j,\mu} + \mathcal{W} \mathbf{u} \cdot \nabla z_{j,\mu} = l_{j,\mu}$  and  $z_{j,\mu}|_{\Gamma^-} = 0$ . Thus, in this first case, we have proven that

$$z_{j,\mu} \in H^1(\Omega) \text{ is the solution of Problem } (P_{j,\mu}). \quad (3.18)$$

2) **Second case:**  $\mathbf{S}_j^{-1} \notin \gamma_{j-1}$ ,  $\mathbf{S}_j^{-1} \notin E$ ,  $\mathbf{S}_j^1 \in \gamma_{j+1}$ .

We can construct a lifting  $z_{0,j,\mu}$  as in the first case. Since  $l_{j,\mu} = 0$  on  $\gamma_k$  for  $1 \leq k \leq q$ ,  $k \neq j$ , there exists  $z_{0,j,\mu}$  in  $H^2(\Omega)$  verifying,

$$\begin{cases} (\frac{\partial z_{0,j,\mu}}{\partial n})|_{\gamma_j} = (\frac{l_{j,\mu}}{\mathcal{W} \mathbf{u} \cdot \mathbf{n}_j})|_{\gamma_j} \\ z_{0,j,\mu}|_{\gamma_j} = 0 \end{cases}$$

and, for  $1 \leq k \leq q$ ,  $k \neq j$

$$\begin{cases} (\frac{\partial z_{0,j,\mu}}{\partial n})|_{\gamma_k} = 0 \\ z_{0,j,\mu}|_{\gamma_k} = 0 \end{cases}.$$

Then, in the same way as in the first case,  $z_{j,\mu} = z_{j,\mu}^* + z_{0,j,\mu}$  verifies  $z_{j,\mu} + \mathcal{W} \mathbf{u} \cdot \nabla z_{j,\mu} = l_{j,\mu}$  and  $z_{j,\mu}|_{\Gamma^-} = 0$ . Thus, in this second case, we have proven that

$$z_{j,\mu} \in H^1(\Omega) \text{ is the solution of the problem } (P_{j,\mu}). \quad (3.19)$$

The cases where, for  $i = -1, 1$ ,  $\mathbf{S}_j^i \in \gamma_{j+i}$  or  $\mathbf{S}_j^i \notin \gamma_{j+i}$  with  $\mathbf{S}_j^i \notin E$  can be studied in the same way as in the first two cases.

3) **Third case:**  $\mathbf{S}_j^{-1} \notin \gamma_{j-1}$ ,  $\mathbf{S}_j^{-1} \in (E \cap S)$ ,  $\mathbf{S}_j^1 \in \gamma_{j+1}$ .

Here,  $[\mathbf{S}_j^{-1}, \mathbf{S}_j^1]$  is a side of the polygon  $\Omega$ ,  $\mathbf{S}_j^{-1}$  is an end point of  $\Gamma^-$  such that  $\mathbf{u}(\mathbf{S}_j^{-1}) \cdot \mathbf{n}_j = 0$  and  $\mathbf{S}_j^1$  is located inside  $\Gamma^-$ . First, let us make the change of variables such that the point  $\mathbf{S}_j^{-1}$  is the origin, the x-axis has the direction of  $\mathbf{n}_j$ , oriented towards inside the domain  $\Omega$ , that is to say as the vector  $-\mathbf{n}_j$ , and with the segment  $\gamma_j$  included in the positive y-axis, which is oriented by the tangent vector  $\boldsymbol{\tau}_-(\mathbf{S}_j^{-1})$  (see the figure 3.6 below, where  $\omega_j$  is the inner angle associated to the vertex  $\mathbf{S}_j^{-1}$ ).

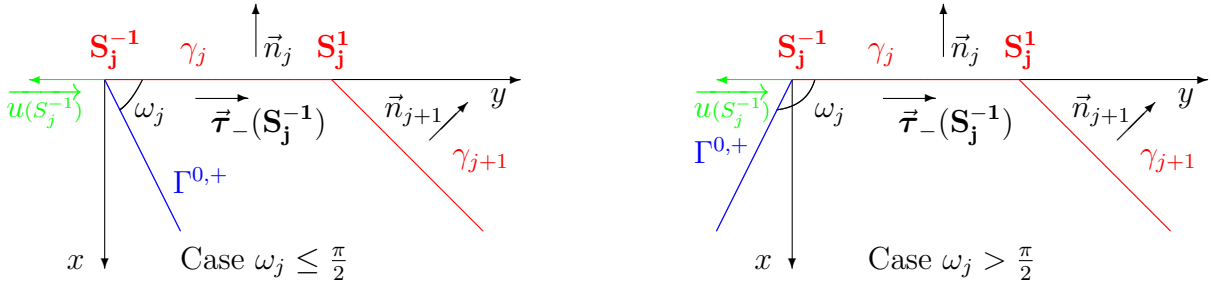


figure 3.6

With these new variables, since  $\mathbf{S}_j^{-1} \in E$ , we have

$$\mathbf{S}_j^{-1} = (0, 0), \quad \mathbf{u}(\mathbf{S}_j^{-1}) \cdot \mathbf{n}_j = -u_1(0, 0) = 0$$

and the assumption (3.5) yields

$$\mathbf{u}(\mathbf{S}_j^{-1}) \cdot \boldsymbol{\tau}_-(\mathbf{S}_j^{-1}) = u_2(0, 0) < 0 \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial \boldsymbol{\tau}_-}(\mathbf{S}_j^{-1}) \cdot \mathbf{n}_-(\mathbf{S}_j^{-1}) = -\frac{\partial u_1}{\partial y}(0, 0) \neq 0.$$

Considering that  $\gamma_j \setminus \{\mathbf{S}_j^{-1}\} \subset \bar{\Gamma}^-$ , we have  $u_1(0, y) > 0$  for  $y > 0$  small enough. Thus, we derive the following properties of  $\mathbf{u}$  in a neighborhood of  $\mathbf{S}_j^{-1} = (0, 0)$  :

$$u_1(0, 0) = 0, \quad u_1(0, y) > 0, \quad u_2(0, 0) < 0 \quad \text{and} \quad \frac{\partial u_1}{\partial y}(0, 0) > 0, \quad (3.20)$$

for  $(0, y) \in \gamma_j \setminus \{\mathbf{S}_j^{-1}\}$ , that is to say for  $y > 0$  small enough.

Next, we are going to split the problem  $(P_{j,\mu})$  into two new problems. In this aim, we define a function  $\lambda_\mu \in \mathcal{D}(\mathbb{R}^2)$  by

$$\forall \mathbf{x} \in \mathbb{R}^2, \quad \lambda_\mu(\mathbf{x}) = \begin{cases} 1 & \text{if } k(x, y) \leq \mu \\ 0 & \text{if } k(x, y) \geq 2\mu, \end{cases} \quad (3.21)$$

where  $k(x, y) = |u_2(0, 0)| |x| + \frac{1}{2} |\frac{\partial u_1}{\partial y}(0, 0)| y^2$ .

Then, we set

$$\tilde{l}_{j,\mu} = (1 - \lambda_\mu)l_{j,\mu} \quad \text{and} \quad \bar{l}_{j,\mu} = \lambda_\mu l_{j,\mu} \quad (3.22)$$

and we define the problem  $(\tilde{P}_{j,\mu})$ , which is associated to the right hand side  $\tilde{l}_{j,\mu}$ , and the problem  $(\bar{P}_{j,\mu})$ , which is associated to the right hand side  $\bar{l}_{j,\mu}$ . Since  $l_{j,\mu} = \tilde{l}_{j,\mu} + \bar{l}_{j,\mu}$ , if we denote by, respectively,  $z_{j,\mu}$ ,  $\tilde{z}_{j,\mu}$  and  $\bar{z}_{j,\mu}$  the unique solutions of, respectively,  $(P_{j,\mu})$ ,  $(\tilde{P}_{j,\mu})$  and  $(\bar{P}_{j,\mu})$ , we have

$$z_{j,\mu} = \tilde{z}_{j,\mu} + \bar{z}_{j,\mu}. \quad (3.23)$$

Thus, to prove that the problem  $(P_{j,\mu})$  has its solution in  $H^1(\Omega)$ , we have only to prove that the problems  $(\tilde{P}_{j,\mu})$  and  $(\bar{P}_{j,\mu})$  have their solutions in  $H^1(\Omega)$ . Note that, extending the function  $l \in H^1(\Omega)$  to  $\mathbb{R}^2$ , from now on, we will consider that the right hand sides  $l$ ,  $l_{j,\mu}$ ,  $\tilde{l}_{j,\mu}$  and  $\bar{l}_{j,\mu}$  belong to  $H^1(\mathbb{R}^2)$

First, we deal with the problem  $(\tilde{P}_{j,\mu})$ . Owing to the definition of the function  $\lambda_\mu$ , we can verify that  $\tilde{l}_{j,\mu}$  vanishes on  $\gamma_j$  on a neighborhood of the point  $\mathbf{S}_j^{-1}$ . So, we can construct a lifting  $\tilde{z}_{0,j,\mu}$  in the same way as in the second case with  $\tilde{l}_{j,\mu}$  in place of  $l_{j,\mu}$ , replacing  $(\frac{\tilde{l}_{j,\mu}}{W \mathbf{u} \cdot \mathbf{n}_j})|_{\gamma_j}$  with 0 in a neighborhood of  $\mathbf{S}_j^{-1}$  on  $\gamma_j$  and  $\tilde{z}_{j,\mu} = \mathbf{z}_{j,\mu}^* + \tilde{z}_{0,j,\mu}$  is the solution of the problem  $(\tilde{P}_{j,\mu})$  in  $H^1(\Omega)$ .

Solving the problem  $(\bar{P}_{j,\mu})$  is much more difficult because  $\mathbf{u}(\mathbf{S}_j^{-1}) \cdot \mathbf{n}_j = 0$  and  $\bar{l}_{j,\mu}$  does not vanish in the neighborhood of  $\mathbf{S}_j^{-1}$ . From now on, we will use the following notation, for  $r > 0$  :

$$B_{j,r} = \{\mathbf{x} \in \mathbb{R}^2, \quad d(\mathbf{S}_j^{-1}, \mathbf{x}) = \sqrt{x^2 + y^2} < r\}, \quad B_{j,r}^+ = B_{j,r} \cap \{(x, y) \in \mathbb{R}^2, \quad x \geq 0\}. \quad (3.24)$$

The proof will be built in several steps. In a first step, we define a local problem, which is the problem  $(\bar{P}_{j,\mu})$  restricted to a neighborhood  $\Omega \cap B_{j,K}$  of  $\mathbf{S}_j^{-1}$ , and we express

this local solution in integral form (see Lemma 3.2). In a second step, we show that, if we choose  $\mu$  small enough, this local solution vanishes in  $\bar{\Omega} \cap C(\mathbf{S}_j^{-1}, r_{1,j}^*, r_{2,j}^*)$  where  $C(\mathbf{S}_j^{-1}, r_{1,j}^*, r_{2,j}^*)$  is a ring centered in  $\mathbf{S}_j^{-1}$  and included in  $B_{j,K}$ . In the third step, using its integral expression, we prove that the local solution belongs to  $H^1(B_{j,r_{1,j}^*} \cap \Omega)$ , which implies, owing to the second step, that its extension by zero is the solution  $H^1$  of  $(\bar{P}_{j,\mu})$ .

### First step

In the following lemma, we give the expression of the local solution.

**Lemma 3.2** *Let  $\mathbf{S}_j^{-1}$  belongs to  $E \cap S$  and the real  $K$  be defined by (3.40). We set  $\Omega_{j,K} = \Omega \cap B_{j,K}$  and  $\Gamma_{j,K}^- = \Gamma^- \cap B_{j,K}$ . The solution of the problem*

$$\begin{cases} z + \mathcal{W} \mathbf{u} \cdot \nabla z = \bar{l}_{j,\mu} & \text{in } \Omega_{j,K} \\ z = 0 & \text{on } \Gamma_{j,K}^- \end{cases}$$

is expressed by

$$z(x, y) = e^{-V(X(x,y),y)} \left( \int_{\alpha^{-1}(X(x,y))}^y \frac{e^{V(X(x,y),t)}}{\mathcal{W}U_2(X(x,y),t)} \bar{L}_{j,\mu}(X(x,y),t) dt \right), \quad (3.25)$$

where  $V$ ,  $U_2$  and  $\bar{L}_{j,\mu}$  are defined in (3.34).

**Proof.** Owing to (3.20), the continuity of  $u_2$  yields that there exists a strictly positive real number  $\mu_2 \leq \mu_0$ , such that

$$\forall \mathbf{x} = (x, y) \in B_{j,\mu_2} \cap \bar{\Omega}, \quad u_2(\mathbf{x}) < 0. \quad (3.26)$$

In the same way, again the continuity of  $u_2$  and the definition of  $\frac{\partial u_1}{\partial y}(0, 0)$  with  $u_1(0, 0) = 0$  imply that there exists a strictly positive real number  $\mu_3 \leq \min(\mu_2, |\gamma_j|)$  such that

$$\begin{aligned} \forall \mathbf{x} \in B_{j,\mu_3} \cap \bar{\Omega}, \quad \frac{3}{2}u_2(0, 0) \leq u_2(\mathbf{x}) \leq \frac{1}{2}u_2(0, 0) < 0 \quad \text{and} \\ \forall y \in [0, \mu_3], \quad \frac{1}{2} \frac{\partial u_1}{\partial y}(0, 0)y \leq u_1(0, y) \leq \frac{3}{2} \frac{\partial u_1}{\partial y}(0, 0)y. \end{aligned} \quad (3.27)$$

For  $0 \leq r_1 < r_2$ , let us define the sets

$$E_{r_1, r_2} = \{(x, y) \in \mathbb{R}^2, r_1 \leq k(x, y) \leq r_2\} \text{ and } C(\mathbf{S}_j^{-1}, r_1, r_2) = \{(x, y) \in \mathbb{R}^2, r_1 \leq \sqrt{x^2 + y^2} \leq r_2\}, \quad (3.28)$$

where  $k(x, y)$  is defined in (3.21). Considering that, for  $r > 0$ ,

$$k(x, y) = r \iff x^2 + y^2 = \frac{(\frac{\partial u_1}{\partial y}(0, 0))^2}{4(u_2(0, 0))^2} y^4 + \left( \frac{r}{|u_2(0, 0)|} \right)^2 + \left( 1 - \frac{r |\frac{\partial u_1}{\partial y}(0, 0)|}{(u_2(0, 0))^2} \right) y^2$$

and that  $y^2 \leq \frac{2r}{|\frac{\partial u_1}{\partial y}(0, 0)|}$ , we can verify, for  $0 < r \leq \frac{(u_2(0, 0))^2}{|\frac{\partial u_1}{\partial y}(0, 0)|}$ , the following inclusions :

$$B_{j, \frac{r}{|u_2(0, 0)|}} \subset E_{0, r} \subset B_{j, 2} \sqrt{\frac{r}{|\frac{\partial u_1}{\partial y}(0, 0)|}} \quad (3.29)$$

and if  $(r_1, r_2)$  verifies

$$\left\{ \begin{array}{l} 0 \leq r_1 < r_2 \leq \frac{(u_2(0,0))^2}{|\frac{\partial u_1}{\partial y}(0,0)|} \\ r_1 < \frac{|\frac{\partial u_1}{\partial y}(0,0)|r_2^2}{4(u_2(0,0))^2} \end{array} \right. \text{ then } C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_1}{|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{r_2}{|u_2(0,0)|}) \subset E_{r_1, r_2}. \quad (3.30)$$

Let us consider the transport equation  $z + \mathcal{W}\mathbf{u} \cdot \nabla z = \bar{l}_{j,\mu}$  of the problem  $(\bar{P}_{j,\mu})$  and the following change of variables : we set for all  $(x, y) \in \bar{\Omega}$

$$\left\{ \begin{array}{l} X(x, y) = -\int_0^x u_2(t, y) dt + \int_0^y u_1(0, t) dt \\ Y(x, y) = y \end{array} \right. . \quad (3.31)$$

Note that it is more convenient, especially in the case where  $\omega_j > \frac{\pi}{2}$  ( $\omega_j$  is the inner angle of the polygon  $\Omega$  associated to the vertex  $\mathbf{S}_j^{-1}$ ), to define  $X(x, y)$  for  $y \leq 0$  and  $x \geq 0$  small enough. So, we will replace  $u_2$  with an extension of  $u_2$ , defined for example by symmetries, in (3.31), this extension of  $u_2$ , still denoted  $u_2$ , verifying (3.26), respectively (3.27), in  $B_{j,\mu_2}^+$ , respectively  $B_{j,\mu_3}^+$ . Then, we define an extension of  $u_1$ , for  $y \leq 0$  and  $x \geq 0$  small enough, by

$$u_1(x, y) = -\int_0^x \frac{\partial u_2}{\partial y}(t, y) dt,$$

such that  $\text{div } \mathbf{u} = 0$  and  $u_1(0, y) = 0$ , for  $y \leq 0$  small enough. We can verify, in view of  $\text{div } \mathbf{u} = 0$ , that

$$X'_x = -u_2 \text{ and } X'_y = u_1 \text{ in } \bar{\Omega}. \quad (3.32)$$

Let us show that the mapping

$$\varphi : \begin{array}{l} B_{j,\mu_2}^+ \\ (x, y) \end{array} \longrightarrow \begin{array}{l} \varphi(B_{j,\mu_2}^+) \\ (X, Y) \end{array} \text{ is one-to-one,} \quad (3.33)$$

where  $\mu_2$  is defined in (3.26). Let us assume that

$$X(x, y) = X(x', y') \text{ and } Y(x, y) = Y(x', y') \text{ with } (x, y) \in B_{j,\mu_2}^+ \text{ and } (x', y') \in B_{j,\mu_2}^+.$$

Then, the second equation gives directly  $y = y'$  and we obtain  $X(x, y) = X(x', y)$ . Since  $X'_x(x, y) = -u_2(x, y) > 0$  for  $(x, y) \in B_{j,\mu_2}^+$ , we derive  $x = x'$ .

Since  $\varphi$  is of class  $C^1$  in  $B_{j,\mu_2}^+$  and since the jacobian of the mapping  $\varphi$  is  $-u_2$ , which is strictly positive in  $B_{j,\mu_2}^+$ , we can define an inverse function  $\varphi^{-1}$  of class  $C^1$  in  $\varphi(B_{j,\mu_2}^+)$ . Then, in view of the definition of  $\mu_3$  in (3.27), we define the functions  $Z$ ,  $U_2$ ,  $\bar{L}_{j,\mu}$  and  $V$  on  $\varphi(B_{j,\mu_3}^+)$  by

$$Z = z \circ \varphi^{-1}, \mathbf{U}_2 = u_2 \circ \varphi^{-1}, \bar{L}_{j,\mu} = \bar{l}_{j,\mu} \circ \varphi^{-1} \text{ and } V : (X, Y) \mapsto \int_0^Y \frac{1}{\mathcal{W}U_2(X, t)} dt. \quad (3.34)$$

Let us show that for  $\mathbf{x} = (x, y)$  in a neighborhood of  $(0, 0)$  and  $0 \leq |t| \leq |Y|$  with  $tY \geq 0$ , then  $(X(x, y), t)$  belongs to  $\varphi(B_{j,\mu_2}^+)$ . First, for  $(x, y) \in B_{j,\frac{\mu_3}{2}}^+$  and  $0 \leq t \leq Y = y$  or  $y = Y \leq t \leq 0$  (case where  $\omega_j > \frac{\pi}{2}$ ), owing to (3.27), in view of  $u_1(0, y) = 0$  for  $y \leq 0$ , we have

$$X(0, t) \leq X(x, y) \leq \frac{3}{2}k(x, y) \text{ and } X(\frac{\mu_3}{2}, t) \geq \frac{\mu_3}{4}|u_2(0, 0)|.$$

Then,

$$\forall (x, y) \in \mathbb{R}^2 \text{ such that } \begin{cases} k(x, y) \leq \frac{\mu_3 |u_2(0, 0)|}{6} \\ (x, y) \in B_{j, \frac{\mu_3}{2}}^+ \end{cases} \quad (3.35)$$

we have  $X(0, t) \leq X(x, y) \leq X(\frac{\mu_3}{2}, t)$  and there exists a real number  $x_t \in [0, \frac{\mu_3}{2}]$  such that  $X(x_t, t) = X(x, y)$  and, therefore,  $(X(x, y), t)$  belongs to  $\varphi(B_{j, \mu_3}^+)$ . Finally, we set

$$\mu_4 = \frac{|u_2(0, 0)|}{|\frac{\partial u_1}{\partial y}(0, 0)|} \text{ and } \tilde{B}_j = B_{j, \min(\frac{\mu_3}{6}, \mu_4)}^+ \cap \Omega. \quad (3.36)$$

Since  $\tilde{B}_j = B_{j, \frac{r}{|u_2(0, 0)|}}^+ \cap \Omega$  with  $r = \min(\frac{\mu_3 |u_2(0, 0)|}{6}, \frac{(u_2(0, 0))^2}{|\frac{\partial u_1}{\partial y}(0, 0)|})$ , in view of (3.29), all  $\mathbf{x} = (x, y) \in \tilde{B}_j$  verifies (3.35) and, consequently,  $(X(x, y), t)$  belongs to  $\varphi(B_{j, \mu_3}^+)$ . Then, with the new functions defined in (3.34), in view of  $\operatorname{div} \mathbf{u} = 0$ , we have the following equivalence :

$$z + \mathcal{W} \mathbf{u} \cdot \nabla z = \bar{l}_{j, \mu} \text{ a.e. in } \tilde{B}_j \iff Z + \mathcal{W} U_2 \frac{\partial Z}{\partial Y} = \bar{L}_{j, \mu} \text{ a.e. in } \varphi(\tilde{B}_j).$$

Solving this last equation yields

$$\begin{aligned} \forall (X, Y) \in \varphi(\tilde{B}_j), \quad Z(X, Y) &= e^{-V(X, Y)} \left( \int_0^Y \frac{e^{V(X, t)}}{\mathcal{W} U_2(X, t)} \bar{L}_{j, \mu}(X, t) dt + C(X) \right) \iff \\ \forall (x, y) \in \tilde{B}_j, \quad z(x, y) &= e^{-V(X(x, y), y)} \left( \int_0^y \frac{e^{V(X(x, y), t)}}{\mathcal{W} U_2(X(x, y), t)} \bar{L}_{j, \mu}(X(x, y), t) dt + C(X(x, y)) \right), \end{aligned}$$

where  $C$  is a function of  $L^2$ . We have to compute the function  $C$  so that the solution  $Z$  verifies the boundary condition on  $\Gamma^-$ .

Let us define the real number  $y_M > 0$  by

$$y_M = \sup\{y, \mathbf{m}(0, y) \in \tilde{B}_j \cap \gamma_j\}$$

and the function  $\alpha$  on the set  $[0, y_M]$  by

$$\forall y \in [0, y_M], \quad \alpha(y) = X(0, y). \quad (3.37)$$

Note that

$$y_M = \min\left(\frac{\mu_3}{6}, \mu_4, |\gamma_j|\right). \quad (3.38)$$

Considering that,  $\forall y \in ]0, y_M]$ ,

$$\alpha'(y) = u_1(0, y) > 0,$$

the mapping  $\alpha$  from  $[0, y_M]$  to  $[0, \alpha(y_M)]$  is one-to-one and we can define the inverse function  $\alpha^{-1}$  from  $[0, \alpha(y_M)]$  to  $[0, y_M]$ . Moreover,  $\alpha^{-1}$  is strictly positive on  $]0, \alpha(y_M)]$ . Then, the continuity of the functions  $X$  and  $Y$  yields that there exist a real number  $\mu_5 > 0$  such that

$$\forall (x, y) \in B_{j, \mu_5}^+, \quad X(x, y) \in [0, \alpha(y_M)]. \quad (3.39)$$

Finally, we set

$$K = \min\left(\frac{\mu_3}{6}, \mu_4, \mu_5, |\gamma_j|\right), \quad (3.40)$$

where the constants  $\mu_3$ ,  $\mu_4$  and  $\mu_5$  are defined, respectively, by (3.27), (3.36) and (3.39). Then, the boundary condition  $z|_{\Gamma_j} = 0$  allows us to compute the function  $C$ . Indeed, setting  $s = X(0, y) = \alpha(y) \Leftrightarrow y = \alpha^{-1}(s)$ , we have

$$\begin{aligned} z|_{\Gamma_{j,K}^-} = 0 &\Leftrightarrow \forall y, 0 \leq y \leq K, z(0, y) = 0 \Leftrightarrow \forall s, 0 \leq s \leq \alpha(K), Z(s, \alpha^{-1}(s)) = 0 \\ &\Leftrightarrow \forall s, 0 \leq s \leq \alpha(K), C(s) = - \int_0^{\alpha^{-1}(s)} \frac{e^{V(s,t)}}{\mathcal{W}U_2(s,t)} \bar{L}_{j,\mu}(s,t) dt \end{aligned}$$

and we obtain that the solution  $z$  is expressed in  $\Omega_{j,K}$  as (3.25).  $\diamond$

### Second step

Let us show that, for  $\mu$  small enough and  $\mathbf{x}$  far enough from  $\mathbf{S}_j^{-1}$ , then  $z(x, y) = 0$ . More precisely, let us prove the following lemma.

**Lemma 3.3** *Let  $r_{1,j}$  and  $r_{2,j}$  be defined by (3.44), let  $\mu > 0$  such that  $\mu \leq \frac{r_{1,j}}{6}$  and let the local solution  $z$  of the problem  $(\bar{P}_{j,\mu})$  be expressed by (3.25). Then,*

$$\forall (x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}, \frac{r_{2,j}}{|u_2(0,0)|}}) \cap \bar{\Omega}, z(x, y) = 0. \quad (3.41)$$

**Proof.** Let us note that, if  $(x, y) \in B_{j,K}^+$ , then  $y \leq \alpha^{-1}(X(x, y))$ . Indeed, if  $y < 0$ , then  $y < 0 \leq \alpha^{-1}(X(x, y))$  and if  $y \geq 0$ , then  $\alpha(y_M) \geq X(x, y) \geq X(0, y) = \alpha(y) \geq 0$ , which implies  $y \leq \alpha^{-1}(X(x, y))$ , since  $\alpha^{-1}$  is strictly increasing on  $[0, \alpha(y_M)]$ . Thus, we distinguish two cases :

a) First case :  $(x, y) \in B_{j, \frac{K}{6}}^+$  and  $0 \leq y \leq t \leq \alpha^{-1}(X(x, y))$ .

Note that, since  $\frac{K}{6} \leq \mu_5$ , then  $\alpha^{-1}(X(x, y)) \leq y_M$ , which implies  $t \leq \frac{\mu_3}{6}$ . On the one hand, we have

$$\alpha(t) = X(0, t) \leq X(x, y).$$

On the other hand, since  $|x| \leq \frac{K}{6}$ , we derive

$$X(x, y) \leq \frac{3}{2}|u_2(0,0)||x| + \int_0^y u_1(0, \theta) d\theta \leq \frac{1}{4}|u_2(0,0)|K + \int_0^t u_1(0, \theta) d\theta \leq X(\frac{K}{2}, t).$$

Therefore, if  $(x, y) \in B_{j, \frac{K}{6}}^+$  with  $y \geq 0$ , there exists  $x_t \in [0, \frac{K}{2}]$  such that

$$X(x, y) = X(x_t, t) \text{ with } (x_t, t) \in B_{j, \mu_3}^+. \quad (3.42)$$

Then, the inequalities (3.27) yield

$$\frac{1}{2}k(x, y) \leq X(x, y) = X(x_t, t) \leq \frac{3}{2}k(x_t, t) \implies \frac{1}{3}k(x, y) \leq k(x_t, t). \quad (3.43)$$

We set

$$r_{1,j} = \min\left(\frac{|u_2(0,0)|K}{12}, \frac{|\frac{\partial u_1}{\partial y}(0,0)|K^2}{288}\right) \text{ and } r_{2,j} = \frac{K|u_2(0,0)|}{6}. \quad (3.44)$$

Choosing the real number  $\mu > 0$  such that

$$\mu \leq \frac{r_{1,j}}{6} \iff 6\mu \leq r_{1,j}, \quad (3.45)$$



we can verify that  $0 < r_{1,j} < r_{2,j} \leq \frac{(u_2(0,0))^2}{|\frac{\partial u_1}{\partial y}(0,0)|}$  and  $r_{1,j} < \frac{|\frac{\partial u_1}{\partial y}(0,0)|r_{2,j}^2}{4(u_2(0,0))^2}$  and, owing to (3.30), we obtain,

$$\forall(x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{r_{2,j}}{|u_2(0,0)|}) \cap \bar{\Omega} \text{ with } y \geq 0, \quad r_{1,j} \leq k(x, y) \leq r_{2,j}.$$

Hence, in view of (3.43), we derive

$$\begin{aligned} \forall(x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{r_{2,j}}{|u_2(0,0)|}) \cap \bar{\Omega} \text{ with } y \geq 0, \\ k(x_t, t) \geq \frac{1}{3}k(x, y) \geq \frac{1}{3}r_{1,j} \geq 2\mu. \end{aligned}$$

Finally, since  $\bar{L}_{j,\mu}(X(x, y), t) = \bar{L}_{j,\mu}(X(x_t, t), t) = \bar{l}_{j,\mu}(x_t, t)$ , considering (3.21), (3.22) and (3.25), we obtain

$$\forall(x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{r_{2,j}}{|u_2(0,0)|}) \cap \bar{\Omega} \text{ with } y \geq 0, \quad z(x, y) = 0,$$

where  $r_{1,j}$ ,  $r_{2,j}$  and  $\mu$  are given by (3.44) and (3.45).

b) Second case ( $\omega_j > \frac{\pi}{2}$ ) :  $(x, y) \in B_{j, \frac{K}{6}}^+$ ,  $y < 0$  and  $y \leq t \leq \alpha^{-1}(X(x, y))$ .

Choosing first  $t \in [0, \alpha^{-1}(X(x, y))]$  and second  $t \in [y, 0]$ , considering that  $u_1(0, y) = 0$  when  $y < 0$ , we proceed in the same way as previously and we obtain, as in the case where  $y \geq 0$ , that there exists  $x_t \in [0, \frac{K}{2}]$  such that

$$X(x, y) = X(x_t, t) \text{ with } (x_t, t) \in B_{j, \mu_3}^+ \quad (3.46)$$

and  $\forall(x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{r_{2,j}}{|u_2(0,0)|}) \cap \bar{\Omega} \text{ with } y < 0,$

$$\int_{\alpha^{-1}(X(x,y))}^0 \frac{e^{V(X(x,y),t)}}{\mathcal{W}U_2(X(x,y),t)} \bar{L}_{j,\mu}(X(x,y),t) dt = 0, \quad \int_0^y \frac{e^{V(X(x,y),t)}}{\mathcal{W}U_2(X(x,y),t)} \bar{L}_{j,\mu}(X(x,y),t) dt = 0,$$

which implies  $z(x, y) = 0$ . Finally, gathering the cases  $y \geq 0$  and  $y < 0$ , we derive (3.41).  $\diamond$

### Third step

Next, we will prove the following lemma that gives the regularity  $H^1$  of the local solution of the problem  $(\bar{P}_{j,\mu})$ .

**Lemma 3.4** *Let  $z$  be defined by (3.25) and let  $r_j^*$  be defined by*

$$r_j^* = 2\sqrt{\frac{r_{1,j}}{|\frac{\partial u_1}{\partial y}(0,0)|}} = \min\left(\sqrt{\frac{|u_2(0,0)K}{3|\frac{\partial u_1}{\partial y}(0,0)|}}, \frac{K}{6\sqrt{2}}\right), \quad (3.47)$$

where  $r_{1,j}$  and  $K$  are defined in (3.44) and (3.40). Then  $z$  belongs to  $H^1(B_{j,r_j^*} \cap \Omega)$ .

**Proof.** Let us prove first that  $z$  belongs to  $L^2(B_{j,r_j^*} \cap \Omega)$ . Using the change of variables defined in (3.31) yields  $z(x, y) = Z(X, Y)$  with

$$Z(X, Y) = e^{-V(X,Y)} \int_{\alpha^{-1}(X)}^Y \frac{e^{V(X,t)}}{\mathcal{W}U_2(X,t)} \bar{L}_{j,\mu}(X,t) dt \quad (3.48)$$

and, in view of the jacobian  $\frac{D(x,y)}{D(X,Y)} = -\frac{1}{U_2}$  and the inequality of Cauchy-Schwarz, we obtain

$$\begin{aligned} \int \int_{B_{j,r_j^*} \cap \Omega} z^2 dx dy &= \int \int_{\varphi(B_{j,r_j^*} \cap \Omega)} Z^2 \frac{1}{|U_2|} dX dY \\ &\leq \int \int_{\varphi(B_{j,r_j^*} \cap \Omega)} \frac{e^{2V}}{|U_2|} \left( \int_Y^{\alpha^{-1}(X)} \frac{e^{-2V(X,t)}}{\mathcal{W}^2(U_2(X,t))^2} dt \right) \left( \int_Y^{\alpha^{-1}(X)} (\bar{L}_{j,\mu}(X,t))^2 dt \right) dX dY. \end{aligned} \quad (3.49)$$

We have to estimate the terms of the previous integral. Since  $r_j^* \leq \frac{K}{6}$ , in view of (3.27), we derive

$$\forall (x, y) \in B_{j,r_j^*} \cap \Omega, \quad \frac{1}{|U_2(X, Y)|} = \frac{1}{|u_2(x, y)|} \leq \frac{2}{|u_2(0, 0)|}.$$

Owing to (3.36) and  $(B_{j,r_j^*} \cap \Omega) \subset \widetilde{B}_j$ , for  $(X, Y) \in \varphi(B_{j,r_j^*} \cap \Omega)$  and  $|t| \leq |Y|$ , we have  $U_2(X, t) = u_2(x_t, t)$  with  $x_t \in [0, \frac{\mu_3}{2}]$  and, considering that  $|Y| \leq \frac{K}{6}$ , we obtain

$$|V(X, Y)| = \left| \int_0^Y \frac{1}{\mathcal{W}u_2(x_t, t)} dt \right| \leq \frac{K}{3|\mathcal{W}||u_2(0, 0)|}.$$

In the same way, for  $(X, Y) \in \varphi(B_{j,r_j^*} \cap \Omega)$  and  $Y \leq t \leq \alpha^{-1}(X)$ , in view of (3.42) and (3.46), we have  $U_2(X, t) = U_2(X(x_t, t), t) = u_2(x_t, t)$  with  $(x_t, t) \in B_{j,\mu_3}^+$ , which implies

$$\frac{1}{|U_2(X, t)|} \leq \frac{2}{|u_2(0, 0)|}$$

and, for  $V(X, t) = \int_0^t \frac{1}{\mathcal{W}U_2(X, \theta)} d\theta$ , we prove that  $X = X(x_\theta, \theta)$  with  $(x_\theta, \theta) \in B_{j,\mu_3}^+$ , which gives, since  $-\frac{K}{6} \leq Y \leq t \leq \alpha^{-1}(X) \leq y_M \leq \frac{\mu_3}{6}$ , the following estimate

$$|V(X, t)| \leq \frac{\mu_3}{3|\mathcal{W}||u_2(0, 0)|}.$$

Substituting these bounds in (3.49) yields that there exists a strictly positive constant  $C$  such that

$$\int \int_{B_{j,r_j^*} \cap \Omega} z^2 dx dy \leq C \int \int_{\varphi(B_{j,r_j^*} \cap \Omega)} \left( \int_Y^{\alpha^{-1}(X)} (\bar{L}_{j,\mu}(X,t))^2 dt \right) dX dY. \quad (3.50)$$

Since  $\forall (X, Y) \in \varphi(B_{j,r_j^*} \cap \Omega)$ , we have  $-\frac{K}{6} \leq Y \leq \frac{K}{6}$  and  $X(0, Y) \leq X \leq X(\frac{K}{6}, Y)$ , we obtain

$$\begin{aligned} \int \int_{B_{j,r_j^*} \cap \Omega} z^2 dx dy &\leq C \int_{-\frac{K}{6}}^{\frac{K}{6}} dY \left( \int_{X(0,Y)}^{X(\frac{K}{6},Y)} dX \left( \int_Y^{\alpha^{-1}(X)} (\bar{L}_{j,\mu}(X,t))^2 dt \right) \right), \\ &\leq C \int_{-\frac{K}{6}}^{\frac{K}{6}} dY \left( \int \int_{D_Y} (\bar{L}_{j,\mu}(X,t))^2 dX dt \right), \end{aligned}$$

where  $D_Y = \{(X, t) \in \mathbb{R}^2, X(0, Y) \leq X \leq X(\frac{K}{6}, Y), Y \leq t \leq \alpha^{-1}(X)\}$ . Next, we compute the integral on  $D_Y$  by making the substitution  $\begin{cases} X = X(\tilde{x}, t) \\ t = t \end{cases}$ , the jacobian of

which is  $-u_2(\tilde{x}, t)$ . Indeed, the mapping  $\psi : D_Y \rightarrow \psi(D_Y)$  is one-to-one and of class  $C^1$  on  $D_Y$ , as we proved previously by (3.42) and (3.46), with  $\tilde{x} \in [0, \frac{K}{2}]$  and  $t \in [-\frac{K}{6}, \frac{\mu_3}{6}]$ . Thus, the jacobian is strictly positive and bounded by  $\frac{3}{2}|u_2(0, 0)|$ , and we obtain

$$\int \int_{B_{j,r_j^*} \cap \Omega} z^2 dx dy \leq \frac{1}{2} CK |u_2(0, 0)| \int \int_{[0, \frac{K}{2}] \times [-\frac{K}{6}, \frac{\mu_3}{6}]} \bar{l}_{j,\mu}(\tilde{x}, t) d\tilde{x} dt < +\infty, \quad (3.51)$$

which proves that  $z$  belongs to  $L^2(B_{j,r_j^*} \cap \Omega)$ , since  $\bar{l}_{j,\mu}$  belongs to  $H^1(\mathbb{R}^2)$ .

It remains to prove that  $\nabla z$  belongs to  $L^2(B_{j,r_j^*} \cap \Omega)$ . Again, we use the change of variables defined in (3.31). Computing the partial derivatives yields

$$\frac{\partial z}{\partial x} = -U_2 \frac{\partial Z}{\partial X} \quad \text{and} \quad \frac{\partial z}{\partial y} = U_1 \frac{\partial Z}{\partial X} + \frac{\partial Z}{\partial Y}.$$

Then, the inequality  $2|ab| \leq a^2 + b^2$  implies  $|\nabla z|^2 \leq (1 + U_1^2 + U_2^2)|\nabla Z|^2$ , where  $|\cdot|$  represents the euclidian norm in  $\mathbb{R}^2$ . Hence, we derive

$$\int \int_{B_{j,r_j^*} \cap \Omega} |\nabla z|^2 dx dy \leq \frac{2 \max_{x \in \Omega} (1 + u_1^2 + u_2^2)}{|u_2(0, 0)|} \int \int_{\varphi(B_{j,r_j^*} \cap \Omega)} |\nabla Z|^2 dX dY.$$

Since  $Z + \mathcal{W}U_2 \frac{\partial Z}{\partial Y} = \bar{L}_{j,\mu}$ , owing to (3.27), we obtain that  $\frac{\partial Z}{\partial Y}$  belongs to  $L^2(\varphi(B_{j,r_j^*} \cap \Omega))$ .

Next, we now come to the crucial point, which is to prove that the other partial derivative  $\frac{\partial Z}{\partial X}$  belongs to  $L^2(\varphi(B_{j,r_j^*} \cap \Omega))$ . From (3.48), computing this derivative yields

$$\begin{aligned} \frac{\partial Z}{\partial X}(X, Y) &= -\frac{\partial V}{\partial X}(X, Y)Z(X, Y) + e^{-V(X, Y)} \\ &\left( \int_{\alpha^{-1}(X)}^Y \frac{\partial(\frac{e^{V(X, t)}}{\mathcal{W}U_2(X, t)} \bar{L}_{j,\mu}(X, t))}{\partial X} dt - (\alpha^{-1})'(X) \left( \frac{e^{V(X, \alpha^{-1}(X))}}{\mathcal{W}U_2(X, \alpha^{-1}(X))} \bar{L}_{j,\mu}(X, \alpha^{-1}(X)) \right) \right) \end{aligned} \quad (3.52)$$

and

$$\frac{\partial(\frac{e^{V(X, t)}}{\mathcal{W}U_2(X, t)} \bar{L}_{j,\mu}(X, t))}{\partial X} = e^{V(X, t)} \frac{\partial V}{\partial X}(X, t) U_2(X, t) - \frac{\partial U_2}{\partial X}(X, t) \bar{L}_{j,\mu}(X, t) + \frac{e^{V(X, t)}}{U_2(X, t)} \frac{\partial \bar{L}_{j,\mu}}{\partial X}(X, t).$$

Then, for  $(X, Y) \in \varphi(B_{j,r_j^*} \cap \Omega)$  and  $|t| \leq |Y|$  or  $Y \leq t \leq \alpha^{-1}(X)$ , we prove that  $(X, t)$  belongs to  $\varphi(B_{j,\mu_3}^+)$ . Considering that  $U_2$  is strictly negative and of class  $C^1$  on  $\varphi(B_{j,\mu_3}^+)$ , we derive that  $U_2$  and  $V$  are of class  $C^1$  on  $\varphi(B_{j,\mu_3}^+)$ , which implies that the functions  $\frac{\partial U_2}{\partial X}$  and  $\frac{\partial V}{\partial X}$  are bounded on  $\varphi(B_{j,\mu_3}^+)$ . Hence, there exist strictly positive constants  $C_1, C_2, C_3$  and  $C_4$  such that

$$\begin{aligned} \left| \frac{\partial Z}{\partial X}(X, Y) \right| &\leq C_1 |Z(X, Y)| + C_2 \int_Y^{\alpha^{-1}(X)} |\bar{L}_{j,\mu}(X, t)| dt \\ &+ C_3 \int_Y^{\alpha^{-1}(X)} \left| \frac{\partial \bar{L}_{j,\mu}}{\partial X}(X, t) \right| dt + C_4 |(\alpha^{-1})'(X)| |\bar{L}_{j,\mu}(X, \alpha^{-1}(X))|. \end{aligned} \quad (3.53)$$

In view of the equalities  $(X, \alpha^{-1}(X)) = (\alpha(\alpha^{-1}(X)), \alpha^{-1}(X)) = (X(0, \alpha^{-1}(X)), \alpha^{-1}(X))$ , we derive  $\bar{L}_{j,\mu}(X, \alpha^{-1}(X)) = \bar{l}_{j,\mu}(0, \alpha^{-1}(X))$ . Next, owing to  $|\alpha^{-1}(X) - Y| \leq \frac{\mu_3 + K}{6}$ , using inequalities of Cauchy-Schwarz and setting  $C_5 = C_1^2 + C_2^2 + C_3^2 + C_4^2$  and  $C_6 = \frac{\mu_3 + K}{6} C_5$  yield

$$\begin{aligned} \left(\frac{\partial Z}{\partial X}(X, Y)\right)^2 &\leq C_5((Z(X, Y))^2 + ((\alpha^{-1})'(X))^2(\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2) \\ &\quad + C_6 \int_Y^{\alpha^{-1}(X)} ((\bar{L}_{j,\mu}(X, t))^2 + \left(\frac{\partial \bar{L}_{j,\mu}}{\partial X}(X, t)\right)^2) dt. \end{aligned} \quad (3.54)$$

There is only one term that is difficult to bound in  $L^2(\varphi(B(\mathbf{S}_j^{-1}, r_j^*) \cap \bar{\Omega}))$ . Indeed, we have just prove that  $Z$  belongs to  $L^2(\varphi(B_{j,r_j^*} \cap \Omega))$  and for

$$\int_Y^{\alpha^{-1}(X)} ((\bar{L}_{j,\mu}(X, t))^2 + \left(\frac{\partial \bar{L}_{j,\mu}}{\partial X}(X, t)\right)^2) dt,$$

we apply the previous method, which allowed us to bound the right hand side of the inequality (3.50), using the same substitution  $\begin{cases} X = X(\tilde{x}, t) \\ t = t \end{cases}$  as previously, since  $\bar{l}_{j,\mu}$  belongs to  $H^1(\mathbb{R}^2)$  and since we have  $\frac{\partial \bar{L}_{j,\mu}}{\partial X}(X, t) = -\frac{1}{u_2(\tilde{x}, t)} \frac{\partial \bar{l}_{j,\mu}}{\partial \tilde{x}}(\tilde{x}, t)$ . It remains to bound the basic term

$$\int \int_{\varphi(B_{j,r_j^*} \cap \Omega)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX dY.$$

Let us recall that, in view of (3.37),(3.39), since  $(x, y)$  belongs to  $B_{j,r_j^*} \cap \bar{\Omega}$ , we have

$$X \in [0, \alpha(y_M)] \iff \alpha^{-1}(X) \leq y_M \leq \min\left(\frac{\mu_3}{6}, \mu_4, |\gamma_j|\right).$$

Since  $\alpha'(\alpha^{-1}(X)) = u_1(0, \alpha^{-1}(X)) > 0$  and considering (3.27), on the one hand, we derive

$$|(\alpha^{-1})'(X)| = \frac{1}{|\alpha'(\alpha^{-1}(X))|} = \frac{1}{|u_1(0, \alpha^{-1}(X))|} \leq \frac{2}{\left|\frac{\partial u_1}{\partial y}(0, 0)\right| |\alpha^{-1}(X)|}. \quad (3.55)$$

On the other hand, owing again to (3.27), we have

$$|X| = |\alpha(\alpha^{-1}(X))| = \left| \int_0^{\alpha^{-1}(X)} u_1(0, \theta) d\theta \right| \leq \frac{3}{4} \left| \frac{\partial u_1}{\partial y}(0, 0) \right| (\alpha^{-1}(X))^2.$$

Substituting this inequality in (3.55) yields the following basic estimate of  $|(\alpha^{-1})'(X)|$

$$|(\alpha^{-1})'(X)| \leq \sqrt{\frac{3}{\left|\frac{\partial u_1}{\partial y}(0, 0)\right|}} \frac{1}{\sqrt{X}}. \quad (3.56)$$

Next, we distinguish two cases : if the angle  $\omega_j \leq \frac{\pi}{2}$ , then

$$\varphi(B_{j,r_j^*} \cap \Omega) \subset E_1 = \left\{ (X, Y) \in \mathbb{R}^2, Y \in \left[0, \frac{K}{6}\right], X \in \left[X(0, Y), X\left(\frac{K}{6}, Y\right)\right] \right\},$$

and, if the angle  $\omega_j > \frac{\pi}{2}$ , then

$$\varphi(B_{j,r_j^*} \cap \Omega) \subset E_1 \cup E_2,$$

where

$$E_2 = \{(X, Y) \in \mathbb{R}^2, Y \in [-\frac{K}{6}, 0], X \in [X((\tan \omega_j)Y, 0), X(\frac{K}{6}, 0)].\}$$

Therefore, in the both case, we have to compute, for  $Y \geq 0$

$$\int_{X(0,Y)}^{X(\frac{K}{6},Y)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX.$$

Considering that  $\bar{l}_{j,\mu}$  belongs to  $H^1(\mathbb{R}^2)$ , we derive that the function  $(0, y) \mapsto \bar{l}_{j,\mu}(0, y)$  belongs to  $H^{\frac{1}{2}}(\gamma_j) \supset L^6(\gamma_j)$ . Hence, in view of (3.56), using the Holder's inequality yields

$$\begin{aligned} & \int_{X(0,Y)}^{X(\frac{K}{6},Y)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX \\ & \leq \frac{3}{|\frac{\partial u_1}{\partial y}(0, 0)|} \left( \int_{X(0,Y)}^{X(\frac{K}{6},Y)} \frac{1}{X^{\frac{3}{2}}} dX \right)^{\frac{2}{3}} \left( \int_{X(0,Y)}^{X(\frac{K}{6},Y)} (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^6 dX \right)^{\frac{1}{3}}. \end{aligned}$$

Setting  $v = \alpha^{-1}(X)$ , with (3.27), we obtain

$$\int_{X(0,Y)}^{X(\frac{K}{6},Y)} (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^6 dX \leq \int_0^{\alpha(y_M)} (\bar{l}_{j,\mu}(0, v))^6 |u_1(0, v)| dv \leq \frac{3}{2} \alpha(y_M) \left| \frac{\partial u_1}{\partial y}(0, 0) \right| \|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^6,$$

which implies

$$\int_{X(0,Y)}^{X(\frac{K}{6},Y)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX \leq \left( \frac{162\alpha(y_M)}{(\frac{\partial u_1}{\partial y}(0, 0))^2} \right)^{\frac{1}{3}} \frac{\|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2}{(X(0, Y))^{\frac{1}{3}}}.$$

Hence, since  $\frac{1}{|X(0, Y)|} \leq \frac{4}{|\frac{\partial u_1}{\partial y}(0, 0)|Y^2}$  (see (3.27)), we derive

$$\int_{X(0,Y)}^{X(\frac{K}{6},Y)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX \leq \frac{(648\alpha(y_M))^{\frac{1}{3}} \|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2}{|\frac{\partial u_1}{\partial y}(0, 0)|Y^{\frac{2}{3}}}. \quad (3.57)$$

In the case where  $\omega_j > \frac{\pi}{2}$ , we have to bound, for  $Y < 0$ ,

$$\int_{X((\tan \omega_j)Y, 0)}^{X(\frac{K}{6}, 0)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX.$$

In the same way as previously, we obtain

$$\int_{X((\tan \omega_j)Y, 0)}^{X(\frac{K}{6}, 0)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX \leq \left( \frac{162\alpha(y_M)}{(\frac{\partial u_1}{\partial y}(0, 0))^2} \right)^{\frac{1}{3}} \frac{\|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2}{(X((\tan \omega_j)Y, 0))^{\frac{1}{3}}}.$$

Since  $|X((\tan \omega_j)Y, 0)| \geq \frac{2}{|u_2(0, 0)| |\tan \omega_j| |Y|}$ , we derive

$$\int_{X((\tan \omega_j)Y, 0)}^{X(\frac{K}{6}, 0)} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX \leq \frac{(324\alpha(y_M))^{\frac{1}{3}} \|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2}{((\frac{\partial u_1}{\partial y}(0, 0))^2 |u_2(0, 0)| |\tan \omega_j|)^{\frac{1}{3}} |Y|^{\frac{1}{3}}}. \quad (3.58)$$

In the case where  $\omega_j \leq \frac{\pi}{2}$ , integrating with respect to  $Y$  on the interval  $[0, \frac{K}{6}]$  the both side of (3.57) yields

$$\int \int_{\varphi(B_{j,r_j^*} \cap \bar{\Omega})} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX dY \leq 9 \frac{(4\alpha(y_M)K)^{\frac{1}{3}}}{|\frac{\partial u_1}{\partial y}(0,0)|} \|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2 < +\infty.$$

In the case where  $\omega_j > \frac{\pi}{2}$ , in addition to the previous integral, we must integrate with respect to  $Y$  on the interval  $[-\frac{K}{6}, 0]$  the both side of (3.58), which gives

$$\begin{aligned} & \int \int_{\varphi(B_{j,r_j^*} \cap \bar{\Omega})} ((\alpha^{-1})'(X))^2 (\bar{l}_{j,\mu}(0, \alpha^{-1}(X)))^2 dX dY \\ & \leq \left(9 \frac{(4\alpha(y_M)K)^{\frac{1}{3}}}{|\frac{\partial u_1}{\partial y}(0,0)|} + \frac{3}{2} \frac{(9\alpha(y_M)K^2)^{\frac{1}{3}}}{((\frac{\partial u_1}{\partial y}(0,0))^2 |u_2(0,0)| |\tan \omega_j|)^{\frac{1}{3}}}\right) \|\bar{l}_{j,\mu}\|_{L^6(\gamma_j)}^2 < +\infty. \end{aligned}$$

Finally, in view of (3.54), we have obtained that  $\frac{\partial Z}{\partial X}$  belongs to  $L^2(\varphi(B_{j,r_j^*} \cap \Omega))$ , which implies, as we saw previously, that  $\nabla z$  belongs to  $L^2(B_{j,r_j^*} \cap \Omega)$  and, therefore, with (3.51), we derive that  $z$  belongs to  $H^1(B_{j,r_j^*} \cap \Omega)$ , which ends the proof of the lemma.  $\diamond$

Considering that (3.41) implies that  $z$  vanishes in a neighborhood of the boundary of  $B_{j,r_j^*} \cap \Omega$  (see the definition (3.47) of  $r_j^*$ ), we can now construct the solution  $\bar{z}_{j,\mu}$  of the problem  $(\bar{P}_{j,\mu})$ , which belongs to  $H^1(\Omega)$ , by

$$\bar{z}_{j,\mu} = \begin{cases} e^{-V(X(x,y),y)} \left( \int_{\alpha^{-1}(X(x,y))}^y \frac{e^{V(X(x,y),t)}}{\mathcal{W}U_2(X(x,y),t)} \bar{L}_{j,\mu}(X(x,y),t) dt \right) & \text{if } (x,y) \in B_{j,r_j^*} \cap \Omega \\ 0 & \text{if } (x,y) \in \Omega \setminus B_{j,r_j^*}, \end{cases} \quad (3.59)$$

where the function  $X$  is defined by (3.31), the functions  $\bar{L}_{j,\mu}$ ,  $U_2$  and  $V$  are defined by (3.34), the function  $\alpha$  is defined by (3.37) and the real number  $r_j^*$  by (3.47), with a small enough real number  $\mu$  verifying (3.45). Thus, thanks to (3.23), in this third case, we have proven that

$$z_{j,\mu} \in H^1(\Omega) \text{ is the solution of the problem } (P_{j,\mu}). \quad (3.60)$$

4) **Fourth case:**  $\mathbf{S}_j^{-1} \notin \gamma_{j-1}$ ,  $\mathbf{S}_j^{-1} \in (E \cap S^c)$ ,  $\mathbf{S}_j^1 \in \gamma_{j+1}$ .

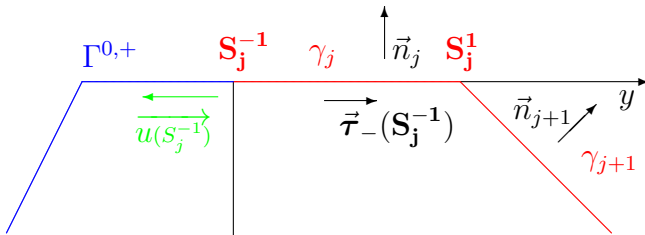


figure 3.7

The fourth case is not very different that the third case :  $\mathbf{S}_j^{-1}$  is still the origin, the x-axis and the y-axis are defined in the same way, but, for  $y < 0$  small enough, the point  $(0, y)$  belongs to  $\Gamma^{0,+}$  and, therefore, we have  $u_1(0, y) \leq 0$ . We denote by  $\Gamma_{k_j}$  the side of

the polygon which contains  $\gamma_j$  and we set  $\eta_j = d(\mathbf{S}_j^{-1}, \partial\Gamma_{k_j})$ , where  $d(\cdot, \cdot)$  is the euclidian distance in  $\mathbb{R}^2$ . According to the assumptions of the fourth case,  $\eta_j > 0$ . Instead of (3.27) which corresponds to the third case, we define  $\mu'_3 \leq \min(\mu_2, \eta_j)$  such that

$$\begin{aligned} \forall \mathbf{x} \in B_{j, \mu'_3} \cap \Omega, \quad \frac{3}{2}u_2(0, 0) \leq u_2(\mathbf{x}) \leq \frac{1}{2}u_2(0, 0) < 0 \quad \text{and} \\ \forall y \in [-\mu'_3, \mu'_3], \quad \frac{1}{2} \frac{\partial u_1}{\partial y}(0, 0)y \leq |u_1(0, y)| \leq \frac{3}{2} \frac{\partial u_1}{\partial y}(0, 0)y. \end{aligned} \quad (3.61)$$

Next, the proof of (3.35)-(3.36) is slightly different in the case where  $y = Y \leq t \leq 0$ . First, we have

$$X(0, t) = \int_0^t u_1(0, \theta) d\theta \leq X(x, y).$$

Second, in view of (3.61), since  $\int_0^t u_1(0, \theta) d\theta \geq 0$ , we still have

$$X(x, y) \leq \frac{3}{2}k(x, y) \quad \text{and} \quad X\left(\frac{\mu'_3}{2}, t\right) \geq \frac{\mu'_3}{4}|u_2(0, 0)|$$

and (3.34), (3.35) and (3.36) are still verified with  $\mu'_3$  in the place of  $\mu_3$ . In the same way, (3.37) and (3.39) run unchanged, while (3.25) is verified with  $K'$  in the place of  $K$  where  $K'$  is defined by

$$K' = \min\left(\frac{\mu'_3}{6}, \mu_4, \mu_5, |\gamma_j|\right). \quad (3.62)$$

Afterwards, the case where  $0 \leq y \leq t \leq \alpha^{-1}(X(x, y))$  remains unchanged and we still have (3.41) for  $y \geq 0$ , with  $r'_{1,j}$  and  $r'_{2,j}$  in the place of  $r_{1,j}$  and  $r_{2,j}$ , where  $r'_{1,j}$  and  $r'_{2,j}$  are defined by

$$r'_{1,j} = \min\left(\frac{|u_2(0, 0)|K'}{12}, \frac{|\frac{\partial u_1}{\partial y}(0, 0)|K'^2}{288}\right) \quad \text{and} \quad r'_{2,j} = \frac{K'|u_2(0, 0)|}{6}. \quad (3.63)$$

When  $y \leq t \leq \alpha^{-1}(X(x, y))$ , with  $y < 0$ , we consider first  $0 \leq t \leq \alpha^{-1}(X(x, y))$  and second  $y \leq t < 0$ .

If  $0 \leq t \leq \alpha^{-1}(X(x, y))$  and  $(x, y) \in B_{j, \frac{K'}{6}}^+$ , we have  $t \leq \frac{\mu'_3}{6}$  and

$$\alpha(t) = X(0, t) \leq X(x, y) \leq \frac{3}{2}k(x, y).$$

Applying (3.29) with  $r = |u_2(0, 0)|\frac{K'}{6}$ , we obtain  $k(x, y) \leq |u_2(0, 0)|\frac{K'}{6}$ . Hence, we derive

$$X(0, t) \leq X(x, y) \leq |u_2(0, 0)|\frac{K'}{4} \leq X\left(\frac{K'}{2}, t\right).$$

Then, there exists  $x_t \in [0, \frac{K'}{2}]$  such that

$$X(x, y) = X(x_t, t) \quad \text{with} \quad (x_t, t) \in B_{j, \mu'_3}^+.$$

Next, as previously, we derive that  $\forall (x, y) \in C(\mathbf{S}_j^{-1}, 2\sqrt{\frac{r'_{1,j}}{|\frac{\partial u_1}{\partial y}(0, 0)|}, \frac{r'_{2,j}}{|u_2(0, 0)|}}) \cap \bar{\Omega}$  with  $y < 0$ ,

$$\int_{\alpha^{-1}(X(x, y))}^0 \frac{e^{V(X(x, y), t)}}{\mathcal{W}U_2(X(x, y), t)} \bar{L}_{j, \mu}(X(x, y), t) dt = 0.$$

Finally, for the case where  $y \leq t \leq 0$ , we proceed in the same way as previously and we obtain that (3.41) is verified for  $y < 0$  with  $r'_{1,j}$  and  $r'_{2,j}$  in the place of  $r_{1,j}$  and  $r_{2,j}$ .

The rest of the proof is the same as in the third case. Thus, in the fourth case, we have proven that

$$z_{j,\mu} \in H^1(\Omega) \text{ is the solution of the problem } (P_{j,\mu}). \quad (3.64)$$

By localization, all the other cases, where  $\mathbf{S}_j^{-1} \in E$  and (or)  $\mathbf{S}_j^1 \in E$ , can be solved as in the third case or the fourth case.  $\diamond$

## 4 Appendix

In the two following examples, the domains  $\Omega$  are no longer a bounded polygon, but domains of class  $C^{1,1}$ . Even if in this article, we mainly deal with bounded polygons, it seems to us interesting to show that the regularity of the solution  $z$  of the transport problem in domains of class  $C^{1,1}$  seems still linked to the multiplicity of the roots of the equation  $\mathbf{u} \cdot \mathbf{n} = 0$  at the end-points of  $\Gamma^-$ , in the case where  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  is negative.

### 4.1 Example 6 : $\Omega = C(I(0, 1), 0.5)$ , $l(x, y) = 1$ , $\mathbf{u}(x, y) = (x, -y)$ .

In this example, the boundary, which is the circle of center  $I(0, 1)$  and of radius  $R = 0.5$ , is very regular, but the function  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the boundary points of  $\Gamma^-$ , which leads to a discontinuity for the partial derivatives of the solution  $z$  in these points.

$$\text{The equation of } \Gamma \text{ is } \begin{cases} x = \frac{1}{2} \cos t \\ y = 1 + \frac{1}{2} \sin t \end{cases}, t \in ]-\pi, \pi] \text{ and}$$

the unit exterior normal is  $\mathbf{n} = (\cos t, \sin t)$ . Let us determine the sets  $\Gamma^-$ ,  $\Gamma^0$  and  $\Gamma^+$ . On  $\Gamma$ , we have

$$(\mathbf{u} \cdot \mathbf{n})(t) = -\sin^2 t - \sin t + \frac{1}{2},$$

that vanishes for  $t_0 = \arcsin(\frac{\sqrt{3}-1}{2})$  and  $t_1 = \pi - \arcsin(\frac{\sqrt{3}-1}{2})$ , and  $\Gamma^-$  is the open arc of the circle  $\Gamma$  defined by  $t_0 < t < t_1$ ,  $\Gamma^+$  is  $\Gamma \setminus \overline{\Gamma^-}$ ,  $\Gamma^0 = \emptyset$ .

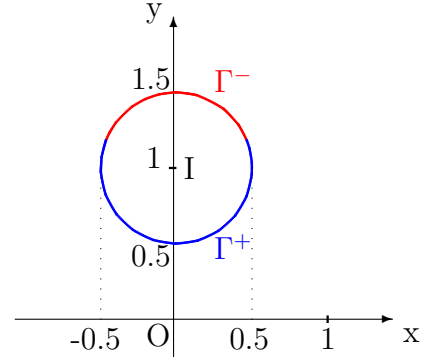


figure 4.8

Note that, we can easily verify that  $(\mathbf{u} \cdot \boldsymbol{\tau}_-)(t)$  is negative for  $t = t_0$  and  $t = t_1$ , so that assumptions analogous to the assumptions of (3.5) are verified at points where  $\mathbf{u} \cdot \mathbf{n}$  vanishes in this example. As in the first three examples, we have

$$\forall (x, y) \in \Omega, z(x, y) = 1 - 2y + yC(xy), \forall (x, y) \in \Gamma^-, C(xy) = 2 - \frac{1}{y}. \quad (4.1)$$

Setting  $X = xy$ , we must compute the function  $\alpha$  such that  $y = \alpha(X)$ , for  $(x, y) \in \Gamma^-$ .

1) **First case** :  $t_0 < t \leq \frac{\pi}{2}$ ,  $(x, y) \in \Gamma^- \cap \mathbb{R}_+^2$ .



$x = \frac{1}{2}\sqrt{1 - (2y - 2)^2} = \frac{1}{2}\sqrt{(2y - 1)(3 - 2y)}$ , which imply  $X = \frac{y}{2}\sqrt{(2y - 1)(3 - 2y)}$ . We compute  $y(t_0) = \frac{3+\sqrt{3}}{4}$  and  $X(t_0) = \frac{\sqrt{2\sqrt{3}(3+\sqrt{3})}}{16}$ . Considering the function

$$g : y \mapsto X = \frac{y}{2}\sqrt{(2y - 1)(3 - 2y)}, \quad (4.2)$$

which is defined on the set  $[\frac{1}{2}, \frac{3}{2}]$ . Since we have,  $\forall y \in ]\frac{1}{2}, \frac{3}{2}[$ ,  $g'(y) = -\frac{4(y - \frac{3+\sqrt{3}}{4})(y - \frac{3-\sqrt{3}}{4})}{\sqrt{(2y - 1)(3 - 2y)}}$ ,

the statement of changes of  $g$  is

|         |               |                        |               |
|---------|---------------|------------------------|---------------|
| $y$     | $\frac{1}{2}$ | $\frac{3+\sqrt{3}}{4}$ | $\frac{3}{2}$ |
| $g'(y)$ | $   +\infty$  | $+ \quad 0 \quad -$    | $-\infty   $  |
| $g$     | $0$           | $X(t_0)$               | $0$           |

Since  $g$  is strictly decreasing from  $[y(t_0), \frac{3}{2}]$  to  $[0, X(t_0)]$ , therefore  $g|_{[y(t_0), \frac{3}{2}]}$  has an inverse function and we have

$$\alpha_{|[0, X(t_0)]} = (g|_{[y(t_0), \frac{3}{2}]})^{-1}. \quad (4.3)$$

Finally, we obtain, for  $(x, y) \in \Gamma^- \cap \mathbb{R}_+^2$ ,  $y = \alpha(X) \iff y = g(X)$ .

2) **Second case** :  $\frac{\pi}{2} \leq t \leq \pi - t_0$ ,  $(x, y) \in \Gamma^- \cap \mathbb{R}_- \times \mathbb{R}_+$ .

In the same way, we have  $X = -\frac{y}{2}\sqrt{(2y - 1)(3 - 2y)}$  and we define  $\alpha$  on  $[-X(t_0), 0]$  by

$$\alpha_{|[-X(t_0), 0]} = ((-g)|_{[y(t_0), \frac{3}{2}]})^{-1}. \quad (4.4)$$

The statement of changes of the even function  $\alpha$  is :

|              |                        |                     |                        |
|--------------|------------------------|---------------------|------------------------|
| $X$          | $-X(t_0)$              | $0$                 | $+X(t_0)$              |
| $\alpha'(X)$ | $   +\infty$           | $+ \quad 0 \quad -$ | $-\infty   $           |
| $\alpha$     | $\frac{3+\sqrt{3}}{4}$ | $\frac{3}{2}$       | $\frac{3+\sqrt{3}}{4}$ |

From (4.1), we derive the solution of the example 6

$$\forall (x, y) \in \Omega, \quad z(x, y) = 1 - \frac{y}{\alpha(xy)}. \quad (4.5)$$

Let us show that  $z$  belongs to  $H^1(\Omega)$ . We compute

$$\forall (x, y) \in \Omega \setminus \{(\pm x(t_0), y(t_0))\}, \quad z'_x(x, y) = \frac{y^2 \alpha'(xy)}{(\alpha(xy))^2}, \quad z'_y(x, y) = \frac{xy \alpha'(xy)}{(\alpha(xy))^2} - \frac{1}{\alpha(xy)},$$

with  $x(t_0) = \frac{\sqrt{2\sqrt{3}}}{4}$  and  $y(t_0) = \frac{3+\sqrt{3}}{4}$ . Hence, we derive

$$z \text{ belongs to } H^1(\Omega) \iff \int \int_{\Omega} (\alpha'(xy))^2 dx dy < +\infty. \quad (4.6)$$

Let us set  $\Omega_+ = \Omega \cap \mathbb{R}_+^2$  and note that  $\int \int_{\Omega} (\alpha'(xy))^2 dx dy = 2 \int \int_{\Omega_+} (\alpha'(xy))^2 dx dy$ . In order to show that the last integral converges, we split  $\Omega_+$  in two subdomains:

$$\Omega_+^1 = \Omega_+ \cap (\mathbb{R}_+ \times [\frac{3+\sqrt{3}}{4}, \frac{3}{2}]) \text{ and } \Omega_+^2 = \Omega_+ \cap (\mathbb{R}_+ \times [\frac{1}{2}, \frac{3+\sqrt{3}}{4}]).$$

1) We compute  $\int \int_{\Omega_+^1} (\alpha'(xy))^2 dx dy$  by making the substitution  $\begin{cases} u = \alpha(xy) \\ y = y \end{cases} \iff \begin{cases} x = \frac{g(u)}{y} \\ y = y \end{cases}$ , the jacobian of which is  $\frac{g'(u)}{y}$ . Since  $\forall u \in ]\frac{3+\sqrt{3}}{4}, \frac{3}{2}]$ ,  $\alpha'(g(u)) = \frac{1}{g'(u)}$ , we obtain

$$\int \int_{\Omega_+^1} (\alpha'(xy))^2 dx dy = \int_{\frac{3+\sqrt{3}}{4}}^{\frac{3}{2}} \frac{1}{y} dy \left( \int_y^{\frac{3}{2}} \frac{1}{|g'(u)|} du \right).$$

Considering that,  $\forall u \in ]\frac{3+\sqrt{3}}{4}, 1.5[$ ,  $g'(u) = -\frac{4(u - \frac{3+\sqrt{3}}{4})(u - \frac{3-\sqrt{3}}{4})}{\sqrt{(2u-1)(3-2u)}}$ , we can verify

$$\int \int_{\Omega_+^1} (\alpha'(xy))^2 dx dy \leq K_1 \int_{\frac{3+\sqrt{3}}{4}}^{\frac{3}{2}} dy \left( \int_y^{\frac{3}{2}} \frac{1}{u - \frac{3+\sqrt{3}}{4}} du \right) = \frac{3 - \sqrt{3}}{4} K_1 < +\infty, \quad (4.7)$$

with  $K_1 = \frac{\sqrt{3} - 1}{3}$ .

2) For  $\int \int_{\Omega_+^2} (\alpha'(xy))^2 dx dy$ , it is more complicated. Setting  $X = xy$ , we obtain

$$\int \int_{\Omega_+^2} (\alpha'(xy))^2 dx dy = \int_{\frac{1}{2}}^{\frac{3+\sqrt{3}}{4}} dy \left( \int_0^{\frac{g(y)}{y}} (\alpha'(xy))^2 dx \right) = \int_{\frac{1}{2}}^{\frac{3+\sqrt{3}}{4}} \frac{1}{y} dy \left( \int_0^{g(y)} (\alpha'(X))^2 dX \right),$$

where the function  $g$  is defined by (4.2). Next, making the substitution  $X = g(u)$ , for  $u \in [\frac{1}{2}, \frac{3+\sqrt{3}}{4}]$ , we derive

$$\int \int_{\Omega_+^2} (\alpha'(xy))^2 dx dy = \int_{\frac{1}{2}}^{\frac{3+\sqrt{3}}{4}} \frac{1}{y} dy \left( \int_{\frac{1}{2}}^y (\alpha'(g(u)))^2 g'(u) du \right).$$

However, the complication comes from the fact that, for  $\frac{1}{2} \leq u < \frac{3+\sqrt{3}}{4}$ ,  $\alpha(g(u)) \neq u$  since  $\alpha(g(u)) > \frac{3+\sqrt{3}}{4}$ . Let us define the function  $\beta$  on the set  $[\frac{1}{2}, \frac{3+\sqrt{3}}{4}]$  by

$$\forall u \in [\frac{1}{2}, \frac{3+\sqrt{3}}{4}], \beta(u) = \alpha(g(u)).$$

Since  $g(\beta(u)) = g(u)$ , then, for  $\frac{1}{2} < u < \frac{3+\sqrt{3}}{4}$ ,

$$g'(u) = g'(\beta(u))\beta'(u) \text{ and } \alpha'(g(u)) = \alpha'(g(\beta(u))) = \frac{1}{g'(\beta(u))} = \frac{\beta'(u)}{g'(u)}.$$

Hence, we can write

$$\int \int_{\Omega_+^2} (\alpha'(xy))^2 dx dy = \int_{\frac{1}{2}}^{\frac{3+\sqrt{3}}{4}} \frac{1}{y} dy \left( \int_{\frac{1}{2}}^y \frac{(\beta'(u))^2}{g'(u)} du \right).$$

Let us show that  $\beta'$  is bounded on the set  $]\frac{1}{2}, \frac{3+\sqrt{3}}{4}[$  and that we can extend  $\beta'$  on  $[\frac{1}{2}, \frac{3+\sqrt{3}}{4}]$  by continuity. Computing  $\beta'(u)$  yields

$$\beta'(u) = \frac{u(u - \frac{3+\sqrt{3}}{4})(u - \frac{3-\sqrt{3}}{4})}{\beta(u)(\beta(u) - \frac{3+\sqrt{3}}{4})(\beta(u) - \frac{3-\sqrt{3}}{4})}.$$

Note that the right hand previous expression extends  $\beta'$  by continuity in  $\frac{1}{2}$ . It remains to compute the limit of  $\beta'$  in  $\frac{3+\sqrt{3}}{4}$ . Applying to the function  $g$  the Taylor-Lagrange formula in the neighborhood of  $\frac{3+\sqrt{3}}{4}$ , we obtain

$$g(u) = g\left(\frac{3+\sqrt{3}}{4}\right) + \frac{1}{2}g''(c)\left(u - \frac{3+\sqrt{3}}{4}\right)^2 = g(\beta(u)) = g\left(\frac{3+\sqrt{3}}{4}\right) + \frac{1}{2}g''(d)\left(\beta(u) - \frac{3+\sqrt{3}}{4}\right)^2,$$

with  $c \in [u, \frac{3+\sqrt{3}}{4}]$  and  $d \in [\frac{3+\sqrt{3}}{4}, \beta(u)]$ , which implies  $\lim_{u \rightarrow \frac{3+\sqrt{3}}{4}^-} \frac{(u - \frac{3+\sqrt{3}}{4})^2}{(\beta(u) - \frac{3+\sqrt{3}}{4})^2} = 1$  and

$$\lim_{u \rightarrow \frac{3+\sqrt{3}}{4}^-} \frac{u - \frac{3+\sqrt{3}}{4}}{\beta(u) - \frac{3+\sqrt{3}}{4}} = -1, \text{ since } \frac{u - \frac{3+\sqrt{3}}{4}}{\beta(u) - \frac{3+\sqrt{3}}{4}} \leq 0. \text{ Hence, we derive } \lim_{u \rightarrow \frac{3+\sqrt{3}}{4}^-} \beta'(u) = -1 \text{ and,}$$

therefore, there exists a constant  $K_2 > 0$ , such that, for  $u \in [\frac{1}{2}, \frac{3+\sqrt{3}}{4}]$ ,  $|\beta'(u)| \leq K_2$ . Then, we have

$$\int \int_{\Omega_+^2} (\alpha'(xy))^2 dx dy \leq (\sqrt{3} + 1)K_2^2 \int_{\frac{1}{2}}^{\frac{3+\sqrt{3}}{4}} dy \left( \int_{\frac{1}{2}}^y \frac{1}{\frac{3+\sqrt{3}}{4} - u} du \right) \leq \left( \frac{(\sqrt{3} + 1)K_2}{2} \right)^2 < +\infty$$

and, with (4.6 and (4.7), we derive that the solution  $z$  belongs to  $H^1(\Omega)$ . Finally, although  $\mathbf{u} \cdot \mathbf{n}$  vanishes at the end points of  $\Gamma^-$ , the problem (1.1) is well-posed, probably because the function  $\mathbf{u} \cdot \mathbf{n}$  has only simple roots at the end points of  $\Gamma^-$  with, in addition,  $\mathbf{u} \cdot \boldsymbol{\tau}_-$  negative at these end points.

#### 4.2 Example 7 : $\Omega = \Omega_7$ , $l(x, y) = 1$ , $\mathbf{u}(x, y) = (x, -y)$ .

The boundary of  $\Omega_7$  is composed of two half semicircles, linked up by two segments (see the figure 4.9). The boundary is of class  $C^{1,1}$  but the arc of circle  $\Gamma^-$  is adjacent to the segment  $\Gamma^0$ , which leads to a discontinuity for the partial derivatives of the solution  $z$ . But, as in the example 4, **this discontinuity is such that the solution  $z$  does not belong to  $\mathbf{H}^1(\Omega)$ , as we shall see further.** The parametric equation of the up-

$$\text{per semicircle is } \begin{cases} x = \frac{1}{2} + \frac{1}{2} \cos t \\ y = 1 + \frac{1}{2} \sin t \end{cases}, \quad t \in [0, \pi].$$

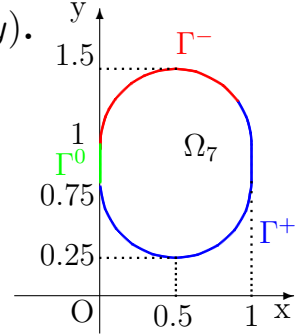


figure 4.9

Let us determine the sets  $\Gamma^-$ ,  $\Gamma^0$  and  $\Gamma^+$ . Again, we have  $\mathbf{n} = (\cos t, \sin t)$ . On the upper semicircle, we compute  $(\mathbf{u} \cdot \mathbf{n})(t) = \cos(\frac{t}{2})(\cos(\frac{t}{2}) - 2 \sin(\frac{t}{2})(\sin t + 1))$ . In view of  $\cos(\frac{t}{2}) > 0$  for  $0 \leq t < \pi$ ,  $(\mathbf{u} \cdot \mathbf{n})(t)$  has the same sign that  $f(t) = \cos(\frac{t}{2}) - 2 \sin(\frac{t}{2})(\sin t + 1)$ .

- 1) For  $\frac{\pi}{2} \leq t \leq \pi$ ,  $f(t) = \cos(\frac{t}{2})(1 - 4 \sin^2(\frac{t}{2})) - 2 \sin(\frac{t}{2}) < 0$ .
- 2) For  $0 \leq t \leq \frac{\pi}{2}$ ,  $f(t) = \cos^3(\frac{t}{2})(-2 \tan^3(\frac{t}{2}) - 3 \tan^2(\frac{t}{2}) - 2 \tan(\frac{t}{2}) + 1)$ . Setting  $\theta = \tan(\frac{t}{2})$ , we must study the sign of the polynomial  $g(\theta) = -2\theta^3 - 3\theta^2 - 2\theta + 1$ , for  $0 \leq \theta \leq 1$ . We have  $g'(\theta) = -6\theta^2 - 6\theta - 2 < 0$ ,  $g(0) = 1$ ,  $g(1) = -6$ . Then the continuity and the strict decreasing of  $g$  implies that there exists a unique number  $\theta_0 \in ]0, 1[$ , such that  $g(\theta_0) = 0$ . Finally, for  $0 \leq t \leq \pi$ ,  $(\mathbf{u} \cdot \mathbf{n})(t)$  vanishes for two values :

$$t_0 = 2 \arctan(\theta_0) \approx 0.614 \text{ and } \pi \tag{4.8}$$

and

$$\Gamma^- \text{ is the open arc of circle defined by } \begin{cases} x = \frac{1}{2} + \frac{1}{2} \cos t \\ y = 1 + \frac{1}{2} \sin t \end{cases}, t \in ]t_0, \pi[,$$

while the part of the previous semicircle, defined by  $t \in [0, t_0[$ , is included into  $\Gamma^+$ . Next, the part of  $\Gamma$ , defined by  $\begin{cases} x = 0 \\ \frac{3}{4} < y < 1 \end{cases}$  is  $\Gamma^0$  and we are going to show that the lower semicircle is included in  $\Gamma^+$ . Indeed, the parametric equations of the lower semicircle is

$$\begin{cases} x = \frac{1}{2} + \frac{1}{2} \cos t \\ y = \frac{3}{4} + \frac{1}{2} \sin t \end{cases}, t \in [-\pi, 0]. \text{ Considering that, for } t \in [-\pi, 0],$$

$$(\mathbf{u} \cdot \mathbf{n})(t) = \frac{1}{2} \cos t + \frac{1}{2} \cos^2 t - \frac{3}{4} \sin t - \frac{1}{2} \sin^2 t,$$

we distinguish two cases :

$$1) t \in ]-\pi, -\frac{3\pi}{4}[ \cup ]-\frac{\pi}{2}, 0]$$

$$(\mathbf{u} \cdot \mathbf{n})(t) = \frac{1}{2}(\cos t - \sin t)(1 + \cos t + \sin t) - \frac{1}{4} \sin t > 0.$$

$$2) t \in [-\frac{3\pi}{4}, -\frac{\pi}{2}]$$

$$\frac{1}{2} \cos t - \frac{1}{2} \sin^2 t = -\frac{5}{8} + \frac{1}{2}(\cos t + \frac{1}{2})^2 \geq -\frac{5}{8} \text{ and } \frac{1}{2} \cos^2 t - \frac{3}{4} \sin t = \frac{25}{32} - \frac{1}{2}(\sin t + \frac{3}{4})^2 \geq \frac{3}{4}.$$

Therefore,  $(\mathbf{u} \cdot \mathbf{n})(t) \geq \frac{1}{8} > 0$ .

Finally,  $\begin{cases} x = 1 \\ \frac{3}{4} < y < 1 \end{cases}$  is included in  $\Gamma^+$ , which ends the determining of the sets  $\Gamma^-$ ,  $\Gamma^0$  and  $\Gamma^+$ , see figure 4.9.

Again, it is easy to verify that  $(\mathbf{u} \cdot \boldsymbol{\tau}_-)(t)$  is negative for  $t = t_0$  and  $t = \pi$ . As in example 6, we have

$$\forall (x, y) \in \Omega, z(x, y) = 1 - 2y + yC(xy), \forall (x, y) \in \Gamma^-, C(xy) = 2 - \frac{1}{y}. \quad (4.9)$$

Setting  $X = xy$ , we must compute the function  $\alpha$  such that  $y = \alpha(X)$ , for  $(x, y) \in \Gamma^-$ .

1) **First case** :  $(x, y) \in \Gamma^- \cap ([0, \frac{1}{2}] \times \mathbb{R}_+)$ .

$X = \frac{y}{2}(1 - \sqrt{(2y-1)(3-2y)})$ . Considering the function

$$g : y \mapsto X = \frac{y}{2}(1 - \sqrt{(2y-1)(3-2y)}). \quad (4.10)$$

We have, for  $\frac{1}{2} < y < \frac{3}{2}$ ,

$$g'(y) = \frac{8y^2 - 12y + 3 + \sqrt{(2y-1)(3-2y)}}{2\sqrt{(2y-1)(3-2y)}}. \quad (4.11)$$

For  $\frac{1}{2} < y < 1$ ,  $8y^2 - 12y + 3 < -1$ ,  $\sqrt{(2y-1)(3-2y)} < 1$ , therefore,  $g'(y) < 0$ . Moreover,  $g'(1) = 0$ .

For  $\frac{3+\sqrt{3}}{4} < y < \frac{3}{2}$ ,  $8y^2 - 12y + 3 > 0$ , therefore  $g'(y) > 0$ .

For  $1 < y < \frac{3+\sqrt{3}}{4}$ , we note that  $g'(y)$  has the same sign that

$$h(y) = 8y^2 - 12y + 3 + \sqrt{(2y-1)(3-2y)}$$

and  $h'(y) = 16y - 12 - \frac{4(y-1)}{\sqrt{(2y-1)(3-2y)}}$ . In view of  $\frac{4(y-1)}{\sqrt{(2y-1)(3-2y)}} \leq \frac{\sqrt{6} - \sqrt{2}}{3^{0.25}} < 1$  and  $(16y - 12) \geq 4$ , we derive  $h'(y) > 0$ . Since  $h(1) = 0$ , we obtain that  $h(y) > 0$ , that is to say,  $g'(y) > 0$ .

Thus, we obtain the statement of changes of  $g$  :

|         |               |            |   |            |               |
|---------|---------------|------------|---|------------|---------------|
| $y$     | $\frac{1}{2}$ | -          | 0 | +          | $\frac{3}{2}$ |
| $g'(y)$ | $-\infty$     | -          | 0 | +          | $+\infty$     |
| $g$     | $\frac{1}{4}$ | $\searrow$ | 0 | $\nearrow$ | $\frac{3}{4}$ |

Finally, the function  $g$ , defined by (4.10), is strictly increasing from  $[1, \frac{3}{2}]$  to  $[0, \frac{3}{4}]$ , therefore  $g_{|[1, \frac{3}{2}]}$  has an inverse function and we define  $\alpha$  on the set  $[0, \frac{3}{4}]$  by

$$\alpha_{|[0, \frac{3}{4}]} = (g_{|[1, \frac{3}{2}]})^{-1}, \quad (4.12)$$

that verifies  $y = \alpha(X) \iff X = g(y)$  for  $(x, y) \in \Gamma^- \cap ([0, \frac{1}{2}] \times \mathbb{R}_+)$ .

2) **Second case** :  $(x, y) \in \Gamma^- \cap ([\frac{1}{2}, 1] \times \mathbb{R}_+)$ .

$X = \frac{y}{2}(1 + \sqrt{(2y-1)(3-2y)})$ , with  $y(t_0) < y < \frac{3}{2}$ , where  $t_0$  is defined by (4.8) and  $y(t_0) \approx 1.288$ . Considering the function

$$\tilde{g} : y \mapsto X = \frac{y}{2}(1 + \sqrt{(2y-1)(3-2y)}).$$

We compute  $\tilde{g}'(y) = \frac{-8y^2 + 12y - 3 + \sqrt{(2y-1)(3-2y)}}{2\sqrt{(2y-1)(3-2y)}}$ . We can verify that

$$\forall t \in [0, \frac{\pi}{2}], (\mathbf{u} \cdot \mathbf{n})(t) = 0 \iff -8y^2 + 12y - 3 + \sqrt{(2y-1)(3-2y)} = 0.$$

Hence, we derive that  $\tilde{g}'$  vanishes at  $y(t_0)$  and, since the numerator of  $\tilde{g}'$  is a strictly decreasing function on  $[y(t_0), \frac{3}{2}]$ , we obtain that  $\tilde{g}'$  is strictly negative on  $]y(t_0), \frac{3}{2}[$ , which implies that the function  $\tilde{g}$  is strictly decreasing from  $[y(t_0), \frac{3}{2}]$  to  $[\frac{3}{4}, \tilde{g}(y(t_0))]$ , with  $\tilde{g}(y(t_0)) \approx 1.17$ . Therefore  $\tilde{g}_{|[y(t_0), \frac{3}{2}]}$  has an inverse function and we define  $\alpha$  on the set  $[\frac{3}{4}, \tilde{g}(y(t_0))]$  by

$$\alpha_{|[\frac{3}{4}, \tilde{g}(y(t_0))]} = (\tilde{g}_{|[y(t_0), \frac{3}{2}]})^{-1}, \quad (4.13)$$

that verifies  $y = \alpha(X) \iff X = \tilde{g}(y)$  for  $(x, y) \in \Gamma^- \cap ([\frac{1}{2}, 1] \times \mathbb{R}_+)$ .

Finally, from (4.9), we derive the solution of example 7 :

$$\forall (x, y) \in \Omega, z(x, y) = 1 - \frac{y}{\alpha(xy)}, \quad (4.14)$$

where the function  $\alpha$  is defined on the interval  $[0, \tilde{g}(y(t_0))]$  by (4.12) and (4.13).

Let us show that  $z$  does not belongs to  $H^1(\Omega)$ . We again compute

$$\forall (x, y) \in \Omega \setminus \{(0, 1)\}, z'_x(x, y) = \frac{y^2 \alpha'(xy)}{(\alpha(xy))^2}.$$

We integrate  $(z'_x(x, y))^2$  on a domain  $\Omega^* = \{(x, y) \in \mathbb{R}^2, \frac{3}{4} \leq y \leq 1, 0 \leq x \leq \frac{1}{4y}\}$ , which is included in  $\Omega$ . Moreover, since in  $\Omega^*$ , we have  $0 \leq xy \leq \frac{1}{4} \leq \frac{3}{4}$ , we use the expression

of the fonction  $\alpha$  defined by (4.12) from the function  $g$ , defined by (4.10). Clearly,

$$\int \int_{\Omega} (z'_x(x, y))^2 dx dy \geq \int \int_{\Omega^*} (z'_x(x, y))^2 dx dy.$$

Let us show that the integral  $\int \int_{\Omega^*} (z'_x(x, y))^2 dx dy = +\infty$ . First, we compute  $\int \int_{\Omega^*} (z'_x(x, y))^2 dx dy$  by making the substitution  $\begin{cases} X = xy \\ y = y \end{cases} \iff \begin{cases} x = \frac{X}{y} \\ y = y \end{cases}$ , the jacobian of which is  $\frac{1}{y}$ . We obtain

$$\int \int_{\Omega^*} (z'_x(x, y))^2 dx dy = \left( \int_{\frac{3}{4}}^1 y^3 dy \right) \left( \int_0^{\frac{1}{4}} \frac{((\alpha'(X))^2}{(\alpha(X))^4} dX \right).$$

Since the function,  $g$  defined by (4.10), is strictly increasing from  $[1, \frac{3}{2}]$  to  $[0, \frac{3}{4}]$ , making a substitution,  $\begin{cases} X = g(u) \\ 1 \leq u \leq \alpha(\frac{1}{4}) \end{cases} \iff \begin{cases} u = \alpha(X) \\ 0 \leq X \leq \frac{1}{4} \end{cases}$  yields

$$\int \int_{\Omega^*} (z'_x(x, y))^2 dx dy = \left( \int_{\frac{3}{4}}^1 y^3 dy \right) \left( \int_1^{\alpha(\frac{1}{4})} \frac{(\alpha'(g(u)))^2 g'(u)}{(\alpha(g(u)))^4} du \right).$$

Since  $\alpha'(g(u)) = \frac{1}{g'(u)}$  and  $\alpha(g(u)) = u$ , for  $1 < u < \alpha(\frac{1}{4})$ , we derive

$$\int \int_{\Omega^*} (z'_x(x, y))^2 dx dy = \left( \int_{\frac{3}{4}}^1 y^3 dy \right) \left( \int_1^{\alpha(\frac{1}{4})} \frac{1}{u^4 g'(u)} du \right) \geq \frac{175}{1024(\alpha(\frac{1}{4}))^4} \int_1^{\alpha(\frac{1}{4})} \frac{1}{g'(u)} du.$$

In view of

$$\frac{1}{g'(u)} = \frac{\sqrt{(2u-1)(3-2u)}(\sqrt{(2u-1)(3-2u)} - 8u^2 + 12u - 3)}{2(u-1)(-16u^3 + 32u^2 - 17u + 3)} \underset{1}{\sim} \frac{1}{2(u-1)},$$

$$\text{since } \int_1^{\alpha(\frac{1}{4})} \frac{1}{2(u-1)} du = +\infty, \text{ we obtain } \int \int_{\Omega^*} (z'_x(x, y))^2 dx dy = +\infty.$$

Finally, the solution  $z$  of the example 7 does not belong to  $H^1(\Omega)$  and, therefore, the problem (1.1) is not well-posed. The reason why is probably that  $(\mathbf{u} \cdot \mathbf{n})(t)$ , which vanishes in  $t = \pi$ , is not equivalent to  $A(t - \pi)$ , with  $A \neq 0$ , in the neighborhood of  $\pi$ , that is to say it vanishes with an order greater than 1. On the contrary, the assumption  $(\mathbf{u} \cdot \boldsymbol{\tau}_-)(t)$  negative is verified for  $t = t_0$  and  $t = \pi$ .

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