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# On convergence rate estimates for approximations of solution operators of linear non-autonomous evolution equations

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We improve some recent convergence rate estimates for approximations of solution operators of linear non-autonomous evolution equations. The approximation results from the Trotter product formula which is proved to converge in the operator-norm and its convergence can be estimated. The result is applied to a diffusion perturbed by a time-dependent potential.

**Keywords:** Trotter product formula, evolution equations, operator-norm topology, approximation, convergence rate; operator splitting

**Classification:** Primary 34G10, 47D06, 34K30; Secondary 47A55..

## 1. Introduction

The theory of equations of evolution plays an important role in various areas of pure and applied mathematics, physics and other natural sciences, [4, 10, 12]. We focus on a non-autonomous linear Cauchy problem of the form

$$\dot{u}(t) = -(A + B(t))u(t), \quad u(s) = u_s \in X, \quad 0 < s \leq t \leq T, \quad (1.1)$$

where  $\{A + B(t), \text{dom}(A) \cap \text{dom}(B(t))\}_{t \in \mathcal{I}}$  is a family of closed linear operators on the separable Banach space  $X$ ,  $\mathcal{I} = [0, T] \subset \mathbb{R}$ . Let  $\mathcal{I}_0 = (0, T]$ . The solution operator  $\{U(t, s)\}_{0 \in \mathcal{I}}$ , i.e.  $u(t) = U(t, s)u_s$  solves (1.1) in some sense, can be obtained using the Howland-Evans approach. The main idea of this approach is to reformulate the non-autonomous problem (1.1) on  $X$  to an autonomous Cauchy problem on the Banach space  $L^p(\mathcal{I}, X)$  of  $p$ -summable functions on  $\mathcal{I}$  with values in  $X$ . The solution of the autonomous and the non-autonomous Cauchy Problem correspond to each other, and therefore it is equivalent to solve the one or the other equation. Once, the solution is obtained, the problem of a good approximation appears. The Trotter product formula [13] or [2, Theorem 3.5.8] provides an approximation in the strong-topology. In practice, a convergence in the operator-norm is more useful, especially, if the convergence can be estimated. Then, for example, in spite of the initial values, the smallness of the steps can be calculated such that the error rate of the approximation is lower than a given accuracy.

Going to analyze a linear non-autonomous Cauchy problem of the form (1.1) the aim is to find the so-called "solution operator" or propagator  $\{U(t, s)\}_{(t,s) \in \Delta}$ ,  $\Delta = \{(t, s) \in \mathcal{I}_0 \times \mathcal{I}_0 : 0 < s \leq t \leq T\}$ ,  $\mathcal{I}_0 = (0, T]$ , of the Cauchy problem (1.1), which has the property that  $u(t) = U(t, s)u_s$  for  $(t, s) \in \Delta$  is a "solution" of the initial Cauchy problem (1.1) for an appropriate set of initial data  $u_s$ . By definition, a propagator  $\{U(t, s)\}_{(t,s) \in \Delta}$  is a strongly continuous operator-valued function  $U(\cdot, \cdot) : \Delta \rightarrow \mathcal{B}(X)$  satisfying

$$U(t, t) = I \quad \text{for } t \in \mathcal{I}_0, \quad U(t, r)U(r, s) = U(t, s) \quad \text{for } t, r, s \in \mathcal{I}_0 \quad \text{with } s \leq r \leq t, \\ \|U\|_{\mathcal{B}(X)} := \sup_{(t,s) \in \Delta} \|U(t, s)\|_{\mathcal{B}(X)} < \infty.$$

Our goal is to find an approximation  $\{U_n(t, s)\}_{(t,s) \in \Delta}$ ,  $n \in \mathbb{N}$ , of the solution operator  $\{U(t, s)\}_{(t,s) \in \Delta}$  in the operator norm with an explicit convergence rate estimate. Such convergence rate estimates were already found by Ichinose and Tamura for positive self-adjoint operators [3]. Recently, in [6] an error estimate was proved, where the underlying space is a Banach space. In [6] the main technical tool to get such an approximation was the Trotter product formula. It was verified in [6] that under the assumptions made there the Trotter product formula converges not only in the strong topology but actually in the operator norm. In a second step this result was carried over to the solution operator.

Following the ideas of [6] we improve the convergence rate estimate of [6] under assumptions on the involved operators  $A$  and  $B(t)$  which are straightforward generalizations of those of [3] to the Banach space. Despite this straightforward generalization we were unable to reproduce the strong convergence rate of [3] for the Banach space. However, with respect to the Trotter product formula we get a slightly stronger convergence rate estimate than in [1].

## 2. Preliminaries and Assumptions

### 2.1. Preliminaries

Throughout the paper we are dealing with a separable Banach space  $(X, \|\cdot\|_X)$ . For an linear operator  $A : \text{dom}(A) \subset X \rightarrow X$ , we define the resolvent by  $R(\lambda, A) := (A - \lambda)^{-1} : X \rightarrow \text{dom}(A)$ . A family  $\{T(t)\}_{t \geq 0}$  of bounded linear operators on a Banach space  $X$  is called a strongly continuous (one-parameter) semigroup if it satisfies the functional equation

$$T(0) = I, \quad T(t + s) = T(t)T(s), \quad t, s \geq 0,$$

and the orbit maps  $[0, \infty) \ni t \mapsto T(t)x$  are continuous for every  $x \in X$ . In the following we simply call them semigroups. For a given semigroup we define its generator by

$$Ax := \lim_{h \searrow 0} \frac{1}{h} (x - T(h)x)$$

with domain

$$\text{dom}(A) := \{x \in X : \lim_{h \searrow 0} \frac{1}{h} (x - T(h)x) \text{ exists}\}.$$

Note that the definition differs from the standard one by the sign minus. It is well-known that the generator of a semigroup is a closed and densely defined linear operator which uniquely determines the semigroup (see e.g. [2, Theorem I.1.4]). For a given generator  $A$  we will write  $T(\tau) = e^{-\tau A}$ ,  $\tau \geq 0$ .

For any semigroup  $\{T(t)\}_{t \geq 0}$  there are constants  $M_A, \gamma_A$ , such that it holds  $\|T(t)\| \leq M_A e^{\gamma_A t}$  for all  $t \geq 0$ . Such semigroups are called of class  $\mathcal{G}(M_A, \gamma_A)$  and we write  $A \in$

$\mathcal{G}(M_A, \gamma_A)$ . If  $\gamma_A \leq 0$ ,  $\{T(t)\}_{t \geq 0}$  is called a bounded semigroup. If  $\|T(t)\| \leq 1$ , the semigroup is called contractive.

For any semigroup we can construct a bounded semigroup by adding some constant  $\nu \geq \gamma_A$  to its generator: The operator  $\tilde{A} := A + \nu$  generates a semigroup  $\{\tilde{T}(t)\}_{t \geq 0}$  with  $\|\tilde{T}(t)\| \leq M_A$ . It is known that for a generator  $A \in \mathcal{G}(M_A, \gamma_A)$ , the open half plane  $\{z \in \mathbb{C} : \operatorname{Re}(z) < \gamma_A\}$  is contained in the resolvent set  $\varrho(A)$  of  $A$  and the estimate  $\|R(\lambda, A)\| \leq \frac{M_A}{\operatorname{Re}(\lambda) - \gamma_A}$  holds. If  $\tilde{A} = A + \nu$ , then the open half-plane  $\{z \in \mathbb{C} : \operatorname{Re}(z) < \gamma_A - \nu\}$  is contained in the resolvent set of  $\tilde{A}$ .

The semigroup  $\{T(t)\}_{t \geq 0}$  on  $X$  is called a bounded holomorphic semigroup if its generator  $A$  satisfies  $\operatorname{ran}(T(t)) \subset \operatorname{dom}(A)$  for all  $t > 0$  and  $\sup_{t > 0} \|tAT(t)\| < \infty$ . It is well-known, that in this case the semigroup  $\{T(t)\}_{t \geq 0}$  can be extended holomorphically to a sector  $\{z \in \mathbb{C} : |\arg(z)| < \delta\} \cup \{0\} \subset \mathbb{C}$  of angle  $\delta > 0$ . For generators  $A$  of bounded holomorphic semigroups with  $0 \in \varrho(A)$  one can define fractional powers  $A^\alpha$ . Then, for  $\alpha \in (0, 1)$ , it holds  $\operatorname{dom}(A) \subset \operatorname{dom}(A^\alpha) \subset \operatorname{dom}(A^0) = X$ . In the following we need the well-known estimate for generators of a bounded holomorphic semigroup:

$$\sup_{t > 0} \|t^\alpha A^\alpha T(t)\| = M_\alpha^A < \infty. \quad (2.1)$$

## 2.2. Assumptions

Below we made the following assumptions with respect to the operator  $A$  and the family  $\{B(t)\}_{t \in \Delta}$ .

### Assumption 2.1.

(A1) The operator  $A$  is a generator of a bounded holomorphic semigroup of class  $\mathcal{G}(M_A, 0)$  and  $0 \in \varrho(A)$ . Let  $\{B(t)\}_{t \in \mathcal{I}}$  be a family of generators on  $X$  belonging to the same class  $\mathcal{G}(M_B, \beta)$ . The function  $\mathcal{I} \ni t \mapsto (B(t) + \xi)^{-1}x \in X$  is strongly measurable for any  $x \in X$  and any  $\xi > \beta$ .

(A2) There is an  $\alpha \in (\frac{1}{2}, 1)$  such that for a.e.  $t \in \mathcal{I}$  it holds that  $\operatorname{dom}(A^\alpha) \subset \operatorname{dom}(B(t))$  and  $\operatorname{dom}((A^\alpha)^*) \subset \operatorname{dom}(B(t)^*)$ . Moreover, it holds

$$C_\alpha := \operatorname{ess\,sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} < \infty \quad \text{and} \quad C_\alpha^* := \operatorname{ess\,sup}_{t \in \mathcal{I}} \|B(t)^*(A^{-\alpha})^*\|_{\mathcal{B}(X^*)} < \infty, \quad (2.2)$$

where  $A^*$  and  $B(t)^*$  denote the adjoint operators of  $A$  and  $B(t)$ , respectively.

(A3) There is a constant  $L > 0$  such that estimate

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\|_{\mathcal{B}(X)} \leq L|t - s|,$$

holds for a.e.  $t, s \in \mathcal{I}$ .

### Remark 2.2.

(a) In [6] the assumptions are slightly weaker. It is assumed that the domains satisfy  $\operatorname{dom}(A^*) \subset \operatorname{dom}(B(t)^*)$ .

(b) The assumption  $0 \in \varrho(A)$  is just for simplicity. Otherwise, the generator  $A$  can be shifted by a constant  $\eta > 0$ . One can prove that the domain of the fractional power of  $A$  does not change either.

(c) In [3] both operators  $A$  and  $B(t)$  are assumed to be positive selfadjoint operators on a separable Hilbert space. The assumptions made in [3] yield that Assumption 2.1 is valid.

(d) The assumptions above imply that for a.e.  $t \in \mathcal{I}$  the operator  $B(t)$  is infinitesimally small with respect to  $A$ . Indeed, fix  $t \in \mathcal{I}$  and assuming (A1), (A2) we conclude

$$\operatorname{dom}(A + \eta) = \operatorname{dom}(A) \subset \operatorname{dom}(A^\alpha) \subset \operatorname{dom}(B(t))$$

for  $\eta > 0$  and hence

$$\|B(t)(A + \eta)^{-1}\|_{\mathcal{B}(X)} \leq \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} \cdot \|A^\alpha(A + \eta)^{-1}\|_{\mathcal{B}(X)} \leq \frac{C_\alpha C_0}{\eta^{1-\alpha}}.$$

Therefore for any  $x \in \text{dom}(A) \subset \text{dom}(B(t))$ , we get

$$\|B(t)x\|_X \leq \frac{C_\alpha C_0}{\eta^{1-\alpha}} \cdot \|(A + \eta)x\|_X \leq C_\alpha C_0 \eta^\alpha \left( \frac{1}{\eta} \|Ax\|_X + \|x\|_X \right).$$

The relative bound can be chosen arbitrarily small by shifting  $\eta > 0$ . In particular, using standard perturbation results ([5, Corollary IX.2.5]), we conclude that  $A + B(t)$  is the generator of a holomorphic semigroup, i.e. problem (1.1) is a parabolic evolution equation.

### 3. Solving strategy

Let us describe briefly our solving strategy. Details can be found in [6]. The approach to finding the solution operator  $\{U(t, s)\}_{(t,s) \in \Delta}$  of (1.1) leads to a perturbation or extension problem for linear operators. It can be used in very general settings and is quite flexible. The usual idea can be described as follows: The non-autonomous Cauchy problem in  $X$  can be reformulated as an autonomous Cauchy problem in a new Banach space  $L^p(\mathcal{I}, X)$ ,  $p \in [1, \infty)$ , of  $p$ -summable functions on the interval  $\mathcal{I}$  with values in the Banach space  $X$ . An operator family  $\{C(t)\}_{t \in \mathcal{I}}$  on  $X$  induces an multiplication operator  $\mathcal{C}$  on  $L^p(\mathcal{I}, X)$  defined by

$$(\mathcal{C}f)(t) := C(t)f(t),$$

$$\text{dom}(\mathcal{C}) := \left\{ f \in L^p(\mathcal{I}, X) : \begin{array}{l} f(t) \in \text{dom}(C(t)) \text{ for a.e. } t \in \mathcal{I} \\ \mathcal{I} \ni t \mapsto C(t)f(t) \in L^p(\mathcal{I}, X) \end{array} \right\}.$$

**Theorem 3.1** ([6, Theorem 2.8]). *Let  $\{C(t)\}_{t \in \mathcal{I}}$  be a family of generators on  $X$  such that for almost all  $t \in \mathcal{I}$  it holds that  $C(t) \in \mathcal{G}(M, \beta)$  for some  $M \geq 1$  and  $\beta \in \mathbb{R}$ . If the function  $\mathcal{I} \ni t \mapsto (C(t) + \xi)^{-1}x \in X$  is strongly measurable for  $\xi > \beta$ ,  $x \in X$ , then the induced multiplication operator  $\mathcal{C}$  is a generator in  $L^p(\mathcal{I}, X)$  and its semigroup is given by*

$$(e^{-\tau \mathcal{C}} f)(t) = e^{-\tau C(t)} f(t), \quad f \in L^p(\mathcal{I}, X),$$

for a.e.  $t \in \mathcal{I}$ . In particular, for the operator-norms we get

$$\|e^{-\tau \mathcal{C}}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} = \text{ess sup}_{t \in \mathcal{I}} \|e^{-\tau C(t)}\|_{\mathcal{B}(X)}.$$

So the generators  $C(t)$  and  $\mathcal{C}$  belong to the same class.

In particular in our case, the operator family  $\{B(t)\}_{t \in \mathcal{I}}$  induces the generator  $\mathcal{B}$  and  $A$  induces trivially the generator  $\mathcal{A}$  on  $L^p(\mathcal{I}, X)$ . Assuming (A1) and (A2) it turns out that the operators  $\mathcal{B}\mathcal{A}^{-\alpha}$  and  $\overline{\mathcal{A}^{-\alpha}\mathcal{B}}$  are bounded on  $L^p(\mathcal{I}, X)$  and it holds that  $\|\mathcal{B}\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq C_\alpha$  and  $\|\overline{\mathcal{A}^{-\alpha}\mathcal{B}}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq C_\alpha^*$ .

Let us introduce the operator  $D_0 := \partial_t$  on  $L^p(\mathcal{I}, X)$  defined by

$$D_0 f(t) := \partial_t f(t), \quad \text{dom}(D_0) := \{f \in W^{1,p}([0, T], X) : f(0) = 0\}.$$

Then,  $D_0$  is a generator of class  $\mathcal{G}(1, 0)$  of the right-shift semigroup  $\{S(\tau)\}_{\tau \geq 0}$  that has the form

$$(e^{-\tau D_0} f)(t) = (S(\tau)f)(t) := f(t - \tau)\chi_{\mathcal{I}}(t - \tau), \quad f \in L^p(\mathcal{I}, X), \quad \text{a.e. } t \in \mathcal{I}.$$

We note that the generator  $D_0$  has empty spectrum since the semigroup  $\{S(\tau)\}_{\tau \geq 0}$  is nilpotent and therefore the integral  $\int_0^\infty e^{-\tau\lambda} S(\tau) f d\tau$  exists for any  $\lambda \in \mathbb{C}$  and for any  $f \in L^p(\mathcal{I}, X)$ .

Let us look at the operator sum  $D_0$  and  $\mathcal{A}$ . Since  $A$  is time-independent, the operators  $\mathcal{A}$  and  $D_0$  commute, and, hence, also their semigroups commute. So, the operator family  $\{e^{-\tau\mathcal{A}}e^{-\tau D_0}\}_{\tau \geq 0}$  defines a semigroup on  $L^p(\mathcal{I}, X)$ . Its generator is denoted by  $\mathcal{K}_0$ . It is closure of the operator sum  $D_0 + \mathcal{A}$ , i.e.  $\mathcal{K}_0 = \overline{D_0 + \mathcal{A}}$ . We note that all the generators  $\mathcal{K}_0$ ,  $\mathcal{A}$ ,  $A$  belong to the same class.

**Remark 3.2.** By assumption (A1) the operator  $\mathcal{A}$  generates a holomorphic semigroup. Note that the operator  $\mathcal{K}_0$  is not a generator of a holomorphic semigroup. Indeed, if we have

$$(e^{-\tau\mathcal{K}_0} f)(t) = (e^{-\tau D_0} e^{-\tau\mathcal{A}} f)(t) = e^{-\tau A} f(t - \tau) \chi_{\mathcal{I}}(t - \tau), \quad f \in L^p(\mathcal{I}, X).$$

Since the right-hand side is zero for  $\tau \geq t$ , the semigroup can not be extended to the complex plane.

Now, look at the operator sum

$$\tilde{\mathcal{K}} = D_0 + \mathcal{A} + \mathcal{B}, \quad \text{dom}(\tilde{\mathcal{K}}) = \text{dom}(D_0) \cap \text{dom}(\mathcal{A}) \cap \text{dom}(\mathcal{B}). \quad (3.1)$$

In [6], the following theorem is proved.

**Theorem 3.3** ([6, Theorems 4.3 and 4.4]). *Assume (A1) and (A2). Then, the operator closure  $\tilde{\mathcal{K}} =: \mathcal{K}$  is a generator on  $L^p(\mathcal{I}, X)$ , and it holds*

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{B}, \quad \text{dom}(\mathcal{K}) = \text{dom}(\mathcal{K}_0) \cap \text{dom}(\mathcal{B}). \quad (3.2)$$

Moreover, it is an evolution generator, i.e. its semigroup defines the unique solution operator  $U(t, s)$  of problem (1.1) by

$$(e^{-\tau\mathcal{K}} f)(t) = (\mathcal{U}(\tau) f)(t) = U(t, t - \tau) \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad \tau \geq 0, \quad t \in \mathcal{I}.$$

We note that for the proof it is not necessary that the operators  $B(t)$  are generators. After proving the existence of a unique solution, we want to approximate the solution operator  $U(t, s)$ . This will be done by proving an operator-norm convergence for the Trotter product formula for  $\mathcal{K} = \mathcal{K}_0 + \mathcal{B}$ .

## 4. Stability

Proving the Trotter product formula, it is important to establish stability conditions. Notice that stability is satisfied if the contractivity of the involved semigroups is assumed which might be too strong in applications. There are many stability conditions known for evolution equations. In particular, the Kato-stability is of interest, cf. [7, Definition 4.1], which is equivalent to a renormalizability conditions of the underlying Banach space, cf. [7]. We note that our following stability condition is weaker than Kato-stability.

**Definition 4.1.** Let  $A$  be a generator and let  $\{B(t)\}_{t \in \mathcal{I}}$  be a family of generators in  $X$ . The family  $\{B(t)\}_{t \in \mathcal{I}}$  is called  $A$ -stable if there is a constant  $M > 0$  such that

$$\text{ess sup}_{(t,s) \in \Delta} \left\| \prod_{j=1}^{n \leftarrow} G_j(t, s; n) \right\|_{\mathcal{B}(X)} \leq M$$

holds for any  $n \in \mathbb{N}$  where  $G_j(t, s; n) := e^{-\frac{t-s}{n} B(s+j\frac{t-s}{n})} e^{-\frac{t-s}{n} A}$ ,  $j = 0, 1, 2, \dots, n$ , and the product is ordered increasingly in  $j$  from the right to the left.

Let us introduce the notion

$$T(\tau) = e^{-\tau\mathcal{B}}e^{-\tau\mathcal{K}_0}, \quad \tau \geq 0.$$

**Lemma 4.2** ([6, Lemma 5.8]). *If the operator family  $\{B(t)\}_{t \in \mathcal{I}}$  is  $A$ -stable, then*

$$\|T\left(\frac{\tau}{n}\right)^m\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq M$$

for any  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $\tau \geq 0$ . In particular, we have

$$\|T(\tau)^m\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq M$$

for any  $m \in \mathbb{N}$  and  $\tau \geq 0$ .

## 5. Convergence in the operator-norm topology

Theorem 3.3 leads to the problem, how the semigroup of  $\mathcal{K}$  can be approximated in terms of the semigroups generated by  $D_0$ ,  $\mathcal{A}$  and  $\mathcal{B}$ . The classical Trotter product formula gives an approximation in the strong topology. In this section, we establish an approximation in the operator-norm topology on  $L^p(\mathcal{I}, X)$ . This is done in several steps. This approximation in  $L^p(\mathcal{I}, X)$  can be used to prove an convergence rate estimate in  $X$  for the propagators.

### 5.1. Technical Lemmata

In this section, we state and prove all technical lemmas that we used to prove the convergence and estimate of the Trotter product formula in the operator-norm in  $L^p(\mathcal{I}, X)$ . For simplicity in notation, we set  $T(\tau) := e^{-\tau\mathcal{B}}e^{-\tau\mathcal{K}_0}$ ,  $\tau \geq 0$ . Note that  $T(\tau) = 0$  for  $\tau \geq T$ . Similarly,  $e^{-\tau\mathcal{K}} = 0$  for  $\tau \geq T$ .

**Lemma 5.1.** *Let the assumptions (A1) and (A2) be satisfied.*

(i) *Then  $\text{dom}(\mathcal{K}_0) \subset \text{dom}(\mathcal{A}^\alpha)$  and there is a constant  $\Lambda_\alpha > 0$  such that*

$$\|\mathcal{A}^\alpha e^{-\tau\mathcal{K}}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq \frac{\Lambda_\alpha}{\tau^\alpha} \quad (5.1)$$

holds for  $\tau > 0$ .

(ii) *If  $\{B(t)\}_{t \in \mathcal{I}}$  is  $A$ -stable, then there is a constant  $\Pi_\alpha > 0$  such that the estimates*

$$\|(T(\tau) - e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq \Pi_\alpha \tau \quad (5.2)$$

$$\|\mathcal{A}^{-\alpha}(T(\tau) - e^{-\tau\mathcal{K}})\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq \Pi_\alpha \tau \quad (5.3)$$

are valid for  $\tau > 0$ .

(iii) *If  $\{B(t)\}_{t \in \mathcal{I}}$  is  $A$ -stable, then there is a constant  $Y_\alpha > 0$  such that the estimate*

$$\|\overline{T(\tau)^k \mathcal{A}^\alpha}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq Y_\alpha \left( \tau^{1-2\alpha} + \frac{1}{(k\tau)^\alpha} \right), \quad \tau > 0, \quad k \in \mathbb{N}. \quad (5.4)$$

holds for  $\tau > 0$ .

*Proof.* (i)-(ii) The assertions  $\text{dom}(\mathcal{K}_0) \subseteq \text{dom}(\mathcal{A}^\alpha)$  as well as (5.1) and (5.2) follow from Lemma 7.3, Lemma 7.4 and Lemma 7.6 of [6]. To prove (5.3) one has slightly to modify the second part of the proof of Lemma 7.6 of [6].

(iii) For  $k\tau \geq T$  we have  $T(\tau)^k = 0$ . Hence, one has to prove the estimate (5.4) only for  $k\tau \leq T$ . In fact, using Lemma 4.2, we get  $\|T(\tau)^k\| \leq M$ ,  $\tau \in [0, \infty)$ . Hence,

$$\begin{aligned} \|T(\tau)^k \mathcal{A}^\alpha f\| &\leq \|(T(\tau)^k - e^{-k\tau\mathcal{K}_0})\mathcal{A}^\alpha f\| + \|e^{-k\tau\mathcal{K}_0} \mathcal{A}^\alpha f\| \\ &\leq \left\| \sum_{j=0}^{k-1} T(\tau)^{k-1-j} (e^{-\tau\mathcal{B}} - I) e^{-(j+1)\tau\mathcal{K}_0} \mathcal{A}^\alpha f \right\| + \|e^{-k\tau\mathcal{K}_0} \mathcal{A}^\alpha f\| \\ &\leq M \sum_{j=0}^{k-1} \int_0^\tau d\sigma \|e^{-\sigma\mathcal{B}} \mathcal{B} \mathcal{A}^{-\alpha}\| \|\mathcal{A}^{2\alpha} e^{-(j+1)\tau\mathcal{K}_0} f\| + \|e^{-k\tau\mathcal{K}_0} \mathcal{A}^\alpha f\|, \end{aligned}$$

where we have used  $I - e^{-\tau\mathcal{B}} = \int_0^\tau \mathcal{B} e^{-\sigma\mathcal{B}} d\sigma$ . Moreover, from (2.1) we get

$$\|\mathcal{A}^{2\alpha} e^{-(j+1)\tau\mathcal{K}_0} f\| \leq \frac{M_{2\alpha}^A}{((j+1)\tau)^{2\alpha}} \|f\| \quad \text{and} \quad \|\mathcal{A}^\alpha e^{-k\tau\mathcal{K}_0} f\| \leq \frac{M_\alpha^A}{(k\tau)^\alpha} \|f\|$$

for  $\tau > 0$ . Hence, using  $\alpha > \frac{1}{2}$ , we get

$$\begin{aligned} \|T(\tau)^k \mathcal{A}^\alpha f\| &\leq \frac{MM_B^T M_{2\alpha}^A C_\alpha \tau}{\tau^{2\alpha}} \sum_{j=0}^{k-1} \frac{1}{(j+1)^{2\alpha}} \|f\| + \frac{M_\alpha^A}{(k\tau)^\alpha} \|f\| \\ &\leq \frac{MM_B^T M_{2\alpha}^A C_\alpha \zeta(2\alpha)}{\tau^{2\alpha-1}} \|f\| + \frac{M_\alpha^A}{(k\tau)^\alpha} \|f\| \end{aligned}$$

for  $\tau \in \mathcal{I}$ , where  $\zeta(\beta) := \sum_{j=1}^\infty \frac{1}{j^\beta}$ ,  $\beta > 1$ , is the Riemann  $\zeta$ -function and we have set  $M_B^T := \sup_{\tau \in \mathcal{I}} \|e^{-\tau\mathcal{B}}\|$ . Using that  $T(\tau)^k = 0$  for  $\tau k \geq T$  we find

$$\|T(\tau)^k \mathcal{A}^\alpha f\| \leq \frac{MM_B^T M_{2\alpha}^A C_\alpha \zeta(2\alpha)}{\tau^{2\alpha-1}} \|f\| + \frac{M_\alpha^A}{(k\tau)^\alpha} \|f\|, \quad f \in \text{dom}(\mathcal{A}),$$

for  $\tau > 0$ . Taking the supremum over the unit ball in  $\text{dom}(\mathcal{A})$  we prove (5.4).  $\square$

**Lemma 5.2.** *Let the assumptions (A1), (A2) and (A3) be satisfied. Then there is a constant  $Z_\alpha > 0$  such that*

$$\|\mathcal{A}^{-\alpha}(T(\tau) - e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq Z_\alpha \tau^{1+\alpha}, \quad \tau \geq 0. \quad (5.5)$$

*Proof.* Let  $f \in \text{dom}(\mathcal{K}_0) = \text{dom}(\mathcal{K})$ . We have

$$\begin{aligned} \frac{d}{d\sigma} T(\sigma) e^{-(\tau-\sigma)\mathcal{K}} f &= \frac{d}{d\sigma} e^{-\sigma\mathcal{B}} e^{-\sigma\mathcal{K}_0} e^{-(\tau-\sigma)\mathcal{K}} f \\ &= -e^{-\sigma\mathcal{B}} \mathcal{B} e^{-\sigma\mathcal{K}_0} e^{-(\tau-\sigma)\mathcal{K}} f - e^{-\sigma\mathcal{B}} e^{-\sigma\mathcal{K}_0} \mathcal{K}_0 e^{-(\tau-\sigma)\mathcal{K}} f + e^{-\sigma\mathcal{B}} e^{-\sigma\mathcal{K}_0} \mathcal{K} e^{-(\tau-\sigma)\mathcal{K}} f \\ &= -e^{-\sigma\mathcal{B}} \mathcal{B} e^{-\sigma\mathcal{K}_0} e^{-(\tau-\sigma)\mathcal{K}} f + e^{-\sigma\mathcal{B}} e^{-\sigma\mathcal{K}_0} \mathcal{B} e^{-(\tau-\sigma)\mathcal{K}} f \\ &= e^{-\sigma\mathcal{B}} \{e^{-\sigma\mathcal{K}_0} \mathcal{B} f - \mathcal{B} e^{-\sigma\mathcal{K}_0}\} e^{-(\tau-\sigma)\mathcal{K}} f, \end{aligned}$$

which yields

$$T(\tau)f - e^{-\tau\mathcal{K}} f = \int_0^\tau \frac{d}{d\sigma} T(\sigma) e^{-(\tau-\sigma)\mathcal{K}} f d\sigma = \int_0^\tau e^{-\sigma\mathcal{B}} \{e^{-\sigma\mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma\mathcal{K}_0}\} e^{-(\tau-\sigma)\mathcal{K}} f d\sigma. \quad (5.6)$$



Now, we have the following identity

$$\begin{aligned}
& e^{-\sigma\mathcal{B}} (e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}) e^{-(\tau-\sigma)\mathcal{K}} f \\
&= (e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})f + \\
&+ (e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}_0}f + \\
&+ \{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})f + \{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}_0}f.
\end{aligned}$$

which yields for  $f = \mathcal{A}^{-\alpha}g$

$$\begin{aligned}
& \mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}} (e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}) e^{-(\tau-\sigma)\mathcal{K}} \mathcal{A}^{-\alpha}g = \\
&= \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g + \\
&+ \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}\mathcal{A}^{-\alpha}e^{-(\tau-\sigma)\mathcal{K}_0}g + \\
&+ \mathcal{A}^{-\alpha}\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g + \\
&+ \mathcal{A}^{-\alpha}\{(e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{D}_0})\mathcal{B} - \mathcal{B}(e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{D}_0})\}e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha}g + \\
&+ \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{D}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{D}_0})\mathcal{A}^{-\alpha}e^{-(\tau-\sigma)\mathcal{K}_0}g.
\end{aligned} \tag{5.7}$$

In the following, we estimate the five terms separately.

At first we use the fact that  $\mathcal{A}$  and  $\mathcal{K}_0$  commute and conclude that

$$(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g = \int_0^{\tau-\sigma} e^{-(\tau-\sigma-r)\mathcal{K}} \mathcal{B}\mathcal{A}^{-\alpha}e^{-r\mathcal{K}_0}gdr.$$

Thus, for the first term we get

$$\begin{aligned}
& \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g \\
&= - \int_0^{\sigma} \mathcal{A}^{-\alpha}\mathcal{B}e^{-r\mathcal{B}}dr [e^{-\sigma\mathcal{K}_0}, \mathcal{B}]\mathcal{A}^{-\alpha} \int_0^{\tau-\sigma} \mathcal{A}^{\alpha}e^{-(\tau-\sigma-r)\mathcal{K}} \mathcal{B}\mathcal{A}^{-\alpha}e^{-r\mathcal{K}_0}gdr.
\end{aligned}$$

where

$$[e^{-\sigma\mathcal{K}_0}, \mathcal{B}]f := \{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}f, \quad f \in \text{dom}(\mathcal{K}_0), \quad \tau \geq 0.$$

Using Lemma 5.1, we obtain the estimate

$$\begin{aligned}
& \|\mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g\| \\
&\leq \sigma \, 2C_{\alpha}^*C_{\alpha}^2\Lambda_{\alpha}M_{\mathcal{B}}^TM_{\mathcal{A}}^2 \int_0^{\tau-\sigma} \frac{1}{(\tau-\sigma-r)^{\alpha}}dr \|g\| \\
&\leq \sigma(\tau-\sigma)^{1-\alpha} \frac{2C_{\alpha}^*C_{\alpha}^2\Lambda_{\alpha}M_{\mathcal{B}}^TM_{\mathcal{A}}^2}{1-\alpha} \|g\|
\end{aligned} \tag{5.8}$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . For the second term, we get the estimate

$$\|\mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}\mathcal{A}^{-\alpha}e^{-(\tau-\sigma)\mathcal{K}_0}g\| \leq \sigma \, 2C_{\alpha}^*C_{\alpha}M_{\mathcal{B}}^TM_{\mathcal{A}}^2\|g\|. \tag{5.9}$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . Since we have

$$e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}h = \int_0^{\tau-\sigma} e^{-(\tau-r-\sigma)\mathcal{K}} \mathcal{B}e^{-r\mathcal{K}_0}hdr, \quad h \in \text{dom}(\mathcal{K}_0),$$

one obtains for the third term the estimate

$$\|\mathcal{A}^{-\alpha}\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha}g\| \leq (\tau-\sigma) \, 2C_{\alpha}^*C_{\alpha}M_{\mathcal{A}}^2M_{\mathcal{K}}\|g\| \tag{5.10}$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . Moreover, using

$$e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0} h = - \int_0^\sigma e^{-r \mathcal{K}_0} \mathcal{A} e^{-(\sigma-r) D_0} h dr,$$

we get for the fourth term

$$\begin{aligned} & \mathcal{A}^{-\alpha} \{ (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \mathcal{B} - \mathcal{B} (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \} e^{-(\tau-\sigma) \mathcal{K}_0} \mathcal{A}^{-\alpha} g = \\ & \left( - \int_0^\sigma \mathcal{A}^{1-\alpha} e^{-r \mathcal{K}_0} e^{-(\sigma-r) D_0} dr \mathcal{B} \mathcal{A}^{-\alpha} + \mathcal{A}^{-\alpha} \mathcal{B} \int_0^\sigma e^{-r \mathcal{K}_0} \mathcal{A}^{1-\alpha} e^{-(\sigma-r) D_0} dr \right) e^{-(\tau-\sigma) \mathcal{K}_0} g, \end{aligned}$$

which yields the estimate

$$\begin{aligned} & \| \mathcal{A}^{-\alpha} \{ (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \mathcal{B} - \mathcal{B} (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \} e^{-(\tau-\sigma) \mathcal{K}_0} \mathcal{A}^{-\alpha} g \| \\ & \leq C_\alpha M_{\mathcal{A}} M_{1-\alpha}^A \int_0^\sigma \frac{1}{r^{1-\alpha}} dr \|g\| + C_\alpha^* M_{\mathcal{A}} M_{1-\alpha}^A \int_0^\sigma \frac{1}{r^{1-\alpha}} dr \|g\| \\ & = \frac{(C_\alpha + C_\alpha^*) M_{\mathcal{A}} M_{1-\alpha}^A}{\alpha} \sigma^\alpha \|g\| \end{aligned} \quad (5.11)$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . To estimate the fifth term, we note that

$$\begin{aligned} (e^{-\sigma D_0} \mathcal{B} - \mathcal{B} e^{-\sigma D_0}) f &= e^{-\sigma D_0} B(\cdot) f(\cdot) - \mathcal{B} \chi_{\mathcal{I}}(\cdot - \sigma) f(\cdot - \sigma) = \\ &= \chi_{\mathcal{I}}(\cdot - \sigma) B(\cdot - \sigma) f(\cdot - \sigma) - B(\cdot) \chi_{\mathcal{I}}(\cdot - \sigma) f(\cdot - \sigma) = \\ &= \chi_{\mathcal{I}}(\cdot - \sigma) \{ B(\cdot - \sigma) - B(\cdot) \} f(\cdot - \sigma), \end{aligned}$$

and therefore

$$\begin{aligned} & \| \mathcal{A}^{-\alpha} (e^{-\sigma D_0} \mathcal{B} - \mathcal{B} e^{-\sigma D_0}) e^{-(\tau-\sigma) \mathcal{K}_0} \mathcal{A}^{-\alpha} g \| \\ & \leq M_{\mathcal{A}} \| \mathcal{A}^{-\alpha} \{ e^{-\sigma D_0} \mathcal{B} - \mathcal{B} e^{-\sigma D_0} \} \mathcal{A}^{-\alpha} g \| \\ & \leq \operatorname{ess\,sup}_{t \in \mathcal{I}} \| \mathcal{A}^{-\alpha} \{ B(t - \sigma) - B(t) \} \mathcal{A}^{-\alpha} \|_{\mathcal{B}(X)} \|g\| \leq L \sigma \|g\|. \end{aligned} \quad (5.12)$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . From (5.7) we find the estimate

$$\begin{aligned} & \| \mathcal{A}^{-\alpha} e^{-\sigma \mathcal{B}} (e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0}) e^{-(\tau-\sigma) \mathcal{K}} \mathcal{A}^{-\alpha} g \| \\ & \leq \| \mathcal{A}^{-\alpha} (e^{-\sigma \mathcal{B}} - I) \{ e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0} \} (e^{-(\tau-\sigma) \mathcal{K}} - e^{-(\tau-\sigma) \mathcal{K}_0}) \mathcal{A}^{-\alpha} g \| \\ & \quad + \| \mathcal{A}^{-\alpha} (e^{-\sigma \mathcal{B}} - I) \{ e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0} \} \mathcal{A}^{-\alpha} e^{-(\tau-\sigma) \mathcal{K}_0} g \| \\ & \quad + \| \mathcal{A}^{-\alpha} \{ e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0} \} (e^{-(\tau-\sigma) \mathcal{K}} - e^{-(\tau-\sigma) \mathcal{K}_0}) \mathcal{A}^{-\alpha} g \| \\ & \quad + \| \mathcal{A}^{-\alpha} \{ (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \mathcal{B} - \mathcal{B} (e^{-\sigma \mathcal{K}_0} - e^{-\sigma D_0}) \} e^{-(\tau-\sigma) \mathcal{K}_0} \mathcal{A}^{-\alpha} g \| \\ & \quad + \| \mathcal{A}^{-\alpha} (e^{-\sigma D_0} \mathcal{B} - \mathcal{B} e^{-\sigma D_0}) \mathcal{A}^{-\alpha} e^{-(\tau-\sigma) \mathcal{K}_0} g \|. \end{aligned}$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . Taking into account (5.8), (5.9), (5.10), (5.11) and (5.12) we find

$$\begin{aligned} & \| \mathcal{A}^{-\alpha} e^{-\sigma \mathcal{B}} (e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0}) e^{-(\tau-\sigma) \mathcal{K}} \mathcal{A}^{-\alpha} g \| \\ & \leq \left\{ \sigma (\tau - \sigma)^{1-\alpha} \frac{2C_\alpha^* C_\alpha^2 \Lambda_\alpha M_{\mathcal{B}}^T M_{\mathcal{A}}^2}{1-\alpha} + \sigma 2C_\alpha^* C_\alpha M_{\mathcal{B}}^T M_{\mathcal{A}}^2 + \right. \\ & \quad \left. (\tau - \sigma) 2C_\alpha^* C_\alpha M_{\mathcal{A}}^2 M_{\mathcal{K}} + \sigma^\alpha \frac{(C_\alpha + C_\alpha^*) M_{\mathcal{A}} M_{1-\alpha}^A}{\alpha} + \sigma L \right\} \|g\| \end{aligned}$$

for  $\sigma \in [0, \tau]$  and  $\tau \geq 0$ . Setting

$$\begin{aligned} Z_1 &:= \frac{2C_\alpha^* C_\alpha^2 \Lambda_\alpha M_{\mathcal{B}}^T M_{\mathcal{A}}^2}{1 - \alpha}, & Z_2 &:= 2C_\alpha^* C_\alpha M_{\mathcal{B}}^T M_{\mathcal{A}}^2 + L \\ Z_3 &:= 2C_\alpha^* C_\alpha M_{\mathcal{A}}^2 M_{\mathcal{K}}, & Z_4 &:= \frac{(C_\alpha + C_\alpha^*) M_{\mathcal{A}} M_{1-\alpha}^A}{\alpha} \end{aligned}$$

we obtain

$$\begin{aligned} & \|\mathcal{A}^{-\alpha} e^{-\sigma \mathcal{B}} (e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0}) e^{-(\tau-\sigma) \mathcal{K}} \mathcal{A}^{-\alpha} g\| \\ & \leq \left\{ Z_1 \sigma (\tau - \sigma)^{1-\alpha} + Z_2 \sigma + Z_3 (\tau - \sigma) + Z_4 \sigma^\alpha \right\} \|g\| \end{aligned} \quad (5.13)$$

From (5.6) we derive the representation

$$\begin{aligned} & \mathcal{A}^{-\alpha} (T(\tau) - e^{-\tau \mathcal{K}}) \mathcal{A}^{-\alpha} g \\ & = \int_0^\tau \mathcal{A}^{-\alpha} e^{-\sigma \mathcal{B}} \{e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0}\} e^{-(\tau-\sigma) \mathcal{K}} \mathcal{A}^{-\alpha} g \, d\sigma. \end{aligned}$$

which yields the estimate

$$\begin{aligned} & \|\mathcal{A}^{-\alpha} (T(\tau) - e^{-\tau \mathcal{K}}) \mathcal{A}^{-\alpha} g\| \\ & \leq \int_0^\tau \|\mathcal{A}^{-\alpha} e^{-\sigma \mathcal{B}} \{e^{-\sigma \mathcal{K}_0} \mathcal{B} - \mathcal{B} e^{-\sigma \mathcal{K}_0}\} e^{-(\tau-\sigma) \mathcal{K}} \mathcal{A}^{-\alpha} g\| \, d\sigma. \end{aligned}$$

Inserting (5.13) into this estimate and using

$$\int_0^\tau \sigma (\tau - \sigma)^{1-\alpha} d\sigma = \tau^{3-\alpha} \int_0^1 x (1-x)^{1-\alpha} dx = \tau^{3-\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)},$$

we find the estimate

$$\|\mathcal{A}^{-\alpha} (T(\tau) - e^{-\tau \mathcal{K}}) \mathcal{A}^{-\alpha} g\| \leq Z_1 \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \tau^{3-\alpha} + \frac{Z_2 + Z_3}{2} \tau^2 + \frac{Z_4}{1+\alpha} \tau^{1+\alpha}$$

for  $\tau \geq 0$ . We have

$$\|\mathcal{A}^{-\alpha} (T(\tau) - e^{-\tau \mathcal{K}}) \mathcal{A}^{-\alpha} g\| \leq \left( Z_1 \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \tau^{2-2\alpha} + \frac{Z_2 + Z_3}{2} \tau^{1-\alpha} + Z_4 \right) \tau^{1+\alpha}$$

for  $\tau \geq 0$ . Since  $T(\tau) = 0$  and  $e^{-\tau \mathcal{K}} = 0$  for  $\tau \geq T$  we finally obtain

$$\|\mathcal{A}^{-\alpha} (T(\tau) - e^{-\tau \mathcal{K}}) \mathcal{A}^{-\alpha} g\| \leq \left( Z_1 \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} T^{2-2\alpha} + \frac{Z_2 + Z_3}{2} T^{1-\alpha} + Z_4 \right) \tau^{1+\alpha}$$

which proves the lemma.  $\square$

**Lemma 5.3.** *Let  $\alpha \in [0, 1)$ . Then the estimates*

$$\sum_{m=1}^{n-1} \frac{1}{m^\alpha} \leq \frac{n^{1-\alpha}}{1-\alpha} \quad \text{and} \quad \sum_{m=1}^{n-1} \frac{1}{(n-m)^\alpha m^\alpha} \leq \frac{2}{1-\alpha} n^{1-2\alpha} \quad (5.14)$$

are valid for  $n = 2, 3, \dots$ .

*Proof.* The function  $f(x) = x^{-\alpha}$ ,  $x > 0$ , is decreasing. Hence

$$\sum_{m=1}^{n-1} \frac{1}{m^\alpha} \leq \int_0^{n-1} \frac{1}{x^\alpha} dx \leq \frac{(n-1)^{1-\alpha}}{1-\alpha} \leq \frac{n^{1-\alpha}}{1-\alpha}$$

for  $n = 2, 3, \dots$ . Further, we have

$$\sum_{m=1}^{n-1} \frac{1}{(n-m)^\alpha m^\alpha} \leq 2 \frac{1}{n^\alpha} \sum_{m=1}^{n-1} \frac{1}{m^\alpha} \leq 2 \frac{1}{n^\alpha} \frac{n^{1-\alpha}}{1-\alpha} = \frac{2}{1-\alpha} n^{1-2\alpha},$$

and the claim follows.  $\square$

## 5.2. The Trotter product formula in operator-norm topology

Now, we are able to prove and estimate operator-norm convergence of the Trotter product formula.

**Theorem 5.4.** *Let the assumptions (A1), (A2) and (A3) be satisfied. If the family of generators  $\{B(t)\}_{t \in \mathcal{I}}$  is  $A$ -stable, then there is a constant  $C_{\alpha, \mathcal{I}} > 0$  (depending on  $\alpha \in (\frac{1}{2}, 1)$  and on the compact interval  $\mathcal{I}$ ) such that*

$$\|(e^{-\tau B/n} e^{-\tau K_0/n})^n - e^{-\tau K}\|_{\mathcal{B}(L^p(\mathcal{I}, X))} \leq C_{\alpha, \mathcal{I}} \frac{1}{n^{1-\alpha}} \quad (5.15)$$

for  $\tau \geq 0$  and  $n = 2, 3, \dots$

*Proof.* Let  $T(\sigma) := e^{-\sigma B} e^{-\sigma K_0}$  and  $U(\sigma) := e^{-\sigma K}$ ,  $\sigma \geq 0$ . Then the following identity holds

$$\begin{aligned} T(\sigma)^n - U(\sigma)^n &= \sum_{m=0}^{n-1} T(\sigma)^{n-m-1} (T(\sigma) - U(\sigma)) U(\sigma)^m \\ &= T(\sigma)^{n-1} (T(\sigma) - U(\sigma)) + (T(\sigma) - U(\sigma)) U(\sigma)^{n-1} + \\ &\quad + \sum_{m=1}^{n-2} T(\sigma)^{n-m-1} (T(\sigma) - U(\sigma)) U(\sigma)^m \\ &= T(\sigma)^{n-1} \mathcal{A}^\alpha \mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma)) + (T(\sigma) - U(\sigma)) \mathcal{A}^{-\alpha} \mathcal{A}^\alpha U(\sigma)^{n-1} + \\ &\quad + \sum_{m=1}^{n-2} T(\sigma)^{n-m-1} \mathcal{A}^\alpha \mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma)) \mathcal{A}^{-\alpha} \mathcal{A}^\alpha U(\sigma)^m. \end{aligned}$$

which yields the estimate

$$\begin{aligned} \|T(\sigma)^n - U(\sigma)^n\| &\leq \|\overline{T(\sigma)^{n-1} \mathcal{A}^\alpha}\| \|\mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma))\| + \|(T(\sigma) - U(\sigma)) \mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha U(\sigma)^{n-1}\| + \\ &\quad + \sum_{m=1}^{n-2} \|\overline{T(\sigma)^{n-m-1} \mathcal{A}^\alpha}\| \|\mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma)) \mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha U(\sigma)^m\|. \end{aligned}$$

From Lemma 5.1 we get the estimates

$$\|\overline{T(\sigma)^{n-1} \mathcal{A}^\alpha}\| \leq Y_\alpha \left( \sigma^{1-2\alpha} + \frac{1}{((n-1)\sigma)^\alpha} \right), \quad n \geq 2,$$

as well as

$$\|\mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma))\| \leq \Pi_\alpha \sigma \quad \text{and} \quad \|(T(\sigma) - U(\sigma)) \mathcal{A}^{-\alpha}\| \leq \Pi_\alpha \sigma$$

for  $\sigma \in (0, \tau]$ . Hence

$$\|\overline{T(\sigma)^{n-1} \mathcal{A}^\alpha}\| \|\mathcal{A}^{-\alpha} (T(\sigma) - U(\sigma))\| \leq \Pi_\alpha Y_\alpha \sigma^{1-\alpha} \left( \sigma^{1-\alpha} + \frac{1}{(n-1)^\alpha} \right)$$

and

$$\|(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha U(\sigma)^{n-1}\| \leq \frac{\Pi_\alpha \Lambda_\alpha}{(n-1)^\alpha} \sigma^{1-\alpha}$$

where we have used (5.1). Since

$$\|\mathcal{A}^{-\alpha}(T(\sigma) - e^{-\sigma\mathcal{K}})\mathcal{A}^{-\alpha}\|_{B(L^p(\mathcal{I}, X))} \leq Z_\alpha \sigma^{1+\alpha}, \quad \tau \in [0, \tau_0),$$

by Lemma 5.2 we have

$$\begin{aligned} & \|\overline{T(\sigma)^{n-m-1}\mathcal{A}^\alpha}\| \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha U(\sigma)^m\| \\ & \leq Y_\alpha \left( \sigma^{1-2\alpha} + \frac{1}{((n-m-1)\sigma)^\alpha} \right) Z_\alpha \sigma^{1+\alpha} \Lambda_\alpha \frac{1}{(\sigma m)^\alpha} \\ & \leq Y_\alpha Z_\alpha \Lambda_\alpha \left( \sigma^{2-2\alpha} \frac{1}{m^\alpha} + \sigma^{1-\alpha} \frac{1}{(n-m-1)^\alpha m^\alpha} \right) \end{aligned}$$

Using Lemma 5.3 we get

$$\begin{aligned} & \sum_{m=1}^{n-2} \|\overline{T(\sigma)^{n-m-1}\mathcal{A}^\alpha}\| \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha U(\sigma)^m\| \\ & \leq Z_\alpha \Lambda_\alpha Y_\alpha \sigma^{2-2\alpha} \sum_{m=1}^{n-2} \frac{1}{m^\alpha} + Z_\alpha \Lambda_\alpha Y_\alpha \sigma^{1-\alpha} \sum_{m=1}^{n-2} \frac{1}{(n-m-1)^\alpha m^\alpha} \\ & \leq \frac{Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} (n^{1-\alpha} \sigma^{2-2\alpha} + 2n^{1-2\alpha} \sigma^{1-\alpha}). \end{aligned}$$

Summing up we get the estimate

$$\begin{aligned} \|T(\sigma)^n - U(\sigma)^n\| & \leq \Pi_\alpha Y_\alpha \sigma^{1-\alpha} \left( \sigma^{1-\alpha} + \frac{1}{(n-1)^\alpha} \right) + \frac{\Pi_\alpha \Lambda_\alpha}{(n-1)^\alpha} \sigma^{1-\alpha} + \\ & \quad \frac{Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} n^{1-\alpha} \sigma^{2-2\alpha} + \frac{2Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} n^{1-2\alpha} \sigma^{1-\alpha}. \end{aligned}$$

Setting  $\sigma := \tau/n$ , we obtain

$$\begin{aligned} & \|T(\tau/n)^n - U(\tau/n)^n\| \\ & \leq \frac{\Pi_\alpha \Lambda_\alpha T^{2-2\alpha}}{(n-1)^{2-2\alpha}} + \frac{\Pi_\alpha \Lambda_\alpha}{n-1} + \frac{\Pi_\alpha \Lambda_\alpha T^{1-\alpha}}{(n-1)} + \frac{Z_\alpha \Lambda_\alpha Y_\alpha T^{2-2\alpha}}{1-\alpha} \frac{1}{n^{1-\alpha}} + \frac{2Z_\alpha \Lambda_\alpha Y_\alpha T^{1-\alpha}}{1-\alpha} \frac{1}{n^\alpha}. \end{aligned}$$

for  $\tau \geq 0$  and  $n = 2, 3, \dots$ . Hence there is a constant  $C_{\alpha, \mathcal{I}} > 0$  such that (5.15) holds.  $\square$

**Remark 5.5.** It is worth saying that the result only depends on the domains of the operators  $A$  and  $B(t)$  and not on the concrete form of them.

### 5.3. Norm convergence for propagators

Let us investigate the consequences of Theorem 5.4 for the approximation of the solution operator  $\{U(t, s)\}_{(t,s) \in \Delta}$ . We have

$$(\{(e^{-\frac{\tau}{n}\mathcal{B}} e^{-\frac{\tau}{n}\mathcal{K}_0})^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\}g)(t) = \{U_n(t, t-\tau) - U(t, t-\tau)\} \chi_{\mathcal{I}}(t-\tau) g(s-\tau).$$

for  $(t, t-\tau) \in \Delta$  and  $g \in L^p(\mathcal{I}, X)$  where

$$U_n(t, s) := \prod_{j=1}^n e^{-\frac{t-s}{n} B(s+(n-j+1)\frac{t-s}{n})} e^{-\frac{t-s}{n} A}, \quad (t, s) \in \Delta.$$

Let us introduce the left-shift semigroup on  $L^p(\mathcal{I}, X)$

$$(L(\tau)f)(t) := \chi_{\mathcal{I}}(t + \tau)f(t + \tau), \quad f \in L^p(\mathcal{I}, X).$$

**Theorem 5.6.** *Let the assumptions (A1), (A2) and (A3) be satisfied. If the family of generators  $\{B(t)\}_{t \in \mathcal{I}}$  is  $A$ -stable, then there is a constant  $C_{\alpha, \mathcal{I}} > 0$*

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \|U_n(t, s) - U(t, s)\|_{\mathcal{B}(X)} \leq \frac{C_{\alpha, \mathcal{I}}}{n^{1-\alpha}}, \quad n = 2, 3, \dots, \quad (5.16)$$

where the constant  $C_{\alpha, \mathcal{I}}$  coincides with that one of Theorem 5.4.

*Proof.* We set

$$S_n(t, s) := U_n(t, s) - U(t, s), \quad (t, s) \in \Delta, \quad n \in \mathbb{N},$$

and

$$\mathcal{S}_n(\tau) := L(\tau)\{(e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0})^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\} : L^p(\mathcal{I}, X) \rightarrow L^p(\mathcal{I}, X).$$

for  $\tau \geq 0$  and  $n = 2, 3, \dots$ . We have

$$(\mathcal{S}_n(\tau)g)(t) = S_n(t + \tau, t)\chi_{\mathcal{I}}(t + \tau)g(t), \quad t \in \mathcal{I}_0, \quad g \in L^p(\mathcal{I}, X).$$

Hence, for any  $\tau \in \mathcal{I}$  and  $n \in \mathbb{N}$ , the operator  $\mathcal{S}_n(\tau)$  is a multiplication operator on  $L^p(\mathcal{I}, X)$  induced by the family  $\{S_n(\cdot + \tau, \cdot)\chi_{\mathcal{I}}(\cdot + \tau)\}_{\tau \in \mathcal{I}}$  of bounded operators. Applying [6, Lemma ?], we conclude for  $\tau \geq 0$

$$\begin{aligned} & \| (e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0})^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)} \|_{\mathcal{B}(L^p(\mathcal{I}, X))} \geq \| V(\tau) \{ (e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0})^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)} \} \|_{\mathcal{B}(L^p(\mathcal{I}, X))} \\ & = \| \mathcal{S}_n(\tau) \|_{\mathcal{B}(L^p(\mathcal{I}, X))} = \operatorname{ess\,sup}_{t \in \mathcal{I}_0} \| S_n(t + \tau, t)\chi_{\mathcal{I}}(t + \tau) \|_{\mathcal{B}(X)} = \\ & = \operatorname{ess\,sup}_{t \in \mathcal{I}_0} \| \{ U_n(t + \tau, s) - U(t + \tau, s) \} \chi_{\mathcal{I}}(t + \tau) \|_{\mathcal{B}(X)} = \\ & = \operatorname{ess\,sup}_{t \in (0, T-\tau]} \| U_n(t + \tau, s) - U(t + \tau, s) \|_{\mathcal{B}(X)}. \end{aligned}$$

Taking into account Theorem 5.4 we find

$$\operatorname{ess\,sup}_{t \in (0, T-\tau]} \| U_n(t + \tau, t) - U(t + \tau, t) \|_{\mathcal{B}(X)} \leq \frac{C_{\alpha, \mathcal{I}}}{n^{1-\alpha}}, \quad \tau \geq 0, \quad n = 2, 3, \dots,$$

which yields (5.16). □

**Remark 5.7.**

(i) Ichinose and Tamura proved in [3] a convergence rate of  $O(\frac{\ln(n)}{n})$  assuming that the operators  $A$  and  $B(t)$  are positive and self-adjoint. In [6], the authors proved the convergence rate of  $O(n^{-(\beta-\alpha)})$  for any  $\beta \in (\alpha, 1)$  assuming  $\operatorname{dom}(A^*) \subset \operatorname{dom}(B(t)^*)$ .

(ii) Moreover, we note that indeed, the estimates (5.15) and (5.16) are equivalent.

(iii) We note that a priori the operator family  $\{U_n(t, s)\}_{(t,s) \in \Delta}$  do not define a propagator since the co-cycle equation is in general not satisfied. But one can check that

$$U_n(t, s) = U_{n-k} \left( t, s + \frac{k}{n}(t - s) \right) U_k \left( s + \frac{k}{n}(t - s), s \right),$$

is satisfied for  $0 < s \leq t \leq T$ ,  $n \in \mathbb{N}$  and any  $k \in \{0, 1, \dots, n\}$ .

## 6. Example: Diffusion equation perturbed by a time-dependent potential

We investigate the diffusion equation perturbed by a time-dependent potential. On the Banach space  $X = L^q(\Omega)$ , where  $\Omega \subset \mathbb{R}^d$  is a bounded domain with  $C^2$ -boundary ( $d \geq 2$ ) and  $q \in (1, \infty)$ , the equation reads

$$\dot{u}(t) = \Delta u(t) - B(t)u(t), \quad u(s) = u_s \in L^q(\Omega), \quad t, s \in \mathcal{I}_0. \quad (6.1)$$

$\Delta$  denotes the Laplace operator on  $L^q(\Omega)$  with Dirichlet boundary conditions defined on

$$\Delta : \text{dom}(\Delta) = H_q^2(\Omega) \cap \dot{H}_q^1(\Omega) \rightarrow L^q(\Omega).$$

It turns out that  $-\Delta$  is the generator of a holomorphic contraction semigroup on  $L^q(\Omega)$  (cf. [9, Theorem 7.3.5/6]).  $B(t)$  denotes a time-dependent scalar-valued multiplication operator given by

$$(B(t)f)(x) = V(t, x)f(x), \quad \text{dom}(B(t)) = \{f \in L^q(\Omega) : V(t, x)f(x) \in L^q(\Omega)\},$$

where

$$V : \mathcal{I} \times \Omega \rightarrow \mathbb{C}, \quad V(t, \cdot) \in L^\varrho(\Omega).$$

For  $\alpha \in (0, 1)$ , the fractional power of  $-\Delta$  are defined on the domain

$$(-\Delta)^\alpha : \dot{H}_q^{2\alpha}(\Omega) \rightarrow L^q(\Omega).$$

Note, that for  $2\alpha < \frac{1}{q}$ , it holds that  $\dot{H}_q^{2\alpha}(\Omega) = H_q^{2\alpha}(\Omega)$ . The adjoint operator of  $\Delta$  is denoted by  $\Delta^*$  and it is defined on the domain  $\text{dom}(\Delta^*) = H_{q'}^2(\Omega) \cap \dot{H}_{q'}^1(\Omega) \subset L^{q'}(\Omega)$ , where  $\frac{1}{q} + \frac{1}{q'} = 1$ . It is well-known that also the operator  $(-\Delta^*)$  generates a bounded holomorphic semigroup. The operators  $B(t)$  are scalar-valued and hence  $B(t)^* = \overline{B(t)} : \text{dom}(B(t)) \subset L^{q'}(\Omega) \rightarrow L^{q'}(\Omega)$ . Moreover, one can show that  $\mathcal{K}_0 = D_0 + \mathcal{A}$ , i.e. the operator sum  $D_0 + \mathcal{A}$  is already closed.

Now, we are going to verify the assumptions (A1) - (A3) in order to approximate the solution of (6.1). This means, we determine the required regularity of  $V(t, \cdot) \in L^\varrho(\Omega)$  to ensure  $\text{dom}((-\Delta)^\alpha) \subset \text{dom}(B(t))$  and  $\text{dom}((-\Delta^*)^\alpha) \subset \text{dom}(B(t)^*)$ , i.e.

$$H_q^{2\alpha}(\Omega), \quad H_{q'}^{2\alpha}(\Omega) \subset \text{dom}(B(t)). \quad (6.2)$$

Using Sobolev embeddings, one obtains the general embedding

$$H_{\gamma_1}^s(\Omega) \subset L^{\gamma_2}(\Omega) \quad \text{for} \quad \begin{cases} \gamma_2 \in [\gamma_1, \frac{d}{s-\gamma_1}], & \text{if } \gamma_1 \in (1, \frac{d}{s}) \\ \gamma_2 \in [\gamma_1, \infty), & \text{if } \gamma_1 \in [\frac{d}{s}, \infty) \end{cases}. \quad (6.3)$$

For our case (6.2), we obtain  $H_q^{2\alpha}(\Omega) \subset L^r(\Omega)$  and  $H_{q'}^{2\alpha}(\Omega) \subset L^\rho(\Omega)$ , for some constants  $r, \rho \in (1, \infty]$ . Hence, it suffices to ensure  $L^r(\Omega), L^\rho(\Omega) \subset \text{dom}(B(t))$ . The parameters  $r, \rho$  define  $\tilde{r}, \tilde{\rho}$  via

$$\frac{1}{r} + \frac{1}{\tilde{r}} = \frac{1}{q}, \quad \frac{1}{\rho} + \frac{1}{\tilde{\rho}} = \frac{1}{q'} \quad (6.4)$$

and since the operator  $B(t)$  is a multiplication operator defined by  $V(t, \cdot)$ , the regularity of  $V(t, \cdot)$  has to be at least  $\varrho := \max\{\tilde{r}, \tilde{\rho}\}$ .

Moreover, let

$$F(t) := (-\Delta)^{-\alpha} B(t) (-\Delta)^{-\alpha} : L^q(\Omega) \rightarrow \dot{H}_q^{2\alpha}(\Omega) \subset L^q(\Omega).$$

We have to ensure Lipschitz-continuity for the operator-valued function  $t \mapsto F(t)$ . So, let  $f \in L^q(\Omega)$  and  $g \in L^{q'}(\Omega)$ . Define  $\tilde{f} = \Delta^{-\alpha} f \in \dot{H}_q^{2\alpha}(\Omega) \subset L^r(\Omega)$  and  $\tilde{g} = (\Delta^{-\alpha})^* g = (\Delta^*)^{-\alpha} g \in \dot{H}^{2\alpha, q'}(\Omega) \subset L^\rho(\Omega)$ . Then, we have for  $t \in \mathcal{I}$

$$\langle F(t)f, g \rangle = \langle (-\Delta)^{-\alpha} B(t) (-\Delta)^{-\alpha} f, g \rangle = \langle (-\Delta)^{-\alpha} f, B(t)^* (-\Delta^*)^{-\alpha} g \rangle = \langle \tilde{f}, B(t)^* \tilde{g} \rangle.$$

The boundedness of  $\langle \tilde{f}, B(t)^* \tilde{g} \rangle$  can be ensured by  $V(t, \cdot) \in L^\tau(\Omega)$ , where  $\tau \in (1, \infty)$  is defined via

$$\frac{1}{r} + \frac{1}{\tau} + \frac{1}{\rho} = 1. \quad (6.5)$$

For fixed  $d \geq 2$  and  $\alpha \in (\frac{1}{2}, 1)$ , the following table turns out for the parameters  $\tilde{r}, \tilde{\rho}, \tau$ :

	$q \in (1, \frac{d}{2\alpha})$	$q \in [\frac{d}{2\alpha}, \infty)$
$q' \in (1, \frac{d}{2\alpha})$	$\tilde{r} \in [\frac{d}{2\alpha}, \infty], \tilde{\rho} \in [\frac{d}{2\alpha}, \infty],$ $\tau \in [\frac{d}{4\alpha}, \infty]$	$\tilde{r} \in (q, \infty], \tilde{\rho} \in [\frac{d}{2\alpha}, \infty],$ $\tau \in [\frac{d}{2\alpha+dq}, \infty]$
$q' \in [\frac{d}{2\alpha}, \infty)$	$\tilde{r} \in [\frac{d}{2\alpha}, \infty], \tilde{\rho} \in (q', 2\alpha, \infty],$ $\tau \in [\frac{d}{2\alpha+dq'}, \infty]$	$\tilde{r} \in (q, \infty], \tilde{\rho} \in (q', \infty],$ $\tau \in (1, \infty]$

We remark that since we have  $r \geq q$ , it holds that  $\tau \leq \tilde{\rho}$  and hence,  $\tau \leq \varrho = \max\{\tilde{r}, \tilde{\rho}\}$ . To guarantee, that the operators  $B(t)$  are generators, we assume that the potential  $V(t, x)$  is positive, i.e.

$$\operatorname{Re}(V(t, x)) \geq 0, \text{ for a.e. } (t, x) \in \mathcal{I} \times \Omega$$

Then, for any  $t \in \mathcal{I}$  the operator  $V(t, x)$  is a generator of a contraction semigroup on  $X = L^q(\Omega)$  (cf. [2, Theorem I.4.11-12]). In particular, the operator family  $B(t)$  is  $A$ -stable.

**Theorem 6.1.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with  $C^2$ -boundary, let  $q \in (1, \infty)$  and let  $\alpha \in (\frac{1}{2}, 1)$ . Let  $B(t)f = V(t, \cdot)f$  define a scalar valued multiplication operator on  $L^q(\Omega)$  with*

$$V \in L^\infty(\mathcal{I}, L^q(\Omega)) \cap C^{\text{Lip}}(\mathcal{I}, L^\tau(\Omega)),$$

where  $\varrho = \max\{\tilde{r}, \tilde{\rho}\}$  and  $\tilde{r}, \tilde{\rho}, \tau$  is chosen from the above table. Moreover, let  $\operatorname{Re}(V(t, x)) \geq 0$  for  $t \in \mathcal{I}$  and for a.e.  $x \in \Omega$ .

Then, the evolution problem (6.1) has a unique solution operator  $U(t, s)$  which can be approximated in operator-norm by

$$\sup_{(t,s) \in \Delta} \|U_n(t, s) - U(t, s)\|_{\mathcal{B}(L^q(\Omega))} = O(n^{-(1-\alpha)}),$$

where

$$U_n(t, s) = \prod_{j=1}^{\rightarrow n} e^{-\frac{t-s}{n} V(\frac{n-j+1}{n}t + \frac{j-1}{n}s, \cdot)} e^{\frac{t-s}{n} \Delta}. \quad (6.6)$$

*Proof.* Using relations (6.4), (6.5) and Sobolev embeddings (6.3), it is easy to see that the required inclusions hold. The claim follows, using Theorem 3.3 and Theorem 5.6. The “ess sup” becomes a “sup”, since the solution operator and the approximating operator are continuous.  $\square$

**Remark 6.2.**

(i) In [11], the existence of a solution operator for equation (6.1) is shown assuming weaker regularity in space and time for the potential. We assumed uniform boundedness of the function  $t \mapsto \|B(t)(-\Delta)^\alpha\|_{\mathcal{B}(X)}$ , which is indeed too strong but important for the considerations.



(ii) We focused on domains, which are bounded and have  $C^2$ -boundaries. Our considerations can be extended to other domains, too.

(iii) Although the approximating propagator  $\{U_n(t, s)\}_{(t,s) \in \Delta}$  defined in (6.6) looks elaborate, it has a simple structure. The semigroup of the Laplace operator on  $L^q(\mathbb{R}^d)$  is given by the Gauss-Weierstrass semigroup (see for example [2, Chapter 2.13]) defined via

$$(e^{t\Delta}u)(x) = (T(t)u)(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u(y) dy.$$

The terms  $e^{-\tau V(t_j)}$  are scalar valued and can be easily computed.

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