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► **To cite this version:**

Damir Filipović, Martin Larsson, Sergio Pulido. Markov cubature rules for polynomial processes. 2016. hal-01418755v1

HAL Id: hal-01418755

<https://hal.science/hal-01418755v1>

Preprint submitted on 16 Dec 2016 (v1), last revised 8 Aug 2019 (v2)

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Markov cubature rules for polynomial processes*

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December 16, 2016

Abstract

We study discretizations of polynomial processes using finite state Markov processes satisfying suitable moment-matching conditions. The states of these Markov processes together with their transition probabilities can be interpreted as Markov cubature rules. The polynomial property allows us to study the existence of such rules using algebraic techniques. These rules aim to improve the tractability and ease the implementation of models where the underlying factors are polynomial processes.

1 Introduction

Polynomial processes have recently gained popularity thanks to their tractability and flexibility. For instance, they have been applied in financial market models for interest rates (Delbaen and Shirakawa, 2002; Zhou, 2003; Filipović et al., 2014), credit risk (Ackerer and Filipović, 2016), variance swaps (Filipović et al., 2016), stochastic volatility (Ackerer et al., 2016), stochastic portfolio theory (Cuchiero, 2016), and life insurance liabilities (Biagini and Zhang, 2016). Polynomial processes, as considered in Cuchiero et al. (2012) and Filipović and Larsson (2016), are stochastic processes with the property that the conditional expectation of a polynomial is a polynomial of the same or lower degree. This implies that conditional moments can be computed efficiently and accurately, which can be exploited to construct tractable models. The polynomial class contains any affine jump-diffusion whose jump measure admits moments of all orders, but enjoys greater flexibility as it can accommodate more general semialgebraic state spaces; see Filipović and Larsson (2016). In this paper we study *discretizations of polynomial processes*, via moment-matching conditions, using finite state

*Martin Larsson gratefully acknowledges support from SNF Grant 205121_163425. The research of Sergio Pulido benefited from the support of the Chair Markets in Transition (Fédération Bancaire Française) and the project ANR 11-LABX-0019. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 307465-POLYTE

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Markov polynomial processes. We call such a finite state polynomial process a *Markov cubature rule* because the states of the process together with their transition probabilities can be interpreted as cubature rules for the law of the original polynomial process at different times. Markov cubature rules aim to facilitate the implementation of polynomial models in order to simplify costly computational tasks such as Monte-Carlo simulation, and pricing of path-dependent and American options.

Discretizations of stochastic models using finite state Markov processes appear regularly in the numerical methods literature. In finance, these techniques have been used in order to price and hedge exotic and American options via finite state Markov chain and tree approximations; see e.g. Gruber and Schweizer (2006); Kifer (2006); Dolinsky (2016). As explained in Kushner (1984) and Kushner and Dupuis (2013), these approximations are linked to numerical analysis techniques such as the finite difference method. It is also relevant to mention quantization methods, as for instance in Bally et al. (2005), that address the optimal choice of the approximation grid on a finite time domain and in higher dimensional state spaces. In all these cases, discretization happens at two levels: the discretization of the time domain, as it is performed in simulation algorithms, and the discretization of the space domain using grids.

Cubature methods also play a crucial role in numerous numerical algorithms. For instance, classical cubature techniques have been applied within the context of filtering in Arasaratnam and Haykin (2009). Additionally, the cubature formulas on Wiener space, developed by Lyons and Victoir (2004), have been used in multiple applications: in filtering problems in Lee and Lyons (2013), to calculate the greeks of financial options in Teichmann (2006), and to numerically approximate solutions of Stochastic Differential Equations in Bayer and Teichmann (2008) and Doersek et al. (2013), Backward Stochastic Differential Equations (BSDEs) in Crisan and Manolarakis (2012, 2014), and Forward-Backward Stochastic Differential Equations (FBSDEs) in Chaudru de Raynal and Garcia Trillos (2015). Cubature methods ease the calculation of conditional expectations, which are at the core of the above mentioned numerical problems. Contrary to the techniques mentioned in the previous paragraph where discretization is performed in the time and space domains, cubature on Wiener space discretizes path space directly. These cubature rules extend Tchakaloff's cubature theorem, as studied in Putinar (1997) and Bayer and Teichmann (2006), to the Wiener space of continuous paths.

In this paper we study *cubature-based discretizations of polynomial processes* in the state space variable, in continuous and discrete time domains. These approximations, which we call Markov cubature rules, correspond to Markov polynomial processes that match the moments of the original process up to a given order. The polynomial property allows us to study the existence of such rules using algebraic techniques. Contrary to the classical cubature problem, we look for cubature rules that use the same set of cubature points at all times and the moments to be matched depend on those points. As explained in Section 2.1, it turns out that in continuous time the notion of Markov cubature rule is too stringent. In continuous time we instead solve a suitably relaxed version of the Markov cubature problem. Our two main existence results are Theorem 4.4 and Theorem 5.1. Theorem 4.4 shows the existence of what we call continuous time lifted Markov cubature rules. Theorem 5.1 shows the existence of Markov cubature rules in discrete time for an appropriately chosen time grid.

The existence of asymptotic moments is a crucial hypothesis and lies at the core of the proofs of these theorems.

Our paper is organized as follows. In Section 2 we set the stage by providing the definitions of Markov and lifted Markov cubature rules as well as some basic facts about polynomial processes. In particular, in Section 2.1, we explain why the notion of continuous time Markov cubature rule is too stringent, and introduce a relaxed version of the problem. In Section 3, we give algebraic and geometric characterizations of continuous time Markov cubature rules for polynomial processes and at the same time illustrate their complexity through examples. The arguments and concepts introduced in this section facilitate and motivate the presentation of lifted Markov cubature rules in Section 4. Theorem 4.4 shows the existence of continuous time lifted Markov cubature rules for polynomial processes under the hypothesis of existence of asymptotic moments. Similar assumptions allow us to prove, in Section 5, the existence of discrete time Markov cubature rules on a conveniently chosen grid; see Theorem 5.1. Appendix A presents the results on asymptotic moments of polynomial processes needed throughout the paper and Appendix B contains the proofs of all results in the article.

We adopt the following notation throughout the manuscript: We write \mathbb{R}_+ for the set of nonnegative real numbers and \mathbb{R}_{++} for the set of positive real numbers. We denote by \mathbb{N} the set of positive natural numbers. For $N, M \in \mathbb{N}$, $\mathbb{R}^{N \times M}$ designates the vector space of matrices of size $N \times M$ and by convention $\mathbb{R}^N = \mathbb{R}^{N \times 1}$ is the vector space of column vectors. Given $d \in \mathbb{N}$ and a set $E \subseteq \mathbb{R}^d$, we say that q is a polynomial on E if there exists p , a polynomial on \mathbb{R}^d , such that $q = p|_E$. Its degree is defined by $\deg q = \min\{\deg p : q = p|_E\}$. We let $\text{Pol}(E)$ and $\text{Pol}_n(E)$ denote the ring of polynomials on E and the vector space of polynomials on E whose degree is less than or equal to n , respectively. For $N \in \mathbb{N}$ and a set $A \subseteq \mathbb{R}^N$ we write $\text{conv}(A)$ for the convex hull of A and $\text{span}(A)$ for the linear span of A .

2 Setup and overview

Fix a state space $E \subseteq \mathbb{R}^d$. We consider a càdlàg adapted process X defined on a filtered measurable space $(\Omega, \mathcal{F}, \mathcal{F}_t)$, along with a family of probability measures \mathbb{P}_x , $x \in E$, such that X is an E -valued Markov processes under each \mathbb{P}_x , starting at $X_0 = x$. We assume that X admits an extended generator \mathcal{G} , whose domain contains all polynomials. That is, we assume

$$p(X_t) - \int_0^t \mathcal{G}p(X_s) ds \quad \text{is a } \mathbb{P}_x\text{-local martingale}$$

for every $x \in E$ and every $p \in \text{Pol}(\mathbb{R}^d)$. By taking $p(x) = x_i$, this immediately implies that X is a semimartingale under each \mathbb{P}_x . Moreover, the positive maximum principle holds, in the sense that for any $p \in \text{Pol}(\mathbb{R}^d)$,

$$\text{if } p(x) = \max_E p \text{ for some } x \in E, \text{ then } \mathcal{G}p(x) \leq 0.^1 \tag{2.1}$$

¹Indeed, suppose $p(x) = \max_E p$, and assume for contradiction $\mathcal{G}p(x) = \delta > 0$. Define $M_t = p(X_t) - \int_0^t \mathcal{G}p(X_s) ds$ and $\tau = \inf\{t : \mathcal{G}p(X_t) \leq \delta/2\}$. Then, under \mathbb{P}_x , M^τ is a nonpositive local martingale with $M_0^+ = 0$, hence $M^\tau = 0$. On the other hand, $M_{t \wedge \tau} \leq -\int_0^{t \wedge \tau} \mathcal{G}p(X_s) ds \leq -(\delta/2)(t \wedge \tau)$, which is strictly negative for $t > 0$. This contradiction proves $\mathcal{G}p(x) \leq 0$.

In particular, $\mathcal{G}p = 0$ on E whenever $p = 0$ on E , which implies that \mathcal{G} is well-defined as an operator on $\text{Pol}(E)$.²

2.1 Markov and lifted Markov cubature rules

Definition 2.1. *We say that a time-homogeneous Markov process Y with finite state space $E^Y = \{x_1, \dots, x_M\} \subseteq E$ defines an n -Markov cubature rule for X on $\mathbf{T} \subseteq [0, \infty)$ if*

$$\mathbb{E}_{x_i}[p(X_t)] = \mathbb{E}_{x_i}[p(Y_t)] \quad (2.2)$$

holds for all $i = 1, \dots, M$, $t \in \mathbf{T}$, and $p \in \text{Pol}_n(E)$.

Remark 2.2. *In condition (2.2), $\mathbb{E}_{x_i}[p(X_t)]$ denotes the expectation with respect to the probability measure \mathbb{P}_{x_i} while $\mathbb{E}_{x_i}[p(Y_t)]$ denotes the expectation with respect to the probability measure $\mathbb{P}_{x_i}^Y$ associated to the finite state Markov process Y . We adopt this convention throughout the paper.*

Suppose that Y is a n -Markov cubature rule for X on \mathbf{T} . The moment-matching condition (2.2) can be rewritten as

$$\mathbb{E}_{x_i}[p(X_t)] = \sum_{j=1}^M p(x_j) \mathbb{P}_{x_i}^Y(Y_t = x_j)$$

for all $i = 1, \dots, M$, $t \in \mathbf{T}$, and $p \in \text{Pol}_n(E)$. Hence, for any $i = 1, \dots, M$ and $t \in \mathbf{T}$, the points x_1, \dots, x_M together with the transition probabilities $\mathbb{P}_{x_i}^Y(Y_t = x_1), \dots, \mathbb{P}_{x_i}^Y(Y_t = x_M)$ define an n -cubature rule for the law of X_t with respect to \mathbb{P}_{x_i} . We observe that in this case, contrary to classical cubature rules, the matched moments depend on the cubature points and these points are used for all times $t \in \mathbf{T}$. In addition, as stated in Proposition 2.7 below, the properties of the weights inherited by the Markov property of Y guarantee time-consistency features of these cubature rules.

We will also consider a relaxed version of n -Markov cubature. Indeed, it turns out that the notion of an n -Markov cubature rule is too stringent in general. To see why, suppose X is given as the solution of an SDE of the form

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t.$$

Under linear growth conditions on the coefficients, one has the estimate

$$\mathbb{E}_x[\|X_t - x\|^4] \leq \kappa(1 + \|x\|^4) t^2, \quad 0 \leq t \leq 1,$$

for all $x \in E$, where κ is a constant that only depends on μ and σ ; see Problem 5.3.15 in Karatzas and Shreve (1991). If Y is a 4-Markov cubature rule for X on $[0, \infty)$, this estimate carries over to Y , which in conjunction with the Markov property yields

$$\mathbb{E}_x[\|Y_t - Y_s\|^4] = \mathbb{E}_x[\mathbb{E}_{Y_s}[\|Y_{t-s} - Y_0\|^4]] \leq \kappa(t-s)^2$$

²Indeed, if $p \in \text{Pol}(\mathbb{R}^d)$ is a representative of $q = p|_E \in \text{Pol}(E)$, we define $\mathcal{G}q = \mathcal{G}p|_E$, which is independent of the choice of representative p .

for any $x \in E^Y$ and any $s \leq t$ with $t - s \leq 1$. By Kolmogorov's continuity lemma, Y then has continuous paths as well, which forces it to be constant. Consequently, in the generic case, the diffusion X will not admit any *non-trivial* n -Markov cubature rule on $[0, \infty)$, unless $n < 4$. Moreover, by the same argument and the intermediate value theorem, unless X exhibits jumps, it is impossible to construct a non-trivial Markov process Y with countable state space such that (2.2), with $n \geq 4$, holds for all initial conditions. This is a rather severe restriction.

One way to avoid this obstruction is to replace $[0, \infty)$ with a discrete time set \mathbf{T} , in which case one remains within the framework of Definition 2.1. This approach is pursued in Section 5. Another possibility is to consider a suitable relaxation of Definition 2.1, which we now describe.

To set the stage, motivated by Bayer and Teichmann (2006), observe that n -Markov cubature rules for X correspond in a natural way to 1-Markov cubature rules for an associated process \bar{X} . To see this, fix n and let N_n denote the dimension of $\text{Pol}_n(E)$. Let h_1, \dots, h_{N_n} be a basis for $\text{Pol}_n(E)$, and define

$$\mathcal{H}_n(x) = (h_1(x), \dots, h_{N_n}(x))^\top. \quad (2.3)$$

Since each polynomial x_i on E , $i = 1, \dots, d$, has a unique coordinate representation in terms of this basis, there exists a unique matrix $A_n \in \mathbb{R}^{d \times N_n}$ such that

$$A_n \mathcal{H}_n(x) = x, \quad x \in E.$$

Thus the map $\mathcal{H}_n : E \rightarrow \mathcal{H}_n(E)$ is a bijection, and it follows that the process $\bar{X} = \mathcal{H}_n(X)$ is a time-homogeneous Markov process with state space $\bar{E} = \mathcal{H}_n(E)$. From this one deduces the following lemma.

Lemma 2.3. *The time-homogeneous Markov process Y with state space $E^Y = \{x_1, \dots, x_M\}$ is an n -Markov cubature rule for X on \mathbf{T} if and only if $\bar{Y} = \mathcal{H}_n(Y)$ is a 1-Markov cubature rule for \bar{X} on \mathbf{T} .*

Consider now the dynamics of \bar{X} . Its drift function is given by

$$B(\bar{x}) = \mathcal{G} \mathcal{H}_n(A_n \bar{x}),$$

where \mathcal{G} acts componentwise on \mathcal{H}_n , i.e. $\mathcal{G} \mathcal{H}_n = (\mathcal{G} h_1, \dots, \mathcal{G} h_{N_n})^\top$. Note that this relies on the fact that \mathcal{G} is well-defined as an operator on $\text{Pol}(E)$. Therefore, one has

$$d\bar{X}_t = B(\bar{X}_t) dt + d\bar{M}_t,$$

where \bar{M} is an \mathbb{R}^{N_n} -valued local martingale. If $B(\bar{x})$ is linear in \bar{x} , \bar{M} is a true martingale, and \bar{X} satisfies mild integrability conditions, then the first moment $\mathbb{E}_{\bar{x}}[\bar{X}_t]$ is equal to $Z_t^{\bar{x}}$, where $Z^{\bar{x}}$ is the solution of the ODE

$$dZ_t = B(Z_t) dt, \quad Z_0 = \bar{x}. \quad (2.4)$$

As we shall see later in Proposition 2.11, this holds if X is a polynomial process.

While \overline{X} is only well-defined for initial conditions $\overline{x} \in \overline{E} = \mathcal{H}_n(E)$, which can be interpreted as a subset of the moment curve and whose geometry is highly complex in general, the solution $Z^{\overline{x}}$ of (2.4) admits any point $\overline{x} \in \mathbb{R}^{N_n}$ as initial condition. This is the motivation behind the following definition, where the state space of \overline{Y} is no longer restricted to be a subset of \overline{E} .

Definition 2.4. *We say that a time-homogeneous Markov process \overline{Y} with finite state space $E^{\overline{Y}} = \{\overline{x}_1, \dots, \overline{x}_M\} \subseteq \mathbb{R}^{N_n}$ defines a lifted n -Markov cubature rule for X on $\mathbf{T} \subseteq [0, \infty)$ if*

$$\mathbb{E}_{\overline{x}_i}[\overline{Y}_t] = Z_t^{\overline{x}_i}$$

holds for all $i = 1, \dots, M$ and $t \in \mathbf{T}$.

A lifted n -Markov cubature rule whose state space $E^{\overline{Y}}$ is contained in \overline{E} gives rise to a bona fide n -Markov cubature rule due to Lemma 2.3. Thus the notion of a lifted n -Markov cubature rule is a relaxation of the notion of an n -Markov cubature rule. The adjective *lifted* comes from the fact that the Markov process \overline{Y} can be viewed as a Markov process taking values in the set of signed measures supported on a suitable set of points x_1, \dots, x_M in E . This is described in detail in Section 4. As we will show in Theorem 4.4, contrary to n -Markov cubature rules, if the process X has asymptotic moments, then for arbitrary $n \in \mathbb{N}$, it is always possible to construct non-trivial lifted n -Markov cubature rules. As shown in Remark 4.10, these non-trivial lifted n -Markov cubature rules can be interpreted as relaxations of Markov cubature rules by allowing negative transition probabilities. Hence, the limitation posed by Kolmogorov's continuity lemma disappears in a framework with negative probabilities.

2.2 Polynomial processes

In what follows we will study Markov cubature rules for a particular class of Markov process, namely polynomial processes. In this section we establish the necessary definitions and basic results.

Definition 2.5. *The operator \mathcal{G} is called polynomial if $\mathcal{G}\text{Pol}_n(E) \subseteq \text{Pol}_n(E)$ for all $n \in \mathbb{N}$. In this case X is called a polynomial process.*

Remark 2.6. *In the present paper, \mathcal{G} is assumed to be the extended generator of some given Markov process X . We are not concerned with the question of existence of such a process given a candidate operator \mathcal{G} . This issue is discussed in Filipović and Larsson (2016) for polynomial diffusions.*

If X is a polynomial process, then all mixed moments of X_t are polynomial functions of the initial state. More precisely, fix n and let $\mathcal{H}_n(x)$ be as in (2.3). If \mathcal{G} is polynomial, one has

$$\mathcal{G}\mathcal{H}_n(x) = G_n^\top \mathcal{H}_n(x) \tag{2.5}$$

for some matrix $G_n \in \mathbb{R}^{N_n \times N_n}$, where \mathcal{G} acts componentwise on \mathcal{H}_n . From this one obtains the following lemma.

Lemma 2.7. *Assume X is a polynomial process. Then for any polynomial $p \in \text{Pol}_n(E)$ with coordinate representation $\vec{p} \in \mathbb{R}^{N_n}$, that is, $p(x) = \mathcal{H}_n(x)^\top \vec{p}$, one has*

$$\mathbb{E}_x[p(X_t)] = \mathcal{H}_n(x)^\top e^{tG_n} \vec{p}. \quad (2.6)$$

Thus the left-hand side is a polynomial in $\text{Pol}_n(E)$ with coordinate representation $e^{tG_n} \vec{p}$.

Remark 2.8. *As a consequence of Lemma 2.7, Markov cubature rules for polynomial processes are polynomial processes as well.*

We say that the time set \mathbf{T} is stable under differences, if $t - s \in \mathbf{T}$ for all $s, t \in \mathbf{T}$ such that $s \leq t$. It turns out that if \mathbf{T} is stable under differences, Markov cubature rules for a polynomial process X on \mathbf{T} can also be used for polynomials of the process X evaluated at different times. This is one of the principal features of Markov cubature rules for polynomial processes and the content of the following proposition.

Proposition 2.9. *Suppose that X is a polynomial process and that \mathbf{T} is stable under differences. Let Y be a time-homogeneous Markov process with state space $E^Y = \{x_1, \dots, x_M\}$. Then the process Y is an n -Markov cubature rule for X on \mathbf{T} if and only if given $t_1, \dots, t_l \in \mathbf{T}$ such that $0 \leq t_1 \leq \dots \leq t_l$ and polynomials $p_1, \dots, p_l \in \text{Pol}_n(E)$ with $\prod_i p_i \in \text{Pol}_n(E)$, we have*

$$\mathbb{E}_x \left[\prod_{i=1}^l p_i(X_{t_i}) \right] = \mathbb{E}_x \left[\prod_{i=1}^l p_i(Y_{t_i}) \right] \quad (2.7)$$

for all $x \in E^Y$.

Remark 2.10. *Assume that Y is an n -Markov cubature rule for a polynomial process X on \mathbf{T} . Set $\overline{\mathbf{T}} = \{\sum_{i=1}^l t_i : t_i \in \mathbf{T}, l \in \mathbb{N}\}$. The time set $\overline{\mathbf{T}}$ is the smallest subset of $[0, \infty)$ that is stable under sums and contains \mathbf{T} . The argument in the proof of Proposition 2.9 shows that Y is also an n -Markov cubature rule for X on $\overline{\mathbf{T}}$.*

We now recall the notation $\overline{X} = \mathcal{H}_n(X)$, with \mathcal{H}_n given by (2.3). The following proposition is crucial for the study of lifted Markov cubature rules for polynomial processes.

Proposition 2.11. *Suppose that X is a polynomial process and fix $n \in \mathbb{N}$. Then the process \overline{X} is a polynomial process on $\mathcal{H}_n(E)$ with dynamics of the form*

$$d\overline{X}_t = G_n^\top \overline{X}_t dt + dM_t \quad (2.8)$$

with G_n as in (2.5) and M a martingale. Consequently, if $Z_t^{\overline{x}}$ is the solution of (2.4) with $B(z) = G_n^\top z$ then $Z_t^{\overline{x}} = \mathbb{E}_{\overline{x}}[\overline{X}_t]$.

To prove this proposition we need the following lemma from algebra.

Lemma 2.12. *Let $k \in \mathbb{N}$. Then $p \in \text{Pol}_{kn}(E)$ if and only if $p(x) = q(\mathcal{H}_n(x))$ for some $q \in \text{Pol}_k(\mathcal{H}_n(E))$.*

3 Continuous time Markov cubature

We assume hereafter that X is polynomial process and fix $n \in \mathbb{N}$. In this section we will study characterizations of continuous time n -Markov cubatures rules for X , namely n -Markov cubature rules on $[0, \infty)$. Even though, as explained in Section 2.1, these cubature rules turn out to be too stringent in general, the results of this section motivate and facilitate the study in Section 4 of the relaxed notion of lifted Markov cubature rule.

We adopt the notation of Section 2 but for simplicity we often omit the index n . Given points $x_1, \dots, x_M \in E$ we shall denote by $H = H(x_1, \dots, x_M)$ the $M \times N_n$ -matrix whose elements are

$$H_{ij} = h_j(x_i) \quad (3.1)$$

for all $i = 1, \dots, M$ and $j = 1, \dots, N_n$. Notice that the i -th row of the matrix $H \in \mathbb{R}^{M \times N_n}$ is equal to $\mathcal{H}_n(x_i)^\top$ as defined in (2.3).

By (2.5) and (2.6) we have

$$\mathcal{G}h_j(x_i) = (HG)_{ij}, \quad (3.2)$$

$$\mathbb{E}_{x_i}[h_j(X_t)] = (H \exp(tG))_{ij} \quad (3.3)$$

for all $i = 1, \dots, M$ and $j = 1, \dots, N_n$. Equations (3.2)-(3.3) establish a relationship between the analytical calculation of the generator and semigroup acting on the functional space of polynomials, and an algebraic calculation involving matrix multiplication.

Theorem 3.2 below is the main characterization theorem for the existence of a continuous time n -Markov rule. Before stating the theorem we recall that a *transition rate matrix* is a matrix whose columns add up to zero and off-diagonal elements are nonnegative. We also need the following definition.

Definition 3.1. *We say that a vector $v \in \mathbb{R}^m$ points into $\text{conv}(\{v_1, \dots, v_n\}) \subset \mathbb{R}^m$ at v_i if there exist $(L_{i,j})_{j \neq i} \in \mathbb{R}_+^{m-1}$ such that*

$$v = \sum_{j \neq i} L_{i,j}(v_j - v_i).$$

Theorem 3.2. *Given a set of points $E^Y = \{x_1, \dots, x_M\} \subseteq E$ the following statements are equivalent.*

- (i) *There exists a continuous time n -Markov cubature rule Y with state space E^Y ; see Definition 2.1.*
- (ii) *Given H as in (3.1), $HG = LH$, for some transition rate matrix $L \in \mathbb{R}^{M \times M}$.*
- (iii) *Given H as in (3.1), $HG = LH$, for some matrix $L \in \mathbb{R}^{M \times M}$ with nonnegative off-diagonal elements.*
- (iv) *For each $x \in E^Y$ the vector $\mathcal{G}\mathcal{H}_n(x)$ points into $\text{conv}(\{\mathcal{H}_n(x_1), \dots, \mathcal{H}_n(x_M)\})$ at the point $\mathcal{H}_n(x)$; see Definition 3.1.*

If in addition $M = N_n$ and the matrix H is invertible, there exists a Lagrange basis of $\text{Pol}_n(E)$, $\tilde{\beta} = (\tilde{h}_1, \dots, \tilde{h}_{N_n})$, i.e. a basis with $\tilde{h}_j(x_i) = \delta_{ij}$, and the above statements are equivalent to:

$$(v) \mathcal{G}\tilde{h}_j(x_i) \geq 0 \text{ for } i \neq j.$$

Moreover, when condition (ii) is satisfied, L can be taken as the transition rate matrix of the n -Markov cubature rule Y .

For the proof of Theorem 3.2 we will need the following lemma.

Lemma 3.3. *Suppose that L is a matrix such that $HG = LH$. Then the columns of L add up to zero.*

As the proof shows, the conditions in Theorem 3.2 imply that if Y is an n -Markov cubature rule then, for each $x \in E^Y$, the flow $(\mathbb{E}_x[\mathcal{H}_n(X_t)])_{t \geq 0}$ never leaves $\text{conv}(\{\mathcal{H}_n(x_1), \dots, \mathcal{H}_n(x_M)\})$. Indeed, notice that $(\exp(tL))_{t \geq 0}$ is a transition semigroup and for all $i = 1, \dots, M$ we have

$$\mathbb{E}_{x_i}[\mathcal{H}_n(X_t)] = \exp(tG^\top) \mathcal{H}_n(x_i) = i\text{-th column of } H^\top \exp(tL^\top).$$

The points $\{\mathcal{H}_n(x_1), \dots, \mathcal{H}_n(x_M)\}$ lie on the *moment curve* $\mathcal{H}_n(E)$ and correspond to the rows of H . Their convex hull represents all the possible initial distributions of a Markov chain with state space $\{\mathcal{H}_n(x_1), \dots, \mathcal{H}_n(x_M)\}$.

In view of Lemma 2.3, this is not surprising as n -Markov cubatures rules for the process X correspond to 1-Markov cubature rules for the process $\bar{X} = \mathcal{H}_n(X)$ and the state space of \bar{X} lies on the moment curve. As explained in Section 2, due to the fact that the notion of n -Markov cubature rule is too stringent in general, in Section 4 we work with the relaxed notion of *lifted n -Markov cubature rule*. Lifted Markov cubature rules satisfy a related geometric condition but are no longer bound to live on the moment curve.

Example 3.4 below illustrates the complexity of the algebraic conditions (ii)-(iii) and the geometric condition (iv) of Theorem 3.2 for 2-Markov cubature rules when $d = 1$.

Example 3.4. *Suppose that $d = 1$ and \mathcal{G} is of the form*

$$\mathcal{G}f(x) = \kappa(\Theta - x)f'(x) + \frac{1}{2}\kappa(\alpha + ax + Ax^2)f''(x).$$

Consider the canonical basis $(1, x)$ of $\text{Pol}_1(E)$. For $n = 1$ the matrix of \mathcal{G} restricted to $\text{Pol}_1(E)$ with respect to this basis is

$$G_1 = \begin{pmatrix} 0 & \kappa\Theta \\ 0 & -\kappa \end{pmatrix}.$$

Let $x_1 < x_2$ be fixed. In this case the matrix

$$H = H(x_1, x_2) = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

is invertible and

$$HG_1H^{-1} = \frac{\kappa}{x_2 - x_1} \begin{pmatrix} x_1 - \Theta & \Theta - x_1 \\ x_2 - \Theta & \Theta - x_2 \end{pmatrix}.$$

Hence, if $\kappa > 0$, condition (ii) of Theorem 3.2 is satisfied if and only if

$$x_1 \leq \Theta \leq x_2.$$

In other words the ‘‘cubature points’’ x_1, x_2 have to be around the asymptotic mean Θ .

Similarly, for $n = 2$, the matrix of \mathcal{G} with respect to $(1, x, x^2)$, the canonical basis of $\text{Pol}_2(E)$, is given by

$$G_2 = \kappa \begin{pmatrix} 0 & \Theta & \alpha \\ 0 & -1 & 2\Theta + a \\ 0 & 0 & A - 2. \end{pmatrix}.$$

Let $x_1 < x_2 < x_3$ be arbitrary. As before the matrix $H = H(x_1, x_2, x_3)$ is invertible and

$$HG_2H^{-1} = \kappa \Xi \Lambda,$$

with Λ a diagonal matrix with positive entries on the diagonal and

$$\begin{aligned} \Xi_{11} &= \sigma^2(x_1) + (\Theta - x_1)(x_1 - x_2) - (\Theta - x_1)(x_3 - x_1) \\ \Xi_{12} &= -\sigma^2(x_1) + (\Theta - x_1)(x_3 - x_1) \\ \Xi_{13} &= \sigma^2(x_1) + (\Theta - x_1)(x_1 - x_2) \\ \Xi_{21} &= \sigma^2(x_2) + (x_2 - \Theta)(x_3 - x_2) \\ \Xi_{22} &= -\sigma^2(x_2) + (x_2 - \Theta)(x_2 - x_1) - (x_2 - \Theta)(x_3 - x_2) \\ \Xi_{23} &= \sigma^2(x_2) - (x_2 - \Theta)(x_2 - x_1) \\ \Xi_{31} &= \sigma^2(x_3) - (x_3 - \Theta)(x_3 - x_2) \\ \Xi_{32} &= -\sigma^2(x_3) + (x_3 - \Theta)(x_3 - x_1) \\ \Xi_{33} &= \sigma^2(x_3) - (x_3 - \Theta)(x_3 - x_1) - (x_3 - \Theta)(x_3 - x_2), \end{aligned} \tag{3.4}$$

where

$$\sigma^2(x) = \alpha + ax + Ax^2 \geq 0.$$

The condition (iii) of Theorem 3.2 is satisfied when the off-diagonal elements of Ξ are non-negative. In particular, necessarily we should have that $x_1 \leq \Theta \leq x_3$.

As explained after the proof of Theorem 3.2, if Y is an n -Markov cubature rule then, for all $x \in E^Y$, the flow $(\mathbb{E}_x[\mathcal{H}_n(X_t)])_t$ should not leave the convex hull of a finite set of points in \mathbb{R}^{N_n} . Therefore, a natural question is whether the existence of asymptotic moments is sufficient to guarantee the existence of n -Markov cubature rules.

Example 3.5 below shows that in general, even for $n = 2$, this is not the case. This highlights the complexity of n -Markov cubature rules even for lower orders. We will see, however, in Sections 4 and 5, that conditions on the asymptotic moments (see conditions (A1), (A2) and (A3)) play a crucial role in order to guarantee the existence of relaxed versions of Markov cubatures.

Example 3.5. We adopt the same notation than in Example 3.4. Assume that $\kappa > 0$ and that

$$\Theta = 0; A = \alpha = 1; \sqrt{2} \leq a < 2.$$

In this case the matrix G_2 has negative nonzero eigenvalues. Additionally, the eigenvalue 0 has algebraic multiplicity 1. Corollary A.3 shows that the asymptotic moments exist and they are independent of the state variable.

We show that in this case it is impossible to find 2-Markov cubature rules whose state space consists of three points. Indeed, suppose that $x_1 < x_2 < x_3$ are the states of a 2-Markov cubature rule.

As explained in Example 3.4 this implies that $HG_2H^{-1} = \kappa \Xi \Lambda$ with Λ a diagonal matrix with positive entries on the diagonal and Ξ as in (3.4), a matrix with nonnegative off-diagonal elements. The condition $\Xi_{32} \geq 0$ implies simultaneously that $x_3 > 0$ and that

$$x_1 < -a < 0.$$

This together with the condition $\Xi_{13} \geq 0$ implies that

$$x_2 \leq -\frac{1}{x_1} - a < \frac{1}{a} - a < 0.$$

On the other hand, using the condition $\Xi_{21} \geq 0$, we deduce that $1 + ax_2 > 0$ and hence

$$-\frac{1}{a} < x_2.$$

In conclusion

$$-\frac{1}{a} < x_2 < \frac{1}{a} - a,$$

which contradicts the fact that $a \geq \sqrt{2}$.

Examples 3.4 and 3.5 together with the discussion in Section 2 show that the notion of n -Markov cubature rule in continuous time is too stringent in general. This motivates the study of relaxed notions in Sections 4 and 5.

4 Lifted continuous time Markov cubature

We first recall the notion of lifted n -Markov cubature rule from Definition 2.4. Trivial lifted n -Markov cubature rules always exist.

Example 4.1. By Proposition 2.11, if we take $M = 1$ and $\bar{x}_1 = 0$, then trivially the constant process $\bar{Y} = \bar{x}_1$ defines a continuous time lifted n -Markov cubature rule.

The next example illustrates other types of trivial cubature rules in continuous time.

Example 4.2. Suppose that G^T has a real negative eigenvalue λ with eigenvector v . Let $\bar{x}_1 = v$ and $\bar{x}_2 = -v$. Define \bar{Y} to be the Markov process on $E^{\bar{Y}} = \{\bar{x}_1, \bar{x}_2\}$ with transition rate matrix

$$L = \frac{\lambda}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

A simple calculation shows that

$$\mathbb{E}_{\bar{x}_i}[\bar{Y}_t] = e^{t\lambda} \bar{x}_i,$$

for $i = 1, 2$ and for all $t \geq 0$. On the other hand, by Proposition 2.11 we get

$$Z_t^{\bar{x}_i} = e^{tG^T} \bar{x}_i = e^{t\lambda} \bar{x}_i.$$

for $i = 1, 2$ and for all $t \geq 0$. Therefore, \bar{Y} is a continuous time lifted n -Markov cubature rule.

Observe that in the previous examples if $E^{\bar{Y}}$ is the state space of the lifted Markov cubature rule, then we do not have $\text{span}(E^{\bar{Y}}) = \mathbb{R}^{N_n}$. As we will highlight in Remark 4.10 below, this is a desirable property and motivates the following definition.

Definition 4.3. We say that a lifted n -Markov cubature rule \bar{Y} with state space $E^{\bar{Y}}$ is non-trivial if $\text{span}(E^{\bar{Y}}) = \mathbb{R}^{N_n}$.

Theorem 4.4 below is the main result of this section. It is a positive result as it shows the existence of non-trivial lifted Markov cubature rules in continuous time. The main assumption in the theorem is given by the following condition

(A1) For all nonzero eigenvalues λ of G , we have that $\text{Re}(\lambda) < 0$ and for the eigenvalue 0, the algebraic and geometric multiplicities coincide.

Theorem A.1 shows that this condition is equivalent to the existence of asymptotic moments of order n . This is a natural condition in view of the discussion in Section 3.

Theorem 4.4. Let $G \in \mathbb{R}^{N_n \times N_n}$ be a matrix that satisfies (A1). Then there exists a non-trivial lifted n -Markov cubature rule for X on $[0, \infty)$.

In order to prove Theorem 4.4 we need the following characterization lemma.

Lemma 4.5. The following statements are equivalent.

(i) There exists a matrix $S \in \mathbb{R}^{R \times N_n}$ with $\text{rank}(S) = N_n$ and a transition rate matrix $L \in \mathbb{R}^{R \times R}$ such that

$$SG = LS. \tag{4.1}$$

(ii) There exist points $E^{\bar{Y}} = \{\bar{x}_1, \dots, \bar{x}_R\}$ in \mathbb{R}^{N_n} and a Markov Process \bar{Y} on $E^{\bar{Y}}$ such that \bar{Y} is a non-trivial lifted n -Markov cubature rule for X on $[0, \infty)$; see Definition 2.4.

Additionally, in the statements above the matrix L can be chosen as the transition rate matrix of the process \bar{Y} and the points $\bar{x}_1, \dots, \bar{x}_R$ can taken as the rows of the matrix S .

Remark 4.6. The proof of Theorem 4.4 is constructive and provides an algorithm to find the lifted Markov cubature rules by using the Jordan normal form of the matrix G^T .

The next proposition motivates the terminology *lifted* Markov cubature rules. It shows that a lifted Markov cubature rule can be viewed as a Markov process taking values in the set of signed measures supported on a suitable set of points x_1, \dots, x_M in E . By considering the set of signed measures, we are in some sense lifting the state space of the process X .

Proposition 4.7. Let $E^Y = \{x_1, \dots, x_M\} \subset E$ be a set of points such that $\text{rank}(H) = N_n$, where $H \in \mathbb{R}^{M \times N_n}$ is the matrix defined by (3.1). If $M \leq R$, the statements in Lemma 4.5 are equivalent to:

(iii) There exist signed measures $E^{\tilde{Y}} = \{\nu_1, \dots, \nu_R\}$ on E^Y of the form

$$\nu_i = \sum_{j=1}^M \tilde{S}_{ij} \delta_{x_j}, \quad (4.2)$$

where $\tilde{S} \in \mathbb{R}^{R \times M}$ has rank M , and a Markov Process \tilde{Y} on $E^{\tilde{Y}}$ such that

$$\mathbb{E}_\nu[f(X_t)] = \int_{E^Y} \mathbb{E}_x[f(X_t)] \nu(dx) = \mathbb{E}_\nu \left[\int_{E^Y} f(x) \tilde{Y}_t(dx) \right] \quad (4.3)$$

for all $\nu \in E^{\tilde{Y}}$ and $f \in \text{Pol}_n(E)$.

Additionally, in the statements above the matrix L can be chosen as the transition rate matrix of the process \tilde{Y} .

In order to prove this proposition we need the following lemma of linear algebra.

Lemma 4.8. Let $S \in \mathbb{R}^{R \times N}$. Suppose that $M \leq R$ and $H \in \mathbb{R}^{M \times N}$ satisfies $\text{rank}(H) = N$. Then $\text{rank}(S) = N$ if and only if there exists $\tilde{S} \in \mathbb{R}^{R \times M}$, with $\text{rank}(\tilde{S}) = M$, such that $S = \tilde{S}H$.

Remark 4.9. The conditions $\text{rank}(\tilde{S}) = M$ and $\text{rank}(H) = N_n$ imply that $\text{rank}(\tilde{S}H) = N_n$. This in turn implies that for any two polynomials $p, q \in \text{Pol}_n(E)$, $p = q$ if and only if for all $i = 1, \dots, R$

$$\int_{E^Y} p(x) \nu_i(dx) = \int_{E^Y} q(x) \nu_i(dx). \quad (4.4)$$

In other words, the polynomials in $\text{Pol}_n(E)$ are completely determined by their averages with respect to the finitely supported signed measures ν_1, \dots, ν_R . Indeed, (4.4) is equivalent to $\tilde{S}H\vec{p} = \tilde{S}H\vec{q}$, where \vec{p} and \vec{q} are the coordinates, with respect to the basis h_1, \dots, h_{N_n} , of the polynomials p and q , respectively.

Non-trivial lifted Markov cubature rules are useful to calculate expectations of polynomials of the original process X . Additionally, lifted Markov cubature rules can be interpreted as relaxations of Markov cubature rules by allowing negative transition probabilities. These are the contents of the following remark.

Remark 4.10. Suppose that the conclusion of Proposition 4.7 holds. Since \tilde{S} has rank M , there exists $A = (A_{ij}) \in \mathbb{R}^{M \times R}$ such that

$$\delta_{x_i} = \sum_{j=1}^R A_{ij} \nu_j$$

for all $i = 1, \dots, M$. Then we have

$$\begin{aligned}
\mathbb{E}_{x_i}[p(X_t)] &= \sum_{j=1}^R A_{ij} \mathbb{E}_{\nu_j}[p(X_t)] \\
&= \sum_{j=1}^R A_{ij} \mathbb{E}_{\nu_j} \left[\int_{E^Y} p(x) \tilde{Y}_t(dx) \right] \\
&= \sum_{j,k=1}^R A_{ij} (e^{tL})_{jk} \int_{E^Y} p(x) \nu_k(x) \\
&= \sum_{k=1}^R (Ae^{tL})_{ik} \int_{E^Y} p(x) \nu_k(x) \\
&= \sum_{k=1}^R \sum_{l=1}^M (Ae^{tL})_{ik} \tilde{S}_{kl} p(x_l) \\
&= \sum_{l=1}^M (Ae^{tL} \tilde{S})_{il} p(x_l)
\end{aligned} \tag{4.5}$$

for all $t \geq 0$, $p \in \text{Pol}_n(E)$ and $i = 1, \dots, M$. Hence, in this case the matrix

$$W(t) = Ae^{tL} \tilde{S} \in \mathbb{R}^{M \times M} \tag{4.6}$$

can be seen as a matrix of weights (not necessarily nonnegative) for the cubature points x_1, \dots, x_M at time t . From the identity

$$W(t)H = A\tilde{S}H \exp(tG) = H \exp(tG)$$

we deduce that the sums over the rows of $W(t)$ are equal to 1. Therefore, the matrix $W(t)$ could also be interpreted as a “transition probability matrix” of a cubature rule with possibly negative transition probabilities. Under this interpretation, Theorem 4.4 shows that, for polynomial processes, if asymptotic moments of order n exist then, by allowing negative probabilities, it is possible to find a relaxed version of a continuous time n -Markov cubature rule.

In general, given $0 \leq s \leq t$, $p, q \in \text{Pol}_n(E)$ with $pq \in \text{Pol}_n(E)$, and $x_i \in E^Y$

$$\mathbb{E}_{x_i}[p(X_t)q(X_s)] \neq \sum_{j=1}^R A_{ij} \mathbb{E}_{\nu_j} \left[\int_{E^Y} p(x) \tilde{Y}_t(dx) \int_{E^Y} q(x) \tilde{Y}_s(dx) \right].$$

Hence, we do not have a simple time-consistency property as the one of Proposition 2.9. Notice however that, under the assumptions above, if we define $\tilde{p}(x) = \mathbb{E}_x[p(X_{t-s})]$ then

$$\mathbb{E}_{x_i}[p(X_t)q(X_s)] = \mathbb{E}_{x_i}[\tilde{p}(X_s)q(X_s)] = \sum_{j=1}^R A_{ij} \mathbb{E}_{\nu_j} \left[\int_{E^Y} \tilde{p}(x) q(x) \tilde{Y}_s(dx) \right].$$

Applying (4.5) twice we obtain

$$\mathbb{E}_{x_i}[p(X_t)q(X_s)] = \sum_{l,m=1}^M p(x_m)q(x_l)W(t-s)_{ml}W(s)_{il}$$

with $W(t-s), W(s)$ as in (4.6). Therefore, allowing possibly negative transition probabilities, non-trivial lifted Markov cubature rules also match expectations of polynomials of the process X evaluated at different times.

Theorem 4.4 guarantees the existence of lifted Markov cubature rules under the assumption of existence of asymptotic moments. This asymptotic assumption together with classical cubature results facilitates, as explained in the next section, the construction of Markov cubature rules in discrete time.

5 Discrete time Markov cubature

The construction of a discrete time n -Markov cubature rule for X (see Theorem 5.1 below) will use cubature methods over the asymptotic moments. According to Theorem A.1, under (A1), all the asymptotic moments of order less than or equal to n exist. We denote these *asymptotic moments* by

$$\mu_j(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x[h_j(X_t)]. \quad (5.1)$$

To use classical cubature rules, we would like the asymptotic moments (5.1) to be independent of x . According to Corollary A.3, this is the case under the following assumption, which is a stronger condition than (A1).

(A2) For all nonzero eigenvalues λ of G , we have that $Re(\lambda) < 0$ and the eigenvalue 0 is a simple eigenvalue.

In this case we write the asymptotic moments (5.1) simply as μ_1, \dots, μ_{N_n} . In conjunction with (A2), we will make the following assumption throughout this section.

(A3) There exist points $x_1, \dots, x_M \in E$ and $w \in \mathbb{R}_{++}^M$ such that

$$\mu_j = \sum_{i=1}^M w_i h_j(x_i) \quad (5.2)$$

for all $j = 1, \dots, N_n$.

This condition states that the asymptotic moments (5.1) belong to $\text{conv}(\mathcal{H}_n(E))$. As $\mathbb{E}_x[h_j(X_t)] \in \text{conv}(\mathcal{H}_n(E))$ for all $x \in E, t \geq 0$ and $j = 1, \dots, N_n$ (see Putinar (1997), Bayer and Teichmann (2006)), this would be the case if for instance $\text{conv}(\mathcal{H}_n(E))$ is closed. It would also hold if the asymptotic moments are the moments of a probability distribution; see Proposition A.6. Additionally, as the weights w in (A3) are strictly positive, there does not exist a strict subset $C \subsetneq \{x_1, \dots, x_M\}$ such that $\text{conv}(\mathcal{H}_n(C))$ contains all the asymptotic moments (5.1).

Theorem 5.1 below is the main theorem of this section.

Theorem 5.1. *Assume that (A2) and (A3) hold. Suppose additionally that for the points x_1, \dots, x_M in (A3), the matrix H given by (3.1) satisfies $\text{rank}(H) = N_n$. Then, for Δ large enough, there exists a n -Markov cubature rule for X on $\{l\Delta : l \in \mathbb{N}\}$ with state space $E^Y = \{x_1, \dots, x_M\}$.*

To prove Theorem 5.1 we need the following lemma.

Lemma 5.2. *Suppose the same hypotheses of Theorem 5.1 hold. Then, for t sufficiently large, there exists a probability matrix with positive entries $Q(t)$ such that $H \exp(tG) = Q(t)H$.*

The following remark shows that the existence of discrete Markov cubature rules is true under more general hypotheses.

Remark 5.3. *Assume that (A1) holds. Suppose additionally that there exist points*

$$x_1, \dots, x_M \in E$$

and

$$W = (w_{ij})_{i,j=1}^M \in \mathbb{R}_{++}^{M \times M}$$

such that

$$\mu_j(x_k) = \sum_{i=1}^M w_{ki} h_j(x_i)$$

for all $j = 1, \dots, N_n$ and $k = 1, \dots, M$, with μ_j as in (5.1). The proof of Theorem 5.1 shows that, if the matrix $H = H(x_1, \dots, x_M)$, defined in (3.1), satisfies $\text{rank}(H) = N_n$, then the conclusion of Theorem 5.1 holds.

A Asymptotic moments of polynomial processes

Suppose that X is a polynomial process with extended generator \mathcal{G} and state space E . Fix $n \in \mathbb{N}$ and let G be the matrix of \mathcal{G} restricted to $\text{Pol}_n(E)$ with respect to a basis $\beta = (h_1, \dots, h_{N_n})$ of $\text{Pol}_n(E)$.

The following theorem shows that Assumption (A1) is equivalent to the existence of asymptotic moments of order n .

Theorem A.1. *The following are equivalent:*

- (i) *Assumption (A1) holds.*
- (ii) *The sequence of matrices $(\exp(tG))_{t \geq 0}$ converges as $t \rightarrow \infty$.*
- (iii) *$\mathbb{E}_x[h_j(X_t)]$ converges as $t \rightarrow \infty$ for all $x \in E$ and $j = 1, \dots, N_n$.*
- (iv) *$\mathbb{E}_x[p(X_t)]$ converges as $t \rightarrow \infty$ for all $x \in E$ and $p \in \text{Pol}_n(E)$.*

Proof. (i) \Leftrightarrow (ii) Suppose that $G = VJV^{-1}$, where J is the (complex) Jordan normal form of G . We have that $(\exp(tG))_{t \geq 0}$ converges as $t \rightarrow \infty$ if and only if $(\exp(tJ))_{t \geq 0}$ converges as $t \rightarrow \infty$. Additionally, $(\exp(tJ))_{t \geq 0}$ converges as $t \rightarrow \infty$ if and only if $\exp(tJ_i)$ converges for all i , where the J_i 's are the Jordan blocks of the matrix J .

Each J_i is of the form $J_i = \lambda_i Id + N_i$ where λ_i is an eigenvalue of G and N_i is a nilpotent matrix. Therefore, $\exp(tJ_i) = \exp(t\lambda_i)p_i(tN_i)$, with p_i a polynomial.

Assumption (A1) holds if and only if $Re(\lambda_i) < 0$ for all i such that $\lambda_i \neq 0$ and if $\lambda_i = 0$, $N_i = 0$. These observations imply the equivalence between (i) and (ii).

(ii) \Rightarrow (iii) Suppose that the matrices $(\exp(tG))_{t \geq 0}$ converge to a matrix $\tilde{P} \in \mathbb{R}^{N_n \times N_n}$ as $t \rightarrow \infty$. By (2.6), we have that

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[h_j(X_t)] = \sum_{i=1}^{N_n} \tilde{P}_{ij} h_i(x)$$

for all $j = 1, \dots, N_n$ and $x \in E$. Hence (iii) holds.

(iii) \Leftrightarrow (iv) This follows from the fact that h_1, \dots, h_{N_n} is a basis of $\text{Pol}_n(E)$.

(iii) \Rightarrow (ii) Suppose now that $\mathbb{E}_x[h_j(X_t)]$ converges for all $x \in E$ and $j = 1, \dots, N_n$, as t goes to infinity.

We claim that there exists N_n points, $x_1, \dots, x_{N_n} \in E$, such that for all $p \in \text{Pol}_n(E)$

$$p(x_i) = 0 \text{ for all } i \Rightarrow p \equiv 0. \quad (\text{A.1})$$

Assume for the sake of contradiction that there are no points $x_1, \dots, x_{N_n} \in E$ such that (A.1) holds. Let $p_1(x) \neq 0$ be a polynomial on E and $x_1 \in E$ such that $p_1(x_1) \neq 0$. By assumption, we can find $p_2 \in \text{Pol}_n(E)$ and $x_2 \in E$ such that $p_2(x_1) = 0$ and $p_2(x_2) \neq 0$. Recursively, we would be able to construct points x_1, \dots, x_{N_n} , and polynomials p_1, \dots, p_{N_n} such that

$$p_i(x_i) \neq 0 \text{ and } p_i(x_j) = 0 \text{ for } j < i. \quad (\text{A.2})$$

These polynomials would be linearly independent and hence a basis of $\text{Pol}_n(E)$.

Assume that $p \in \text{Pol}_n(E)$ satisfies $p(x_i) = 0$ for all i . As p is a linear combination of the polynomials p_i we would conclude by (A.2) that all the coefficients of the linear combination are equal to zero and p is zero everywhere, a contradiction.

Hence we can always find $x_1, \dots, x_{N_n} \in E$ such that (A.1) holds. These points allow us to define a norm on the space $\text{Pol}_n(E)$ by

$$\|p\|_1 = \sup_i |p(x_i)|.$$

Another norm is given by

$$\|p\|_2 = \sup_i |\lambda_i|$$

where $p = \sum_j \lambda_j h_j$. As these norms are equivalent, convergence of a sequence of polynomials on x_1, \dots, x_{N_n} implies convergence of the coefficients. The coefficients of the polynomials of the form $\mathbb{E}_x[h_j(X_t)]$ are entries of the matrix $\exp(tG)$. Hence (ii) holds. \square

In general, these asymptotic moments might depend on x . In fact we have the following proposition.

Proposition A.2. *Suppose that Assumption (A1) holds. Let $G = VJV^{-1}$ be the canonical Jordan decomposition of G , with V the matrix of generalized eigenvectors. Then*

$$\lim_{t \rightarrow \infty} e^{tG} = \sum_{i=1}^l v_i r_i, \quad (\text{A.3})$$

where the vectors v_1, \dots, v_l are the eigenvectors of G corresponding to the eigenvalue 0 and r_1, \dots, r_l are the rows of V^{-1} . Moreover, the asymptotic moments (5.1) are given by

$$(\mu_1(x), \dots, \mu_{N_n}(x)) = \sum_{i=1}^l \mathcal{H}_n(X)^\top v_i r_i. \quad (\text{A.4})$$

Proof. The proof of the equivalence between (i) and (ii) in Theorem A.1 shows that Assumption (A1) implies that

$$\lim_{t \rightarrow \infty} e^{tG} = V \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix} V^{-1},$$

where the identity matrix Id comes from the block corresponding to the eigenvalue 0. This implies (A.3). Moreover, (A.3), (3.3) and (5.1) imply (A.4). \square

An immediate corollary of these results characterizes the case when the asymptotic moments are independent of x .

Corollary A.3. *Assumption (A2) holds if and only if the asymptotic moments $\mu_1(x), \dots, \mu_{N_n}(x)$ as defined in (5.1) exist and they are independent of x , i.e. constant on E .*

Proof. We already have the equivalence between Assumption (A1) and the existence of the asymptotic moments by Theorem A.1. Moreover, observe that (A.4) in the previous proposition implies that for all $j = 1, \dots, N_n$

$$\mu_j(x) = \sum_{i=1}^l r_i(j) \tilde{h}_i(x),$$

where the eigen-polynomials $\tilde{h}_1, \dots, \tilde{h}_l$ (corresponding to the eigenvalue 0 of \mathcal{G}) are given by

$$\tilde{h}_i(x) = \mathcal{H}_n(x)^\top v_i,$$

for all $i = 1, \dots, l$. These polynomials are linearly independent, as polynomials in $\text{Pol}_n(E)$. This linear independence implies that $\mu_j(x)$ is constant on E for all j if and only if $l = 1$. \square

Example A.4. *Suppose that X is a polynomial martingale. This holds when $G_1 = 0$, where G_1 is the matrix of the generator restricted to the space $\text{Pol}_1(E)$. A particular example is geometric Brownian motion. In this case we have that $\mathbb{E}_x[X_t] = x$ for all $t \geq 0$ and $x \in E$, and hence,*

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[X_t] = x.$$

In this example, 0, as an eigenvalue of G_1 , has algebraic multiplicity 2.

Example A.5. Suppose that $d = 1$ and $\mathcal{G}f(x) = -xf'(x) + xf''(x)$. Then

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[X_t] = 0; \quad \lim_{t \rightarrow \infty} \mathbb{E}_x[X_t^2] = x^2.$$

In this example, 0 has multiplicity 1 as an eigenvalue of G_1 (the matrix of the generator \mathcal{G} restricted to $\text{Pol}_1(E)$.) However, 0 has algebraic multiplicity 2 as an eigenvalue of G_2 (the matrix of the generator \mathcal{G} restricted to $\text{Pol}_2(E)$).

The following proposition gives sufficient conditions under which the limiting moments $\mu_j(x)$ are moments of a positive Borel measure.

Proposition A.6. Let G_{n+1} be the matrix of the generator restricted to the space $\text{Pol}_{n+1}(E)$ with respect to an extended basis $\tilde{\beta} = (h_1, \dots, h_{N_n}, \dots, h_R)$ of $\text{Pol}_{n+1}(E)$. Assume that (A1) holds for G_{n+1} . Then for all $x \in E$ there exists a positive Borel measure π_x such that

$$\int_E h_j(y) \pi_x(dy) = \mu_j(x) \tag{A.5}$$

for all $j = 1, \dots, N_n$.

Proof. Let $x \in E$ and $j = 1, \dots, N_n$ be fixed. We have by Theorem A.1 that $\mathbb{E}_x[f(X_t)]$ converges as $t \rightarrow \infty$ for any polynomial $f \in \text{Pol}_{n+1}(E)$. Define $Y_t = h_j(X_t)$. De La Vallée-Poussin's theorem implies that $(Y_t)_{t \geq 0}$ is uniformly integrable. Additionally, we have that the sequence of Borel probability measures on E given by $(\mathbb{P}_x \circ X_t^{-1})_{t \geq 0}$ is tight.

Let π_x be an accumulation Borel probability measure of this sequence. We conclude that (A.5) holds. Indeed, assume with out loss of generality that $\mathbb{P}_x \circ X_t^{-1}$ converges in distribution to π_x . By Fatou's lemma

$$\begin{aligned} \int_E |h_j(y)| \pi(dy) &= \int_0^\infty \pi(|h_j(y)| > z) dz \\ &\leq \liminf_{t \rightarrow \infty} \int_0^\infty \mathbb{P}_x(|Y_t| > z) dz \\ &= \liminf_{t \rightarrow \infty} \mathbb{E}[|Y_t|] < \infty. \end{aligned}$$

Therefore, given $j = 1, \dots, N_n$ and $\epsilon > 0$, there exist constants $C, T > 0$ such that $\mathbb{E}_x[|Y_t| 1_{|Y_t| > C}] < \epsilon$ for all $t \geq 0$, $\int_{|h_j(y)| > C} |h_j(y)| \pi(dy) < \epsilon$ and for $t \geq T$

$$\left| \mathbb{E}_x[Y_t 1_{|Y_t| \leq C}] - \int_{|h_j(y)| \leq C} h_j(y) \pi(dy) \right| < \epsilon.$$

Hence, for $t \geq T$

$$\begin{aligned} \left| \mathbb{E}_x[Y_t] - \int_E H_j(y) \pi_x(dy) \right| &\leq \left| \mathbb{E}_x[Y_t 1_{|Y_t| \leq C}] - \int_{|h_j(y)| \leq C} h_j(y) \pi(dy) \right| \\ &\quad + \mathbb{E}_x[|Y_t| 1_{|Y_t| > C}] + \int_E |h_j(y)| 1_{|h_j(y)| > C} \pi(dy) \\ &\leq 3\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we obtain (A.5); see also Theorem 3.5 in Billingsley (1995). \square

In some cases the measure π_x of the proposition above is not necessarily an invariant measure.

Example A.7. *Suppose that X is an exponential Brownian motion. In particular X is a martingale and $\mathbb{E}_x[X_t] = x$ for all $t, x \geq 0$. Hence, $\mu_1(x) = x$, where μ_1 is the asymptotic mean. In this case $\pi_x = \delta_x$ which is not an invariant measure for $x > 0$.*

B Proofs

Proof of Lemma 2.7. In view of (2.5) we obtain the vector equation

$$\mathcal{H}_n(X_t) = \mathcal{H}_n(x) + \int_0^t G_n^\top \mathcal{H}_n(X_s) ds + M_t, \quad t \geq 0, \quad (\text{B.1})$$

for some local martingale M with N_n components. We claim that the expectation $\mathbb{E}[\|\mathcal{H}_n(X_t)\|]$ is locally bounded in t . This follows from the inequality

$$\mathbb{E}_x[1 + \|X_t\|^{2k}] \leq (1 + \|x\|^{2k})e^{Ct}, \quad t \geq 0,$$

which holds for some constant $C > 0$ that depends on \mathcal{G} but not on t or x . This inequality is proved using the argument in Cuchiero et al. (2012). Furthermore, in conjunction with Lemma B.1 below, this also implies that M is a true martingale. Taking expectations on both sides of (B.1) thus yields the integral equation

$$\mathbb{E}[\mathcal{H}_n(X_t)] = \mathcal{H}_n(x) + \int_0^t G_n^\top \mathbb{E}[\mathcal{H}_n(X_s)] ds, \quad t \geq 0,$$

whose solution is $\mathbb{E}[\mathcal{H}_n(X_t)] = e^{tG_n^\top} \mathcal{H}_n(x)$. This yields (2.6). \square

Lemma B.1. *Let $p \in \text{Pol}(E)$. The local martingale $M_t = p(X_t) - \int_0^t \mathcal{G}p(X_s) ds$ admits a predictable quadratic variation process, given by $\langle M, M \rangle_t = \int_0^t (\mathcal{G}p^2 - 2p\mathcal{G}p)(X_s) ds$.*

Proof. Squaring the expression for M_t and rearranging yields

$$\begin{aligned}
p(X_t)^2 - M_t^2 &= 2p(X_t) \int_0^t \mathcal{G}p(X_s) ds - \left(\int_0^t \mathcal{G}p(X_s) ds \right)^2 \\
&= 2 \left(M_t + \int_0^t \mathcal{G}p(X_s) ds \right) \int_0^t \mathcal{G}p(X_s) ds - \left(\int_0^t \mathcal{G}p(X_s) ds \right)^2 \\
&= 2M_t \int_0^t \mathcal{G}p(X_s) ds + \left(\int_0^t \mathcal{G}p(X_s) ds \right)^2 \\
&= 2 \int_0^t M_s \mathcal{G}p(X_s) ds + \left(\int_0^t \mathcal{G}p(X_s) ds \right)^2 + (\text{local martingale}) \\
&= 2 \int_0^t \left(p(X_s) - \int_0^s \mathcal{G}p(X_u) du \right) \mathcal{G}p(X_s) ds \\
&\quad + \left(\int_0^t \mathcal{G}p(X_s) ds \right)^2 + (\text{local martingale}) \\
&= 2 \int_0^t p(X_s) \mathcal{G}p(X_s) ds + (\text{local martingale}),
\end{aligned}$$

where the last equality follows from the identity $(\int_0^t g(s) ds)^2 = 2 \int_0^t g(s) \int_0^s g(u) du ds$ with $g(t) = \mathcal{G}p(X_t)$. Therefore, since p^2 is also a polynomial and hence in the domain of \mathcal{G} , we obtain

$$M_t^2 - \int_0^t (\mathcal{G}p^2(X_s) - 2p(X_s)\mathcal{G}p(X_s)) ds = (\text{local martingale}).$$

This implies the assertion of the lemma. \square

Proof of Proposition 2.9. Clearly if (2.7) holds then Y is an n -Markov cubature rule for X on \mathbf{T} . Conversely, suppose that Y is an n -Markov cubature rule for X on \mathbf{T} . By an induction argument it is enough to show (2.7) with $l = 2$. To this end, fix $p, q \in \text{Pol}_n(E)$ with $pq \in \text{Pol}_n(E)$ and let $s, t \in \mathbf{T}$ be such that $0 \leq s \leq t$. Define the function

$$\tilde{p}(x) = \mathbb{E}_x[p(X_{t-s})].$$

Since X is a polynomial process, by Lemma 2.7 the function \tilde{p} is a polynomial and $\tilde{p}q \in \text{Pol}_n(E)$. On the other hand, by the definition of a Markov cubature rule and the stability under differences of \mathbf{T} we have

$$\begin{aligned}
\mathbb{E}_x[\tilde{p}(X_s)q(X_s)] &= \mathbb{E}_x[\tilde{p}(Y_s)q(Y_s)], \\
\tilde{p}(x) &= \mathbb{E}_x[p(Y_{t-s})]
\end{aligned}$$

for all $x \in E^y$. Therefore, as X and Y are Markov processes, we conclude that

$$\begin{aligned}
\mathbb{E}_x[p(X_t)q(X_s)] &= \mathbb{E}_x[q(X_s)\mathbb{E}_{X_s}[p(X_{t-s})]] \\
&= \mathbb{E}_x[\tilde{p}(X_s)q(X_s)] \\
&= \mathbb{E}_x[\tilde{p}(Y_s)q(Y_s)] \\
&= \mathbb{E}_x[q(Y_s)\mathbb{E}_{Y_s}[p(Y_{t-s})]] \\
&= \mathbb{E}_x[p(Y_t)q(Y_s)]
\end{aligned}$$

for all $x \in E^Y$. □

Proof of Lemma 2.12. To simplify the notation throughout the proof for $\alpha = (\alpha_1, \dots, \alpha_d)^\top \in (\mathbb{N} \cup \{0\})^d$ we set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $|\alpha| = \sum_i \alpha_i$.

Given a polynomial $q \in \text{Pol}_k(\mathcal{H}_n(E))$, it is clear that $p(x) = q(\mathcal{H}_n(x))$ belongs to $\text{Pol}_{kn}(E)$. Conversely, suppose that $p \in \text{Pol}_{kn}(E)$. We want to show the existence of $q \in \text{Pol}_k(\mathcal{H}_n(E))$ such that $p(x) = q(\mathcal{H}_n(x))$. Without loss of generality we can assume that p is a monomial of the form $p(x) = x^\alpha$ with $\alpha \in (\mathbb{N} \cup \{0\})^d$ such that $|\alpha| \leq kn$. We use an inductive argument over k . For $k = 1$ the existence of q follows from the fact that h_1, \dots, h_{N_n} constitute a basis of $\text{Pol}_n(E)$.

Suppose now that for any $m \leq k$ and $\beta \in (\mathbb{N} \cup \{0\})^d$ with $|\beta| \leq mn$ we can find $q \in \text{Pol}_m(\mathcal{H}_n(E))$ such that $x^\beta = q(\mathcal{H}_n(x))$ on E . Assume that $\alpha \in (\mathbb{N} \cup \{0\})^d$ and $|\alpha| \leq (k+1)n$. There exist $\beta, \gamma \in (\mathbb{N} \cup \{0\})^d$ such that $\alpha = \beta + \gamma$, $|\beta| \leq kn$ and $|\gamma| \leq n$.

By induction hypothesis we find $q_1 \in \text{Pol}_k(\mathcal{H}_n(E))$ and $q_2 \in \text{Pol}_1(\mathcal{H}_n(E))$ such that $x^\beta = q_1(\mathcal{H}_n(x))$ and $x^\gamma = q_2(\mathcal{H}_n(x))$ on E . Since $x^\alpha = x^\beta x^\gamma = q_1(\mathcal{H}_n(x))q_2(\mathcal{H}_n(x))$, and $q_1 q_2$ has degree at most $k+1$, the conclusion follows. □

Proof of Proposition 2.11. Since X is a polynomial process, \overline{X} is also a polynomial process thanks to Lemma 2.12. The proof of Lemma 2.7 shows that the drift of $\mathcal{H}(X)$ is $G^\top \mathcal{H}(X)$ and the process M given in (2.8) is a martingale. □

Proof of Lemma 3.3. Denote by $v \in \mathbb{R}^{N_n}$ the coordinates of the constant polynomial $\mathbf{1}$ with respect to the basis $h_1(x), \dots, h_{N_n}(x)$. We have that

$$Hv = \mathbf{1}_M,$$

the vector of 1's in \mathbb{R}^M . Additionally by (3.2),

$$HGv = (\mathcal{G}\mathbf{1}(x_i))_{i=1}^M = 0.$$

Hence

$$L\mathbf{1}_M = LHv = HGv = 0.$$

□

Proof of Theorem 3.2. (i) \Rightarrow (ii) Let L be the transition rate matrix of the n -Markov cubature rule Y . Equations (3.3) and (2.2) imply that for all $i = 1, \dots, M$, $j = 1, \dots, N_n$ and $t \geq 0$

$$(H \exp(tG))_{ij} = (\exp(tL)H)_{ij}.$$

Hence, $H \exp(tG) = \exp(tL)H$ for all $t \geq 0$. Differentiating with respect to t and evaluating at $t = 0$ we obtain (ii).

(ii) \Leftrightarrow (iii) This follows directly from Lemma 3.3.

(ii) \Rightarrow (iv) By (3.2)

$$\text{the } i\text{-th row of } HG = \mathcal{G}\mathcal{H}_n(x_i)^\top$$

for all $i = 1, \dots, M$. On the other hand, the i -th row of LH can be written as a cone combination of the form

$$\sum_{j \neq i} L_{ij} (\mathcal{H}_n^\top(x_j) - \mathcal{H}_n^\top(x_i)), \quad (\text{B.2})$$

where the coefficients L_{ij} are nonnegative. Since $HG = LH$ we conclude (iv).

(iv) \Rightarrow (i) Condition (iv) implies the existence of coefficients $L_{ij} \geq 0$ for $i \neq j$ such that (B.2) is equal to the i -th row of HG for all i . Hence, we can find a transition rate matrix L such that $HG = LH$. This implies, by an induction argument, that

$$HG^l = L^l H \text{ for all } l \in \mathbb{N}.$$

This in turn implies that

$$H \exp(tG) = \exp(tL)H. \quad (\text{B.3})$$

Since $(\exp(tL))_{t \geq 0}$ defines a transition semigroup, we can define a Markov process with state space E^Y by

$$\mathbb{P}_{x_i}^Y(Y_t = x_j) = (\exp(tL))_{ij}.$$

Equations (3.3) and (B.3) imply that $\mathbb{E}_x[h_j(X_t)] = \mathbb{E}_x[h_j(Y_t)]$ for all $x \in E$ and $j = 1, \dots, N_n$, i.e. Y defines a continuous time n -Markov cubature rule.

Suppose now that $M = N_n$ and the matrix H is invertible. For all $j = 1, \dots, N_n$ define \tilde{h}_j as the polynomial whose coordinates with respect to the basis (h_1, \dots, h_{N_n}) are the j -th column of H^{-1} . We have that $\tilde{\beta} = (\tilde{h}_1, \dots, \tilde{h}_{N_n}) \subset \text{Pol}_n(E)$ is a basis that satisfies $\tilde{h}_j(x_i) = \delta_{ij}$.

Given a Markov cubature rule Y , Y is a cubature rule with respect to any basis of $\text{Pol}_n(E)$. In particular with respect to the basis $\tilde{\beta}$. Observe that in this case

$$\tilde{H} = (\tilde{h}_j(x_i))_{ij} = I_{N_n},$$

the identity matrix. Hence, the equivalence between (i) and (v) follows from the equivalence between (i) and (iii). \square

Proof of Lemma 4.5. (i) \Rightarrow (ii) Define $E^{\bar{Y}} = \{\bar{x}_1, \dots, \bar{x}_R\}$ as the rows of S . Let $Z_t = \mathbb{E}[\bar{X}_t]$ and denote by \mathcal{G}^Z its extended generator. By Proposition 2.11 we have that

$$(\mathcal{G}^Z p)(\bar{x}_i) = (SG\vec{p})_i.$$

for any $p \in \text{Pol}_1(\mathbb{R}^{N_n})$ and $i = 1, \dots, R$, where \vec{p} are the coordinates of p with respect to the canonical basis of $\text{Pol}_1(\mathbb{R}^{N_n})$. Hence since $SG = LS$ we get

$$(\mathcal{G}^Z \vec{p})(\bar{x}_i) = (LS\vec{p})_i,$$

for all $p \in \text{Pol}_1(\mathbb{R}^{N_n})$ and $i = 1, \dots, R$. Let \bar{Y} be the Markov Process on $E^{\bar{Y}}$ with transition rate matrix L . A similar argument as in the proof of Theorem 3.2 shows that \bar{Y} is a lifted n -Markov cubature for X on $[0, \infty)$. Additionally, since $\text{rank}(S) = N_n$ we have that $\text{span}(E^{\bar{Y}}) = \mathbb{R}^{N_n}$, and Y is a non-trivial lifted n -Markov cubature rule.

(ii) \Rightarrow (i) Define S with rows given by the points $\bar{x}_1, \dots, \bar{x}_R$. By the same argument as above, since \bar{Y} is a lifted n -Markov cubature for X on $[0, \infty)$, we deduce that $SG = LS$ where L is the transition rate matrix of the process \bar{Y} . Since $\text{span}(E^{\bar{Y}}) = \mathbb{R}^{N_n}$ we have $\text{rank}(S) = N_n$. \square

Proof of Theorem 4.4. We would like to show that the conditions of the theorem imply Lemma 4.5(i). To simplify the notation throughout the proof we set 0_k the vector of zeros in \mathbb{R}^k for $k \in \mathbb{N}$.

Without loss of generality via a similarity transformation we can assume that G^T is in real Jordan normal form. In other words G^T has the form

$$\begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & J_q \end{pmatrix}$$

where $J_1 = 0 \in \mathbb{R}^{l \times l}$ and each J_i for $i \geq 2$ is either a matrix of the form

$$\begin{pmatrix} \lambda & c & 0 & \cdots & 0 \\ 0 & \lambda & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & c \\ \cdots & \cdots & \cdots & \cdots & \lambda \end{pmatrix} \tag{B.4}$$

with $\lambda < 0$ and $c \in \{0, 1\}$, or a matrix of the form

$$\begin{pmatrix} a & b & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b & a & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a & b & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -b & a & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a & b & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -b & a & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & a & b \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -b & a \end{pmatrix} \tag{B.5}$$

with $a < 0$, $b \in \mathbb{R}$.

We can reduce the problem to the construction of vectors $\bar{x}_1, \dots, \bar{x}_R \in \mathbb{R}^{N_n}$, such that $\text{span}(\bar{x}_1, \dots, \bar{x}_R) = \mathbb{R}^{N_n}$, and nonnegative weights $(L_{ij})_{1 \leq i \neq j \leq R}$ such that

$$G^T \bar{x}_i = \sum_{j \neq i} L_{ij} (\bar{x}_j - \bar{x}_i) \tag{B.6}$$

for all $i = 1, \dots, R$.

We will do the construction by induction on the number of Jordan blocks. For the Jordan blocks of zeros, $0 \in \mathbb{R}^{l \times l}$, we consider the canonical basis of \mathbb{R}^l with weights all equal to 0 –this corresponds to the base case in the induction argument.

Now suppose that we have q -blocks. Assume that the vectors $\bar{x}_1, \dots, \bar{x}_{R_{q-1}}$ and weights $(L_{ij})_{i \neq j \leq R_{q-1}}$ satisfy (B.6) for the first $q-1$ blocks. Suppose first that J_q has size $s \times s$ and that

$$\text{span}(\bar{x}_1, \dots, \bar{x}_{R_{q-1}}) = \mathbb{R}^{N_n - s}.$$

We consider different cases.

- (i) Assume first that J_q has the form (B.4). We can take in a recursive manner $u(1) = 1$ and $u(j) > 0$, for all $j = 2, \dots, s$, such that

$$u(j)\lambda + cu(j+1) < 0$$

for all $j < s$.

We make the following observation: suppose that $f(j), f(j+1) \in \{-1, 1\}$ are arbitrary. Then

$$\text{sign}(f(j)u(j)\lambda + f(j+1)cu(j+1)) = -f(j).$$

Hence, in this case

$$\text{sign}((J_q u_f)(i)) = -f(i),$$

where

$$u_f = (f(j)u(j))_j.$$

We have that for each $f \in \{-1, 1\}^s$

$$J_q u_f = (-f(j)v_f(j))_j$$

for some $v(f) = (v_f(j))_j$ with positive entries. Define $g(f, j)(j) = -f(j)$ and $g(f, j)(i) = f(i)$ for $i \neq j$. We can write then

$$J_q u_f = \sum_{j \leq s} \frac{v_f(j)}{2u(j)} (u_{g(f, j)} - u_f).$$

This expression is of the form

$$J_q u_f = \sum_{g \neq f} w_{g, f} (u_g - u_f),$$

with the weights $w_{g, f} \geq 0$.

If we consider the following vectors in \mathbb{R}^{N_n} :

$$\left\{ \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{r_q} \\ \vdots \\ 0_s \end{pmatrix}, \dots, \begin{pmatrix} \bar{x}_{r_q} \\ \vdots \\ \bar{x}_1 \\ \vdots \\ 0_s \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0_{N_n - s} \\ u_f \end{pmatrix} : f \in \{-1, 1\}^s \right\},$$

we deduce that (B.6) is satisfied for some weights. Additionally, by induction hypothesis and the configuration of the vectors of the form u_f , which form a hypercube in \mathbb{R}^s , we also deduce that this set of vectors spans \mathbb{R}^{N_n} .

- (ii) Suppose that J_q of size $s \times s$ has the form (B.5). We first consider the case when $s = 2$ and J_q has the form

$$J_q = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \kappa \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix},$$

with $\kappa > 0$ and $\varphi \in (\pi/2, \pi]$, so that $\cos \varphi < 0$. This matrix represents a counter clockwise rotation by φ followed by a scaling of κ – the clockwise rotation case can be treated similarly.

Let m be the smallest positive integer greater than 2 such that

$$\pi(m+2)/2m \leq \varphi.$$

One can show that in this case if we consider the regular polygon of m -sides in \mathbb{R}^2 centred at the origin, then at all the vertices in the polygon v_1, \dots, v_m , $J_q v_i$ points inside the polygon.

Therefore in this case

$$\left\{ \begin{pmatrix} \bar{x}_1 \\ 0_s \end{pmatrix}, \dots, \begin{pmatrix} \bar{x}_{r_q} \\ 0_s \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0_{n-s} \\ v_i \end{pmatrix} : i = 1, \dots, m \right\},$$

satisfies (B.6) for some weights and this set of vectors spans \mathbb{R}^{N_n} .

- (iii) Next, assume that J_q has the form

$$\begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}$$

with the submatrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ as before.

Let $v_1, \dots, v_m \in \mathbb{R}^2$ be as before. For each $i = 1, \dots, m$ consider the transformation $T_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T_i u = \begin{pmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v_i \end{pmatrix}.$$

For $\|u\|$ sufficiently large we can bound the direction of the vector $T_i u$ with respect to the vector u by angles $\varphi_1 < \varphi_2 \in (\pi/2, \pi]$.

By a similar argument as the one used before, we can construct an m' -regular polygon with vertices $u_1, \dots, u_{m'}$ such that $T_i u_k$ points inside the polygon for all $k \leq m'$.

We conclude that at each the points

$$\left\{ \begin{pmatrix} u_i \\ v_j \end{pmatrix} : i \leq m', j \leq m \right\},$$

$J_q \begin{pmatrix} u_i \\ v_j \end{pmatrix}$ points inside the convex hull of these points.

Additionally these points span \mathbb{R}^4 . Indeed, the differences between the vertices of the regular polygon span \mathbb{R}^2 and we can span these differences concatenated with 0's by using the set specified above.

Therefore

$$\left\{ \begin{pmatrix} \bar{x}_1 \\ 0_s \end{pmatrix}, \dots, \begin{pmatrix} \bar{x}_{r_q} \\ 0_s \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0_{N_n-s} \\ u_i \\ v_j \end{pmatrix} : i \leq m', j \leq m \right\},$$

satisfies (B.6) for some weights and spans \mathbb{R}^{N_n} .

- (iv) The construction of the points of the form $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ presented above can be carried out recursively for a matrix of the form (B.5) to complete the induction argument.

□

Proof of Lemma 4.8. It is clear that if there exists $\tilde{S} \in \mathbb{R}^{R \times M}$, with $\text{rank}(\tilde{S}) = M$, such that $S = \tilde{S}H$, then $\text{rank}(S) = N$. Suppose now that $\text{rank}(S) = N$. Assume without loss of generality that if $v_1, \dots, v_R \in \mathbb{R}^N$ are the columns of S^\top , then v_1, \dots, v_N are linearly independent. By assumption there exist $w_1, \dots, w_N \in \mathbb{R}^M$ such that $H^\top w_i = v_i$ for all $i = 1, \dots, N$. We have that $w_1, \dots, w_N \in \mathbb{R}^M$ are linearly independent as well. Let $x_{N+1} \dots x_M$ be a basis for the kernel of H^\top . Suppose that $v_i = \sum_{k=1}^N \alpha_k^i v_k$ for $i = N+1, \dots, M$. We define $w_i = (\sum_{k=1}^N \alpha_k^i v_k) + x_i$ for $i = N+1, \dots, M$. It can be shown that w_1, \dots, w_M are linearly independent. Finally, for $i = M+1, \dots, R$ we define w_i as any solution of the equation $H^\top w_i = v_i$. Let $\tilde{S} \in \mathbb{R}^{R \times M}$ be the matrix with rows $w_1^\top, \dots, w_R^\top$. Then $\text{rank}(\tilde{S}) = M$ and $S = \tilde{S}H$. □

Proof of Proposition 4.7. Suppose that Lemma 4.5 (i) holds. Since H has rank N_n , by Lemma 4.8, there exists $\tilde{S} \in \mathbb{R}^{R \times M}$, with $\text{rank}(\tilde{S})$, such that $S = \tilde{S}H$.

For $i = 1, \dots, R$, define ν_i as in (4.2). We have that $\tilde{S}HG = L\tilde{S}H$, with L a transition rate matrix. Let $E^{\tilde{Y}} = \{\nu_1, \dots, \nu_R\}$ and \tilde{Y} be the Markov process on $E^{\tilde{Y}}$ with transition rate matrix L . By similar arguments as in the proof of Theorem 3.2 we conclude that (4.3) holds.

Assume now that (iii) holds. Suppose that the signed measure ν_1, \dots, ν_R are given by (4.2) and define $\tilde{S} = (\tilde{S}_{ij}) \in \mathbb{R}^{R \times M}$.

A similar argument as in the proof of Theorem 3.2 shows that (4.3) implies that $\tilde{S}HG = L\tilde{S}H$, where L is the transition rate matrix of \tilde{Y} . We set $S = \tilde{S}H$ and deduce (4.1). Additionally, $\text{rank}(S) = N_n$ by Lemma 4.8. □

Proof of Lemma 5.2. Equations (3.3), (5.1) and (5.2) imply that

$$\lim_{t \rightarrow \infty} He^{tG} = W^\top H,$$

where $W \in \mathbb{R}^{M \times M}$ is the matrix with all columns equal to w . Since H has rank N_n the set

$$\mathcal{B} = \{\widetilde{W}H : \widetilde{W} \in \mathbb{R}^{M \times M} \text{ has positive entries}\}$$

is an open set in $\mathbb{R}^{M \times N_n}$. Then, for t large enough

$$HP(t) = Q(t)H \in \mathcal{B}. \quad (\text{B.7})$$

The argument to show that the rows of $Q(t)$ add up to 1 is similar to that of Lemma 3.3. Suppose that $v \in \mathbb{R}^{N_n}$ are the coordinates of the constant polynomial $\mathbf{1}$ with respect to the basis h_1, \dots, h_{N_n} . We have that $\mathbf{1}_M = Hv$, where $\mathbf{1}_M \in \mathbb{R}^M$ is the vector of 1's. Since $\mathbf{1}$ is an eigenvalue of \mathcal{G} with corresponding eigenvalue 0, v is an eigenvector of $P(t)$ with eigenvalue 1. Hence, $HP(t)v = \mathbf{1}_M$. This observation together with (B.7) implies that $Q(t)\mathbf{1}_M = Q(t)Hv = \mathbf{1}_M$ and the columns of $Q(t)$ add up to 1, i.e. $Q(t)$ is a probability matrix. \square

Proof of Theorem 5.1. Lemma 5.2 guarantees that for Δ large enough, there exists a probability matrix $Q \in \mathbb{R}^{M \times M}$ such that

$$He^{\Delta G} = QH, \quad (\text{B.8})$$

with H defined in (3.1).

Let Y be the time-homogeneous Markov Process with transition probability matrix Q as in (B.8) and state space $E^Y = \{x_1, \dots, x_M\}$. By (3.3), Y is an n -Markov cubature for X on $\{\Delta\}$. Remark 2.10 implies that Y is also an n -Markov cubature for X $\{l\Delta : l \in \mathbb{N}\}$. \square

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