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# Optimal non-outage probability maximizing the outage rate in Rayleigh channel

Vincent Savaux, Isabelle Siaud, Rodolphe Legouable

In this paper, we derive an analytic closed-form of the optimal non-outage probability value which maximizes the outage rate in a Rayleigh channel. Furthermore, asymptotic approximations of both the optimal non-outage probability and the maximum outage rate can be deduced from the proposed analysis.

**Introduction:** The outage capacity is defined as the maximum data rate which is achieved for a specified outage probability [1, 2]. Thus, the outage capacity, unlike the ergodic capacity, takes into account the decoding errors due to the channel. From this definition of the capacity, the outage rate is expressed as

$$C_o = \mathbb{P}_{no} B \log_2(1 + \gamma_{min}), \quad (1)$$

where  $B$  is the bandwidth of the received signal, and

$$\mathbb{P}_{no} = \mathbb{P}(\gamma > \gamma_{min}), \quad (2)$$

is the non-outage probability<sup>1</sup>, i.e. the probability that the transmitted signal is properly decoded, where  $\gamma$  is the signal-to-noise ratio (SNR), and  $\gamma_{min}$  is a fixed threshold. In [2], the maximization of the outage rate in presence of Rayleigh channel is performed through simulations results only.

In this paper, we extend the work in [2], since we provide an analytical expression of the non-outage probability  $\mathbb{P}_{no}$  which maximizes the outage rate in Rayleigh channel. Furthermore, we develop an asymptotic expansion of the obtained solution, which reveals that the optimal  $\mathbb{P}_{no}$  value is bounded by 1 and  $1/e$ . Next Section is devoted to the proposed analysis.

**Optimal Non-Outage Probability:** In this paper, the SNR, which is measured at a distance  $r$  from the transmit antenna, is defined by the log-distance path loss model as

$$\gamma = \gamma_0 \left(\frac{r}{d_0}\right)^{-\alpha}, \quad (3)$$

where  $\gamma_0$  is the SNR measured at distance  $d_0$  from the transmit antenna, and obeys a given probability process according to the channel models,  $\alpha$  is the path-loss exponent. In Rayleigh channel,  $\gamma_0$  is a random variable, the probability density function (pdf) of which is given by the exponential distribution:

$$f_{\gamma_0}(\gamma_0) = \frac{1}{2\bar{\gamma}_0} \exp\left(-\frac{\gamma_0}{2\bar{\gamma}_0}\right), \quad (4)$$

where  $\bar{\gamma}_0$  is the average SNR value. Other models as Nakagami (for the multipath fading), log-normal (for the shadowing), and a composite multipath/shadowing are provided in [3]. From (4) and (2), the non-outage probability  $\mathbb{P}_{no}$  can be expressed as

$$\begin{aligned} \mathbb{P}_{no} &= \mathbb{P}\left(\gamma_0 \left(\frac{r}{d_0}\right)^{-\alpha} > \gamma_{min}\right) \\ &= \mathbb{P}\left(\gamma_0 > \underbrace{\gamma_{min} \left(\frac{r}{d_0}\right)^\alpha}_A\right) \\ &= \int_A^{+\infty} \frac{\gamma_0}{2\bar{\gamma}_0} \exp\left(-\frac{\gamma_0}{2\bar{\gamma}_0}\right) d\gamma_0 \\ &= \exp\left(-\frac{\gamma_{min} \left(\frac{r}{d_0}\right)^\alpha}{2\bar{\gamma}_0}\right). \end{aligned} \quad (5)$$

If a target non-outage probability, which is denoted by  $\mathbb{P}_{no}^t$ , is set, then the threshold  $\gamma_{min}$  can be written as a function of  $\mathbb{P}_{no}^t$ . Thus, we have:

<sup>1</sup> Note that, in this paper, we use the non-outage probability  $\mathbb{P}_{no}$  instead of  $(1 - \mathbb{P}_o)$  in the definition (1), where  $\mathbb{P}_o$  is the outage probability.

$$\gamma_{min} = -2\bar{\gamma}_0 \left(\frac{r}{d_0}\right)^{-\alpha} \ln\left(\mathbb{P}_{no}^t\right). \quad (6)$$

The non-outage rate can be rewritten by substituting (6) into (1):

$$C_o = \mathbb{P}_{no}^t B \log_2\left(1 - 2\bar{\gamma}_0 \left(\frac{r}{d_0}\right)^{-\alpha} \ln\left(\mathbb{P}_{no}^t\right)\right). \quad (7)$$

It can be noted that  $C_o$  is a strictly decreasing function with the variable  $r$ , and a concave function with the variable  $\mathbb{P}_{no}^t$  such that  $C_o(\mathbb{P}_{no}^t = 0) = C_o(\mathbb{P}_{no}^t = 1) = 0$ . Furthermore, it is straightforwardly shown in Appendix that, by solving the equation  $\frac{\partial C_o}{\partial \mathbb{P}_{no}^t} = 0$ , the maximum outage rate is achieved for

$$\mathbb{P}_{no}^t = \exp\left(\frac{1 - \exp(W(X))}{X}\right), \quad (8)$$

where  $X = 2\bar{\gamma}_0 \left(\frac{r}{d_0}\right)^{-\alpha}$ , and  $W$  is the Lambert W function, which is defined as the solution of the equation  $a = xe^x$  [4]. The substitution of (8) into (7) leads to the maximum outage rate:

$$C_o = \frac{B}{\ln(2)} W(X) \exp\left(\frac{1 - \exp(W(X))}{X}\right). \quad (9)$$

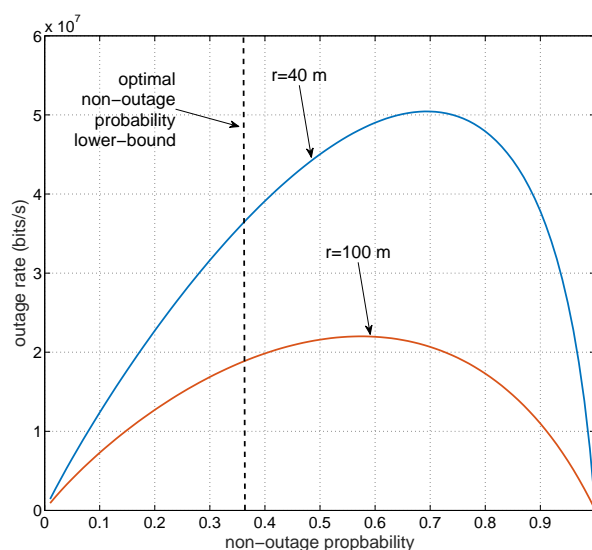
The asymptotic expansions of  $W(X)$  in 0 and  $+\infty$  (corresponding to cases where the SNR  $\gamma$  is weak and large, respectively) leads to the following limits (see Appendix for more details):

$$\lim_{X \rightarrow 0} \mathbb{P}_{no}^t = e^{-1}, \quad (10)$$

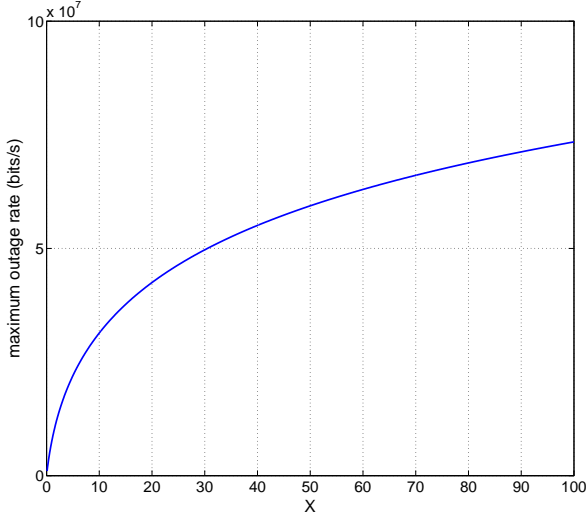
$$\lim_{X \rightarrow \infty} \mathbb{P}_{no}^t = 1. \quad (11)$$

Result in (11) reveals that a very large  $\gamma$  value (similarly  $X$  value) leads to an optimal non-outage probability equal to 1, i.e. the signal can be perfectly decoded when the SNR tends to infinity. On the contrary, the asymptotic analysis in (10) reveals that the optimal non-outage probability is not 0 but  $e^{-1}$  when the SNR tends to zero. This reflects the fact that the term  $\mathbb{P}_{no}^t B$  is dominating compared with the  $\log_2(\cdot)$  in (7), and therefore a non-zero  $\mathbb{P}_{no}^t$  value is mandatory in order to maximize the outage rate  $C_o$ .

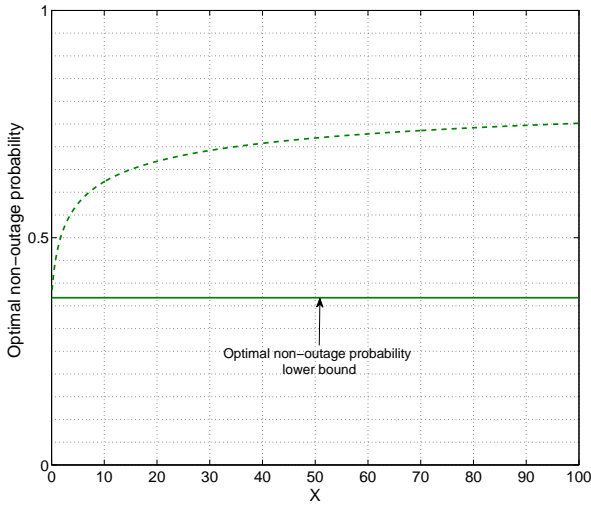
**Numerical Results:** In all the presented simulations results, the following parameters have been chosen:  $B = 20$  MHz,  $d_0 = 5$  m,  $\bar{\gamma}_0 = 30$  dB, and  $\alpha = 2$ . Fig. 1 depicts the outage rate as a function of  $\mathbb{P}_{no}^t$ , considering a Rayleigh channel, and for  $r = 40$  m and  $r = 100$  m. It can be seen that, as expected, the outage rate is a concave function, which reaches 0 at  $\mathbb{P}_{no}^t = 0$  and  $\mathbb{P}_{no}^t = 1$ . In the latter case when the non-outage probability tends to 1, only low rate signals (robust) can be transmitted in order to be properly decoded. Furthermore, one can observe that the lowest the distance  $r$  (namely the higher  $X$ ), the largest the optimal  $\mathbb{P}_{no}^t$  value.



**Fig. 1** Outage rate versus the non-outage probability  $\mathbb{P}_{no}$ , in Rayleigh channels.



(a) Maximum outage rate versus  $X$ .



(b) Optimal  $\mathbb{P}_{no}$  value versus  $X$ .

**Fig. 2** Maximum outage rate and optimal  $\mathbb{P}_{no}$  value versus  $X$ , in Rayleigh channels.

Fig. 2-(a) shows the maximum outage rate (9) and Fig. 2-(b) the optimal  $\mathbb{P}_{no}^t$  value (8) versus  $X$ . The lowest  $X$  values correspond to cases where  $\bar{\gamma}_0$  is weak or  $r$  is large, whereas the largest  $X$  values correspond to the opposite cases. From the asymptotic expansion given in Appendix, we can deduce that  $\mathbb{P}_{no}^t \approx \exp\left(\frac{1-X/\ln(X)}{X}\right)$  and  $C_0 \approx \frac{B}{\ln(2)} \ln(X)$  when  $X$  tends to infinity. Furthermore, once again, it can be noticed that the optimal  $\mathbb{P}_{no}^t$  is lower-bounded by  $1/e$ .

**Conclusion:** In this letter, we have proposed an analysis of the maximum achievable outage rate. To this end, we have derived the closed-form expression of the corresponding optimal non-outage probability. Asymptotic expansions allow us to approximate these expressions in limit cases. The provided theoretical results have been illustrated through simulations.

**Appendix:**

#### Derivation of $C_0$

The derivative of  $C_0$  with respect to  $\mathbb{P}_{no}^t$  can be written as:

$$\frac{\partial C_0}{\partial \mathbb{P}_{no}^t} = B \ln(1 - X \ln(\mathbb{P}_{no}^t)) - \frac{BX}{1 - X \ln(\mathbb{P}_{no}^t)}, \quad (12)$$

where  $X = 2\bar{\gamma}_0 \left(\frac{r}{d_0}\right)^{-\alpha}$ . By setting  $Y = 1 - X \ln(\mathbb{P}_{no}^t)$ , we deduce that the equality  $\frac{\partial C_0}{\partial \mathbb{P}_{no}^t} = 0$  leads to the following equation:

$$\ln(Y)Y = X. \quad (13)$$

Therefore, the solution of (13) maximizes the outage rate, since  $C_0$  is a concave function. The above equation (13) can be rewritten as  $\ln(Y)e^{\ln(Y)} = X$ , and one can recognize the Lambert's transcendental equation, which is solved by:  $Y = \exp(W(X))$ . Since  $\mathbb{P}_{no}^t = \exp\left(\frac{1-Y}{X}\right)$ , this proves the expression of  $\mathbb{P}_{no}^t$  given in (8).

#### Asymptotic expansion of $\mathbb{P}_{no}$

The series expansion of  $W$  Lambert function in 0 and  $+\infty$  are, respectively:

$$W(x) = \sum_{n=1}^{+\infty} \frac{(-n)^{n-1}}{n!} x^n, \quad (14)$$

and

$$W(x) = \ln(x) - \ln(\ln(x)) + o(\ln(\ln(x))). \quad (15)$$

The substitution of (14) into the exponential function, and the series expansion of the exponential lead to:

$$\begin{aligned} \exp\left(\sum_{n=1}^{+\infty} \frac{(-n)^{n-1}}{n!} X^n\right) &= \prod_{n=1}^{+\infty} \exp\left(\frac{(-n)^{n-1}}{n!} X^n\right) \\ &= \prod_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-n)^{k(n-1)}}{n!^k} X^{nk} \\ &= \prod_{n=1}^{+\infty} \left(1 + \sum_{k=1}^{+\infty} \frac{(-n)^{k(n-1)}}{n!^k} X^{nk}\right) \\ &= 1 + X + R_{n,k}, \end{aligned} \quad (16)$$

with  $R_{n,k}$  is the rest containing all the terms in  $X^p$ , where  $p > 1$ . Therefore, the limit of  $\mathbb{P}_{no}$  in zero can be expressed as

$$\lim_{X \rightarrow 0} \mathbb{P}_{no} = \lim_{X \rightarrow 0} \exp\left(\frac{1 - (1 + X + R_{n,k})}{X}\right) = e^{-1}. \quad (17)$$

The limit of  $\mathbb{P}_{no}$  in  $+\infty$  can be expressed by using (15) as

$$\begin{aligned} \lim_{X \rightarrow +\infty} \mathbb{P}_{no} &= \lim_{X \rightarrow +\infty} \exp\left(\frac{1 - \exp(\ln(X) - \ln(\ln(X)))}{X}\right) \\ &= \lim_{X \rightarrow +\infty} \exp\left(\frac{1 - X/\ln(X)}{X}\right) = 1, \end{aligned} \quad (18)$$

which concludes the proof of (11).

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