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NON-COOPERATIVE FISHER–KPP SYSTEMS: TRAVELING WAVES AND LONG-TIME BEHAVIOR

LÉO GIRARDIN

ABSTRACT. This paper is concerned with non-cooperative parabolic reaction–diffusion systems which share structural similarities with the scalar Fisher–KPP equation. These similarities make it possible to prove, among other results, an extinction and persistence dichotomy and, when persistence occurs, the existence of a positive steady state, the existence of traveling waves with a half-line of possible speeds and a positive minimal speed and the equality between this minimal speed and the spreading speed for the Cauchy problem. Non-cooperative KPP systems can model various phenomena where the following three mechanisms occur: local diffusion in space, linear cooperation and superlinear competition.

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Laboratoire Jacques-Louis Lions, CNRS UMR 7598, Université Pierre et Marie Curie, 4 place Jussieu, 75005 Paris, France.

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1. INTRODUCTION

In this paper, we study a large class of parabolic reaction–diffusion systems whose prototype is the so-called Lotka–Volterra mutation–competition–diffusion system:

$$\left\{ \begin{array}{l} \partial_t u_1 - d_1 \partial_{xx} u_1 = r_1 u_1 - \left(\sum_{j=1}^N c_{1,j} u_j \right) u_1 - \mu u_1 + \mu u_2 \\ \partial_t u_2 - d_2 \partial_{xx} u_2 = r_2 u_2 - \left(\sum_{j=1}^N c_{2,j} u_j \right) u_2 - 2\mu u_2 + \mu u_1 + \mu u_3 \\ \vdots \\ \partial_t u_N - d_N \partial_{xx} u_N = r_N u_N - \left(\sum_{j=1}^N c_{N,j} u_j \right) u_N - \mu u_N + \mu u_{N-1} \end{array} \right.$$

where N is an integer larger than or equal to 2 and the coefficients d_i , r_i , $c_{i,j}$ (with $i, j \in \{1, \dots, N\}$) and μ are positive real numbers.

This system can be understood as an ecological model, where (u_1, \dots, u_N) is a metapopulation density phenotypically structured, $\mu u_{i-1} - \mu u_i$ and $\mu u_{i+1} - \mu u_i$ are the step-wise mutations of the i -th phenotype with a mutation rate μ , d_i is its dispersal rate, r_i is its growth rate per capita in absence of mutation, $c_{i,j}$ is the rate of the competition exerted by the j -th phenotype on the i -th phenotype, $\frac{r_i}{c_{i,i}}$ is the carrying capacity of the i -th phenotype in absence of mutation and interphenotypic competition.

We are especially interested in spreading properties which describe the invasion of the population in an uninhabited environment and which are expected to involve so-called traveling wave solutions. Such solutions were first studied, independently and both in 1937, by Fisher [28] on one hand and by Kolmogorov, Petrovsky and Piskunov [35] on the other hand for the equation that is now well-known as the Fisher–KPP equation, Fisher equation or KPP equation:

$$\partial_t u - \partial_{xx} u = u(1 - u).$$

While a lot of work has been accomplished about traveling waves and spreading properties for scalar reaction–diffusion equations, the picture is much less complete regarding coupled systems of reaction–diffusion equations. In particular, almost nothing is known about non-cooperative systems like the system above.

Before going any further, let us introduce more precisely the problem.

1.1. Notations. Let $(n, n') \in (\mathbb{N} \cap [1, +\infty))^2$. The set of the first n positive integers $[1, n] \cap \mathbb{N}$ is denoted $[n]$ (and $[0] = \emptyset$ by convention).

1.1.1. Typesetting conventions. In order to ease the reading, we reserve the italic typeface (x, f, X) for reals, real-valued functions or subsets of \mathbb{R} , the bold typeface (\mathbf{v}, \mathbf{A}) for euclidean vectors or vector-valued functions, in lower case for column

vectors and in upper case for other matrices¹, the sans serif typeface in upper case (B, K) for subsets of euclidean spaces² and the calligraphic typeface in upper case (\mathcal{C} , \mathcal{L}) for functional spaces and operators.

1.1.2. Linear algebra notations.

- The canonical basis of \mathbb{R}^n is denoted $(\mathbf{e}_{n,i})_{i \in [n]}$. The euclidean norm of \mathbb{R}^n is denoted $|\bullet|_n$. The open euclidean ball of center $\mathbf{v} \in \mathbb{R}^n$ and radius $r > 0$ and its boundary are denoted $\mathbf{B}_n(\mathbf{v}, r)$ and $\mathbf{S}_n(\mathbf{v}, r)$ respectively.
- The space \mathbb{R}^n is equipped with one partial order \geq_n and two strict partial orders $>_n$ and \gg_n , defined as:

$$\mathbf{v} \geq_n \hat{\mathbf{v}} \text{ if } v_i \geq \hat{v}_i \text{ for all } i \in [n],$$

$$\mathbf{v} >_n \hat{\mathbf{v}} \text{ if } \mathbf{v} \geq_n \hat{\mathbf{v}} \text{ and } \mathbf{v} \neq \hat{\mathbf{v}},$$

$$\mathbf{v} \gg_n \hat{\mathbf{v}} \text{ if } v_i > \hat{v}_i \text{ for all } i \in [n].$$

The strict orders $>_n$ and \gg_n coincide if and only if $n = 1$.

A vector $\mathbf{v} \in \mathbb{R}^n$ is *nonnegative* if $\mathbf{v} \geq_n \mathbf{0}$, *nonnegative nonzero* if $\mathbf{v} >_n \mathbf{0}$, *positive* if $\mathbf{v} \gg_n \mathbf{0}$. The sets of all nonnegative, nonnegative nonzero and positive vectors are respectively denoted \mathbf{K}_n , \mathbf{K}_n^+ and \mathbf{K}_n^{++} .

- The sets $\mathbf{K}_n^+ \cap \mathbf{S}_n(\mathbf{0}, 1)$ and $\mathbf{K}_n^{++} \cap \mathbf{S}_n(\mathbf{0}, 1)$ are respectively denoted $\mathbf{S}_n^+(\mathbf{0}, 1)$ and $\mathbf{S}_n^{++}(\mathbf{0}, 1)$.
- For any $X \subset \mathbb{R}$, the sets of X -valued matrices of dimension $n \times n'$ and $n \times n$ are respectively denoted $\mathbf{M}_{n,n'}(X)$ and $\mathbf{M}_n(X)$. If $X = \mathbb{R}$ and if the context is unambiguous, we simply write $\mathbf{M}_{n,n'}$ and \mathbf{M}_n . As usual, the entry at the intersection of the i -th row and the j -th column of the matrix $\mathbf{A} \in \mathbf{M}_{n,n'}$ is denoted $a_{i,j}$ and the i -th component of the vector $\mathbf{v} \in \mathbb{R}^n$ is denoted v_i . For any vector $\mathbf{v} \in \mathbb{R}^n$, $\text{diag} \mathbf{v}$ denotes the diagonal matrix whose i -th diagonal entry is v_i .
- Matrices are vectors and consistently we may apply the notations $\geq_{nn'}$, $>_{nn'}$ and $\gg_{nn'}$ as well as the vocabulary nonnegative, nonnegative nonzero and positive to matrices. We emphasize this convention because of the possible confusion with the notion of ‘‘positive definite square matrix’’.
- A matrix $\mathbf{A} \in \mathbf{M}_n$ is *essentially nonnegative*, *essentially nonnegative nonzero*, *essentially positive* if $\mathbf{A} - \min_{i \in [n]} (a_{i,i}) \mathbf{I}_n$ is nonnegative, nonnegative nonzero, positive respectively.
- The identity of \mathbf{M}_n and the element of $\mathbf{M}_{n,n'}$ whose every entry is equal to 1 are respectively denoted \mathbf{I}_n and $\mathbf{1}_{n,n'}$ ($\mathbf{1}_n$ if $n = n'$).
- We recall the definition of the Hadamard product of a pair of matrices $(\mathbf{A}, \mathbf{B})^2 \in (\mathbf{M}_{n,n'})^2$:

$$\mathbf{A} \circ \mathbf{B} = (a_{i,j} b_{i,j})_{(i,j) \in [n] \times [n']}.$$

The identity matrix under Hadamard multiplication is $\mathbf{1}_{n,n'}$.

- The spectral radius of any $\mathbf{A} \in \mathbf{M}_n$ is denoted $\rho(\mathbf{A})$. Recall from the Perron–Frobenius theorem that if \mathbf{A} is nonnegative and irreducible, $\rho(\mathbf{A})$ is the dominant eigenvalue of \mathbf{A} , called the *Perron–Frobenius eigenvalue*

¹This convention being superseded by the previous one when the dimension is specifically equal to 1.

²Same exception.

$\lambda_{PF}(\mathbf{A})$, and is the unique eigenvalue associated with a positive eigenvector. Recall also that if $\mathbf{A} \in M_n$ is essentially nonnegative and irreducible, the Perron–Frobenius theorem can still be applied. In such a case, the unique eigenvalue of \mathbf{A} associated with a positive eigenvector is $\lambda_{PF}(\mathbf{A}) = \rho\left(\mathbf{A} - \min_{i \in [n]}(a_{i,i})\mathbf{I}_n\right) + \min_{i \in [n]}(a_{i,i})$. Any eigenvector associated with $\lambda_{PF}(\mathbf{A})$ is referred to as a *Perron–Frobenius eigenvector* and the unit one is denoted $\mathbf{n}_{PF}(\mathbf{A})$.

1.1.3. Functional analysis notations.

- We will consider a parabolic problem of two real variables, the “time” t and the “space” x . A (straight) *parabolic cylinder* in \mathbb{R}^2 is a subset of the form $(t_0, t_f) \times (a, b)$ with $(t_0, t_f, a, b) \in \overline{\mathbb{R}}^4$, $t_0 < t_f$ and $a < b$. The parabolic boundary $\partial_P Q$ of a bounded parabolic cylinder Q is defined classically. A *classical solution* of some second-order parabolic problem of dimension n set in a parabolic cylinder $Q = (t_0, t_f) \times (a, b)$ is a solution in:

$$\mathcal{C}^1((t_0, t_f), \mathcal{C}^2((a, b), \mathbb{R}^n)) \cap \mathcal{C}(Q \cup \partial Q, \mathbb{R}^n).$$

Similarly, a classical solution of some second-order elliptic problem of dimension n set in an interval $(a, b) \subset \overline{\mathbb{R}}$ is a solution in:

$$\mathcal{C}^2((a, b), \mathbb{R}^n) \cap \mathcal{C}((a, b) \cup \partial(a, b), \mathbb{R}^n).$$

- Consistently with \mathbb{R}^n , the set of functions $(\mathbb{R}^n)^{(\mathbb{R}^{n'})}$ is equipped with:

$$\mathbf{f} \geq_{\mathbb{R}^{n'}, \mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f}(\mathbf{v}) - \hat{\mathbf{f}}(\mathbf{v}) \in K_n \text{ for all } \mathbf{v} \in \mathbb{R}^{n'},$$

$$\mathbf{f} >_{\mathbb{R}^{n'}, \mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f} \geq_{\mathbb{R}^{n'}, \mathbb{R}^n} \hat{\mathbf{f}} \text{ and } \mathbf{f} \neq \hat{\mathbf{f}},$$

$$\mathbf{f} \gg_{\mathbb{R}^{n'}, \mathbb{R}^n} \hat{\mathbf{f}} \text{ if } \mathbf{f}(\mathbf{v}) - \hat{\mathbf{f}}(\mathbf{v}) \in K_n^{++} \text{ for all } \mathbf{v} \in \mathbb{R}^{n'}.$$

We define consistently nonnegative, nonnegative nonzero and positive functions³.

- The composition of two compatible functions \mathbf{f} and $\hat{\mathbf{f}}$ is denoted $\mathbf{f} \left[\hat{\mathbf{f}} \right]$, the usual \circ being reserved for the Hadamard product.
- If the context is unambiguous, a functional space $\mathcal{F}(X, \mathbb{R})$ is denoted $\mathcal{F}(X)$.
- For any smooth open bounded connected set $\Omega \subset \mathbb{R}^{n'}$ and any second order linear elliptic operator $\mathcal{L} : \mathcal{C}^2(\Omega, \mathbb{R}^n) \rightarrow \mathcal{C}(\Omega, \mathbb{R}^n)$ with coefficients in $\mathcal{C}_b(\Omega, \mathbb{R}^n)$, the *Dirichlet principal eigenvalue* of \mathcal{L} in Ω , denoted $\lambda_{1,Dir}(-\mathcal{L}, \Omega)$, is well-defined if \mathcal{L} is order-preserving in Ω . Recall from the Krein–Rutman theorem that $\lambda_{1,Dir}(-\mathcal{L}, \Omega)$ is the unique eigenvalue associated with a principal eigenfunction positive in Ω and null on $\partial\Omega$. Sufficient conditions for the order-preserving property are:
 - $n = 1$;
 - $n \geq 2$ and the system is *weakly coupled* (the coupling occurs only in the zeroth order term) and *fully coupled* (the zeroth order coefficient is an essentially nonnegative irreducible matrix). When $n \geq 2$, order-preserving operators are also referred to as *cooperative* operators.

³Regarding functions, some authors use $>$ to denote what is here denoted \gg . Thus the use of these two functional notations will be as sparse as possible and we will prefer the less ambiguous expressions “nonnegative nonzero” and “positive”.

1.2. Setting of the problem. From now on, an integer $N \in \mathbb{N} \cap [2, +\infty)$ is fixed. For the sake of brevity, the subscripts depending only on 1 or N in the various preceding notations will be omitted when the context is unambiguous.

We also fix $\mathbf{d} \in \mathbb{K}^{++}$, $\mathbf{D} = \text{diag} \mathbf{d}$, $\mathbf{L} \in M$ and $\mathbf{c} \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$.

The semilinear parabolic evolution system under scrutiny is the following:

$$\partial_t \mathbf{u} - \mathbf{D} \partial_{xx} \mathbf{u} = \mathbf{L} \mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}, \quad (E_{KPP})$$

the unknown being $\mathbf{u} : \mathbb{R}^2 \rightarrow \mathbb{R}^N$ (although (E_{KPP}) might occasionally be restricted to a parabolic cylinder).

The associated semilinear elliptic stationary system is:

$$-\mathbf{D} \mathbf{u}'' = \mathbf{L} \mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}, \quad (S_{KPP})$$

the unknown being $\mathbf{u} : \mathbb{R} \rightarrow \mathbb{R}^N$ (although (S_{KPP}) might occasionally be restricted to an interval).

1.2.1. Restrictive assumptions. The main restrictive assumptions are the following ones.

- (H₁) \mathbf{L} is essentially nonnegative and irreducible.
- (H₂) $\mathbf{c}(\mathbb{K}) \subset \mathbb{K}$.
- (H₃) $\mathbf{c}(\mathbf{0}) = \mathbf{0}$.
- (H₄) There exists:

$$(\underline{\alpha}, \delta, \underline{\mathbf{c}}) \in [1, +\infty)^2 \times \mathbb{K}^{++}$$

such that:

$$\sum_{j=1}^N l_{i,j} n_j \geq 0 \implies \alpha^\delta \underline{\mathbf{c}}_i \leq c_i(\alpha \mathbf{n})$$

for all:

$$(\mathbf{n}, \alpha, i) \in \mathbf{S}^+(\mathbf{0}, 1) \times [\underline{\alpha}, +\infty) \times [N].$$

A few immediate consequences of these assumptions deserve to be pointed out.

- (E_{KPP}) and (S_{KPP}) are not cooperative and do not satisfy a comparison principle.
- The Perron–Frobenius eigenvalue $\lambda_{PF}(\mathbf{L})$ is well-defined and the system $\mathbf{u}' = \mathbf{L} \mathbf{u}$ is cooperative.
- For all $\mathbf{v} \in \mathbb{R}^N$, the Jacobian matrix of $\mathbf{w} \mapsto \mathbf{c}(\mathbf{w}) \circ \mathbf{w}$ at \mathbf{v} is:

$$\text{diag} \mathbf{c}(\mathbf{v}) + (\mathbf{v} \mathbf{1}_{1,N}) \circ D \mathbf{c}(\mathbf{v}).$$

In particular, at $\mathbf{v} = \mathbf{0}$, this Jacobian is null if and only if (H_3) is satisfied. Also, if $D \mathbf{c}(\mathbf{v}) \geq \mathbf{0}$ for all $\mathbf{v} \in \mathbb{K}$, then the system $\mathbf{u}' = -\mathbf{c}[\mathbf{u}] \circ \mathbf{u}$ is competitive.

- This framework contains both the Lotka–Volterra linear competition $\mathbf{c}(\mathbf{u}) = \mathbf{C} \mathbf{u}$ and the Gross–Pitaevskii quadratic competition $\mathbf{c}(\mathbf{u}) = \mathbf{C}(\mathbf{u} \circ \mathbf{u})$ (with, in both cases, $\mathbf{C} \gg \mathbf{0}$).

1.2.2. KPP property. The system (E_{KPP}) is, in some sense, a “multidimensional KPP equation”. Let us recall the main features of scalar KPP nonlinearities:

- (1) $f'(0) > 0$ (instability of the null state),
- (2) $f'(0)v \geq f(v)$ for all $v \geq 0$ (no Allee effect),
- (3) there exists $K > 0$ such that $f(v) < 0$ if and only if $v > K$ (saturation).

Of course, our assumptions (H_1) – (H_4) aim to put forward a possible generalization of these features. A few comments are in order.

Regarding the saturation property, (H_4) will imply an analogous statement. Ensuring uniform \mathcal{L}^∞ estimates is really the main mathematical role of the competitive term.

Regarding the presence of an Allee effect, (H_2) and (H_3) clearly yield that $\partial_t \mathbf{u} - \mathbf{D}\partial_{xx} \mathbf{u} = \mathbf{L}\mathbf{u}$ is the linearization at $\mathbf{0}$ of (E_{KPP}) and moreover that $\mathbf{f} : \mathbf{v} \mapsto \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ satisfies:

$$D\mathbf{f}(\mathbf{0})\mathbf{v} \geq \mathbf{f}(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbf{K}.$$

Regarding the instability of the null state, we stress here that the notion of positivity of matrices is somewhat ambiguous and, consequently, finding a natural generalization of $f'(0) > 0$ is not completely straightforward.

In order to decide which positivity sense is the right one, we offer the following criterion. On one hand, a suitable multidimensional generalization of the KPP equation should enable generalizations of the striking results concerning its scalar counterpart. On the other hand, the most remarkable result about the KPP equation is that the answer to many natural questions (value of the spreading speed, persistence in bounded domains, etc.) only depends on $f'(0)$ (the importance of $f'(0)$ can already be seen in the features above). Thus, in our opinion, a KPP system should also be linearly determinate regarding these questions.

With this criterion in mind, let us explain for instance why positivity understood as positive definite matrices (i.e. positive spectrum) is not satisfying. In such a case, Lotka–Volterra competition–diffusion nonlinearities, whose linearization at $\mathbf{0}$ has the form $\text{diag } \mathbf{r}$ with $\mathbf{r} \in \mathbf{K}^{++}$, would be KPP nonlinearities. Nevertheless, it is known that the spreading speed of a competition–diffusion system is not necessarily linearly determinate (for instance, see Lewis–Li–Weinberger [37]).

On the contrary, the main results of the present paper show unambiguously that (H_1) supplemented with $\lambda_{PF}(\mathbf{L}) > 0$, that is existence and positivity of the Perron–Frobenius eigenvalue, is the right notion.

1.3. Statement of the results.

Theorem 1.1. *[Strong positivity] For all nonnegative classical solutions \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$, if $x \mapsto \mathbf{u}(0, x)$ is nonnegative nonzero, then \mathbf{u} is positive in $(0, +\infty) \times \mathbb{R}$.*

Consequently, all nonnegative nonzero classical solutions of (S_{KPP}) are positive.

Theorem 1.2. *[Absorbing set and upper estimates] There exists a positive function $\mathbf{g} \in \mathcal{C}([0, +\infty), \mathbf{K}^{++})$, component-wise non-decreasing, such that all nonnegative classical solutions \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ satisfy:*

$$\mathbf{u}(t, x) \leq \left(g_i \left(\sup_{x \in \mathbb{R}} u_i(0, x) \right) \right)_{i \in [N]} \text{ for all } (t, x) \in [0, +\infty) \times \mathbb{R}$$

and furthermore, if $x \mapsto \mathbf{u}(0, x)$ is bounded, then:

$$\left(\limsup_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) \right)_{i \in [N]} \leq \mathbf{g}(0).$$

Consequently, all bounded nonnegative classical solutions \mathbf{u} of (S_{KPP}) satisfy:

$$\mathbf{u} \leq \mathbf{g}(0).$$

Theorem 1.3. *[Extinction or persistence dichotomy]*

- i) Assume $\lambda_{PF}(\mathbf{L}) < 0$. Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ converge asymptotically in time, exponentially fast, and uniformly in space to $\mathbf{0}$.
- ii) Conversely, assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists $\nu > 0$ such that all bounded positive classical solutions \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ satisfy, for all bounded intervals $I \subset \mathbb{R}$:

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in I} u_i(t, x) \right)_{i \in [N]} \geq \nu \mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative classical solutions of (S_{KPP}) are valued in:

$$\prod_{i=1}^N [\nu, g_i(0)].$$

As will be explained later on, the critical case $\lambda_{PF}(\mathbf{L}) = 0$ is more challenging than expected and is not solved here, in spite of the following extinction conjecture.

Conjecture. Assume $\lambda_{PF}(\mathbf{L}) = 0$ and, for all $\mathbf{v} \in \text{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathbf{K}^{++}$, $\mathbf{c}(\mathbf{v}) > \mathbf{0}$.

Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ converge asymptotically in time and locally uniformly in space to $\mathbf{0}$.

Although Theorem 1.3 proves that the attractor of the induced semiflow is reduced to $\{\mathbf{0}\}$ in the extinction case, in the persistence case the asymptotic behavior is unclear and might not be reduced to a locally uniform convergence toward a unique stable steady state. This direct consequence of the multidimensional structure of (E_{KPP}) is a major difference with the scalar KPP equation. Still, the following theorem provides some additional information about the steady states of (E_{KPP}) and confirms in some sense the preceding conjecture.

Theorem 1.4. *[Existence of steady states]*

- i) If $\lambda_{PF}(\mathbf{L}) < 0$, there exists no positive classical solution of (S_{KPP}) .
- ii) If $\lambda_{PF}(\mathbf{L}) = 0$, then the set of bounded positive classical solutions of (S_{KPP}) is:

$$\{\mathbf{v} \in \text{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathbf{K}^{++} \mid \mathbf{c}(\mathbf{v}) = \mathbf{0}\}.$$

- iii) If $\lambda_{PF}(\mathbf{L}) > 0$, there exists a constant positive classical solution of (S_{KPP}) .

Having in mind the possibly complicated asymptotic behavior of (E_{KPP}) when $\lambda_{PF}(\mathbf{L}) > 0$, we introduce the following definition of traveling wave, more flexible than the usual definition of monostable traveling wave for the scalar KPP equation.

Definition. A traveling wave solution of (E_{KPP}) is a pair:

$$(\mathbf{p}, c) \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^N) \times [0, +\infty)$$

which satisfies:

- (1) $\mathbf{u} : (t, x) \mapsto \mathbf{p}(x - ct)$ is a bounded positive classical solution of (E_{KPP}) ;
- (2) $\left(\liminf_{\xi \rightarrow -\infty} p_i(\xi) \right)_{i \in [N]} \in \mathbf{K}^+$;
- (3) $\lim_{\xi \rightarrow +\infty} \mathbf{p}(\xi) = \mathbf{0}$.

We refer to \mathbf{p} as the *profile* of the traveling wave and to c as its *speed*.

Theorem 1.5. *[Traveling waves] Assume $\lambda_{PF}(\mathbf{L}) > 0$.*

i) There exists $c^ > 0$ such that:*

- (a) there exists no traveling wave solution of (E_{KPP}) with speed c for all $c \in [0, c^*]$;*
- (b) if, furthermore,*

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{K},$$

then there exists a traveling wave solution of (E_{KPP}) with speed c for all $c \geq c^$.*

ii) All profiles \mathbf{p} satisfy:

$$\mathbf{p} \leq \mathbf{g}(0).$$

iii) All profiles \mathbf{p} satisfy:

$$\left(\liminf_{\xi \rightarrow -\infty} p_i(\xi) \right)_{i \in [N]} \geq \nu \mathbf{1}_{N,1}.$$

iv) All profiles are component-wise decreasing in a neighborhood of $+\infty$.

When traveling waves exist for all speeds $c \geq c^*$, c^* is called the minimal wave speed.

Theorem 1.6. *[Spreading speed] Assume $\lambda_{PF}(\mathbf{L}) > 0$. For all $x_0 \in \mathbb{R}$ and all bounded nonnegative nonzero $\mathbf{v} \in \mathcal{C}(\mathbb{R}, \mathbb{R}^N)$, the classical solution \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ with initial data $\mathbf{v}\mathbf{1}_{(-\infty, x_0)}$ satisfies:*

$$\left(\lim_{t \rightarrow +\infty} \sup_{x \in (y, +\infty)} u_i(t, x + ct) \right)_{i \in [N]} = \mathbf{0} \text{ for all } c \in (c^*, +\infty) \text{ and all } y \in \mathbb{R},$$

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in [-R, R]} u_i(t, x + ct) \right)_{i \in [N]} \in \mathbf{K}^{++} \text{ for all } c \in [0, c^*) \text{ and all } R > 0.$$

Of course, by well-posedness of (E_{KPP}) , the solution with initial data $x \mapsto \mathbf{v}(-x)\mathbf{1}_{(-x_0, +\infty)}$ is precisely $(t, x) \mapsto \mathbf{u}(t, -x)$ (\mathbf{u} being the solution with initial data $\mathbf{v}\mathbf{1}_{(-\infty, x_0)}$). This gives the expected symmetrical spreading result (the solution with initial data $x \mapsto \mathbf{v}(-x)\mathbf{1}_{(-x_0, +\infty)}$ spreads on the left at speed $-c^*$). Moreover, since these two spreading results with front-like initial data actually cover compactly supported \mathbf{v} , we also get straightforwardly the spreading result for compactly supported initial data (the solution spreads on the right at speed c^* and on the left at speed $-c^*$).

Consequently, c^* is also called the spreading speed associated with front-like or compactly supported initial data. We recall that for generic KPP problems these two spreading speeds are different as soon as the spatial domain is multidimensional. In such a case, the spreading speed associated with front-like initial data generically coincides with the minimal wave speed whereas the spreading speed associated with compactly supported initial data is smaller.

Theorem 1.7. *[Characterization and estimates for c^*] Assume $\lambda_{PF}(\mathbf{L}) > 0$. We have:*

$$c^* = \min_{\mu > 0} \frac{\lambda_{PF}(\mu^2 \mathbf{D} + \mathbf{L})}{\mu}$$

and this minimum is attained at a unique $\mu_{c^*} > 0$.

Consequently, if we assume (without loss of generality):

$$d_1 \leq d_2 \leq \dots \leq d_N,$$

the following estimates hold.

i) We have:

$$2\sqrt{d_1 \lambda_{PF}(\mathbf{L})} \leq c^* \leq 2\sqrt{d_N \lambda_{PF}(\mathbf{L})}.$$

If $d_1 < d_N$, both inequalities are strict. If $d_1 = d_N$, both inequalities are equalities.

ii) For all $i \in [N]$ such that $l_{i,i} > 0$, we have:

$$c^* > 2\sqrt{d_i l_{i,i}}.$$

iii) Let $\alpha > 0$ such that $\alpha \mathbf{1}_{1,N} \mathbf{n}_{PF}(\mu_{c^*}^2 \mathbf{D} + \mathbf{L}) = 1$. Let $\mathbf{r} \in \mathbb{R}^N$ and $\mathbf{M} \in \mathbf{M}$ be given by the unique decomposition of \mathbf{L} of the form:

$$\mathbf{L} = \text{diagr} + \mathbf{M} \text{ with } \mathbf{M}^T \mathbf{1}_{N,1} = \mathbf{0}.$$

Let $(\langle d \rangle, \langle r \rangle) \in (0, +\infty) \times \mathbb{R}$ be defined as:

$$(\langle d \rangle, \langle r \rangle) = (\alpha \mathbf{d}^T \mathbf{n}_{PF}(\mu_{c^*}^2 \mathbf{D} + \mathbf{L}), \alpha \mathbf{r}^T \mathbf{n}_{PF}(\mu_{c^*}^2 \mathbf{D} + \mathbf{L})).$$

If $\langle r \rangle \geq 0$, then:

$$c^* \geq 2\sqrt{\langle d \rangle \langle r \rangle}.$$

It now makes sense to introduce the following definition, which we expect to be optimal with respect to the preceding collection of theorems.

Definition 1.8. A nonlinear function $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ is a *KPP nonlinearity* if:

- (1) $\mathbf{f}(\mathbf{0}) = \mathbf{0}$;
- (2) $D\mathbf{f}(\mathbf{0}) - \min_{i \in [N]} \left(\frac{\partial f_i}{\partial i}(\mathbf{0}) \right) \mathbf{I}$ is nonnegative, irreducible and $\lambda_{PF}(D\mathbf{f}(\mathbf{0})) > 0$;
- (3) $D\mathbf{f}(\mathbf{0}) \mathbf{v} \geq \mathbf{f}(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{K}$;
- (4) the semiflow induced by $\partial_t \mathbf{u} = \mathbf{D} \partial_{xx} \mathbf{u} + \mathbf{f}[\mathbf{u}]$ with sufficiently regular initial data admits an absorbing set bounded in $\mathcal{L}^\infty(\mathbb{R})$.

Remark. Of course, defining $\mathbf{L} = D\mathbf{f}(\mathbf{0})$ and:

$$\mathbf{c} : \mathbf{v} \mapsto \left(\frac{1}{v_i} \left(\sum_{j=1}^N \frac{\partial f_i}{\partial j}(0) v_j - f_i(\mathbf{v}) \right) \right)_{i \in [N]},$$

we recover the specific form $\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ and the assumptions (H_1) and (H_2) follow directly. (H_3) is satisfied as soon as \mathbf{c} is sufficiently regular at $\mathbf{0}$. Still, let us emphasize that $\mathbf{c} \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ and (H_4) might very well not be satisfied in general. Although they are trivially satisfied for the main applications we have in mind (which will be exposed in a moment), it is mathematically interesting to look at possible relaxations. For instance, let us discuss briefly (H_4) .

The only forthcoming result whose proof depends directly on (H_4) is Lemma 3.1 (which is remarkably one of the main assumptions of a related paper by Barles, Evans and Souganidis [6, (F3)]). It is easily seen that if \mathbf{c} grows sublinearly, we cannot hope in general to recover Lemma 3.1 (in other words, under some reasonable assumptions, Barles–Evans–Souganidis’s (F3) is satisfied if and only if (H_4) ; of course this makes (H_4) even more interesting).

Nevertheless, this lemma is not a result in itself but a tool used for the proofs of Theorem 1.2 as well as the existence results of Theorem 1.4 and Theorem 1.5. Hence relaxing (H_4) mainly means finding new proofs of these results.

Now, without entering into too much details, we point out that if there exists $\eta > 0$ such that the following dissipative assumption:

$$(H_{diss,\eta}) \quad \begin{cases} \exists C_1 \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^N & (\mathbf{f}(\mathbf{v}) + \eta \mathbf{v})^T \mathbf{v} \leq C_1 \\ \exists C_2 \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^N & D\mathbf{f}(\mathbf{v}) + \eta \mathbf{I} \leq C_2 \mathbf{I} \\ \exists (C_3, p) \in [0, +\infty)^2 \quad \forall \mathbf{v} \in \mathbb{R}^N & |\mathbf{f}(\mathbf{v}) + \eta \mathbf{v}| \leq C_3 (1 + |\mathbf{v}|^p), \end{cases}$$

holds, then the semiflow induced by $\partial_t \mathbf{u} = \mathbf{D} \partial_{xx} \mathbf{u} + \mathbf{f}[\mathbf{u}]$ admits an attractor in some locally uniform topology which is bounded in $\mathcal{C}_b(\mathbb{R}, \mathbb{R}^N)$ (see Zelik [49]). If the semiflow leaves \mathbf{K} invariant and if we only consider nonnegative initial data, then the quantifiers $\forall \mathbf{v} \in \mathbb{R}^N$ above can all be replaced by $\forall \mathbf{v} \in \mathbf{K}$.

In particular, $\mathbf{v} \mapsto \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ supplemented with (H_1) – (H_3) and:

$$(H'_4) \quad \lim_{|\mathbf{v}| \rightarrow +\infty, \mathbf{v} \in \mathbf{K}} |\mathbf{c}(\mathbf{v})| = +\infty \text{ with at most algebraic growth}$$

satisfies $(H_{diss,\eta})$ for any $\eta > 0$. (Clearly, $(H_4) \cup (H'_4)$ contains every choice of \mathbf{c} such that $\lim_{|\mathbf{v}| \rightarrow +\infty, \mathbf{v} \in \mathbf{K}} |\mathbf{c}(\mathbf{v})| = +\infty$.)

Consequently, dissipative theory provides for some slowly decaying KPP nonlinearities a proof of Theorem 1.2. It should also provide a proof of Proposition 3.5, which is the key estimate to derive the existence of steady states and of traveling waves. With these proofs at hand, all of our results are recovered.

1.4. Related results in the literature.

1.4.1. *Cooperative or almost cooperative systems.* The bibliography about weakly and fully coupled elliptic and parabolic linear systems is of course extensive. It is possible, for instance, to define principal eigenvalues and eigenfunctions (Sweers *et al.* [13, 46]), to prove the weak maximum principle (the classical theorems of Protter–Weinberger [41] were refined in the more involved elliptic case by Figueiredo *et al.* [23, 24] and Sweers [46]) or Harnack inequalities (Chen–Zhao [20] or Arapostathis–Gosh–Marcus [3] for the elliptic case⁴, Földes–Poláčik [29] for the parabolic case) and to use the super- and sub-solution method to deduce existence of solutions (Pao [44] among others). In some sense, weakly and fully coupled systems form the “right”, or at least the most straightforward, generalization of scalar equations.

For (possibly nonlinear) cooperative systems, results analogous to Theorem 1.5 i), iii), Theorem 1.6 and Theorem 1.7 were established by Lewis, Li and Weinberger [38, 48]. Recently, Al-Kiffai and Crooks [1] introduced a convective term into a two-species cooperative system to study its influence on linear determinacy.

For non-cooperative systems that can still be controlled from above and from below by weakly and fully coupled systems whose linearizations at $\mathbf{0}$ coincide with that of the non-cooperative system, Wang [47] recovered the results of Lewis–Li–Weinberger by comparison arguments. Before going any further, let us point out that we will use extensively comparison arguments as well, nevertheless we will not need equality of the linearizations at $\mathbf{0}$. This is a crucial difference between the two sets of assumptions. To illustrate this claim, let us present an explicit example of system

⁴They both prove the same type of results but we will refer hereafter only to the latter because the former does not cover, as stated, the one-dimensional space case.

covered by our assumptions and not by Wang's ones: take any $N \geq 3$, $r > 0$, $\mu \in (0, \frac{r}{2})$ and define \mathbf{L} and \mathbf{c} as follows:

$$\mathbf{L} = r\mathbf{I} + \mu \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix},$$

$$\mathbf{c} : \mathbf{v} \mapsto \mathbf{1}\mathbf{v}.$$

On one hand, (H_1) – (H_4) are easily verified, but on the other hand, the function $\mathbf{f} : \mathbf{v} \mapsto \mathbf{L}\mathbf{v} - \mathbf{c}[\mathbf{v}] \circ \mathbf{v}$ is such that, for all $i \in [N] \setminus \{1, N\}$ and all $\mathbf{v} \in \mathbf{K}^{++}$:

$$\frac{\partial \mathbf{f}_i}{\partial v_j}(\mathbf{v}) = -v_i < 0 \text{ for all } j \in [N] \setminus \{i-1, i, i+1\}.$$

Consequently, the application $v \mapsto \mathbf{f}_i(v\mathbf{e}_j)$ is decreasing in $[0, +\infty)$. This clearly violates Wang's assumptions: this instance of (E_{KPP}) cannot be controlled from below by a cooperative system whose linearization at $\mathbf{0}$ is $\partial_t \mathbf{u} - \mathbf{D}\partial_{xx} \mathbf{u} = \mathbf{L}\mathbf{u}$.

Even if \mathbf{L} is essentially positive and the cooperative functions \mathbf{f}^- , \mathbf{f}^+ satisfying:

$$\begin{cases} \mathbf{f}^-(\mathbf{v}) \leq \mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v} \leq \mathbf{f}^+(\mathbf{v}) \\ \mathbf{f}^-(\mathbf{0}) = \mathbf{f}^+(\mathbf{0}) = \mathbf{0} \\ D\mathbf{f}^-(\mathbf{0}) = D\mathbf{f}^+(\mathbf{0}) = \mathbf{L} \end{cases}$$

are constructible, in general it is difficult to verify that \mathbf{f}^- and \mathbf{f}^+ have each a minimal positive zero (another requirement of Wang). Our setting needs not such a verification.

Furthermore, even if these minimal zeros exist, several results presented here are still new.

- (1) Theorem 1.5 i) adds to [47, Theorem 2.1 iii)–v)] the existence of a critical traveling wave (Wang obtained the existence of a bounded non-constant nonnegative solution traveling at speed c^* but the limit at $+\infty$ of its profile was not addressed).
- (2) Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 as well as Theorem 1.5 ii), iv) rely more deeply on the KPP structure and are completely new to the best of our knowledge.

1.4.2. KPP systems. Regarding weakly coupled systems equipped with KPP nonlinearities, as far as we know most related works assume the essential positivity of \mathbf{L} , some even requiring its positivity. Our results tend to show that this collection of results should be generalizable under the weaker assumption (H_1) with $\lambda_{PF}(\mathbf{L}) > 0$.

Dockery, Hutson, Mischaikow and Pernarowski [25] studied in a celebrated paper the solutions of (S_{KPP}) in a bounded and smooth domain with Neumann boundary conditions. Their matrix \mathbf{L} had the specific form $a(x)\mathbf{I} + \mu\mathbf{M}$ where a is a non-constant function of the space variable and with minimal assumptions on the constant matrix \mathbf{M} . They also assumed strict ordering of the components of \mathbf{d} , explicit and symmetric Lotka–Volterra competition, vanishingly small μ . They proved the existence of a unique positive steady state, globally attractive for the Cauchy

problem with positive initial data, and which converges as $\mu \rightarrow 0$ to a steady state where only u_1 persists.

More recently, the solutions of (S_{KPP}) , still in a bounded and smooth domain with Neumann boundary conditions, were studied under the assumptions of essential positivity of \mathbf{L} and small Lipschitz constant of $\mathbf{v} \mapsto \mathbf{c}(\mathbf{v}) \circ \mathbf{v}$ by Hei and Wu [34]. They established by means of super- and sub-solutions the equivalence between the negativity of the principal eigenvalue of $-\mathbf{D} \frac{d^2}{dx^2} - \mathbf{L}$ and the existence of a positive steady state.

Provided the positivity of \mathbf{L} , the vanishing viscosity limit of (E_{KPP}) is the object of a work by Barles, Evans and Souganidis [6]. Although their paper and the present one differ both in results and in techniques, they share the same ambition: describing the spreading phenomenon for KPP systems. Therefore our feeling is that together they give a more complete answer to the problem.

For two-component systems with explicit Lotka–Volterra competition, $\mathbf{D} = \mathbf{I}_2$ and symmetric and positive \mathbf{L} , Theorem 1.4 and Theorem 1.5 i), iii), iv) reduce to the results of Griette and Raoul [31] (see Alfaro–Griette [2] for a partial extension to space-periodic media). Their paper uses very different arguments (topological degree, explicit computations involving in particular the sum of the equations, weak mutation limit, phase plane analysis) but was our initial motivation to work on this question: our intent is really to extend their result to a larger setting by changing the underlying mathematical techniques. Let us emphasize that they obtained an algebraic formula for the minimal wave speed, $c^* = 2\sqrt{\lambda_{PF}(\mathbf{L})}$, that we are able to generalize (Theorem 1.7). The case $\mathbf{D} \neq \mathbf{I}_2$ has been investigated heuristically and numerically by Elliott and Cornell [26], who considered the weak mutation limit as well and obtained further results.

Let us point out that the problem of the spreading speed for the Cauchy problem for the two-component system with explicit Lotka–Volterra competition was formulated but left open by Elliott and Cornell [26] as well as by Cosner [21] and not considered by Griette and Raoul [31]. This problem is completely solved here (see Theorem 1.6).

Just after the submission of this paper, a paper by Moris, Börger and Crooks [40] submitted concurrently and devoted to the analytical confirmation of Elliott and Cornell’s numerical observations was brought to our attention. By applying’s successfully Wang’s framework, they obtained the existence of traveling waves as well as the spreading speed for the Cauchy problem. However, in order to apply Wang’s framework, they had to make additional assumptions (roughly speaking, small interphenotypic competition and small mutations) and which are in fact, in view of our results, unnecessary. They also obtained very interesting results regarding the dependency on the mutation rate μ of the spreading speed $\lambda_{PF}(\mu_{c^*}\mathbf{D} + \mu_{c^*}^{-1}(\text{diagr} + \mu\mathbf{M}))$ and the associated distribution $\mathbf{n}_{PF}(\mu_{c^*}\mathbf{D} + \mu_{c^*}^{-1}(\text{diagr} + \mu\mathbf{M}))$.

1.5. From systems to non-local equations. It is well-known that systems can be seen as discretizations of continuous models. In this subsection, we present briefly some equations structured not only in time and space but also with a third variable and whose natural discretizations are particular instances of our system (E_{KPP}) satisfying the criterion $\lambda_{PF}(\mathbf{L}) > 0$. Our results bring therefore indirect insight into the spreading properties of these equations.

Notice that these examples provide also examples of biomathematical applications of our results and prove that, although most of the literature about KPP

systems is only concerned with Lotka–Volterra mutation–competition–diffusion settings, other interesting applications exist.

1.5.1. *The cane toads equation with non-local competition.* We define the matrix $\mathbf{M}_{Lap,N}$ as:

$$\mathbf{M}_{Lap,N} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix} \text{ if } N \geq 3,$$

$$\mathbf{M}_{Lap,2} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ if } N = 2.$$

With this notation, the Lotka–Volterra mutation–competition–diffusion system exhibited earlier can be written in vector form as follows:

$$\partial_t \mathbf{u} - \mathbf{D} \partial_{xx} \mathbf{u} = \text{diag}(\mathbf{r}) \mathbf{u} + \mu \mathbf{M}_{Lap} \mathbf{u} - (\mathbf{C} \mathbf{u}) \circ \mathbf{u}.$$

An especially interesting instance of it is the system where:

- for all $i \in [N]$, $d_{N,i} = \underline{\theta} + (i-1)\theta_N$ with $\theta_N = \frac{\bar{\theta} - \underline{\theta}}{N-1}$ and with some fixed $\bar{\theta} > \underline{\theta} > 0$;
- $\mathbf{r}_N = r \mathbf{1}_{N,1}$ with some fixed $r > 0$;
- $\mu_N = \frac{\alpha}{\theta_N^2}$ with some fixed $\alpha > 0$;
- $\mathbf{C}_N = \theta_N \mathbf{1}_N$.

Since $\lambda_{PF}(\mathbf{M}_{Lap,N}) = 0$ (because $\mathbf{M}_{Lap,N} \mathbf{1}_{N,1} = \mathbf{0}$), the Perron–Frobenius eigenvalue of \mathbf{L} is positive indeed:

$$\lambda_{PF} \left(r \mathbf{I}_N + \frac{\alpha}{\theta_N^2} \mathbf{M}_{Lap,N} \right) = r + \lambda_{PF} \left(\frac{\alpha}{\theta_N^2} \mathbf{M}_{Lap,N} \right) = r > 0.$$

As $N \rightarrow +\infty$, this system converges (at least formally) to the now classical non-local cane toads equation with bounded phenotypes [7, 14, 15, 16]:

$$\begin{cases} \partial_t n - \theta \partial_{xx} n - \alpha \partial_{\theta\theta} n = n(t, x, \theta) \left(r - \int_{\underline{\theta}}^{\bar{\theta}} n(t, x, \theta') d\theta' \right) \\ \partial_{\theta} n(t, x, \underline{\theta}) = \partial_{\theta} n(t, x, \bar{\theta}) = 0 \text{ for all } (t, x) \in \mathbb{R}^2 \end{cases}$$

where n is a function of (t, x, θ) , $\theta \in [\underline{\theta}, \bar{\theta}]$ is the motility trait, α is the mutation rate and $\int_{\underline{\theta}}^{\bar{\theta}} n(t, x, \theta') d\theta'$ is the total population present at (t, x) . The mutation operator $\alpha \partial_{\theta\theta}$ is local in θ .

1.5.2. *The doubly non-local cane toads equation.* It seems that, historically, the cane toads equation comes from a doubly non-local model due to Prévost *et al.* [4, 45] (see also the earlier individual-based model by Champagnat and Méléard [19]). Since the non-local mutation operator was too difficult to handle mathematically, the cane toads equation with local mutations was favored as a simplified first approach. But it remains unsatisfying from the modeling point of view for various reasons and non-local kernels are the real aim.

Defining as above $\theta_N = \frac{\bar{\theta} - \underline{\theta}}{N-1}$ and $(\theta_i)_{i \in [N]} = (\underline{\theta} + (i-1)\theta_N)_{i \in [N]}$, the natural discretization of the following doubly non-local cane toads equation:

$$\partial_t n - d(\theta) \partial_{xx} n = rn + \alpha (K \star_{\theta} n - n) - n \int_{\underline{\theta}}^{\bar{\theta}} n(t, x, \theta') d\theta'$$

with $d \in \mathcal{C}([\underline{\theta}, \bar{\theta}], (0, +\infty))$ and $K \in \mathcal{C}(\mathbb{R}, [0, +\infty))$ is the following system:

$$\partial_t \mathbf{u} - \mathbf{D}_N \partial_{xx} \mathbf{u} = \mathbf{L}_N \mathbf{u} - (\theta_N \mathbf{1}_N \mathbf{u}) \circ \mathbf{u},$$

with:

$$\mathbf{d}_N = (d(\theta_i))_{i \in [N]},$$

$$\begin{aligned} \mathbf{L}_N &= r \mathbf{I}_N + \alpha \left(\theta_N (K(\theta_i - \theta_j))_{(i,j) \in [N]^2} - \mathbf{I}_N \right) \\ &= (r - \alpha) \mathbf{I}_N + \alpha \theta_N (K((i-j)\theta_N))_{(i,j) \in [N]^2}. \end{aligned}$$

(H_2) – (H_4) are obviously satisfied and, as soon as, say, K is positive, (H_1) is satisfied as well. Subsequently, $\lambda_{PF}(\mathbf{L}_N) \geq r - \alpha$, whence $r > \alpha$ is a sufficient condition to ensure $\lambda_{PF}(\mathbf{L}_N) > 0$ for all $N \in \mathbb{N}$.

More generally, the system corresponding to the following equation (see Prévost *et al.* [4, 45]):

$$\begin{aligned} \partial_t n - d(\theta) \partial_{xx} n &= r(\theta) n(t, x, \theta) + \int_{\underline{\theta}}^{\bar{\theta}} n(t, x, \theta') K(\theta, \theta') d\theta' \\ &\quad - n(t, x, \theta) \int_{\underline{\theta}}^{\bar{\theta}} n(t, x, \theta') C(\theta, \theta') d\theta' \end{aligned}$$

with $d \in \mathcal{C}([\underline{\theta}, \bar{\theta}], (0, +\infty))$, $r \in \mathcal{C}([\underline{\theta}, \bar{\theta}], [0, +\infty))$, $K, C \in \mathcal{C}([\underline{\theta}, \bar{\theta}]^2, [0, +\infty))$ is:

$$\partial_t \mathbf{u} - \mathbf{D}_N \partial_{xx} \mathbf{u} = \mathbf{L}_N \mathbf{u} - (\mathbf{C}_N \mathbf{u}) \circ \mathbf{u},$$

with:

$$\begin{aligned} \mathbf{d}_N &= (d(\theta_i))_{i \in [N]}, \\ \mathbf{L}_N &= \text{diag}(r(\theta_i))_{i \in [N]} + \theta_N (K(\theta_i, \theta_j))_{(i,j) \in [N]^2}, \\ \mathbf{C}_N &= \theta_N (C(\theta_i, \theta_j))_{(i,j) \in [N]^2}. \end{aligned}$$

Again, (H_3) and (H_4) are clearly satisfied, (H_2) is satisfied if C is nonnegative and both (H_1) and $\lambda_{PF}(\mathbf{L}_N) > 0$ are satisfied if, say, K is positive.

In both cases, of course, the positivity of K is a far from necessary condition and might be relaxed.

1.5.3. *The Gurtin–MacCamy equation with diffusion and overcrowding effect.* Consider the following age-structured equation with diffusion [32, 33]:

$$\begin{cases} \partial_t n + \partial_a n - d(a) \partial_{xx} n = -n(t, x, a) \left(r(a) + \int_0^A n(t, x, a') C(a, a') da' \right) \\ n(t, x, 0) = \int_{a_m}^A n(t, x, a') K(a') da' \text{ for all } (t, x) \in \mathbb{R}^2 \\ n(t, x, A) = 0 \text{ for all } (t, x) \in \mathbb{R}^2 \end{cases}$$

where n is a function of (t, x, a) , $a \in [0, A]$ is the age variable, $a_m \geq 0$ is the maturation age, $A > a_m$ is the maximal age, $d \in \mathcal{C}([0, A], (0, +\infty))$ is the diffusion rate, $r \in \mathcal{C}([0, A], (0, +\infty))$ is the mortality rate, $C \in \mathcal{C}([0, A]^2, [0, +\infty))$ is the competition kernel and $K \in \mathcal{C}([0, A], [0, +\infty))$ is the birth rate.

Defining:

$$\begin{aligned}
a_{N+1} &= \frac{A}{N}, \\
(a_i)_{i \in [N]} &= ((i-1)a_{N+1})_{i \in [N]}, \\
j_{m,N} &= \min \{j \in [N] \mid a_j \geq a_m\}, \\
\mathbf{u}(t, x) &= (n(t, x, a_i))_{i \in [N]}, \\
\mathbf{d}_N &= (d(a_i))_{i \in [N]}, \\
\mathbf{L}_{mortality, N} &= -\text{diag} \left(r(a_i)_{i \in [N]} \right), \\
\mathbf{L}_{birth, N} &= a_{N+1} \begin{pmatrix} 0 & \dots & 0 & K(a_{j_{m,N}}) & \dots & K(a_N) \\ 0 & & & \dots & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & \dots & & 0 \end{pmatrix}, \\
\mathbf{L}_{aging, N} &= \frac{1}{a_{N+1}} \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}, \\
\mathbf{L}_N &= \mathbf{L}_{mortality, N} + \mathbf{L}_{birth, N} + \mathbf{L}_{aging, N}, \\
\mathbf{C}_N &= a_{N+1} (C(a_i, a_j))_{(i,j) \in [N]^2},
\end{aligned}$$

it follows again that:

$$\partial_t \mathbf{u} - \mathbf{D}_N \partial_{xx} \mathbf{u} = \mathbf{L}_N \mathbf{u} - (\mathbf{C}_N \mathbf{u}) \circ \mathbf{u}$$

is the natural discretization with (H_3) and (H_4) automatically satisfied. K non-negative nonzero and C nonnegative are sufficient conditions to enforce (H_1) and (H_2) .

Since we have:

$$\lambda_{PF}(\mathbf{L}_N) \geq \lambda_{PF}(\mathbf{L}_{birth, N} + \mathbf{L}_{aging, N}) - \max_{[0, A]} r$$

and since $\lambda_{PF}(\mathbf{L}_{birth, N} + \mathbf{L}_{aging, N})$ is bounded from below by a positive constant independent of N (the proof of this claim being deliberately not detailed here for the sake of brevity), if $\max_{[0, A]} r$ is small enough, then $\lambda_{PF}(\mathbf{L}_N) > 0$ for all $N \in \mathbb{N}$.

2. STRONG POSITIVITY

Theorem 1.1 is mainly straightforward. It follows from the following local result.

Proposition 2.1. *Let $Q \subset \mathbb{R}^2$ be a bounded parabolic cylinder and \mathbf{u} be a classical solution of (E_{KPP}) set in Q .*

If \mathbf{u} is nonnegative on $\partial_P Q$, then it is either null or positive in Q .

Proof. Let $K = \max_{\bar{Q}} |\mathbf{u}|$ and observe that, for all $i \in [N]$ and all $(t, x) \in Q$:

$$|l_{i,i} - c_i(\mathbf{u}(t, x))| \leq |l_{i,i}| + \frac{\max_{\mathbf{v} \in B(\mathbf{0}, K)} |c_i(\mathbf{v})|}{1}.$$

Then, defining:

$$\mathbf{C} : (t, x) \mapsto \mathbf{L} - \text{diag}(\mathbf{c}(\mathbf{u}(t, x))),$$

\mathbf{u} is a solution of the following linear weakly and fully coupled equation with bounded coefficients:

$$\partial_t \mathbf{u} - \mathbf{D} \partial_{xx} \mathbf{u} - \mathbf{C} \mathbf{u} = \mathbf{0}.$$

By virtue of Protter–Weinberger’s strong maximum principle [41, Chapter 3, Theorem 13], \mathbf{u} is indeed either null or positive in Q . \square

Before going any further, let us state the analogous strong positivity for the elliptic problem. Actually, noticing that the previous proof holds without any modification if we add to (E_{KPP}) a constant drift term $c \partial_x \mathbf{u}$ (with $c \in \mathbb{R}$, but in fact, it holds for any diagonal drift $\mathbf{b} \circ \partial_x \mathbf{u}$ with $\mathbf{b} \in \mathbb{R}^N$ as well), we state it directly in a slightly more general form that will be quite useful later on. Of course, since solutions of the elliptic problem are nothing but time-independent solutions of the parabolic one, the following result is a direct corollary of the preceding one. It can also be deduced from Arapostathis–Gosh–Marcus’s Harnack inequality [3] applied to the operator $\mathbf{D} \frac{d^2}{dx^2} + \mathbf{C}$.

Proposition 2.2. *Let $a, b, c \in \mathbb{R}$ such that $a < b$. Let \mathbf{u} be a nonnegative classical solution of:*

$$-\mathbf{D} \mathbf{u}'' - c \mathbf{u}' = \mathbf{L} \mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \text{ in } (a, b).$$

Then \mathbf{u} is either null or positive in (a, b) .

Remark. This statement does not establish the non-negativity of all solutions of $-\mathbf{D} \mathbf{u}'' - c \mathbf{u}' = \mathbf{L} \mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}$; it only enforces the interior positivity of the nonnegative nonzero solutions. Regarding the weak maximum principle, we refer among others to Figueiredo [23], Figueiredo–Mitidieri [24], Sweers [46]. In view of what is known about the simpler scalar case, it is to be expected that, for some small values of c and a large enough interval (a, b) , sign-changing solutions exist.

3. ABSORBING SET AND UPPER ESTIMATES

On the contrary, Theorem 1.2 requires some work.

3.1. Saturation of the reaction term. For all $i \in [N]$, let $H_i \subset \mathbb{R}^N$ be the closed half-space defined as:

$$H_i = \{\mathbf{v} \in \mathbb{R}^N \mid (\mathbf{L} \mathbf{v})_i \geq 0\}.$$

Lemma 3.1. *There exists $\mathbf{k} \in \mathbb{K}^{++}$ such that, for all $i \in [N]$ and for all $\mathbf{v} \in \mathbb{K} \setminus \mathbf{e}_i^\perp$:*

$$(\mathbf{L}(\mathbf{v} + k_i \mathbf{e}_i) - \mathbf{c}(\mathbf{v} + k_i \mathbf{e}_i) \circ (\mathbf{v} + k_i \mathbf{e}_i))_i < 0.$$

Proof. Let $i \in [N]$ and let:

$$F_i = (\mathbb{S}^+(\mathbf{0}, 1) \cap H_i) \setminus \mathbf{e}_i^\perp.$$

Let:

$$\begin{aligned} f_i : (0, +\infty) \times \mathbb{S}(\mathbf{0}, 1) &\rightarrow \mathbb{R} \\ (\alpha, \mathbf{n}) &\mapsto \sum_{j=1}^N l_{i,j} n_j - c_i(\alpha \mathbf{n}) n_i. \end{aligned}$$

Notice that for all $\mathbf{n} \in \mathbf{S}^+(\mathbf{0}, 1) \setminus \mathbf{F}_i$, either $\sum_{j=1}^N l_{i,j} n_j < 0$ and then $f_i(\alpha, \mathbf{n}) < 0$ for all $\alpha > 0$ or $n_i = 0$ and then $f_i(\alpha, \mathbf{n}) = \sum_{j=1}^N l_{i,j} n_j \geq 0$ does not depend on α .

Let $\mathbf{n} \in \mathbf{F}_i$. By virtue of (H_4) and since $\mathbf{n} \notin \mathbf{e}_i^\perp$:

$$\lim_{\alpha \rightarrow +\infty} f_i(\alpha, \mathbf{n}) = -\infty.$$

Therefore the following quantity is finite and nonnegative:

$$\alpha_{i,\mathbf{n}} = \inf \{ \alpha \geq 0 \mid \forall \alpha' \in (\alpha, +\infty) \quad f_i(\alpha', \mathbf{n}) < 0 \}.$$

Now, the set:

$$\{ \alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathbf{F}_i \} = \{ \alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathbf{F}_i, \alpha_{i,\mathbf{n}} > \underline{\alpha} \} \cup \{ \alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathbf{F}_i, \alpha_{i,\mathbf{n}} \leq \underline{\alpha} \}$$

is bounded if and only if the set $\{ \alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathbf{F}_i, \alpha_{i,\mathbf{n}} > \underline{\alpha} \}$ is bounded. Recall that $\underline{\alpha} \geq 1$ and $\delta \geq 1$. For all $\mathbf{n} \in \mathbf{F}_i$ such that $\alpha_{i,\mathbf{n}} > \underline{\alpha}$, thanks to (H_4) , we have by virtue of the discrete Cauchy-Schwarz inequality:

$$\begin{aligned} |\alpha_{i,\mathbf{n}} n_i| &= \alpha_{i,\mathbf{n}} n_i \\ &\leq \alpha_{i,\mathbf{n}}^\delta n_i \\ &\leq \frac{\sum_{j=1}^N l_{i,j} n_j}{\underline{c}_i} \\ &\leq \frac{|(l_{i,j})_{j \in [N]}|}{\underline{c}_i}, \end{aligned}$$

whence the finiteness of:

$$k_i = \sup \{ \alpha_{i,\mathbf{n}} n_i \mid \mathbf{n} \in \mathbf{F}_i \}$$

is established. Its positivity follows from (H_3) which implies that for all $\mathbf{n} \in \text{int} \mathbf{F}_i$, $\alpha_{i,\mathbf{n}} > 0$.

The result about $\mathbf{v} + k_i \mathbf{e}_i$ with $\mathbf{v} \in \mathbf{K} \setminus \mathbf{e}_i^\perp$ is a direct consequence. \square

Assuming in addition strict monotonicity of $\alpha \mapsto c_i(\alpha \mathbf{n})$ (which is for instance satisfied if $\mathbf{c}(\mathbf{v}) = \mathbf{C}\mathbf{v}$ with $\mathbf{C} \gg \mathbf{0}$, that is in the Lotka-Volterra competition case), we can obtain the following more precise geometric description of the reaction term. The proof is quite straightforward and is not detailed here.

Lemma 3.2. *Assume in addition that $\alpha \mapsto c_i(\alpha \mathbf{n})$ is increasing for all $\mathbf{n} \in \mathbf{H}_i$.*

Then there exists a collection of connected \mathcal{C}^1 -hypersurfaces:

$$(\mathbf{Z}_i)_{i \in [N]} \subset \prod_{i=1}^N ((\mathbf{K}^+ \cap \mathbf{H}_i) \setminus \mathbf{e}_i^\perp)$$

such that, for any $i \in [N]$ and any $\mathbf{v} \in (\mathbf{K}^+ \cap \mathbf{H}_i) \setminus \mathbf{e}_i^\perp$:

$$(\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v})_i = 0 \text{ if and only if } \mathbf{v} \in \mathbf{Z}_i.$$

For all $i \in [N]$, \mathbf{Z}_i satisfies the following properties.

- (1) *For all $\mathbf{n} \in (\mathbf{S}^+(\mathbf{0}, 1) \cap \mathbf{H}_i) \setminus \mathbf{e}_i^\perp$, $\mathbf{Z}_i \cap \mathbb{R}\mathbf{n}$ is a singleton.*
- (2) *The function \mathbf{z}_i which associates with any $\mathbf{n} \in (\mathbf{S}^+(\mathbf{0}, 1) \cap \mathbf{H}_i) \setminus \mathbf{e}_i^\perp$ the unique element of $\mathbf{Z}_i \cap \mathbb{R}\mathbf{n}$ is continuous and is a \mathcal{C}^1 -diffeomorphism of $(\mathbf{S}^{++}(\mathbf{0}, 1) \cap \text{int} \mathbf{H}_i) \setminus \mathbf{e}_i^\perp$ onto $\text{int} \mathbf{Z}_i$.*

(3) For any $\mathbf{v} \in \mathbb{K}^+ \setminus \mathbf{e}_i^\perp$, $(\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v})_i > 0$ if and only if:

$$\mathbf{v} \in \mathbb{H}_i \text{ and } |\mathbf{v}| < \left| \mathbf{z}_i \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \right|.$$

3.2. Absorbing set and upper estimates. Define for all $i \in [N]$:

$$\begin{aligned} g_i : [0, +\infty) &\rightarrow (0, +\infty) \\ \mu &\mapsto \max(\mu, k_i). \end{aligned}$$

The function g_i is non-decreasing and piecewise affine (whence Lipschitz-continuous).

The following local in space \mathcal{L}^∞ estimate for the parabolic problem is due to Barles–Evans–Souganidis [6]. We repeat its proof for the sake of completeness.

Lemma 3.3. *Let $\mathbb{Q} \subset \mathbb{R}^2$ be a parabolic cylinder bounded in space and bounded from below in time.*

Let \mathbf{u} be a nonnegative classical solution of (E_{KPP}) set in \mathbb{Q} such that:

$$\mathbf{u}|_{\partial_P \mathbb{Q}} \in \mathcal{L}^\infty(\partial_P \mathbb{Q}, \mathbb{R}^N).$$

Then we have:

$$\left(\sup_{\mathbb{Q}} u_i \right)_{i \in [N]} \leq \left(g_i \left(\sup_{\partial_P \mathbb{Q}} u_i \right) \right)_{i \in [N]}.$$

Proof. Let $t_0 \in \mathbb{R}$, $T \in (0, +\infty]$ and $(a, b) \in \mathbb{R}^2$ such that $\mathbb{Q} = (t_0, t_0 + T) \times (a, b)$. Let $i \in [N]$.

Define a smooth convex function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ which satisfies:

$$\begin{cases} \eta(u) = 0 & \text{if } u \in (-\infty, g_i \left(\sup_{\partial_P \mathbb{Q}} u_i \right)] \\ \eta(u) > 0 & \text{otherwise.} \end{cases}$$

For all $t \in (t_0, t_0 + T)$, let:

$$\Xi_i(t) = \left\{ x \in (a, b) \mid u_i(t, x) > g_i \left(\sup_{\partial_P \mathbb{Q}} u_i \right) \right\}.$$

This set is measurable and, by integration by parts, for all $t \in (t_0, t_0 + T)$:

$$\begin{aligned} \partial_t \left(\int_a^b \eta(u_i(t, x)) \, dx \right) &= \int_a^b \eta'(u_i(t, x)) \partial_t u_i(t, x) \, dx \\ &= -d_i \int_a^b \eta''(u_i(t, x)) (\partial_x u_i(t, x))^2 \, dx \\ &\quad + \int_a^b \eta'(u_i(t, x)) \left(\sum_{j=1}^N l_{i,j} u_j(t, x) - c_i(\mathbf{u}(t, x)) u_i(t, x) \right) \, dx \\ &= -d_i \int_{\Xi_i(t)} \eta''(u_i(t, x)) (\partial_x u_i(t, x))^2 \, dx \\ &\quad + \int_{\Xi_i(t)} \eta'(u_i(t, x)) \left(\sum_{j=1}^N l_{i,j} u_j(t, x) - c_i(\mathbf{u}(t, x)) u_i(t, x) \right) \, dx \\ &\leq 0 \end{aligned}$$

Since $\int_a^b \eta(u_i(t_0, x)) dx = 0$, we deduce:

$$u_i \leq g_i \left(\sup_{\partial_P \mathbb{Q}} u_i \right) \text{ in } \mathbb{Q},$$

whence:

$$\sup_{\mathbb{Q}} u_i \leq g_i \left(\sup_{\partial_P \mathbb{Q}} u_i \right).$$

□

As a corollary of this local estimate, we get Theorem 1.2.

Proposition 3.4. *Let $\mathbf{u}_0 \in \mathcal{C}_b(\mathbb{R}, \mathbb{K})$. Then the unique classical solution \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ with initial data \mathbf{u}_0 satisfies:*

$$\left(\sup_{(0, +\infty) \times \mathbb{R}} u_i \right)_{i \in [N]} \leq \left(g_i \left(\sup_{\mathbb{R}} u_{0,i} \right) \right)_{i \in [N]}$$

and furthermore:

$$\left(\limsup_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) \right)_{i \in [N]} \leq \mathbf{g}(0).$$

Consequently, all bounded nonnegative classical solutions of (S_{KPP}) are valued in:

$$\prod_{i=1}^N [0, g_i(0)].$$

Proof. To get the global in space \mathcal{L}^∞ estimate, apply the local one to the family $(\mathbf{u}_R)_{R>0}$, where \mathbf{u}_R is the solution of (E_{KPP}) set in $(0, +\infty) \times (-R, R)$ with:

$$\begin{cases} \mathbf{u}_R(0, x) = \mathbf{u}_0(x) & \text{for all } x \in [-R, R], \\ \mathbf{u}_R(t, \pm R) = \mathbf{u}_0(\pm R) & \text{for all } t \geq 0, \end{cases}$$

and recall that, by classical parabolic estimates (Lieberman [39]) and a diagonal extraction process, $(\mathbf{u}_R)_{R>0}$ converges up to extraction in $\mathcal{C}_{loc}^1((0, +\infty), \mathcal{C}_{loc}^2(\mathbb{R}, \mathbb{R}^N))$ to the solution of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ with initial data \mathbf{u}_0 .

Next, let us prove that the invariant set:

$$\prod_{i=1}^N [0, g_i(0)] = \prod_{i=1}^N [0, k_i]$$

is in fact an absorbing set.

Assume by contradiction that there exists a bounded nonnegative classical solution \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ such that there exists $i \in [N]$ such that:

$$\limsup_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) > g_i(0).$$

Since $[0, g_i(0)]$ is invariant, it implies directly:

$$\sup_{x \in \mathbb{R}} u_i(t, x) > g_i(0) \text{ for all } t \geq 0.$$

Using the classical second order condition at any local maximum, it is easily seen that at any local maximum in space of u_i , the time derivative is negative. At any $t > 0$ such that there is no local maximum in space, by \mathcal{C}^1 regularity of u_i , $x \mapsto u_i(t, x)$ is either strictly monotonic or piecewise strictly monotonic with one unique local minimum and consequently it converges to some constant as $x \rightarrow \pm\infty$.

At least one of these constants is $\sup_{x \in \mathbb{R}} u_i(t, x)$. For instance, assume it is the limit at $+\infty$. By classical parabolic estimates and a diagonal extraction process, there exists $(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $x_n \rightarrow +\infty$ and such that the following sequence converges in $\mathcal{C}_{loc}^1((0, +\infty), \mathcal{C}_{loc}^2(\mathbb{R}))$:

$$((t', x) \mapsto u_i(t + t', x + x_n))_{n \in \mathbb{N}}.$$

Let v be its limit; by construction,

$$v(0, x) = \sup_{x \in \mathbb{R}} u_i(t, x) \text{ for all } x \in \mathbb{R},$$

so that:

$$\partial_{xx} v(0, x) = 0 \text{ for all } x \in \mathbb{R}.$$

Using the equation satisfied by u_i , we obtain:

$$\partial_t v(0, x) < 0 \text{ for all } x \in \mathbb{R}.$$

Since this argument does not depend on the choice of the sequence $(x_n)_{n \in \mathbb{N}}$, we deduce:

$$\limsup_{x \rightarrow +\infty} \partial_t u_i(t, x) < 0.$$

In all cases,

$$t \mapsto \|x \mapsto u_i(t, x)\|_{\mathcal{L}^\infty(\mathbb{R})}$$

is a decreasing function, and using the global \mathcal{L}^∞ estimate derived earlier, we deduce that:

$$t \mapsto \|u_i\|_{\mathcal{L}^\infty((t, +\infty) \times \mathbb{R})}$$

is a decreasing function as well. Therefore:

$$\limsup_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) = \liminf_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) = \lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u_i(t, x) > g_i(0).$$

Now, the following sequence:

$$((t, x) \mapsto u_i(t + n, x))_{n \in \mathbb{N}}$$

being uniformly bounded in $\mathcal{L}^\infty((0, +\infty) \times \mathbb{R})$, by classical parabolic estimates and a diagonal extraction process, it converges up to extraction in $\mathcal{C}_{loc}^1((0, +\infty), \mathcal{C}_{loc}^2(\mathbb{R}))$ to some limit $u_{\infty, i} \in \mathcal{C}^1((0, +\infty), \mathcal{C}^2(\mathbb{R}))$.

On one hand, by construction, the following function:

$$t \mapsto \|x \mapsto u_{\infty, i}(t, x)\|_{\mathcal{L}^\infty(\mathbb{R})}$$

is constant and larger than $g_i(0)$. But on the other hand, passing also to the limit the other components of $(t, x) \mapsto \mathbf{u}(t + n, x)$ and then repeating the argument used earlier to prove the strict monotonicity of:

$$t \mapsto \|x \mapsto u_i(t, x)\|_{\mathcal{L}^\infty(\mathbb{R})},$$

we deduce the strict monotonicity of:

$$t \mapsto \|x \mapsto u_{\infty, i}(t, x)\|_{\mathcal{L}^\infty(\mathbb{R})},$$

which is an obvious contradiction. □

Quite similarly, we can establish an \mathcal{L}^∞ estimate for (S_{KPP}) , set in a strip, and with an additional drift.

Proposition 3.5. *Let $(a, b, c) \in \mathbb{R}^3$ such that $a < b$ and \mathbf{u} be a nonnegative classical solution of:*

$$-\mathbf{D}\mathbf{u}'' - \mathbf{c}\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \text{ in } (a, b).$$

Then:

$$\left(\max_{[a,b]} u_i \right)_{i \in [N]} \leq \left(g_i \left(\max_{\{a,b\}} u_i \right) \right)_{i \in [N]}.$$

Proof. Assume by contradiction that there exists $i \in [N]$ such that:

$$\max_{[a,b]} u_i > g_i \left(\max_{\{a,b\}} u_i \right).$$

Then there exists $x_0 \in (a, b)$ such that:

$$\max_{[a,b]} u_i = u_i(x_0) > k_i.$$

There exists $(x_1, x_2) \in (a, b)^2$ such that $x_1 < x_0 < x_2$ and:

$$\begin{cases} u_i(x) > k_i & \text{for all } x \in (x_1, x_2) \\ u_i(x) = \frac{1}{2}(k_i + u_i(x_0)) & \text{for all } x \in \{x_1, x_2\}. \end{cases}$$

But then the elliptic strong maximum principle applied to:

$$-d_i \partial_{xx} u_i \ll 0 \text{ in } (x_1, x_2)$$

contradicts the existence of an interior maximum in (x_1, x_2) at x_0 . \square

4. EXTINCTION AND PERSISTENCE

This section is devoted to the proof of Theorem 1.3. The extinction case is mainly straightforward but, because of the lack of comparison principle, the persistence case is more involved.

4.1. Extinction.

Proposition 4.1. *Assume $\lambda_{PF}(\mathbf{L}) < 0$.*

Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ converge asymptotically in time, exponentially fast, and uniformly in space to $\mathbf{0}$.

Proof. It suffices to notice that if \mathbf{u} is a nonnegative bounded solution of (E_{KPP}) , then $\mathbf{v} : (t, x) \mapsto e^{\lambda_{PF}(\mathbf{L})t} \mathbf{n}_{PF}(\mathbf{L})$ satisfies by virtue of (H_2) :

$$\partial_t(\mathbf{v} - \mathbf{u}) - \mathbf{D}\partial_{xx}(\mathbf{v} - \mathbf{u}) - \mathbf{L}(\mathbf{v} - \mathbf{u}) = \mathbf{c}[\mathbf{u}] \circ \mathbf{u} \geq \mathbf{0}.$$

Hence, up to a multiplication of \mathbf{v} by a large constant, the comparison principle applied to the linear weakly and fully coupled operator $\partial_t - \mathbf{D}\partial_{xx} - \mathbf{L}$ in $(0, +\infty) \times \mathbb{R}$ implies that $\mathbf{0} \leq \mathbf{u} \leq \mathbf{v}$. The limit easily follows. \square

4.1.1. *Regarding the critical case.* The proof for the case $\lambda_{PF}(\mathbf{L}) < 0$ clearly cannot be adapted if $\lambda_{PF}(\mathbf{L}) = 0$. In this subsection, we briefly explain why the present paper only conjectures the result in this case.

Let us recall that for the scalar equation $\partial_t u - \partial_{xx} u = -u^2$, the comparison principle ensures extinction (by comparison with a solution of $u'(t) = -u(t)^2$ with large enough initial data). Since the comparison principle is not satisfied by (E_{KPP}) , we cannot hope to generalize this proof and need to find another method. Still, in view of this scalar result, it is natural to aim for a proof of extinction.

As a preliminary observation, if some Perron–Frobenius eigenvectors of \mathbf{L} are zeros of \mathbf{c} , then extinction will not occur in general. Therefore, in order to solve the critical case, it is necessary to rule out this somehow degenerated case. This is of course consistent with the critical case for Theorem 1.4.

For the non-degenerated non-diffusive system $\mathbf{u}' = \mathbf{L}\mathbf{u} - \mathbf{c}[\mathbf{u}] \circ \mathbf{u}$, we know how to handle two particular cases:

- if \mathbf{L} is symmetric, then the classical Lyapunov function $V : \mathbf{u} \mapsto \frac{1}{2} |\mathbf{u}|^2$ ensures extinction;
- if there exists $\mathbf{a} \in \mathbb{K}^{++}$ such that $\mathbf{c}(\mathbf{v}) = (\mathbf{a}^T \mathbf{v}) \mathbf{1}_{N,1}$, the change of unknown:

$$\mathbf{z} : t \mapsto \exp \left(\int_0^t (\mathbf{a}^T \mathbf{u}(\tau)) d\tau \right) \mathbf{u}(t)$$

(exploited for instance by Leman–Mélard–Mirrahimi [?, Theorem 1.4]) ensures extinction.

But even in these special cases, the diffusive system cannot be handled (as far as we know).

The first idea of proof (which would be in the parabolic setting an entropy proof) would involve an integration by parts of $\mathbf{u}^T \mathbf{D} \partial_{xx} \mathbf{u}$ and therefore would have to deal with the unboundedness of the space domain \mathbb{R} . In such a situation, the classical trick (multiplication of (E_{KPP}) by $e^{-\varepsilon|x|} \mathbf{u}(t, x)^T$ instead of $\mathbf{u}(t, x)^T$ so that sufficient integrability is recovered) brings forth a new problematic term (see for instance Zelik [49] where this computation is carried on and only leads to the existence of an absorbing set). Hence, apart from some particular cases (space-periodic solutions or solutions vanishing as $x \rightarrow \pm\infty$) where we do not have to resort to this trick, the entropy method does not establish the extinction.

As for the second idea of proof, it is completely ruined by the space variable: the exponential term now depends also on x and, again, new problematic terms arise in the equation satisfied by \mathbf{z} .

In view of these facts, extinction in the critical case is both a very natural conjecture and a surprisingly challenging problem (which would be way beyond the scope of this article).

4.2. Persistence. The first step toward the persistence result is giving some rigorous meaning to the statement “if $\lambda_{PF}(\mathbf{L}) > 0$, then $\mathbf{0}$ is unstable”.

4.2.1. Slight digression: generalized principal eigenvalues and eigenfunctions for weakly and fully coupled elliptic systems.

Theorem 4.2. *Let $(n, n') \in \mathbb{N} \cap [1, +\infty) \times \mathbb{N} \cap [2, +\infty)$ and $\mathcal{L} : \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{n'}) \rightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R}^{n'})$ be a second-order elliptic operator, weakly and fully coupled, with continuous and bounded coefficients.*

Let:

$$\lambda_1(-\mathcal{L}) = \sup \{ \lambda \in \mathbb{R} \mid \exists \mathbf{v} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{K}_n^{++}) \quad -\mathcal{L}\mathbf{v} \geq \lambda \mathbf{v} \} \in \overline{\mathbb{R}}.$$

Then:

$$\lim_{R \rightarrow +\infty} \lambda_{1,Dir}(-\mathcal{L}, \mathbb{B}_n(\mathbf{0}, R)) = \lambda_1(-\mathcal{L}).$$

Furthermore, $\lambda_1(-\mathcal{L})$ is in fact a finite maximum and there exists a generalized principal eigenfunction, that is a positive solution of:

$$-\mathcal{L}\mathbf{v} = \lambda_1(-\mathcal{L})\mathbf{v}.$$

Remark. The convergence of the Dirichlet principal eigenvalue to the aforementioned generalized principal eigenvalue as $R \rightarrow +\infty$ as well as the existence of a generalized principal eigenfunction are well-known for scalar elliptic equations (see Berestycki–Rossi [12]), but as far as we know these results do not explicitly appear in the literature regarding elliptic systems. Still, the proof of Berestycki–Rossi [12] uses arguments developed in the celebrated article by Berestycki–Nirenberg–Varadhan [11] and which have been generalized to weakly and fully coupled elliptic systems already in order to prove the existence of a Dirichlet principal eigenvalue in non-necessarily smooth but bounded domains by Birindelli–Mitidieri–Sweers [13]. Hence we only briefly outline here the proof so that it can be checked that the generalization to unbounded domains is straightforward.

It begins with the standard verification of the equality between the generalized principal eigenvalue as defined above and the Dirichlet principal eigenvalue for bounded smooth domains (whose existence was proved for instance by Sweers [46]). Then, since the generalized principal eigenvalue is, by definition, non-increasing with respect to the inclusion of the domains, we get that the limit of the Dirichlet principal eigenvalues as $R \rightarrow +\infty$ exists and is larger than or equal to the generalized principal eigenvalue. It remains to prove that it is also smaller than or equal to it. This is done thanks to the family of Dirichlet eigenfunctions $(\mathbf{v}_R)_{R>0}$ associated with the family of Dirichlet principal eigenvalues normalized by:

$$\min_{i \in [n']} v_{i,R}(\mathbf{0}) = 1.$$

Thanks to Arapostathis–Gosh–Marcus’s Harnack inequality [3], we obtain a locally uniform \mathcal{L}^∞ estimate, whence, by virtue of classical elliptic estimates (Gilbarg–Trudinger [30]) and a diagonal extraction process, the existence of a limit, up to extraction, for the family $(\mathbf{v}_R)_{R>0}$ as $R \rightarrow +\infty$. This limit \mathbf{v}_∞ is nonnegative nonzero and satisfies:

$$-\mathcal{L}\mathbf{v}_\infty = \left[\lim_{R \rightarrow +\infty} \lambda_{1,Dir}(-\mathcal{L}, \mathbf{B}_n(\mathbf{0}, R)) \right] \mathbf{v}_\infty.$$

Thanks again to Arapostathis–Gosh–Marcus’s Harnack inequality, \mathbf{v}_∞ is in fact positive in \mathbb{R}^n . Thus, by definition of the generalized principal eigenvalue, the limit as $R \rightarrow +\infty$ is indeed smaller than or equal to it, and in the end the equality is proved as well as the existence of a generalized principal eigenfunction \mathbf{v}_∞ .

4.2.2. *Local instability and persistence.* Noticing:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - (\mathbf{L} - \gamma \lambda_{PF}(\mathbf{L}) \mathbf{I}) \right) = -(1 - \gamma) \lambda_{PF}(\mathbf{L}) \text{ for all } \gamma \in [0, 1]$$

and using either Dancer [22] or Lam–Lou [36], we get the following lemma.

Lemma 4.3. *Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists $(R_0, R_{1/2}) \in (0, +\infty)^2$ such that:*

$$\begin{aligned} \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \mathbf{L}, (-R_0, R_0) \right) &< 0, \\ \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right), (-R_{1/2}, R_{1/2}) \right) &< 0. \end{aligned}$$

Remark. In fact, much more precisely, it can be shown that, for all $\gamma \in [0, 1]$,

$$R \mapsto \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - (\mathbf{L} - \gamma \lambda_{PF}(\mathbf{L}) \mathbf{I}), (-R, R) \right)$$

is a decreasing homeomorphism from $(0, +\infty)$ onto $(-(1-\gamma) \lambda_{PF}(\mathbf{L}), +\infty)$.

By continuity of \mathbf{c} and (H_3) , as soon as $\lambda_{PF}(\mathbf{L}) > 0$, the following quantity:

$$\alpha_{1/2} = \max \left\{ \alpha > 0 \mid \forall \mathbf{v} \in [0, \alpha]^N \quad \mathbf{c}(\mathbf{v}) \leq \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{1}_{N,1} \right\}$$

is well-defined in \mathbb{R} and is positive. The pair $(R_{1/2}, \alpha_{1/2})$ will be used repeatedly up to the end of this section.

Lemma 4.4. *Assume $\lambda_{PF}(\mathbf{L}) > 0$. For all $\mu \in (0, \alpha_{1/2})$, let:*

$$T_\mu = \frac{\ln \alpha_{1/2} - \ln \mu}{-\lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right), (-R_{1/2}, R_{1/2}) \right)} > 0.$$

For all $(t_0, T, a, b) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}^2$ such that $\frac{b-a}{2} = R_{1/2}$ and for all nonnegative classical solutions \mathbf{u} of (E_{KPP}) set in the bounded parabolic cylinder $(t_0, t_0 + T) \times (a, b)$, if:

$$\begin{aligned} \min_{i \in [N]} \min_{x \in [a, b]} u_i(t_0, x) &= \mu, \\ \max_{i \in [N]} \max_{[t_0, t_0 + T] \times [a, b]} u_i &\leq \alpha_{1/2}, \end{aligned}$$

then $T < T_\mu$.

Proof. Let:

$$\Lambda = \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right), (-R_{1/2}, R_{1/2}) \right) < 0.$$

Let \mathbf{n} be the principal eigenfunction associated with the preceding Dirichlet principal eigenvalue normalized so that:

$$\max_{i \in [N]} \max_{[-R_{1/2}, R_{1/2}]} n_i = 1.$$

The following inequality is satisfied in $(-R_{1/2}, R_{1/2})$ by definition:

$$-\left(-\mathbf{D} \mathbf{n}'' - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right) \mathbf{n} \right) = -\Lambda \mathbf{n} \gg \mathbf{0}.$$

By definition of $\alpha_{1/2}$ and (H_2) , for all $\mathbf{v} \in [0, \alpha_{1/2}]^N$:

$$\mathbf{c}(\mathbf{v}) \circ \mathbf{v} \leq \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{v},$$

whence:

$$-(\mathbf{L} \mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \leq -\left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right) \mathbf{v}.$$

Now, fix $(t_0, T, a, b) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}^2$ such that $\frac{b-a}{2} = R_{1/2}$ and $T \geq T_\mu$. Assume by contradiction that there exists a nonnegative solution $\mathbf{u} : (t_0, t_0 + T) \times (a, b) \rightarrow \mathbb{K}$ of (E_{KPP}) such that the following properties hold:

$$\mu = \min_{i \in [N]} \min_{x \in [a, b]} u_i(t_0, x) > 0,$$

$$\max_{i \in [N]} \max_{[t_0, t_0+T] \times [a, b]} u_i \leq \alpha_{1/2}.$$

In particular, since $\mu > 0$, \mathbf{u} is nonnegative nonzero.

To simplify the notations, hereafter we assume that $t_0 = 0$ and $\frac{a+b}{2} = 0$. The general case is only a matter of straightforward translations.

Define the function:

$$\mathbf{v} : (t, x) \mapsto \mu e^{-\Lambda t} \mathbf{n}(x).$$

Clearly:

$$\mathbf{v}(0, x) \leq \mathbf{u}(0, x) \text{ for all } x \in [a, b].$$

It is easily verified as well that \mathbf{v} satisfies in $(0, T_\mu) \times (-R_{1/2}, R_{1/2})$:

$$-\left(\partial_t \mathbf{v} - \mathbf{D} \partial_{xx} \mathbf{v} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right) \mathbf{v} \right) \geq \mathbf{0},$$

whence, by construction of $\alpha_{1/2}$, $\mathbf{w} = \mathbf{u} - \mathbf{v}$ satisfies:

$$\partial_t \mathbf{w} - \mathbf{D} \partial_{xx} \mathbf{w} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right) \mathbf{w} \geq \partial_t \mathbf{u} - \mathbf{D} \partial_{xx} \mathbf{u} - \mathbf{L} \mathbf{u} + \mathbf{c}[\mathbf{u}] \circ \mathbf{u} = \mathbf{0}.$$

Most importantly, since by construction:

$$T_\mu = \max \left\{ t > 0 \mid \max_{i \in [N]} \max_{x \in [-R_{1/2}, R_{1/2}]} v_i(t, x) \leq \alpha_{1/2} \right\},$$

there exists $t^* \leq T_\mu \leq T$ and $x^* \in (-R_{1/2}, R_{1/2})$ such that $\mathbf{w} \gg \mathbf{0}$ in $[0, t^*) \times (-R_{1/2}, R_{1/2})$ and $\mathbf{w}(t^*, x^*) \in \partial \mathcal{K}$.

The strong maximum principle applied to the weakly and fully coupled linear operator $\partial_t - \mathbf{D} \partial_{xx} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right)$ proves then that $\mathbf{w} = \mathbf{0}$ in $[0, t^*) \times (-R_{1/2}, R_{1/2})$, which contradicts $\mathbf{w}(0, \pm R_{1/2}) \gg \mathbf{0}$. \square

The persistence result follows.

Proposition 4.5. *Assume $\lambda_{PF}(\mathbf{L}) > 0$.*

There exists $\nu > 0$ such that all bounded nonnegative nonzero classical solutions \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ satisfy, for all bounded intervals $I \subset \mathbb{R}$:

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in I} u_i(t, x) \right)_{i \in [N]} \geq \nu \mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative classical solutions of (S_{KPP}) are valued in:

$$\prod_{i=1}^N [\nu, g_i(0)].$$

Proof. Let \mathbf{u} be a bounded nonnegative nonzero classical solution of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$. In view of Proposition 3.4, for all $\varepsilon > 0$ there exists $t_\varepsilon \in (0, +\infty)$ such that:

$$\mathbf{u} \leq \left(\max_{i \in [N]} (g_i(0)) + \varepsilon \right) \mathbf{1}_{N,1} \text{ in } (t_\varepsilon, +\infty) \times \mathbb{R}.$$

Let $I \subset \mathbb{R}$ be a bounded interval. Fix temporarily $\varepsilon > 0$ and $x \in I$ and define $I_x = (x - R_{1/2}, x + R_{1/2})$.

A first application of Lemma 4.4 establishes that there exists $\hat{t}_x \in [t_\varepsilon, +\infty)$ such that:

$$\max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(\hat{t}_x, y) = \alpha_{1/2}$$

and that there exists $\tau > 0$ such that:

$$\max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t, y) > \alpha_{1/2} \text{ for all } t \in (\hat{t}_x, \hat{t}_x + \tau).$$

Hence the following quantity is well-defined in $[\hat{t}_x + \tau, +\infty)$:

$$t_1 = \inf \left\{ t \geq \hat{t}_x + \tau \mid \max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t, y) < \alpha_{1/2} \right\}.$$

Assume first $t_1 < +\infty$. Then by continuity,

$$\max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t_1, y) = \alpha_{1/2}.$$

For all functions $\mathbf{w} \in \mathcal{C}_b((0, +\infty) \times \mathbb{R}, [0, k]^N)$, the function:

$$\mathbf{A}_{\mathbf{w}} : x \mapsto \mathbf{L} - \text{diag} \mathbf{c}(\mathbf{w}(t, x))$$

satisfies:

$$\|\mathbf{A}_{\mathbf{w}}\|_{\mathcal{L}^\infty((\hat{t}_x, +\infty) \times I_x, \mathbb{R}^N)} \leq \max_{(i,j) \in [N]^2} |l_{i,j}| + \max_{i \in [N]} c_i(k \mathbf{1}_{N,1}).$$

This upper bound does not depend on \mathbf{w} itself.

By virtue of Földes–Poláčik’s Harnack inequality [29], there exists $\bar{\kappa} > 0$, dependent only on $N, R_{1/2}, \min_{i \in [N]} d_i, \max_{i \in [N]} d_i, \max_{(i,j) \in [N]^2} |l_{i,j}| + \max_{i \in [N]} c_i \left(\left(\max_{i \in [N]} (g_i(0)) + \varepsilon \right) \mathbf{1}_{N,1} \right)$

such that, for all $\mathbf{w} \in \mathcal{C}_b \left((0, +\infty) \times \mathbb{R}, \left[0, \max_{i \in [N]} (g_i(0)) + \varepsilon \right]^N \right)$, all nonnegative classical solutions \mathbf{v} of:

$$\partial_t \mathbf{v} - \mathbf{D} \partial_{xx} \mathbf{v} - \mathbf{A}_{\mathbf{w}} \mathbf{v} = \mathbf{0} \text{ in } I_x,$$

satisfy:

$$\min_{i \in [N]} \min_{y \in \overline{I_x}} v_i(t_1 + 1, y) \geq \bar{\kappa} \max_{i \in [N]} \max_{y \in \overline{I_x}} v_i(t_1, y).$$

We stress that $\bar{\kappa}$ does not depend on \mathbf{w} . In particular, taking $\mathbf{w} = \mathbf{v} = \mathbf{u}$, we deduce:

$$\min_{i \in [N]} \min_{y \in \overline{I_x}} u_i(t_1 + 1, y) \geq \bar{\kappa} \alpha_{1/2}.$$

Of course, up to a shrink of $\bar{\kappa}$, we can assume without loss of generality $\bar{\kappa} \in (0, 1)$. Then let:

$$T = \frac{-\ln \bar{\kappa}}{-\lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \left(\mathbf{L} - \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{I} \right), I_x \right)} > 0.$$

T does not depend on the choice of \mathbf{u} .

A second application of Lemma 4.4 establishes:

$$\max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t_1 + 1 + T) > \alpha_{1/2}.$$

Hence, defining the sequence:

$$t_{n+1} = \inf \left\{ t \geq t_n + 1 + T \mid \max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t, y) < \alpha_{1/2} \right\}$$

and repeating by induction the process, we deduce that any connected component of:

$$\left\{ t \in (\hat{t}_x, +\infty) \mid \max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t, y) < \alpha_{1/2} \right\}$$

is an interval of length smaller than $1 + T$.

A second application of Földes–Poláčik’s Harnack inequality shows that there exists $\overline{\sigma}_\varepsilon > 0$, dependent only on $N, R_{1/2}, T, \min_{i \in [N]} d_i, \max_{i \in [N]} d_i, \max_{(i,j) \in [N]^2} |l_{i,j}| +$

$\max_{i \in [N]} c_i \left(\left(\max_{i \in [N]} (g_i(0)) + \varepsilon \right) \mathbf{1}_{N,1} \right)$ such that, for all $t \in (\hat{t}_x, +\infty)$:

$$\min_{i \in [N]} \min_{y \in \overline{I_x}} u_i(t + T + 2, y) \geq \overline{\sigma}_\varepsilon \max_{i \in [N]} \max_{(t', y) \in [t, t+T+1] \times \overline{I_x}} u_i(t', y),$$

whence:

$$\min_{i \in [N]} \min_{y \in \overline{I_x}} u_i(t, y) \geq \overline{\sigma}_\varepsilon \alpha_{1/2} \text{ for all } t \in (\hat{t}_x + T + 2, +\infty).$$

Assume next $t_1 = +\infty$. Then:

$$\max_{i \in [N]} \max_{y \in \overline{I_x}} u_i(t, y) \geq \alpha_{1/2} \text{ for all } t \in (\hat{t}_x, +\infty),$$

and consequently:

$$\min_{i \in [N]} \min_{y \in \overline{I_x}} u_i(t, y) \geq \overline{\sigma}_\varepsilon \alpha_{1/2} \text{ for all } t \in (\hat{t}_x + T + 2, +\infty).$$

Since I is bounded and $x \mapsto \hat{t}_x$ can be assumed continuous in \mathbb{R} without loss of generality, it follows:

$$\min_{i \in [N]} \inf_{y \in I} u_i(t, y) \geq \overline{\sigma}_\varepsilon \alpha_{1/2} \text{ for all } t \in \left(\max_{x \in \overline{I}} (\hat{t}_x) + T + 2, +\infty \right),$$

whence:

$$\liminf_{t \rightarrow +\infty} \min_{i \in [N]} \inf_{y \in I} u_i(t, y) \geq \overline{\sigma}_\varepsilon \alpha_{1/2}$$

with $\overline{\sigma}_\varepsilon \alpha_{1/2}$ dependent only on ε . The conclusion follows of course by setting:

$$\nu = \sup_{\varepsilon > 0} (\overline{\sigma}_\varepsilon) \alpha_{1/2}.$$

□

Remark. We point out that $\max_{x \in \overline{I}} \hat{t}_x$ is finite because I is bounded. Of course, in $I = \mathbb{R}$, this problem becomes a spreading problem (see Proposition 7.1).

5. EXISTENCE OF POSITIVE STEADY STATES

This section is devoted to the proof of Theorem 1.4 .

Proposition 5.1. *Assume $\lambda_{PF}(\mathbf{L}) < 0$. Then there exists no positive classical solution of (S_{KPP}) .*

Proof. Recall that the Dirichlet principal eigenvalue is non-increasing with respect to the zeroth order coefficient.

On one hand, by virtue of (H_2) , we have for all $R > 0$ and all $\mathbf{v} \in \mathcal{C}_b(\mathbb{R}, \mathbf{K}^{++})$:

$$\lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - (\mathbf{L} - \text{diag}[\mathbf{v}]), (-R, R) \right) \geq \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{dx^2} - \mathbf{L}, (-R, R) \right),$$

whence as $R \rightarrow +\infty$:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - (\mathbf{L} - \text{diagc}[\mathbf{v}]) \right) \geq -\lambda_{PF}(\mathbf{L}) > 0.$$

On the other hand, any positive steady state \mathbf{v} is also a generalized principal eigenfunction for the generalized principal eigenvalue:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - (\mathbf{L} - \text{diagc}[\mathbf{v}]) \right) = 0.$$

□

Proposition 5.2. *Assume $\lambda_{PF}(\mathbf{L}) = 0$. Then the set of bounded positive classical solutions of (S_{KPP}) is:*

$$\{\mathbf{v} \in \text{span}(\mathbf{n}_{PF}(\mathbf{L})) \cap \mathbf{K}^{++} \mid \mathbf{c}(\mathbf{v}) = \mathbf{0}\}.$$

In particular, if the latter set is empty, then there exists no bounded positive classical solution of (S_{KPP}) .

Proof. We use $\mathbf{n}_{PF}(\mathbf{L})$ multiplied by a large positive constant as a super-solution and conclude with Arapostathis–Gosh–Marcus’s Harnack inequality [3] applied to the linear elliptic operator $-\mathbf{D} \frac{d^2}{dx^2} - \mathbf{L}$. □

Finally, recall that if $\lambda_{PF}(\mathbf{L}) > 0$, then the following quantity is well-defined and positive:

$$\alpha_{1/2} = \max \left\{ \alpha > 0 \mid \forall \mathbf{v} \in [0, \alpha]^N \quad \mathbf{c}(\mathbf{v}) \leq \frac{\lambda_{PF}(\mathbf{L})}{2} \mathbf{1}_{N,1} \right\}.$$

Proposition 5.3. *Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists a solution $\mathbf{v} \in \mathbf{K}^{++}$ of:*

$$\mathbf{L}\mathbf{v} = \mathbf{c}(\mathbf{v}) \circ \mathbf{v}.$$

Proof. By virtue of the Perron–Frobenius theorem, $\mathbf{n}_{PF}(\mathbf{L}^T) \in \mathbf{K}^{++}$.

There exists $\eta > 0$ such that, for all $\mathbf{v} \in \mathbf{K}$, if $\mathbf{n}_{PF}(\mathbf{L}^T)^T \mathbf{v} = \eta$, then $\mathbf{v} \in [0, \alpha_{1/2}]^N$. Defining:

$$\mathbf{A} = \left\{ \mathbf{v} \in \mathbf{K} \mid \mathbf{n}_{PF}(\mathbf{L}^T)^T \mathbf{v} = \eta \right\},$$

it follows that for all $\mathbf{v} \in \mathbf{A}$,

$$\mathbf{n}_{PF}(\mathbf{L}^T)^T (\mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \leq \frac{\lambda_{PF}(\mathbf{L})}{2} \eta,$$

whence:

$$\begin{aligned} \mathbf{n}_{PF}(\mathbf{L}^T)^T (\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) &= \lambda_{PF}(\mathbf{L}^T) \eta - \mathbf{n}_{PF}(\mathbf{L}^T)^T (\mathbf{c}(\mathbf{v}) \circ \mathbf{v}) \\ &\geq \frac{\lambda_{PF}(\mathbf{L})}{2} \eta, \end{aligned}$$

which is positive if $\lambda_{PF}(\mathbf{L}) > 0$ is assumed indeed.

Then, defining the convex compact set:

$$\mathbf{C} = \left\{ \mathbf{v} \in \mathbf{K} \mid \mathbf{n}_{PF}(\mathbf{L}^T)^T \mathbf{v} \geq \eta \text{ and } \mathbf{v} \leq \mathbf{k} + \mathbf{1}_{N,1} \right\},$$

it can easily be verified that for all $\mathbf{v} \in \partial\mathbf{C}$:

$$\mathbf{n}_{\mathbf{v}}^T (\mathbf{L}\mathbf{v} - \mathbf{c}(\mathbf{v}) \circ \mathbf{v}) < 0$$

where \mathbf{n}_v is the outward pointing normal. In particular, there is no solution of $\mathbf{L}v = \mathbf{c}(v) \circ v$ in ∂C . Also, by convexity, for all $v \in \partial C$, there exists a unique $\delta_v > 0$ such that:

$$v + \delta_v (\mathbf{L}v - \mathbf{c}(v) \circ v) \in \partial C.$$

Assume by contradiction that there is no solution of $\mathbf{L}v = \mathbf{c}(v) \circ v$ in $\text{int}C$. Consequently and by convexity again, for all $v \in \text{int}C$, there exists a unique $\delta_v > 0$ such that:

$$v + \delta_v (\mathbf{L}v - \mathbf{c}(v) \circ v) \in \partial C.$$

The function:

$$\begin{array}{ccc} C & \rightarrow & (0, +\infty) \\ v & \mapsto & \delta_v \end{array}$$

is continuous and so is the function:

$$\begin{array}{ccc} C & \rightarrow & \partial C \\ v & \mapsto & v + \delta_v (\mathbf{L}v - \mathbf{c}(v) \circ v). \end{array}$$

According to the Brouwer fixed point theorem, this function has a fixed point, which of course contradicts the assumption.

Hence there exists indeed a solution in $\text{int}C \subset K^{++}$ of:

$$\mathbf{L}v = \mathbf{c}(v) \circ v.$$

□

6. TRAVELING WAVES

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$ and prove Theorem 1.5.

Notice as a preliminary that, for any $(\mathbf{p}, c) \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^N) \times [0, +\infty)$,

$$\mathbf{u} : (t, x) \mapsto \mathbf{p}(x - ct)$$

is a classical solution of (E_{KPP}) if and only if \mathbf{p} is a classical solution of:

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}[\mathbf{p}] \circ \mathbf{p} \text{ in } \mathbb{R}. \quad (TW[c])$$

6.1. The linearized equation. As usual in KPP-type problems, the linearized equation near $\mathbf{0}$:

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} \text{ in } \mathbb{R} \quad (TW_0[c])$$

will bring forth the main informations we need in order to construct and study the traveling wave solutions. Hence we devote this first subsection to its detailed study.

Lemma 6.1. *Let $(c, \lambda) \in \mathbb{R}^2$. We consider the equation:*

$$-\mathbf{D}\mathbf{p}'' - c\mathbf{p}' - (\mathbf{L} + \lambda\mathbf{I})\mathbf{p} = \mathbf{0} \text{ in } \mathbb{R}. \quad (TW_0[c, \lambda])$$

If there exists a classical positive solution \mathbf{p} of $(TW_0[c, \lambda])$, then there exists $(\mu, \mathbf{n}) \in \mathbb{R} \times K^{++}$ such that $\mathbf{q} : \xi \mapsto e^{-\mu\xi}\mathbf{n}$ is a classical solution of $(TW_0[c, \lambda])$.

Remark. This is of course to be related with the notions of generalized principal eigenvalue and generalized principal eigenfunction (see Theorem 4.2). The mere existence of \mathbf{p} enforces:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi} - c \frac{d}{d\xi} - (\mathbf{L} + \lambda\mathbf{I}) \right) \geq 0.$$

The following proof is inspired by Berestycki–Hamel–Roques [9, Lemma 3.1].

Proof. Let \mathbf{p} be a classical positive solution of $(TW_0 [c, \lambda])$.

Let $\mathbf{v} = \left(\frac{p'_i}{p_i} \right)_{i \in [N]}$. By virtue of Arapostathis–Gosh–Marcus’s Harnack inequality [3], classical elliptic estimates (Gilbarg–Trudinger [30]) and invariance by translation of $(TW_0 [c, \lambda])$, \mathbf{v} is globally bounded. Define for all $i \in [N]$:

$$\Lambda_i = \limsup_{\xi \rightarrow +\infty} v_i(\xi).$$

Let $\bar{\Lambda} = \max_{i \in [N]} \Lambda_i$, so that:

$$\left(\limsup_{\xi \rightarrow +\infty} v_i(\xi) \right)_{i \in [N]} \leq \bar{\Lambda} \mathbf{1}_{N,1}.$$

Let $(\xi_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\xi_n \rightarrow +\infty$ and such that there exists $\bar{i} \in [N]$ such that:

$$v_{\bar{i}}(\xi_n) \rightarrow \bar{\Lambda}.$$

On one hand, let:

$$\hat{\mathbf{p}}_n : \xi \mapsto \frac{\mathbf{p}(\xi + \xi_n)}{p_{\bar{i}}(\xi_n)} \text{ for all } n \in \mathbb{N}.$$

Once more by virtue of Arapostathis–Gosh–Marcus’s Harnack inequality, the sequence $(\hat{\mathbf{p}}_n)_{n \in \mathbb{N}}$ is locally uniformly bounded. Since all $\hat{\mathbf{p}}_n$ solve $(TW_0 [c, \lambda])$, by classical elliptic estimates and a diagonal extraction process, $(\hat{\mathbf{p}}_n)_{n \in \mathbb{N}}$ converges as $n \rightarrow +\infty$ in \mathcal{C}_{loc}^2 . Let $\hat{\mathbf{p}}_\infty$ be its limit. Notice by linearity of $(TW_0 [c, \lambda])$ that $\hat{\mathbf{p}}_\infty$ is in fact smooth and all its derivatives satisfy $(TW_0 [c, \lambda])$ as well.

On the other hand, let:

$$\mathbf{w}_n = \bar{\Lambda} \hat{\mathbf{p}}_n - \hat{\mathbf{p}}'_n \text{ for all } n \in \mathbb{N} \cup \{+\infty\}.$$

Notice the following equality:

$$\mathbf{w}_n(\xi) = \hat{\mathbf{p}}_n(\xi) \circ (\bar{\Lambda} \mathbf{1}_{N,1} - \mathbf{v}(\xi + \xi_n)) \text{ for all } n \in \mathbb{N} \text{ and } \xi \in \mathbb{R}.$$

Fix $\xi \in \mathbb{R}$. Recalling the obvious fact:

$$\left(\limsup_{n \rightarrow +\infty} v_i(\xi + \xi_n) \right)_{i \in [N]} \leq \left(\limsup_{\zeta \rightarrow +\infty} v_i(\zeta) \right)_{i \in [N]} \leq \bar{\Lambda} \mathbf{1}_{N,1},$$

it follows that for all $\varepsilon > 0$ there exists $n_{\xi, \varepsilon} \in \mathbb{N}$ such that for all $n \geq n_{\xi, \varepsilon}$,

$$(\bar{\Lambda} + \varepsilon) \mathbf{1}_{N,1} \geq \mathbf{v}(\xi + \xi_n),$$

whence for all $n \geq n_{\xi, \varepsilon}$:

$$\begin{aligned} \mathbf{w}_n(\xi) &\geq -\varepsilon \left(\sup_{m \geq n_{\xi, \varepsilon}} \hat{p}_{m,i}(\xi) \right)_{i \in [N]} \\ &\geq -\varepsilon \left(\sup_{m \in \mathbb{N}} \hat{p}_{m,i}(\xi) \right)_{i \in [N]}, \end{aligned}$$

and consequently, passing to the limit $n \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$, we obtain the non-negativity of $\mathbf{w}_\infty(\xi)$.

Hence \mathbf{w}_∞ is a nonnegative solution of $(TW_0 [c, \lambda])$ satisfying in addition:

$$w_{\infty, \bar{i}}(0) = \hat{p}_{\infty, \bar{i}}(0) \left(\bar{\Lambda} - \lim_{n \rightarrow +\infty} v_{\bar{i}}(\xi_n) \right) = 0,$$

whence, again by Arapostathis–Gosh–Marcus’s Harnack inequality, \mathbf{w}_∞ is in fact the null function.

Consequently, $\overline{\Lambda}\hat{\mathbf{p}}_\infty = \hat{\mathbf{p}}'_\infty$, that is $\hat{\mathbf{p}}_\infty$ has exactly the form:

$$\xi \mapsto e^{\overline{\Lambda}\xi}\mathbf{n} \text{ with } \mathbf{n} \in \mathbb{R}^N.$$

Since $\hat{\mathbf{p}}_\infty$ is nonnegative with $\hat{p}_{\infty,\bar{i}}(0) = 1$ by construction, $\mathbf{n} \in \mathbb{K}^+$, and since any nonnegative nonzero solution of $(TW_0[c, \lambda])$ is positive (Proposition 2.2), $\mathbf{n} \in \mathbb{K}^{++}$. The proof is ended with $\mu = -\overline{\Lambda}$. \square

For all $\mu \in \mathbb{R}$, the matrix $\mu^2\mathbf{D} + \mathbf{L}$ is essentially nonnegative irreducible. Define $\kappa_\mu = -\lambda_{PF}(\mu^2\mathbf{D} + \mathbf{L})$ and $\mathbf{n}_\mu = \mathbf{n}_{PF}(\mu^2\mathbf{D} + \mathbf{L})$.

Of course, the interest of the pair $(\kappa_\mu, \mathbf{n}_\mu)$ lies in the preceding lemma: for all $(\mu, \mathbf{n}) \in \mathbb{R} \times \mathbb{K}^{++}$, $\xi \mapsto e^{-\mu\xi}\mathbf{n}$ is a solution of $(TW_0[c])$ if and only if:

$$-\mu^2\mathbf{D}\mathbf{n} + \mu\mathbf{c}\mathbf{n} - \mathbf{L}\mathbf{n} = \mathbf{0},$$

that is, thanks to the Perron–Frobenius theorem, if and only if $\mu\mathbf{c} = -\kappa_\mu$ and $\frac{\mathbf{n}}{|\mathbf{n}|} = \mathbf{n}_\mu$. This most important observation leads naturally to the following study of the equation $c = -\frac{\kappa_\mu}{\mu}$.

Lemma 6.2. *The quantity:*

$$c^* = \min_{\mu > 0} \left(-\frac{\kappa_\mu}{\mu} \right)$$

is well-defined and positive.

Let $c \in [0, +\infty)$. In $(-\infty, 0)$, the equation $-\frac{\kappa_\mu}{\mu} = c$ admits no solution. In $(0, +\infty)$, it admits exactly:

- (1) *no solution if $c < c^*$;*
- (2) *one solution $\mu_{c^*} > 0$ if $c = c^*$;*
- (3) *two solutions $(\mu_{1,c}, \mu_{2,c})$ if $c > c^*$, which satisfy moreover:*

$$0 < \mu_{1,c} < \mu_{c^*} < \mu_{2,c}.$$

Remark. c^* does not depend on \mathbf{c} and is entirely determined by \mathbf{D} and \mathbf{L} . It will be the minimal speed of traveling waves and this kind of dependency is strongly reminiscent of the scalar Fisher–KPP case, where $c^* = 2\sqrt{rd}$. In fact the following proof is mostly a generalization of scalar arguments.

Proof. Of course, $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is odd in $\mathbb{R} \setminus \{0\}$. It is also positive in $(0, +\infty)$:

$$-\frac{\kappa_\mu}{\mu} = \frac{1}{\mu}\lambda_{PF}(\mu^2\mathbf{D} + \mathbf{L}) > \frac{1}{\mu}\lambda_{PF}(\mathbf{L}) > 0.$$

Therefore it is negative in $(-\infty, 0)$ and in particular there is no solution of $-\frac{\kappa_\mu}{\mu} = c \geq 0$ in $(-\infty, 0)$.

We recall Nussbaum’s theorem [43] which proves the convexity of the function $\mu \mapsto \rho(\mathbf{A}_\mu)$ provided:

- the matrix \mathbf{A}_μ is irreducible,
- its diagonal entries are convex functions of μ ,
- its off-diagonal entries are nonnegative log-convex functions of μ .

These conditions are easily verified for $\mu^2\mathbf{D} + \mathbf{L}$ and $\mu\mathbf{D} + \frac{1}{\mu}\mathbf{L}$ (actually, for all $\mu^{-\gamma}(\mu^2\mathbf{D} + \mathbf{L})$ provided $\gamma \in [0, 2]$). Their spectral radii being respectively $-\kappa_\mu$ and $-\frac{\kappa_\mu}{\mu}$, these are therefore convex functions of μ . Moreover, Nussbaum's result also proves that these convexities are actually strict. Therefore $\mu \mapsto -\kappa_\mu$ and $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ are strictly convex functions in $(0, +\infty)$.

Now, we investigate the behavior of $-\frac{\kappa_\mu}{\mu}$ as $\mu \rightarrow 0$ and $\mu \rightarrow +\infty$.

By continuity,

$$\kappa_\mu \rightarrow \kappa_0 \text{ as } \mu \rightarrow 0,$$

whence $-\frac{\kappa_\mu}{\mu} \rightarrow +\infty$ as $\mu \rightarrow 0$.

Since $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is convex and positive, either it is bounded in a neighborhood of $+\infty$ and then it converges to some nonnegative constant, either it is unbounded in a neighborhood of $+\infty$ and then it converges to $+\infty$. Assume that it converges to a finite constant. Notice:

$$\lim_{\mu \rightarrow +\infty} \frac{1}{\mu^2} (\mu^2\mathbf{D} + \mathbf{L}) = \mathbf{D}.$$

There exists a family of Perron-Frobenius eigenvectors of $\mu\mathbf{D} + \frac{1}{\mu}\mathbf{L}$, $(\mathbf{m}_\mu)_{\mu>0}$, normalized so that $\max_{i \in [N]} m_{\mu,i} = 1$ for all $\mu > 0$. Thanks to classical compactness arguments in \mathbb{R} and \mathbb{R}^N , we can extract a sequence $(\mu_n)_{n \in \mathbb{N}}$ such that $\mu_n \rightarrow +\infty$, $-\frac{\kappa_{\mu_n}}{\mu_n^2}$ converges to 0 and \mathbf{m}_{μ_n} converges to some $\mathbf{m} \in \mathbb{K}^+$. We point out that we do not know if $\mathbf{m} \in \mathbb{K}^{++}$, but from the normalizations, we do know that $\mathbf{m} \in \mathbb{K}^+$. Since \mathbf{m} satisfies $\mathbf{D}\mathbf{m} = \mathbf{0}$ and since \mathbf{D} is invertible, we get a contradiction. Thus:

$$\lim_{\mu \rightarrow +\infty} -\frac{\kappa_\mu}{\mu} = +\infty.$$

Hence $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is a strictly convex positive function which goes to $+\infty$ as $\mu \rightarrow 0$ or $\mu \rightarrow +\infty$: it admits necessarily a unique global minimum in $(0, +\infty)$. The quantity c^* is well-defined.

Define $\mu_{c^*} > 0$ such that:

$$c^* = -\frac{\kappa_{\mu_{c^*}}}{\mu_{c^*}}.$$

The quantity μ_{c^*} is uniquely defined by strict convexity. The function $\mu \mapsto -\frac{\kappa_\mu}{\mu}$ is bijective from $(0, \mu_{c^*})$ to $(c^*, +\infty)$ and from $(\mu_{c^*}, +\infty)$ to $(c^*, +\infty)$ as well. This ends the proof. \square

Putting together Lemma 6.1 and Lemma 6.2, we get the following important result.

Corollary 6.3. *For all $c \in [0, +\infty)$, the set of nonnegative nonzero classical solutions of $(TW_0[c])$ is empty if and only if $c \in [0, c^*)$.*

We can also get the exact values of c for which $\mathbf{0}$ is an unstable steady state of $(TW_0[c])$, in the sense of Lemma 4.3.

Lemma 6.4. *Let $c \in [0, +\infty)$. Then:*

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - c \frac{d}{dx} - \mathbf{L} \right) = \sup_{\mu \in \mathbb{R}} (\kappa_\mu + \mu c).$$

Furthermore:

- (1) $\sup_{\mu \in \mathbb{R}} (\kappa_\mu + \mu c) = \max_{\mu \geq 0} (\kappa_\mu + \mu c)$;
(2) $\max_{\mu \geq 0} (\kappa_\mu + \mu c) < 0$ if and only if $c < c^*$.

Remark. Just as in the case $c = 0$, it can be shown that, for all $c \in [0, +\infty)$,

$$R \mapsto \lambda_{1,Dir} \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L}, (-R, R) \right)$$

is a decreasing homeomorphism from $(0, +\infty)$ onto $\left(\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - c \frac{d}{dx} - \mathbf{L} \right), +\infty \right)$.

Proof. The fact that $\sup_{\mu \in \mathbb{R}} (\kappa_\mu + \mu c)$ is finite and actually a maximum attained in $[0, +\infty)$ is a direct consequence of:

- the evenness of $\mu \mapsto \kappa_\mu$ (whence, for all $\mu > 0$, $\kappa_{-\mu} + (-\mu)c < \kappa_\mu + \mu c$);
- $\kappa_0 < 0$;
- $\frac{\kappa_\mu}{\mu} + c \rightarrow -\infty$ as $\mu \rightarrow +\infty$ (see the proof of Lemma 6.2).

In addition, the sign of this maximum depending on the sign $c - c^*$ is given by Lemma 6.2.

Hence it only remains to prove the equality:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - c \frac{d}{dx} - \mathbf{L} \right) = \max_{\mu \geq 0} (\kappa_\mu + \mu c).$$

To do so, we use and adapt a well-known strategy of proof (see for instance Nadin [42]).

We recall from Theorem 4.2 the definition of the generalized principal eigenvalue:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{dx^2} - c \frac{d}{dx} - \mathbf{L} \right) = \sup \{ \lambda \in \mathbb{R} \mid \exists \mathbf{n} \in \mathcal{C}^2(\mathbb{R}, \mathbf{K}^{++}) \quad -\mathbf{D}\mathbf{n}'' - c\mathbf{n}' - \mathbf{L}\mathbf{n} \geq \lambda \mathbf{n} \}.$$

Also, there exists a generalized principal eigenfunction. We recall from Lemma 6.1 that if there exists a generalized principal eigenfunction, then there exists a generalized principal eigenfunction of the form $\xi \mapsto e^{-\mu^* \xi} \mathbf{m}$ with some constant $\mu^* \geq 0$ and $\mathbf{m} \in \mathbf{K}^{++}$.

Now, $(\mu^*, \mathbf{m}) \in [0, +\infty) \times \mathbf{K}^{++}$ satisfies:

$$-(\mu^*)^2 \mathbf{D}\mathbf{m} + c\mu^* \mathbf{m} - \mathbf{L}\mathbf{m} = \lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) \mathbf{m},$$

that is:

$$-\left((\mu^*)^2 \mathbf{D} + \mathbf{L} \right) \mathbf{m} = \left(\lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) - c\mu^* \right) \mathbf{m},$$

or in other words:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) = \kappa_{\mu^*} + c\mu^* \text{ and } \frac{\mathbf{m}}{|\mathbf{m}|} = \mathbf{n}_{\mu^*}.$$

Finally, the suitable test function to verify:

$$\lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) \geq \kappa_\mu + \mu c \text{ for all } \mu \geq 0$$

is of course $\mathbf{v}_\mu : \xi \mapsto e^{-\mu \xi} \mathbf{n}_\mu$ itself, which satisfies precisely:

$$-\mathbf{D}\mathbf{v}_\mu'' - c\mathbf{v}_\mu' - \mathbf{L}\mathbf{v}_\mu = (\kappa_\mu + \mu c) \mathbf{v}_\mu.$$

□

Corollary 6.5. *The quantity c^* is characterized by:*

$$\begin{aligned} c^* &= \sup \left\{ c \geq 0 \mid \lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) < 0 \right\} \\ &= \inf \left\{ c \geq 0 \mid \lambda_1 \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \mathbf{L} \right) > 0 \right\}. \end{aligned}$$

6.2. Qualitative properties of the traveling solutions. Thanks to Lemma 6.1 and Corollary 6.3, we are now in position to establish a few interesting properties that have direct consequences but will also be used at the end of the construction of the traveling waves.

Lemma 6.6. *Let $c \in \mathbb{R}$ and \mathbf{p} be a bounded nonnegative nonzero classical solution of $(TW [c])$.*

- (1) *If $\left(\liminf_{\xi \rightarrow +\infty} p_i(\xi) \right)_{i \in [N]} \in \partial\mathcal{K}$, then $c \geq c^*$.*
- (2) *If $\left(\liminf_{\xi \rightarrow -\infty} p_i(\xi) \right)_{i \in [N]} \in \partial\mathcal{K}$, then $c \leq -c^*$.*

Remark. The following proof is analogous to that of Berestycki–Nadin–Perthame–Ryzhik [10, Lemma 3.8] for the non-local KPP equation.

Proof. Obviously, the second statement is deduced from the first one by symmetry. Hence we only prove the first one.

Let $(\zeta_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that, as $n \rightarrow +\infty$, $\zeta_n \rightarrow +\infty$ and at least one component of $(\mathbf{p}(\zeta_n))_{n \in \mathbb{N}}$ converges to 0. Define:

$$\mathbf{p}_n : \xi \mapsto \mathbf{p}(\xi + \zeta_n)$$

and observe that \mathbf{p}_n satisfies $(TW [c])$ as well. Thanks to Arapostathis–Gosh–Marcus’s Harnack inequality [3], classical elliptic estimates (Gilbarg–Trudinger [30]) and a diagonal extraction process, $(\mathbf{p}_n)_{n \in \mathbb{N}}$ converges in \mathcal{C}_{loc}^2 to $\mathbf{0}$. This proves that there is no limit point of \mathbf{p} at $+\infty$ in $\partial\mathcal{K} \setminus \{\mathbf{0}\}$.

Next, we define:

$$\tilde{\mathbf{p}}_n : \xi \mapsto \frac{\mathbf{p}(\xi + \zeta_n)}{|\mathbf{p}(\zeta_n)|}$$

and notice, again by Arapostathis–Gosh–Marcus’s Harnack inequality, that $(\tilde{\mathbf{p}}_n)_{n \in \mathbb{N}}$ is locally uniformly bounded. Since, for all $n \in \mathbb{N}$, $\tilde{\mathbf{p}}_n$ solves the following elliptic system:

$$-\mathbf{D}\tilde{\mathbf{p}}_n'' - c\tilde{\mathbf{p}}_n' = \mathbf{L}\tilde{\mathbf{p}}_n - \mathbf{c}[\mathbf{p}_n] \circ \tilde{\mathbf{p}}_n,$$

with, thanks to (H_3) , $\mathbf{c}[\mathbf{p}_n] \rightarrow \mathbf{0}$ locally uniformly, up to extraction $(\tilde{\mathbf{p}}_n)_{n \in \mathbb{N}}$ converges in \mathcal{C}_{loc}^2 to a nonnegative solution $\tilde{\mathbf{p}}$ of $(TW_0 [c])$. Since $\tilde{\mathbf{p}}_n(0) \in \mathbf{S}^{++}(\mathbf{0}, \mathbf{1})$ for all $n \in \mathbb{N}$, $\tilde{\mathbf{p}}$ is nonnegative nonzero, whence positive (Proposition 2.2).

Now, from Corollary 6.3, we deduce indeed that $c \geq c^*$. □

This result implies the nonexistence half of Theorem 1.5 i).

Corollary 6.7. *For all $c \in [0, c^*)$, there is no traveling wave solution of (E_{KPP}) with speed c .*

Now, with Proposition 3.4, $c \geq c^* > 0$ and the fact that $(t, x) \mapsto \mathbf{p}(x - ct)$ solves (E_{KPP}) , we can straightforwardly derive the uniform upper bound Theorem 1.5 ii).

Corollary 6.8. *All profiles \mathbf{p} satisfy:*

$$\mathbf{p} \leq \mathbf{g}(0) \text{ in } \mathbb{R}.$$

Subsequently, using Proposition 4.5 and again $c \geq c^* > 0$ and the fact that $(t, x) \mapsto \mathbf{p}(x - ct)$ solves (E_{KPP}) , we get Theorem 1.5 iii).

Corollary 6.9. *All profiles \mathbf{p} satisfy:*

$$\left(\liminf_{\xi \rightarrow -\infty} p_i(\xi) \right)_{i \in [N]} \geq \nu \mathbf{1}_{N,1}.$$

Now, we establish Theorem 1.5 iv). Its proof is actually mostly a repetition of that of Lemma 6.1.

Proposition 6.10. *Let (\mathbf{p}, c) be a traveling wave solution of (E_{KPP}) .*

Then there exists $\bar{\xi} \in \mathbb{R}$ such that \mathbf{p} is component-wise decreasing in $[\bar{\xi}, +\infty)$.

Proof. Let $\mathbf{v} = \left(\frac{p'_i}{p_i} \right)_{i \in [N]}$. By virtue of Arapostathis–Gosh–Marcus’s Harnack inequality [3], classical elliptic estimates (Gilbarg–Trudinger [30]) and invariance by translation of $(TW[c])$, \mathbf{v} is globally bounded. Define for all $i \in [N]$:

$$\Lambda_i = \limsup_{\xi \rightarrow +\infty} v_i(\xi).$$

Let $\bar{\Lambda} = \max_{i \in [N]} \Lambda_i$, so that:

$$\left(\limsup_{\xi \rightarrow +\infty} v_i(\xi) \right)_{i \in [N]} \leq \bar{\Lambda} \mathbf{1}_{N,1}.$$

Let $(\xi_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\xi_n \rightarrow +\infty$ and such that there exists $\bar{i} \in [N]$ such that:

$$v_{\bar{i}}(\xi_n) \rightarrow \bar{\Lambda} \text{ as } n \rightarrow +\infty.$$

Let:

$$\hat{\mathbf{p}}_n : \xi \mapsto \frac{\mathbf{p}(\xi + \xi_n)}{p_{\bar{i}}(\xi_n)} \text{ for all } n \in \mathbb{N}.$$

and notice, again by Arapostathis–Gosh–Marcus’s Harnack inequality, that $(\hat{\mathbf{p}}_n)_{n \in \mathbb{N}}$ is locally uniformly bounded. Since, for all $n \in \mathbb{N}$, $\hat{\mathbf{p}}_n$ solves the following elliptic system:

$$-\mathbf{D}\hat{\mathbf{p}}_n'' - c\hat{\mathbf{p}}_n' = \mathbf{L}\hat{\mathbf{p}}_n - \mathbf{c}[p_{\bar{i}}(\xi_n)\hat{\mathbf{p}}_n] \circ \hat{\mathbf{p}}_n,$$

and, thanks to (H_3) , $\mathbf{c}[p_{\bar{i}}(\xi_n)\hat{\mathbf{p}}_n]$ converges locally uniformly to $\mathbf{0}$ as $n \rightarrow +\infty$, up to a diagonal extraction process, $(\hat{\mathbf{p}}_n)_{n \in \mathbb{N}}$ converges in \mathcal{C}_{loc}^2 to a nonnegative solution $\hat{\mathbf{p}}_\infty$ of $(TW_0[c])$.

Now we repeat the second part of the proof of Lemma 6.1 and we deduce in the end from Lemma 6.2 that $\hat{\mathbf{p}}_\infty$ has exactly the form:

$$\xi \mapsto Ae^{-\mu_c \xi} \mathbf{n}_{\mu_c},$$

with $\mu_c \in \{\mu_{1,c}, \mu_{2,c}\}$ if $c > c^*$, $\mu_c = \mu_{c^*}$ if $c = c^*$, $A > 0$ and, most importantly, with $\mu_c = -\bar{\Lambda}$.

Thus $\bar{\Lambda} < 0$. This implies that there exists $\bar{\xi} \in \mathbb{R}$ such that, for all $\xi \geq \bar{\xi}$,

$$\mathbf{v}(\xi) \leq -\frac{|\bar{\Lambda}|}{2} \mathbf{1}_{N,1},$$

whence, by positivity of \mathbf{p} :

$$\mathbf{p}'(\xi) \leq -\frac{|\bar{\Lambda}|}{2} \mathbf{p}(\xi).$$

The right-hand side being negative, \mathbf{p} is component-wise decreasing indeed. \square

Lemma 6.11. *Let $c \in \mathbb{R}$ and \mathbf{p} be a bounded nonnegative nonzero classical solution of $(TW[c])$.*

- (1) *If $\left(\liminf_{\xi \rightarrow +\infty} p_i(\xi)\right)_{i \in [N]} \in \partial\mathcal{K}$, then $\lim_{\xi \rightarrow +\infty} \mathbf{p}(\xi) = \mathbf{0}$.*
- (2) *If $\left(\liminf_{\xi \rightarrow -\infty} p_i(\xi)\right)_{i \in [N]} \in \partial\mathcal{K}$, then $\lim_{\xi \rightarrow -\infty} \mathbf{p}(\xi) = \mathbf{0}$.*

Proof. Again, the second statement being deduced from the first one by symmetry, we only prove the first one.

Let $(\zeta_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that, as $n \rightarrow +\infty$, $\zeta_n \rightarrow +\infty$ and at least one component of $(\mathbf{p}(\zeta_n))_{n \in \mathbb{N}}$ converges to 0. The proof of Lemma 6.6 shows that $(\mathbf{p}_n)_{n \in \mathbb{N}}$, defined by $\mathbf{p}_n : \xi \mapsto \mathbf{p}(\xi + \zeta_n)$, converges up to extraction in \mathcal{C}_{loc}^2 to $\mathbf{0}$.

Now, defining:

$$\begin{aligned} \mathbf{v}_n : \xi &\mapsto \left(\frac{p'_{n,i}(\xi)}{p_{n,i}(\xi)} \right)_{i \in [N]}, \\ \Lambda_i &= \limsup_{n \rightarrow +\infty} \max_{[-1,1]} v_{n,i}, \\ \bar{\Lambda} &= \max_{i \in [N]} \Lambda_i, \\ \bar{i} &\in [N] \text{ such that } \Lambda_{\bar{i}} = \bar{\Lambda}, \end{aligned}$$

and $(n_m)_{m \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ an increasing sequence such that $v_{n_m, \bar{i}}(0) \rightarrow \bar{\Lambda}$ as $m \rightarrow +\infty$, we can repeat once more the argument of the proof of Lemma 6.1 and obtain:

$$\bar{\Lambda} \hat{\mathbf{p}}_{\infty} = \hat{\mathbf{p}}'_{\infty} \text{ in } (-1, 1)$$

(notice that, contrarily to the proof of Lemma 6.1 where this equality was proved in \mathbb{R} , here it only holds locally). This brings forth $\bar{\Lambda} = -\mu_c < 0$, as in the proof of Proposition 6.10, whence \mathbf{p}_n is component-wise decreasing in $[-1, 1]$ provided n is large enough.

Now, assuming by contradiction:

$$\left(\limsup_{\xi \rightarrow +\infty} p_i(\xi) \right)_{i \in [N]} \in \mathcal{K}^+,$$

that is:

$$\left(\limsup_{\xi \rightarrow +\infty} p_i(\xi) \right)_{i \in [N]} \in \mathcal{K}^{++},$$

we deduce from the \mathcal{C}^1 regularity of \mathbf{p} that, for any $i \in [N]$, there exists a sequence $(\zeta'_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that:

- $\zeta'_n \rightarrow +\infty$ as $n \rightarrow +\infty$,
- $p_i(\zeta'_n)$ is a local minimum of p_i ,

- $p_i(\zeta'_n) \rightarrow 0$ as $n \rightarrow +\infty$.

Since this directly contradicts the preceding argument, we get indeed:

$$\left(\limsup_{\xi \rightarrow +\infty} p_i(\xi) \right)_{i \in [N]} = \mathbf{0} = \left(\liminf_{\xi \rightarrow +\infty} p_i(\xi) \right)_{i \in [N]}.$$

□

Lemma 6.12. *Let $c \in \mathbb{R}$. There exists $\eta_c > 0$ such that, for all bounded nonnegative classical solutions \mathbf{p} of $(TW[c])$, exactly one of the following properties holds:*

- (1) $\lim_{\xi \rightarrow +\infty} \mathbf{p}(\xi) = \mathbf{0}$ or $\lim_{\xi \rightarrow -\infty} \mathbf{p}(\xi) = \mathbf{0}$;
- (2) $\left(\inf_{\mathbb{R}} p_i \right)_{i \in [N]} \geq \eta_c \mathbf{1}_{N,1}$.

Remark. The following proof is again analogous to that of Berestycki–Nadin–Perthame–Ryzhik [10, Lemma 3.4] for the non-local KPP equation.

Proof. Recall from Proposition 2.2 and Lemma 6.11 that $\left(\inf_{\mathbb{R}} p_i \right)_{i \in [N]} \in \partial\mathcal{K}$ if and only if $\lim_{\xi \rightarrow +\infty} \mathbf{p}(\xi) = \mathbf{0}$ or $\lim_{\xi \rightarrow -\infty} \mathbf{p}(\xi) = \mathbf{0}$. Hence, defining Σ as the set of all bounded nonnegative classical solutions \mathbf{p} of $(TW[c])$ such that:

$$\min_{i \in [N]} \inf_{\mathbb{R}} p_i > 0,$$

this set being nonempty by virtue of Theorem 1.4, we only have to show the positivity of:

$$\eta_c = \inf \left\{ \min_{i \in [N]} \inf_{\mathbb{R}} p_i \mid \mathbf{p} \in \Sigma \right\}.$$

We assume by contradiction the existence of a sequence $(\mathbf{p}_n)_{n \in \mathbb{N}} \in \Sigma^{\mathbb{N}}$ such that:

$$\lim_{n \rightarrow +\infty} \min_{i \in [N]} \inf_{\mathbb{R}} p_{n,i} = 0.$$

For all $n \in \mathbb{N}$, define:

$$\beta_n = \min_{i \in [N]} \inf_{\mathbb{R}} p_{n,i} > 0,$$

fix $\xi_n \in \mathbb{R}$ such that:

$$\min_{i \in [N]} p_{n,i}(\xi_n) \in \left[\beta_n, \beta_n + \frac{1}{n} \right],$$

and define finally:

$$\mathbf{v}_n : \xi \mapsto \frac{1}{\beta_n} \mathbf{p}_n(\xi + \xi_n).$$

By virtue of Arapostathis–Gosh–Marcus’s Harnack inequality [3], classical elliptic estimates (Gilbarg–Trudinger [30]) and invariance by translation of $(TW[c])$, $(\mathbf{v}_n)_{n \in \mathbb{N}}$ is locally uniformly bounded and, up to a diagonal extraction process, converges in \mathcal{C}_{loc}^2 to some bounded limit \mathbf{v}_∞ . As in the proof of Lemma 6.1, it is easily verified that \mathbf{v}_∞ is a bounded positive classical solution of $(TW_0[c])$. Furthermore, by definition of $(\mathbf{v}_n)_{n \in \mathbb{N}}$,

$$\mathbf{v}_\infty \geq \mathbf{1}_{N,1} \text{ in } \mathbb{R}.$$

Repeating once more the argument of the proof of Lemma 6.1, we deduce that \mathbf{v}_∞ is component-wise decreasing in a neighborhood of $+\infty$. Thus its limit at $+\infty$,

say $\mathbf{m} \geq \mathbf{1}_{N,1}$, is well-defined. By classical elliptic estimates, \mathbf{m} satisfies $\mathbf{Lm} = \mathbf{0}$, which obviously contradicts $\lambda_{PF}(\mathbf{L}) > 0$. \square

6.3. Existence of traveling waves. This whole subsection is devoted to the adaptation of a proof of existence due to Berestycki, Nadin, Perthame and Ryzhik [10] and originally applied to the non-local KPP equation.

Remark. There is a couple of slight mistakes in the aforementioned proof.

- (1) Using the notations of [10], the sub-solution is defined as $\bar{r}_c = \max(0, r_c)$, with r_c chosen so that:

$$-cr'_c \leq r''_c + \mu r_c - \mu \bar{q}_c(\phi \star \bar{q}_c)$$

and it is claimed that \bar{r}_c satisfies as well this inequality, in the distributional sense. This is false: in an interval where $\bar{r}_c = 0$, we have:

$$-c\bar{r}'_c - \bar{r}''_c - \mu\bar{r}_c = 0 > -\mu\bar{q}_c(\phi \star \bar{q}_c).$$

As we will show, the correct sub-solution is $\bar{r}_c = \max(0, r_c)$ with r_c chosen so that:

$$-cr'_c \leq r''_c + \mu r_c - \mu r_c(\phi \star \bar{q}_c).$$

Fortunately, the function r_c constructed by the authors satisfies this inequality as well.

- (2) Later on, Φ_a is defined as the mapping which maps u_0 to the solution of:

$$-cu' = u'' + \mu u_0(1 - \phi \star u_0).$$

This mapping does not leave invariant the set of functions R_a defined with the correct sub-solution. It is necessary to change Φ_a and to define it as the mapping which maps u_0 to the solution of:

$$-cu' = u'' + \mu u(1 - \phi \star u_0).$$

Consequently, in order to establish that the set of functions R_a is invariant by Φ_a , the elliptic maximum principle is applied not to $u \mapsto -cu' - u''$ but to:

$$u \mapsto -u'' - cu' - \mu u$$

on one hand and to:

$$u \mapsto -u'' - cu' - \mu(1 - \phi \star \bar{q}_c)u$$

on the other hand.

During the first three subsubsections, we fix $c > c^*$.

6.3.1. Super-solution. We will use $\bar{\mathbf{p}} : \xi \mapsto e^{-\mu_1, c \xi} \mathbf{n}_{\mu_1, c}$ as a super-solution (recall from Lemma 6.2 that it is a solution of $(TW_0[c])$).

6.3.2. Sub-solution.

Proposition 6.13. *There exist $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, there exists $A_\varepsilon \in (0, +\infty)$ such that the function:*

$$\underline{\mathbf{p}} : \xi \mapsto \left(\max \left(e^{-\mu_1, c \xi} n_{\mu_1, c, i} - A_\varepsilon e^{-(\mu_1, c + \varepsilon) \xi} n_{\mu_1, c + \varepsilon, i}, 0 \right) \right)_{i \in [N]},$$

satisfies:

$$-\mathbf{D}\underline{\mathbf{p}}'' - \mathbf{c}\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}} \leq -\mathbf{c}[\bar{\mathbf{p}}] \circ \underline{\mathbf{p}} \text{ in } \mathcal{H}^{-1}(\mathbb{R}, \mathbb{R}^N).$$

Remark. During this quite technical proof, in order to ease the reading, we denote $\langle \bullet, \bullet \rangle_1$ and $\langle \bullet, \bullet \rangle_N$ the duality pairings of $\mathcal{H}^1(\mathbb{R}, \mathbb{R})$ and $\mathcal{H}^1(\mathbb{R}, \mathbb{R}^N)$ respectively, the latter being of course defined by:

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}^{-1}(\mathbb{R}, \mathbb{R}^N) \times \mathcal{H}^1(\mathbb{R}, \mathbb{R}^N)} = \sum_{i=1}^N \langle f_i, g_i \rangle_{\mathcal{H}^{-1}(\mathbb{R}) \times \mathcal{H}^1(\mathbb{R})}.$$

The speed c being fixed, we also omit the subscript c in the notations $\mu_{1,c}$ and $\mu_{2,c}$.

Proof. For the moment, let $A, \varepsilon > 0$ (they will be made precise during the course of the proof) and define:

$$\begin{aligned} \mathbf{v} : \xi &\mapsto e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} - A e^{-(\mu_1 + \varepsilon) \xi} \mathbf{n}_{\mu_1 + \varepsilon}, \\ \underline{\mathbf{p}} : \xi &\mapsto \left(\max_{i \in [N]} \left(e^{-\mu_1 \xi} n_{\mu_1, i} - A_\varepsilon e^{-(\mu_1 + \varepsilon) \xi} n_{\mu_1 + \varepsilon, i}, 0 \right) \right), \\ \Xi_+ &= \underline{\mathbf{p}}^{-1}(\mathbf{K}^{++}), \\ \Xi_0 &= \underline{\mathbf{p}}^{-1}(\mathbf{0}), \\ \Xi_\# &= \mathbb{R} \setminus (\Xi_+ \cup \Xi_0). \end{aligned}$$

Notice that $\Xi_\#$ is a connected compact set.

Fix a positive test function $\varphi \in \mathcal{H}^1(\mathbb{R}, \mathbf{K}^{++})$. We have to verify the following inequality:

$$\langle -\mathbf{D}\underline{\mathbf{p}}'' - c\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}}, \varphi \rangle_N \leq \langle -c[\underline{\mathbf{p}}] \circ \underline{\mathbf{p}}, \varphi \rangle_N.$$

To this end, we distinguish three cases: $\text{supp} \varphi \subset \Xi_+$, $\text{supp} \varphi \subset \Xi_0$ and $\text{supp} \varphi \cap \Xi_\# \neq \emptyset$. The case $\text{supp} \varphi \subset \Xi_0$ is trivial, with the inequality above satisfied in the classical sense.

Regarding the case $\text{supp} \varphi \subset \Xi_+$, we only have to verify the inequality in the classical sense in Ξ_+ for the regular function \mathbf{v} .

Fix temporarily $\xi \in \Xi_+$. We have:

$$\begin{aligned} -\mathbf{D}\mathbf{v}''(\xi) - c\mathbf{v}'(\xi) - \mathbf{L}\mathbf{v}(\xi) &= A e^{-(\mu_1 + \varepsilon) \xi} \left((\mu_1 + \varepsilon)^2 \mathbf{D} - c(\mu_1 + \varepsilon) \mathbf{I} + \mathbf{L} \right) \mathbf{n}_{\mu_1 + \varepsilon}, \\ (-c[\underline{\mathbf{p}}] \circ \mathbf{v})(\xi) &= -e^{-\mu_1 \xi} c \left(e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} \right) \circ (\mathbf{n}_{\mu_1} - A e^{-\varepsilon \xi} \mathbf{n}_{\mu_1 + \varepsilon}). \end{aligned}$$

From the following equalities:

$$\begin{aligned} \left((\mu_1 + \varepsilon)^2 \mathbf{D} + \mathbf{L} \right) \mathbf{n}_{\mu_1 + \varepsilon} &= -\kappa_{\mu_1 + \varepsilon} \mathbf{n}_{\mu_1 + \varepsilon}, \\ -c(\mu_1 + \varepsilon) \mathbf{n}_{\mu_1 + \varepsilon} &= \frac{\kappa_{\mu_1}}{\mu_1} (\mu_1 + \varepsilon) \mathbf{n}_{\mu_1 + \varepsilon}, \end{aligned}$$

and from the following inequality, which is a direct consequence of (H_2) :

$$-c \left(e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} \right) \circ (\mathbf{n}_{\mu_1} - A e^{-\varepsilon \xi} \mathbf{n}_{\mu_1 + \varepsilon}) \geq -c \left(e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} \right) \circ \mathbf{n}_{\mu_1},$$

it follows that it suffices to find A and ε such that:

$$A e^{-\varepsilon \xi} (\mu_1 + \varepsilon) \left(-\frac{\kappa_{\mu_1 + \varepsilon}}{\mu_1 + \varepsilon} + \frac{\kappa_{\mu_1}}{\mu_1} \right) \mathbf{n}_{\mu_1 + \varepsilon} \leq -c \left(e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} \right) \circ \mathbf{n}_{\mu_1}.$$

The right-hand side above being nonnegative, from the properties of $\mu \mapsto \frac{\kappa_\mu}{\mu}$ exhibited during the proof of Lemma 6.2, it follows clearly that such an inequality is never satisfied if $\mu_1 + \varepsilon > \mu_2$, whence a first necessary condition on ε is $\varepsilon \leq \mu_2 - \mu_1$ (notice that if $\varepsilon = \mu_2 - \mu_1$, then the inequality above holds if and only if $c \left(e^{-\mu_1 \xi} \mathbf{n}_{\mu_1} \right) = \mathbf{0}$, which is in general not true). Thus from now on we assume

$\mu < \mu_2 - \mu_1$. This ensures that $\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon} - \frac{\kappa_{\mu_1}}{\mu_1} > 0$, whence we now search for A and ε such that:

$$A \mathbf{n}_{\mu_1+\varepsilon} > \frac{e^{\varepsilon\xi}}{(\mu_1 + \varepsilon) \left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon} - \frac{\kappa_{\mu_1}}{\mu_1} \right)} \mathbf{c} \left(e^{-\mu_1\xi} \mathbf{n}_{\mu_1} \right) \circ \mathbf{n}_{\mu_1}.$$

Define $\bar{\xi} = \min \Xi_{\#}$, so that any $\xi \in \Xi_+$ satisfies necessarily $\xi > \bar{\xi}$. Remark that there exists $\bar{i} \in [N]$ such that:

$$\bar{\xi} = \frac{1}{\varepsilon} \left(\ln A + \ln \left(\frac{n_{\mu_1+\varepsilon, \bar{i}}}{n_{\mu_1, \bar{i}}} \right) \right).$$

Now, defining $\alpha : \xi \mapsto e^{-\mu_1\xi}$, if:

$$A \geq \max_{i \in [N]} \left(\frac{n_{\mu_1+\varepsilon, i}}{n_{\mu_1, i}} \right),$$

then $\bar{\xi} \geq 0$ and $\alpha(\xi) \leq 1$ in $(\bar{\xi}, +\infty)$. Moreover, we have:

$$e^{\varepsilon\xi} = (\alpha(\xi))^{-\frac{\varepsilon}{\mu_1}},$$

whence, for all $i \in [N]$:

$$e^{\varepsilon\xi} c_i \left(e^{-\mu_1\xi} \mathbf{n}_{\mu_1} \right) = \frac{c_i(\alpha(\xi) \mathbf{n}_{\mu_1})}{(\alpha(\xi))^{\frac{\varepsilon}{\mu_1}}},$$

and from the \mathcal{C}^1 regularity of \mathbf{c} as well as (H_3) , the above function of ξ is globally bounded in $(\bar{\xi}, +\infty)$, provided $\frac{\varepsilon}{\mu_1} \leq 1$, by the following positive constant:

$$\begin{aligned} M_i &= \sup_{\xi \in (\bar{\xi}, +\infty)} \frac{c_i(\alpha(\xi) \mathbf{n}_{\mu_1})}{\alpha(\xi)} \\ &= \sup_{\alpha \in (0, 1)} \frac{c_i(\alpha \mathbf{n}_{\mu_1})}{\alpha}. \end{aligned}$$

Subsequently, if A and ε satisfy also:

$$\varepsilon \leq \mu_1,$$

$$A \geq \max_{i \in [N]} \left(\frac{M_i n_{\mu_1, i}}{(\mu_1 + \varepsilon) \left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon} - \frac{\kappa_{\mu_1}}{\mu_1} \right) n_{\mu_1+\varepsilon, i}} \right),$$

then the inequality is established indeed in Ξ_+ . Hence we define:

$$\bar{\varepsilon} = \min(\mu_2 - \mu_1, \mu_1)$$

and, for any $\varepsilon \in (0, \bar{\varepsilon})$:

$$A_\varepsilon = \max_{i \in [N]} \max \left(\frac{n_{\mu_1+\varepsilon, i}}{n_{\mu_1, i}}, \frac{M_i n_{\mu_1, i}}{(\mu_1 + \varepsilon) \left(\frac{\kappa_{\mu_1+\varepsilon}}{\mu_1+\varepsilon} - \frac{\kappa_{\mu_1}}{\mu_1} \right) n_{\mu_1+\varepsilon, i}} \right)$$

and we assume from now on $\varepsilon \in (0, \bar{\varepsilon})$ and $A = A_\varepsilon$.

Let us point out here a fact which is crucial for the next step: choosing $\bar{\xi} = \min \Xi_{\#}$ instead of $\bar{\xi} = \max \Xi_{\#}$ (which might seem more natural at first view) implies that the differential inequality:

$$-\mathbf{D}\mathbf{v}'' - \mathbf{c}\mathbf{v}' - \mathbf{L}\mathbf{v} \leq -\mathbf{c}[\bar{\mathbf{p}}] \circ \mathbf{v}$$

holds classically in $\Xi_{\#} \cup \Xi_{+}$.

To conclude, let us verify the case $\text{supp}\varphi \cap \Xi_{\#} \neq \emptyset$. In order to ease the following computations, we actually assume $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}^N)$ (the result with $\varphi \in \mathcal{H}^1(\mathbb{R}, \mathbb{R}^N)$ can be recovered as usual by density). By definition:

$$\langle -\mathbf{D}\underline{\mathbf{p}}'' - \mathbf{c}\underline{\mathbf{p}}' - \mathbf{L}\underline{\mathbf{p}} + \mathbf{c}[\underline{\mathbf{p}}] \circ \underline{\mathbf{p}}, \varphi \rangle_N = \sum_{i=1}^N \left\langle -d_i \underline{p}_i'' - c \underline{p}_i' - \sum_{j=1}^N l_{i,j} \underline{p}_j + c_i [\underline{\mathbf{p}}] \underline{p}_i, \varphi_i \right\rangle_1.$$

Fix $i \in [N]$ and define $\xi_{0,i}$ as the unique element of $v_i^{-1}(\{0\})$ and:

$$\Psi_i = \left\langle -d_i \underline{p}_i'' - c \underline{p}_i' - \sum_{j=1}^N l_{i,j} \underline{p}_j + c_i [\underline{\mathbf{p}}] \underline{p}_i, \varphi_i \right\rangle_1.$$

Classical integrations by parts yield:

$$\begin{aligned} \int_{\mathbb{R}} \underline{p}_i'' \varphi_i &= \int_{\xi_{0,i}}^{+\infty} v_i'' \varphi_i + v_i'(\xi_{0,i}) \varphi_i(\xi_{0,i}) \geq \int_{\xi_{0,i}}^{+\infty} v_i'' \varphi_i, \\ \int_{\mathbb{R}} \underline{p}_i' \varphi_i &= \int_{\xi_{0,i}}^{+\infty} v_i' \varphi_i, \end{aligned}$$

whence:

$$\Psi_i \leq \int_{\xi_{0,i}}^{+\infty} (-d_i v_i'' - c v_i' + c_i [\underline{\mathbf{p}}] v_i) \varphi_i - \sum_{j=1}^N l_{i,j} \int_{\xi_{0,j}}^{+\infty} v_j \varphi_i.$$

As was pointed out previously, from the construction of ε and A , we know that:

$$-\mathbf{D}\mathbf{v}'' - \mathbf{c}\mathbf{v}' + \mathbf{c}[\underline{\mathbf{p}}] \circ \mathbf{v} \leq \mathbf{L}\mathbf{v} \text{ in } \Xi_{\#},$$

whence, with $J_i = \{j \in [N] \mid \xi_{0,j} < \xi_{0,i}\}$:

$$\Psi_i \leq - \sum_{j \in J_i} \int_{\xi_{0,j}}^{\xi_{0,i}} l_{i,j} v_j \varphi_i + \sum_{j \in [N] \setminus J_i} \int_{\xi_{0,i}}^{\xi_{0,j}} l_{i,j} v_j \varphi_i.$$

Finally, recalling that $v_j(\xi) > 0$ if $\xi > \xi_{0,j}$ and $v_j(\xi) < 0$ if $\xi < \xi_{0,j}$, the inequality above yields $\Psi_i \leq 0$, which ends the proof. \square

6.3.3. *The finite domain problem.* Let $R > 0$ and define the following truncated problem:

$$\begin{cases} -\mathbf{D}\mathbf{p}'' - \mathbf{c}\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}[\mathbf{p}] \circ \mathbf{p} & \text{in } (-R, R), \\ \mathbf{p}(\pm R) = \underline{\mathbf{p}}(\pm R). \end{cases} \quad (TW[R, c])$$

Lemma 6.14. *Assume:*

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{K}.$$

Then there exists a nonnegative nonzero classical solution \mathbf{p}_R of $(TW[R, c])$.

Remark. The new assumption made here ensures that the vector field \mathbf{c} is non-decreasing in \mathbf{K} , in the following natural sense: if $\mathbf{0} \leq \mathbf{v} \leq \mathbf{w}$, then $\mathbf{0} \leq \mathbf{c}(\mathbf{v}) \leq \mathbf{c}(\mathbf{w})$.

Proof. Fix arbitrarily $\varepsilon \in (0, \bar{\varepsilon})$, define consequently $\underline{\mathbf{p}}$ and then define the following convex set of functions:

$$\mathcal{F} = \{ \mathbf{v} \in \mathcal{C}([-R, R], \mathbb{R}^N) \mid \underline{\mathbf{p}} \leq \mathbf{v} \leq \bar{\mathbf{p}} \}.$$

Recall that Figueiredo–Mitidieri [24] establishes that the elliptic weak maximum principle holds for a weakly and fully coupled elliptic operator with null Dirichlet boundary conditions if this operator admits a positive strict super-solution. Since, for all $\mathbf{v} \in \mathcal{C}([-R, R], \mathbb{R}^N)$ such that $\mathbf{0} \leq \mathbf{v} \leq \bar{\mathbf{p}}$, we have by (H_2) :

$$-\mathbf{D}\bar{\mathbf{p}}'' - c\bar{\mathbf{p}}' - \mathbf{L}\bar{\mathbf{p}} + \mathbf{c}[\mathbf{v}] \circ \bar{\mathbf{p}} \geq -\mathbf{D}\bar{\mathbf{p}}'' - c\bar{\mathbf{p}}' - \mathbf{L}\bar{\mathbf{p}} \geq \mathbf{0},$$

$$\bar{\mathbf{p}}(\pm R) \gg \mathbf{0},$$

it follows that every operator of the following family:

$$\left(\mathbf{D} \frac{d^2}{d\xi^2} + c \frac{d}{d\xi} + (\mathbf{L} - \text{diag} \mathbf{c}[\mathbf{v}]) \right)_{\mathbf{0} \leq \mathbf{v} \leq \bar{\mathbf{p}}}$$

supplemented with null Dirichlet boundary conditions at $\pm R$ satisfies the weak maximum principle in $(-R, R)$.

Define the map \mathbf{f} which associates with some $\mathbf{v} \in \mathcal{F}$ the unique classical solution $\mathbf{f}[\mathbf{v}]$ of:

$$\begin{cases} -\mathbf{D}\mathbf{p}'' - c\mathbf{p}' = \mathbf{L}\mathbf{p} - \mathbf{c}[\mathbf{v}] \circ \mathbf{p} & \text{in } (-R, R) \\ \mathbf{p}(\pm R) = \underline{\mathbf{p}}(\pm R). \end{cases}$$

The map \mathbf{f} is compact by classical elliptic estimates (Gilbarg–Trudinger [30]). Let $\mathbf{v} \in \mathcal{F}$. By monotonicity of \mathbf{c} , the function $\mathbf{w} = \mathbf{f}[\mathbf{v}] - \underline{\mathbf{p}}$ satisfies:

$$\begin{aligned} -\mathbf{D}\mathbf{w}'' - c\mathbf{w}' - \mathbf{L}\mathbf{w} &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] + \mathbf{c}[\bar{\mathbf{p}}] \circ \underline{\mathbf{p}} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] + \mathbf{c}[\mathbf{v}] \circ \underline{\mathbf{p}} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{w} \end{aligned}$$

with null Dirichlet boundary conditions at $\pm R$. Therefore, by virtue of the weak maximum principle applied to $\mathbf{D} \frac{d^2}{d\xi^2} + c \frac{d}{d\xi} + (\mathbf{L} - \text{diag} \mathbf{c}[\mathbf{v}])$, $\mathbf{f}[\mathbf{v}] \geq \underline{\mathbf{p}}$ in $(-R, R)$. Next, since it is now established that $\mathbf{f}[\mathbf{v}] \geq \mathbf{0}$, we also have by (H_2) :

$$\begin{aligned} -\mathbf{D}\bar{\mathbf{p}}'' - c\bar{\mathbf{p}}' - \mathbf{L}\bar{\mathbf{p}} &= \mathbf{0} \\ &\geq -\mathbf{c}[\mathbf{v}] \circ \mathbf{f}[\mathbf{v}] \\ &= -\mathbf{D}\mathbf{f}[\mathbf{v}]'' - c\mathbf{f}[\mathbf{v}]' - \mathbf{L}\mathbf{f}[\mathbf{v}], \end{aligned}$$

$$\bar{\mathbf{p}}(\pm R) \geq \underline{\mathbf{p}}(\pm R) = \mathbf{f}[\mathbf{v}](\pm R),$$

whence $\bar{\mathbf{p}} \geq \mathbf{f}[\mathbf{v}]$ follows from the weak maximum principle applied this time to $\mathbf{D} \frac{d^2}{d\xi^2} + c \frac{d}{d\xi} + \mathbf{L}$.

Thus $\underline{\mathbf{p}} \leq \mathbf{f}[\mathbf{v}] \leq \bar{\mathbf{p}}$ and consequently $\mathbf{f}(\mathcal{F}) \subset \mathcal{F}$.

Finally, by virtue of the Schauder fixed point theorem, \mathbf{f} admits a fixed point $\mathbf{p}_R \in \mathcal{F}$, which is indeed a classical solution of $(TW[R, c])$ by elliptic regularity. \square

6.3.4. *The infinite domain limit and the minimal wave speed.* The speed c is not fixed anymore.

The following uniform upper estimate is a direct consequence of Proposition 3.5.

Corollary 6.15. *There exists $R^* > 0$ such that, for any $c > c^*$, any $R \geq R^*$ and any nonnegative classical solution \mathbf{p} of $(TW [R, c])$:*

$$\left(\max_{[-R, R]} p_i \right)_{i \in [N]} \leq \mathbf{g}(0).$$

We are now in position to prove the second half of Theorem 1.5 i).

Proposition 6.16. *Assume:*

$$D\mathbf{c}(\mathbf{v}) \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{K}.$$

Then for all $c \geq c^$, there exists a traveling wave solution of (E_{KPP}) with speed c .*

Remark. Of course, it would be interesting to exhibit other additional assumptions on \mathbf{c} sufficient to ensure existence of traveling waves for all $c \geq c^*$. In view of known results about scalar multistable reaction–diffusion equations (we refer for instance to Fife–McLeod [27]), some additional assumption should in any case be necessary.

Proof. Hereafter, for all $c > c^*$ and all $R > 0$, the triplet $(\bar{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{p}_R)$ constructed in the preceding subsections is denoted $(\bar{\mathbf{p}}_c, \underline{\mathbf{p}}_c, \mathbf{p}_{R,c})$.

For all $c > c^*$, thanks to Corollary 6.15, the family $(\mathbf{p}_{R,c})_{R>0}$ is uniformly globally bounded. By classical elliptic estimates (Gilbarg–Trudinger [30]) and a diagonal extraction process, we can extract a sequence $(R_n, \mathbf{p}_{R_n,c})_{n \in \mathbb{N}}$ such that, as $n \rightarrow +\infty$, $R_n \rightarrow +\infty$ and $\mathbf{p}_{R_n,c}$ converges to some limit \mathbf{p}_c in \mathcal{C}_{loc}^2 . As expected, \mathbf{p}_c is a bounded nonnegative classical solution of $(TW [c])$. The fact that its limit as $\xi \rightarrow +\infty$ is $\mathbf{0}$, as well as the fact that \mathbf{p}_c is nonzero whence positive (Proposition 2.2), are obvious thanks to the inequality $\underline{\mathbf{p}}_c \leq \mathbf{p}_c \leq \bar{\mathbf{p}}_c$. At the other end of the real line, Corollary 6.7 clearly enforces:

$$\left(\liminf_{\xi \rightarrow -\infty} p_{c,i}(\xi) \right)_{i \in [N]} \in \mathbf{K}^{++} \subset \mathbf{K}^+.$$

Thus (\mathbf{p}_c, c) is a traveling wave solution.

In order to construct a critical traveling wave (\mathbf{p}_{c^*}, c^*) , we consider a decreasing sequence $(c_n)_{n \in \mathbb{N}} \in (c^*, +\infty)^{\mathbb{N}}$ such that $c_n \rightarrow c^*$ as $n \rightarrow +\infty$ and intend to apply a compactness argument to a normalized version of the sequence $(\mathbf{p}_{c_n})_{n \in \mathbb{N}}$.

By Corollary 6.9,

$$\liminf_{\xi \rightarrow -\infty} \min_{i \in [N]} p_{c_n,i}(\xi) \geq \nu \text{ for all } n \in \mathbb{N}.$$

Recall from Lemma 6.12 the definition of $\eta_c > 0$. For all $n \in \mathbb{N}$ the following quantity is well-defined and finite:

$$\xi_n = \inf \left\{ \xi \in \mathbb{R} \mid \min_{i \in [N]} p_{c_n,i}(\xi) < \min \left(\frac{\nu}{2}, \frac{\eta_{c^*}}{2} \right) \right\}.$$

We define the normalized sequence of profiles by:

$$\tilde{\mathbf{p}}_{c_n} : \xi \mapsto \mathbf{p}_{c_n}(\xi + \xi_n) \text{ for all } n \in \mathbb{N}.$$

A translation of a profile of traveling wave being again a profile of traveling wave, $(\tilde{\mathbf{p}}_{c_n}, c_n)_{n \in \mathbb{N}}$ is again a sequence of traveling wave solutions. Notice the following two immediate consequences of the normalization:

$$\begin{aligned} \min_{i \in [N]} \tilde{p}_{c_n, i}(0) &= \min\left(\frac{\nu}{2}, \frac{\eta_{c^*}}{2}\right) \text{ for all } n \in \mathbb{N}, \\ \inf_{\xi \in (-\infty, 0)} \min_{i \in [N]} \tilde{p}_{c_n, i}(\xi) &\geq \min\left(\frac{\nu}{2}, \frac{\eta_{c^*}}{2}\right) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

We are now in position to pass to the limit $n \rightarrow +\infty$. The sequence $(\tilde{\mathbf{p}}_{c_n})_{n \in \mathbb{N}}$ being globally uniformly bounded, it admits, up to a diagonal extraction process, a bounded nonnegative limit \mathbf{p}_{c^*} in \mathcal{C}_{loc}^2 . Since $c_n \rightarrow c^*$, \mathbf{p}_{c^*} satisfies $(TW[c^*])$. The normalization yields:

$$\begin{aligned} \min_{i \in [N]} p_{c^*, i}(0) &= \min\left(\frac{\nu}{2}, \frac{\eta_{c^*}}{2}\right), \\ \inf_{\xi \in (-\infty, 0)} \min_{i \in [N]} p_{c^*, i}(\xi) &\geq \min\left(\frac{\nu}{2}, \frac{\eta_{c^*}}{2}\right). \end{aligned}$$

Consequently,

$$\left(\liminf_{\xi \rightarrow -\infty} p_{c^*, i}(\xi) \right)_{i \in [N]} \in \mathbf{K}^{++}$$

and, according to Lemma 6.12,

$$\lim_{\xi \rightarrow +\infty} \mathbf{p}_{c^*}(\xi) = \mathbf{0}.$$

The pair (\mathbf{p}_{c^*}, c^*) is a traveling wave solution indeed and this ends the proof. \square

7. SPREADING SPEED

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$, fix $x_0 \in \mathbb{R}$ and a non-negative non-zero $\mathbf{v} \in \mathcal{C}_b(\mathbb{R}, \mathbb{R}^N)$ and define $\mathbf{u}_0 = \mathbf{v}\mathbf{1}_{(-\infty, x_0)}$ and the unique classical solution \mathbf{u} of (E_{KPP}) set in $(0, +\infty) \times \mathbb{R}$ with initial data \mathbf{u}_0 , in order to prove Theorem 1.6.

Remark. This type of spreading result, as well as its proof by means of super- and sub-solutions, is quite classical (we refer to Aronson–Weinberger [5] and Berestycki–Hamel–Nadin [8] among others). Still, we provide it to make clear that the lack of comparison principle for (E_{KPP}) is not really an issue.

Of course, for the scalar KPP equation, much more precise spreading results exist (for instance the celebrated articles by Bramson [18, 17] using probabilistic methods). Here, our aim is not to give a complete description of the spreading properties of (E_{KPP}) but rather to illustrate that it is, once more, very similar to the scalar situation and that further generalizations should be possible.

7.1. Upper estimate.

Proposition 7.1. *Let $c > c^*$ and $y \in \mathbb{R}$. We have:*

$$\left(\lim_{t \rightarrow +\infty} \sup_{x \in (y, +\infty)} u_i(t, x + ct) \right)_{i \in [N]} = \mathbf{0}.$$

Proof. By definition of \mathbf{u}_0 , there exists $\xi_1 \in \mathbb{R}$ such that the following positive solution of $(TW_0[c^*])$:

$$\bar{\mathbf{p}} : \xi \mapsto e^{-\mu_{c^*}(\xi - \xi_1)} \mathbf{n}_{\mu_{c^*}}$$

(see Lemma 6.2) is additionally such that $\bar{\mathbf{p}} - \mathbf{u}_0$ is positive. Then, defining $\bar{\mathbf{u}} : (t, x) \mapsto \bar{\mathbf{p}}(x - c^*t)$, we obtain by (H_2) :

$$\begin{aligned} \partial_t \bar{\mathbf{u}} - \mathbf{D}\partial_{xx} \bar{\mathbf{u}} - \mathbf{L}\bar{\mathbf{u}} &= \mathbf{0} \\ &\geq -\mathbf{c}[\mathbf{u}] \circ \mathbf{u} \\ &= \partial_t \mathbf{u} - \mathbf{D}\partial_{xx} \mathbf{u} - \mathbf{L}\mathbf{u} \end{aligned}$$

and then, applying the parabolic strong maximum principle to the operator $\partial_t - \mathbf{D}\partial_{xx} - \mathbf{L}$, we deduce that $\bar{\mathbf{u}} - \mathbf{u}$ is nonnegative in $[0, +\infty) \times \mathbb{R}$. Consequently, for all $x \in \mathbb{R}$, $t > 0$ and $c > c^*$:

$$\mathbf{0} \leq \mathbf{u}(t, x + ct) \leq \bar{\mathbf{p}}(x + (c - c^*)t),$$

and by component-wise monotonicity of $\bar{\mathbf{p}}$, for all $y \in \mathbb{R}$ and all $x \geq y$,

$$\mathbf{u}(t, x + ct) \leq \bar{\mathbf{p}}(y + (c - c^*)t),$$

which gives the result. \square

7.2. Lower estimate.

Proposition 7.2. *Let $c \in [0, c^*)$ and $I \subset \mathbb{R}$ be a bounded interval. We have:*

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in I} u_i(t, x + ct) \right)_{i \in [N]} \in \mathbf{K}^{++}.$$

Proof. Recall Lemma 6.4 and define:

$$\lambda_c = -\max_{\mu \geq 0} (\kappa_\mu + \mu c) > 0$$

and, using (H_3) :

$$\alpha_c = \max \left\{ \alpha > 0 \mid \forall \mathbf{n} \in [0, \alpha]^N \quad \max_{i \in [N]} c_i(\alpha \mathbf{n}) \leq \frac{\lambda_c}{2} \right\}.$$

Let R_c be a sufficiently large radius satisfying:

$$\lambda_{1, Dir} \left(-\mathbf{D} \frac{d^2}{d\xi^2} - c \frac{d}{d\xi} - \left(\mathbf{L} - \frac{\lambda_c}{2} \mathbf{I} \right), (-R_c, R_c) \right) < 0.$$

Let $\mathbf{u}_c : (t, y) \mapsto \mathbf{u}(t, y + ct)$. It is a solution of:

$$\partial_t \mathbf{u}_c - \mathbf{D}\partial_{yy} \mathbf{u}_c - c\partial_y \mathbf{u}_c = \mathbf{L}\mathbf{u}_c - \mathbf{c}[\mathbf{u}_c] \circ \mathbf{u}_c \text{ in } (0, +\infty) \times \mathbb{R}$$

with initial data \mathbf{u}_0 . Just as in the proof of Proposition 4.5, we can use R_c , α_c and Földes–Poláčik’s Harnack inequality [29] to deduce the existence of $\nu_c > 0$ such that:

$$\left(\liminf_{t \rightarrow +\infty} \inf_{x \in I} u_i(t, x + ct) \right)_{i \in [N]} \geq \nu_c \mathbf{1}_{N,1}.$$

This ends the proof. \square

Remark. We point out that $R_c \rightarrow +\infty$ as $c \rightarrow c^*$. Hence the proof above cannot be used directly to obtain a lower bound uniform with respect to c . Although we expect indeed the existence of such a bound, we do not know how to obtain it.

8. ESTIMATES FOR THE MINIMAL WAVE SPEED

In this section, we assume $\lambda_{PF}(\mathbf{L}) > 0$,

$$d_1 \leq d_2 \leq \dots \leq d_N,$$

and prove the estimates provided by Theorem 1.7.

Recall as a preliminary that for all $r > 0$ and $d > 0$, the following equality holds:

$$2\sqrt{rd} = \min_{\mu > 0} \left(\mu d + \frac{r}{\mu} \right).$$

Proposition 8.1. *We have:*

$$2\sqrt{d_1 \lambda_{PF}(\mathbf{L})} \leq c^* \leq 2\sqrt{d_N \lambda_{PF}(\mathbf{L})}.$$

If $d_1 < d_N$, both inequalities are strict. If $d_1 = d_N$, both inequalities are equalities.

Proof. Since $d_1 \mathbf{1}_{N,1} \leq \mathbf{d} \leq d_N \mathbf{1}_{N,1}$, we have for all $\mu > 0$:

$$\mu d_1 + \frac{1}{\mu} \lambda_{PF}(\mathbf{L}) \leq \lambda_{PF} \left(\mu \mathbf{D} + \frac{1}{\mu} \mathbf{L} \right) \leq \mu d_N + \frac{1}{\mu} \lambda_{PF}(\mathbf{L}),$$

whence we deduce:

$$2\sqrt{d_1 \lambda_{PF}(\mathbf{L})} \leq c^* \leq 2\sqrt{d_N \lambda_{PF}(\mathbf{L})}.$$

On one hand, it is well-known that if $d_1 < d_N$, then the above inequalities are strict. On the other hand, if $d_1 = d_N$, we have of course:

$$\lambda_{PF} \left(\mu \mathbf{D} + \frac{1}{\mu} \mathbf{L} \right) = \mu d_1 + \frac{1}{\mu} \lambda_{PF}(\mathbf{L}),$$

whence the equality. □

Recall from Lemma 6.2 that $\mathbf{n}_{\mu_{c^*}} = \mathbf{n}_{PF}(\mu_{c^*}^2 \mathbf{D} + \mathbf{L})$.

Proposition 8.2. *For all $i \in [N]$ such that $l_{i,i} > 0$, we have:*

$$c^* > 2\sqrt{d_i l_{i,i}}.$$

Proof. The characterization of c^* (see Lemma 6.2) yields:

$$\mu_{c^*} d_i + \frac{l_{i,i}}{\mu_{c^*}} = c^* - \frac{1}{\mu_{c^*}} \sum_{j \neq i, j=1}^N l_{i,j} \frac{n_{\mu_{c^*},j}}{n_{\mu_{c^*},i}} \text{ for all } i \in [N],$$

whence, if $l_{i,i} > 0$:

$$c^* \geq 2\sqrt{d_i l_{i,i}} + \frac{1}{\mu_{c^*}} \sum_{j \neq i, j=1}^N l_{i,j} \frac{n_{\mu_{c^*},j}}{n_{\mu_{c^*},i}} > 2\sqrt{d_i l_{i,i}}.$$

□

Recall the existence of a unique decomposition of \mathbf{L} of the form:

$$\mathbf{L} = \text{diag} \mathbf{r} + \mathbf{M} \text{ with } \mathbf{r} \in \mathbb{R}^N \text{ and } \mathbf{M}^T \mathbf{1}_{N,1} = \mathbf{0}.$$

Remark. Regarding the Lotka–Volterra mutation–competition–diffusion ecological model, the decomposition $\mathbf{L} = \text{diag} \mathbf{r} + \mathbf{M}$ is ecological meaningful: \mathbf{r} is the vector of the growth rates of the phenotypes whereas \mathbf{M} describes the mutations between the phenotypes.

Proposition 8.3. *Let $\alpha > 0$ such that $\mathbf{1}_{1,N}(\alpha \mathbf{n}_{\mu_{c^*}}) = 1$. Let:*

$$(\langle d \rangle, \langle r \rangle) = (\mathbf{d}^T(\alpha \mathbf{n}_{\mu_{c^*}}), \mathbf{r}^T(\alpha \mathbf{n}_{\mu_{c^*}})) \in (0, +\infty) \times \mathbb{R}.$$

If $\langle r \rangle \geq 0$, then:

$$c^* \geq 2\sqrt{\langle d \rangle \langle r \rangle}.$$

Proof. Using (\mathbf{r}, \mathbf{M}) , the characterization of c^* (see Lemma 6.2) is rewritten as:

$$(\mu_{c^*}^2 \mathbf{D} + \text{diag} \mathbf{r}) \mathbf{n}_{\mu_{c^*}} + \mathbf{M} \mathbf{n}_{\mu_{c^*}} = \mu_{c^*} c^* \mathbf{n}_{\mu_{c^*}}.$$

Summing the lines of this system, dividing by $\sum_{i=1}^N n_{\mu_{c^*}, i}$ and defining $\langle d \rangle$ and $\langle r \rangle$ as in the statement yields:

$$\mu_{c^*}^2 \langle d \rangle + \langle r \rangle = \mu_{c^*} c^*.$$

The equation $\langle d \rangle \mu^2 - c^* \mu + \langle r \rangle = 0$ admits a real positive solution μ if and only if $(c^*)^2 - 4 \langle d \rangle \langle r \rangle \geq 0$. \square

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E-mail address: girardin@ljl1.math.upmc.fr