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Random matrices and the Potts model on random graphs

Alice Guionnet

Abstract

These are the lecture notes for a mini-course given in St. Petersburg School in Probability and Statistical Physics during June 18-29, 2012. The purpose of this course is to explain the uses of random matrices to compute the partition function of the Potts model taken on (certain) random planar maps. We start by explaining how matrix integrals can be used to enumerate maps, that is graphs which are sorted by the genus of the surface on which they can be properly embedded. We then see how this can be elaborated to enumerate loop models, and their relation with the Potts model on some random planar maps. Finally we use this relation to solve the combinatorial problem of computing generating functions of the Potts model.

Introduction

The connection between matrix integrals and the enumeration of maps goes back to the work of 't Hooft and Brézin-Parisi-Itzykson-Zuber in the seventies. The main observation is that if one expands formally Gaussian matrix integrals in terms of Gaussian moments and use Feynman diagrams representation of the moments, then Gaussian matrix integrals can be expressed in terms of numbers which in fact corresponds to the enumeration of certain graphs. Moreover, if one consider the inverse of the dimension of these matrices as a parameter, the coefficients of the series enumerate maps, that is graphs which are sorted by the genus of the surface on which they can be properly embedded. This is often called the “topological expansion”. Topological expansions were since then used in diverse contexts in physics or mathematics ; after the enumeration of planar triangulations following Brézin, Itzykson, Parisi and Zuber (who proposed this matrix approach to give a new proof of a combinatorial result by Tutte on the enumeration of rooted planar triangulation and quadrangulation), it was used to study the enumeration of the Ising model on random graph (Mehta), of meanders (Di Francesco ...), of loop configurations and the $O(n)$ model (Eynard, Kostov), and applied to knot theory (Zinn-Justin, Zuber)... The full topological expansions were used in mathematics in the work of Harer and Zagier (1986)

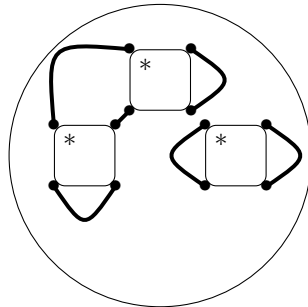
aguionne@ens-lyon.fr, CNRS & École Normale Supérieure de Lyon, Unité de mathématiques pures et appliquées, 46 allée d'Italie, 69364 Lyon Cedex 07, France and MIT, Mathematics department, 77 Massachusetts Av, Cambridge MA 02139-4307, USA..

in their article on the Euler characteristics of the moduli space of curves, and the famous work of Kontsevich. It was also seen as a tool to construct invariants based on its relation with algebraic geometry and topological string theory (the famous Dijkgraaf-Vafa conjecture states that Gromov-Witten invariants generating functions should be matrix integrals).

In these lecture notes we discuss the uses of topological expansions to study the Potts model on random graphs, that is some (shaded) elaboration of the so-called $O(n)$ model. We will start by describing the topological expansion and how they can be derived from Gaussian calculus, that is Wick formula. To enumerate interesting graphs, that is graphs with several vertices, it turns out that one needs to compute Laplace transforms of traces of polynomials of random matrices. Such integrals might then diverge and their relations with maps enumeration be only formal. We shall discuss the relation between topological expansions and matrix integrals via a third mathematical concept, that is the so-called loop (or Schwinger-Dyson's) equations. These equations are simply derived from the matrix integrals via integration by parts. They correspond to induction relation in the enumeration of maps, as first introduced by Tutte to enumerate planar graphs. We shall describe how these equations give an alternative bridge between matrix integrals and the enumeration of maps which in fact allows to turn formal equalities into asymptotic equalities. Based on this point, one can define properly matrix integrals whose asymptotics are given by generating functions for the enumeration of planar connected graphs.

Finally, we will specialize this relation to analyze the Potts model on certain random planar graphs. This connection is based on the interpretation of the Potts model as a loop model. Loop models naturally appear in a variety of statistical models where the loops represent the configuration of boundaries of some random regions. Perhaps the most famous of these is the so-called $O(n)$ loop model which can be described as follows.

Consider the two-sphere S^2 , and fix r disjoint disks $\mathcal{D}_j : 1 \leq j \leq r$ inside of S^2 . Each disk is given an even number of boundary points, one of which is marked. By a *tangle* we mean (the isotopy class of) any possible collection of non-intersecting strings in $S^2 \setminus (\mathcal{D}_1 \cup \dots \cup \mathcal{D}_r)$ joining the boundary points.



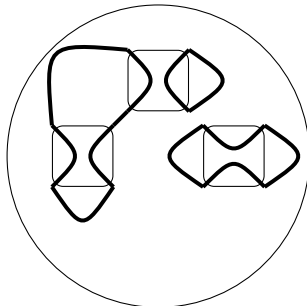
By a *vertex* (also called a Temperley–Lieb diagram) we mean an (isotopy class of) arbitrary non-intersecting collection of strings drawn inside a disk with a given even number of boundary points (one of which is marked).

By a *configuration* P built on vertices D_1, \dots, D_r we mean a tangle Q into which the vertices D_1, \dots, D_r have been inserted (so that D_j is inserted into \mathcal{D}_j in a way that boundary

points match, and marked boundary points match). In the $O(n)$ model, the vertices are all assumed to be copies of the same vertex D : Let D be given by the following picture :



The outer boundary is represented by a thin line. The boundary contains 4 *boundary points*, which are joined inside of the disk by non-intersecting strings (represented by thick lines). One of the boundary points is marked by a * to distinguish it from the others. A possible configuration is drawn below :



Fix a number δ (often called the weight loop, or the *fugacity*) and associate to a configuration P its value given by $\delta^{\#\text{loops}}$ where $\#\text{loops}$ is the number of (closed) loops that can be seen in the configuration after removing the outer boundary of each \mathcal{D}_j . One is interested in understanding the partition function

$$(1) \quad f_\delta(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{P \in P(r)} \delta^{\#\text{ loops in } P}$$

where the sum is taken over the set $P(r)$ of all configurations built on r copies of D (note here that the vertices and boundary points are labeled, so configurations corresponding to different matchings of the labeled boundaries of vertices and the tangle give rise to different terms in the summation). The partition function is simply a generating function for the enumeration of configurations with a given number of loops.

The loop model we described has been widely studied, in part because of its connection with the critical Potts model. We shall consider a slightly more complicated model which is shaded. We will see that the partition function for such a model can be expressed as a matrix integral for a continuum of values of the fugacity δ , hence allowing to identify uniquely this generating function by analyticity. We will finally show that these matrix integrals can be computed, hence giving a way to compute the partition function for the Potts model based on estimating matrix integrals.

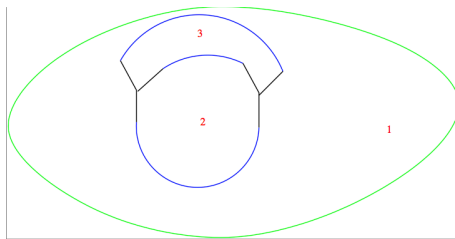
1. Random matrices and the enumeration of maps

The asymptotics of moments of random matrices are well known to be related with the enumeration of non-crossing pair partitions since the work of Wigner. In fact, it turns out that when the entries are Gaussian, the moments can be written as a generating function for the enumeration of pair-partitions sorted by their “topology”, in the sense that they can be represented by a graph with one vertex which can be properly embedded into a surface of given genus. This fact is nicely described in the review of Zvonkin [Zvo97]. We shall summarize this relation in this section. We first specify how we enumerate graphs and define their genus (in particular we shall introduce the notion of maps) and then discuss the relation between their enumeration and matrix moments.

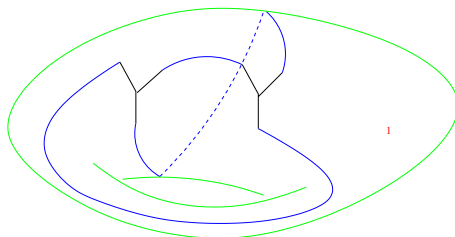
1.1. Maps. A map is a connected graph which is properly embedded into a surface, that is embedded in such a way that its edges do not cross and the faces (obtained by cutting the surface along the edges of the graph) are homeomorphic to disks. The genus of a map is the genus of such a surface. By Euler formula, we have

$$2 - 2g = \#\{\text{vertices}\} + \#\{\text{faces}\} - \#\{\text{edges}\}.$$

Be careful that in the definition above the external face is counted. Here is a genus zero (or planar) map with two vertices, 3 edges and 3 faces



and here is a genus one map with two vertices, 3 edges and one face :



As surfaces come with an orientation, any given cyclic order at the ends of edges of a graph around each vertex uniquely determines the imbedding of the graph into a surface.

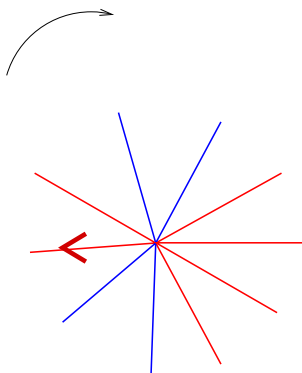
Hence, to enumerate maps, we shall be given vertices equipped with “half-edges” and a cyclic order at the ends of their edges. Edges will just be created by matching the end points of the half-edges. As we shall count labeled maps, we shall assume also that each vertex is given

a root, that is a marked edge. Being given a collection of such vertices with labelled half-edges, we shall count the number of different matchings (different in the sense that they have at least one matching corresponding to different labels) of the end points of these half-edges so that the resulting graph is a map with a given genus.

In fact, to make the connection with matrix moments, it will be convenient to associate these vertices with words in non-commutative variables. This connection is maybe easier to understand directly in the case of vertices with colored half-edges (which will lead to the enumeration of colored maps). One associates (bijectively) to any word in d non-commutative variables either ordered colored points on the real line or a “vertex with colored half-edges” in order to make the difference between the different matrices in the word. Namely, associate to

$$q(X_1, \dots, X_d) = X_{i_1} X_{i_2} \cdots X_{i_p}$$

a “star of type q ” given by the vertex with p colored half-edges drawn on the sphere so that the first branch has color i_1 , the second of color i_2 etc until the last which has color i_p . For instance, if $q(X_1, X_2) = X_1^2 X_2^2 X_1^4 X_2^2$ and 1 is associated to red whereas 2 is associated with blue, the star of type q is a vertex with first two half-edges which are red, then two blue, four red and finally two blue.



We will denote for $\mathbf{k} = (k_1, \dots, k_n)$ and monomials q_1, \dots, q_n ,

$$\mathcal{M}_g((q_i, k_i), 1 \leq i \leq n) = \#\{\text{maps with genus } g$$

and k_i stars of type $q_i, 1 \leq i \leq n\}$.

the number of maps with genus g build on k_i stars of type $q_i, 1 \leq i \leq n$, by matching the half-edges of the stars which have the same color. The enumeration is done up to homeomorphisms. By convention, we will denote $\mathcal{M}_0(1) = 1$.

Note that stars can also be seen by duality as polygons with colored sides and one mark side, where each end point of the half-edge is replaced by a perpendicular segment of the same color. Maps are then “polygonizations” of a surface with given genus by polygons of prescribed nature. For instance, for the matrix model with $q(X) = X^4$, the stars are vertices with valence four, which in the dual picture are just square. We are thus counting quadrangulations of a

surface with given genus and a given number of squares. The counting is done with labeled sides.

1.2. Random matrices and the enumeration of partitions. Let X^N be a matrix following the Gaussian Unitary Ensemble, that is a $N \times N$ Hermitian matrix with i.i.d centered Gaussian entries with covariance N^{-1} :

$$\begin{aligned} X^N(k\ell) &= \bar{X}^N(\ell k) = \frac{1}{\sqrt{2}}(x_{k\ell} + iy_{k\ell}), \quad 1 \leq k \leq \ell \leq N, \\ X^N(kk) &= x_{kk}, \quad 1 \leq k \leq N, \end{aligned}$$

with

$$\begin{aligned} d\mathbb{P}_N(X^N) &= \frac{1}{Z^N} \exp\left\{-\frac{N}{2} \sum_{k \leq \ell} (x_{k\ell}^2 + y_{k\ell}^2)\right\} \prod dx_{k\ell} dy_{k\ell} \\ &= \frac{1}{Z^N} \exp\left\{-\frac{N}{2} \text{Tr}((X^N)^2)\right\} dX_N \end{aligned}$$

The main result of this section is the following

THEOREM 1. *For any integer number $p \geq 0$*

$$(2) \quad \mathbb{E}\left[\frac{1}{N} \text{Tr}((X^N)^p)\right] = \sum_{g \geq 0} \frac{1}{N^{2g}} M_g(p)$$

where $M_g(p) = M_g(x^p)$ is the number of maps with genus g build on one vertex of degree p .

In the definition above, the vertex is given the structure of a star of type x^p , that is, is rooted and with labelled half-edges.

One of the corollary of this theorem, which is valid under much more general assumptions on the entries of X^N , is Wigner theorem [**Wig55**].

COROLLARY 2 (Wigner). *For any $p \geq 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}\left[\frac{1}{N} \text{Tr}((X^N)^p)\right] = C_{p/2} = \int x^p d\sigma(x),$$

with $d\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{|x| \leq 2} dx$ the semi-circle distribution and $C_{p/2} = 0$ if p is odd and otherwise equals the Catalan number, that is the number of non-crossing pair partitions of p points.

The proof of this result relies on Wick calculus which in fact provides not only the asymptotic of traces of words in random matrices but also the whole N expansion. Indeed, Wick formula shows that if (G_1, \dots, G_{2p}) is a centered Gaussian vector, then

$$\mathbb{E}[G_1 \cdots G_{2p}] = \sum_{\pi} \prod_{(s,r) \text{ block of } \pi} \mathbb{E}[G_s G_r],$$

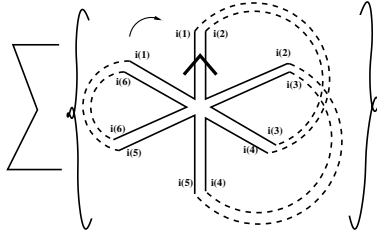
where we sum over all pair partitions π . To prove the theorem, one simply expands the trace in terms of the matrix entries

$$\mathbb{E}\left[\frac{1}{N}\text{Tr}((X^N)^p)\right] = \frac{1}{N} \sum_{i(1), \dots, i(p)=1}^N \mathbb{E}[X_{i(1)i(2)}^N X_{i(2)i(3)}^N \cdots X_{i(p)i(1)}^N].$$

Using Wick formula with $\mathbb{E}[X_{ij}^N X_{k\ell}^N] = N^{-1}1_{ij=\ell k}$, one gets

$$\mathbb{E}[X_{i(1)i(2)}^N \cdots X_{i(p)i(1)}^N] = \frac{1}{N^{p/2}} \sum_{\text{pair partition } \pi} \prod_{(k,\ell) \text{ block of } \pi} 1_{i(k)i(k+1)=i(\ell+1)i(\ell)}$$

The later matchings can be conveniently represented by seeing the Gaussian entries as the end points of half-edges of a vertex with valence p with one marked vertex (that is a star of type x^p)



A face is obtained by cutting the graph along the edges. As $\mathbb{E}[X_{ij}^N X_{k\ell}^N] = N^{-1}1_{ij=\ell k}$, only matchings so that indices are constant along the boundary of the faces contribute to the sum. Since indices are constant along the boundaries of the faces and take any value between 1 and N , we conclude that

$$(3) \quad \mathbb{E}\left[\frac{1}{N}\text{Tr}((X^N)^p)\right] = \sum_{\substack{\text{graph 1 vertex} \\ \text{degree } p}} N^{\#\text{faces}-p/2-1}$$

But, by Euler formula, any connected graph satisfies that its genus is given by

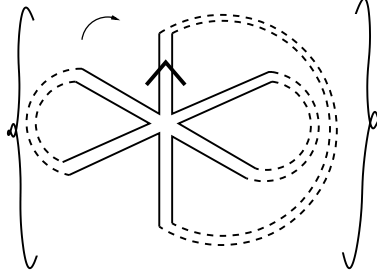
$$2 - 2g = \#\{\text{vertices}\} - \#\{\text{edges}\} = 1 + \#\{\text{faces}\} - p/2$$

so that

$$\#\text{faces} - p/2 - 1 = -2g \leq 0$$

with equality only if the graph is planar. This proves the theorem and then taking the limit N going to infinity yields

$$\lim_{N \rightarrow \infty} \mathbb{E}\left[\frac{1}{N}\text{Tr}((X^N)^p)\right] = \#\{\text{planar graph with 1 vertex with degree } p\}$$



Theorem 1 can be easily generalized to several matrices provided one considers colored maps.

THEOREM 3. *For any monomial q , in d non-commutative letters, we have*

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(q(X_1^N, X_2^N, \dots, X_d^N))\right] = \sum_{g \geq 0} \frac{1}{N^{2g}} M_g(q)$$

where $M(g, q)$ is the number of matching of the edges of the star of type q with the same color so that the resulting graph is properly embedded into a surface of genus g .

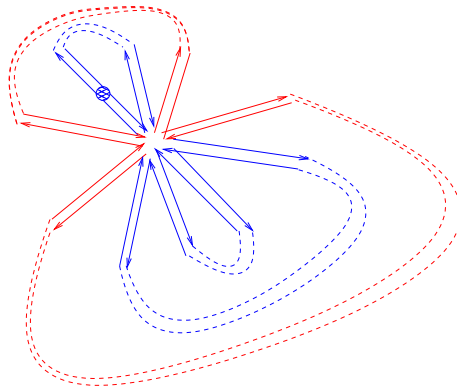
Indeed, the very same argument holds as in the one matrix (one color) case, except that since the covariance between entries of two different matrices vanishes, pairing between half-edges of different colors do not contribute to the sum.

As a corollary we can obtain Voiculescu's theorem :

COROLLARY 4. *For any monomial q ,*

$$\sigma^d(q) = \lim_{N \rightarrow \infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(q(X_1^N, X_2^N, \dots, X_d^N))\right]$$

is the number of planar maps build on one star of type q by matching only branches of the same color.



In fact, Voiculescu's theorem asserts that the limit σ^d is the “non-commutative law of d free semi-circular variables”. This means that if we restrict it to polynomials in one variable, it is given by the semi-circle distribution, whereas more general moments can be computed from the following property which defines freeness :

$$\sigma^d(q_1(X_{i_1})q_2(X_{i_2})\cdots q_\ell(X_{i_\ell})) = 0$$

for any monomials q_i so that $\sigma^d(q_k(X_i)) = \int q_k(x)d\sigma(x)$ vanishes and any indices $i_j \in \{1, \dots, d\}$ so that $i_k \neq i_{k+1}$. It is not hard to verify that this relation is satisfied by the the limit given in the corollary from the property of planar maps as it amounts to show that any planar map build on a star of type q must leave at least one subset of connected one-color half-edges (corresponding to each of the monomials q_i) to have only self-matchings. This can be checked by induction over the degree of q .

1.3. Topological expansions. Topological expansions build upon the formula for moments of Gaussian (GUE) matrices which are given by (2). This formula was proved in (3) based on Wick calculus. Brézin-Itzykson-Parisi-Zuber [BIPZ78], after the work of 't Hooft [Hoo74], had the idea in the seventies to develop further this remarkable identity between matrix moments and the enumeration of graphs to enumerate maps with several vertices. It turns out that such topological expansions are closely related with the so-called loop (or Schwinger-Dyson) equations which are satisfied by matrix models but also can be seen as topological recursion relations. The loop equations describe inductions between matrix moments, which in fact are the same as those satisfied by maps. At the first order, these equations are just given by a kind of non-commutative integration by parts formula. The next orders equations appear as derivatives of the first loop equation taken at finite dimension, and allow to describe the full topological expansion. Hence, as put forward by B. Eynard, the loop equations can be used as the key to construct topological expansions and therefore interesting geometric quantities and invariants.

In this lecture we describe more precisely the relation between matrix integrals, topological expansions and loop equations. Based on this relation, we show that topological expansions can be derived in much greater generality than those related with matrices with Gaussian entries and Feynmann diagrams, namely in models for which loop equations given by a non-commutative derivative are valid. We detail the case of β ensembles and integrals over the unitary group.

1.3.1. *Topological expansions and Wick formula.* The first natural idea to count maps with several vertices is to consider the expectation of a product of traces of words as follows. Let q_1, \dots, q_n be monomials in d non-commutative variables. Then, applying Gaussian calculus (that is Wick formula), we find that if \mathbb{P}_N is the law of the GUE (see section 1)

$$\begin{aligned} & \int \prod_{i=1}^n (N \text{Tr}(q_i(X_1, \dots, X_m))) d\mathbb{P}_N(X_1) \cdots d\mathbb{P}_N(X_m) \\ &= \sum_{g \in \mathbb{N}} \sum_{c \geq 1} \frac{1}{N^{2g-2c}} \# \{G_{g,c}((q_i, 1), 1 \leq i \leq n)\} \end{aligned}$$

Here $\#\{G_{g,c}((q_i, 1), 1 \leq i \leq n)\}$ is the number of graphs (up to homeomorphism) that can be build on stars of type $q_i, 1 \leq i \leq n$ with exactly c connected components so that the sum of their genera is equal to g .

Hence, such expectations are related with the enumeration of graphs with several vertices but unfortunately do not sort the connected graphs. We next show how this can be done.

1.3.2. *Matrix models and topological expansions.* To enumerate connected graphs, and more precisely maps, the idea of Brézin, Itzykson, Parisi and Zuber [BIPZ78] is to consider partition functions instead of moments, that is logarithm of Laplace transforms of traces of monomials.

Consider q_1, \dots, q_n monomials. Then, [BIPZ78] shows that

$$(4) \quad \begin{aligned} \log Z_N^{\sum t_i q_i} &:= \log \left(\int e^{\sum_{i=1}^n t_i N \text{Tr}(q_i(X_1, \dots, X_m))} d\mathbb{P}_N(X_1) \cdots d\mathbb{P}_N(X_m) \right) \\ &= \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{k_1, \dots, k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i, k_i), 1 \leq i \leq n) \end{aligned}$$

where the equality means that derivatives of all orders at $t_i = 0, 1 \leq i \leq n$, match. The proof of this formula is simply done by developing the exponential and recalling that the logarithmic function will yield cumulants, and therefore connected graphs. Adding in the potential a term tq , taking a formal derivative in t at the origin shows that if $V = \sum t_i q_i(X_1, \dots, X_m)$ then for any monomial q

$$(5) \quad \begin{aligned} &\int \frac{1}{N} \text{Tr}(q(X_1^N, \dots, X_d^N)) d\mathbb{P}_N^V(X_1^N, \dots, X_d^N) \\ &= \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{k_1, \dots, k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_g((q, 1); (q_i, k_i), 1 \leq i \leq n) \end{aligned}$$

where $\mathcal{M}_g((q, 1); (q_i, k_i), 1 \leq i \leq n)$ is the number of maps of genus g build on one star of type q and k_i stars of type $k_i, 1 \leq i \leq n$, and

$$d\mathbb{P}_N^V(X_1^N, \dots, X_d^N) = \frac{1}{Z_N^V} \exp\{N \text{Tr}(V(X_1^N, \dots, X_d^N))\} d\mathbb{P}_N(X_1) \cdots d\mathbb{P}_N(X_m).$$

At this point the equality is formal but it can in fact be made asymptotic as soon as reasonable assumptions are made to ensure that the integral converges and that the parameters t_i are small enough to guarantee the convergence of the series. Equality (5) given asymptotically up to any order of correction N^{-k} is called an asymptotic topological expansion. We next discuss how to obtain such asymptotic expansions, and how the loop equations can play a key role in deriving the topological expansions.

1.3.3. *Loop equations and asymptotic expansions.* Let us first describe loop equations for independent GUE matrices and in fact more generally for the distribution

$$d\mathbb{P}_N^V(X_1^N, \dots, X_d^N) = \frac{1}{Z_N^V} \exp\{N \text{Tr}(V(X_1^N, \dots, X_d^N))\} d\mathbb{P}_N(X_1^N) \cdots d\mathbb{P}_N(X_d^N).$$

Hereafter, $\mathbb{P}_N^V(X_1^N, \dots, X_d^N)$ shall be a well defined probability measure on the space of Hermitian matrices so that we will assume that the function ϕ_V which associates to the entries of the self-adjoint matrices X_1^N, \dots, X_d^N the real number

$$(6) \quad \phi_V(X_1^N, \dots, X_d^N) := \frac{1}{2} \text{Tr} \left(\sum (X_i^N)^2 \right) - \text{Tr}(V(X_1^N, \dots, X_d^N))$$

is strictly convex, with Hessian bounded below by cI for some $c > 0$, and this for any integer dimension N . For potentials which could hardly satisfy such an hypothesis (for instance $V(x) = tx^3$) we might add a cutoff on the spectral radius of the matrices and assumed the t_i 's are small enough so that the probability $d\mathbb{P}_N^V(X_1^N, \dots, X_d^N)$ keeps a strictly log-concave distribution. We shall not discuss this point here and refer the interested reader to [GMS06].

The first loop equation under the law \mathbb{P}_V^N can be written as follows. It simply says that for all polynomial function P

$$(7) \quad \begin{aligned} & \int \frac{1}{N} \text{Tr}(P(X_i - \mathcal{D}_i V)) d\mathbb{P}_V^N(X_1^N, \dots, X_d^N) \\ &= \int \frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}(\partial_i P) d\mathbb{P}_V^N(X_1^N, \dots, X_d^N). \end{aligned}$$

Here, \mathcal{D}_i and ∂_i are derivatives on the space $\mathbb{C}\langle X_1, \dots, X_d \rangle$ of polynomials in d non-commutative variables defined as follows. Define the cyclic gradient as the linear derivative on the set of polynomial whose restriction to monomials is given by

$$\mathcal{D}_i(X_{i_1} \cdots X_{i_p}) = \sum_{i_\ell=i} X_{i_{\ell+1}} \cdots X_{i_p} X_{i_1} \cdots X_{i_{\ell-1}}$$

and observe that for any $(sr) \in \{1, \dots, N\}^2$, since in the complex setting $\partial_{X_i(sr)} X_j(s'r') = \mathbf{1}_{i=j} \mathbf{1}_{sr=r's'}$,

$$\partial_{X_i(sr)} \text{Tr}(P(X_1, \dots, X_d)) = (\mathcal{D}_i P)(X_1, \dots, X_d)(sr).$$

Moreover, the non-commutative derivative ∂_i is given by

$$\partial_i(X_{i_1} \cdots X_{i_p}) = \sum_{i_\ell=i} X_{i_1} \cdots X_{i_{\ell-1}} \otimes X_{i_{\ell+1}} \cdots X_{i_p}$$

and observe similarly that

$$\partial_{X_i(sr)} (P(X_1, \dots, X_d))(s'r') = (\partial_i P)(X_1, \dots, X_d)(s'r, sr').$$

Therefore the classical integration by parts yields, at least when no cutoff is present, the following formula for any polynomial function P and any $sr, s'r'$

$$(8) \quad \begin{aligned} N \int P(s'r')(X_i - \mathcal{D}_i V)(sr) d\mathbb{P}_V^N &= \int P(s'r') \partial_{X_i(sr)} N \text{Tr} \left(\frac{1}{2} \sum X_i^2 - V \right) d\mathbb{P}_V^N(X_1^N, \dots, X_d^N) \\ &= \int \partial_{X_i(sr)} P(s'r') d\mathbb{P}_V^N(X_1^N, \dots, X_d^N) \\ &= \int (\partial_i P)(s'r, sr') d\mathbb{P}_V^N(X_1^N, \dots, X_d^N) \end{aligned}$$

Taking $r' = s$ and $s' = r$ and summing over the indices yields the Schwinger-Dyson (or loop) equation (7).

Let us show how to deduce the first order asymptotics of the measure \mathbb{P}_N^V from the first loop equation (7). In the case where V is strictly convex (in the sense described in equation 6), we can argue by standard concentration of measure theory and Brascamp Lieb inequality (see [AGZ10] and [Gu09]) that there exists a finite constant C (which only depends on c) so that for any monomial q of degree less than \sqrt{N}

$$(9) \quad \int \left| \frac{1}{N} \text{Tr}(q(X_1^N, \dots, X_d^N)) \right| d\mathbb{P}_N^V(X_1^N, \dots, X_d^N) \leq C^{\deg q},$$

and

$$(10) \quad \int \left| \frac{1}{N} \text{Tr}(q) - \int \frac{1}{N} \text{Tr}(q) d\mathbb{P}_V^N \right|^2 d\mathbb{P}_V^N \leq \frac{C^{\deg(q)}}{N^2}.$$

As a consequence of (9), the family $\left\{ \int \frac{1}{N} \text{Tr}(q(X_1^N, \dots, X_d^N)) d\mathbb{P}_N^V(X_1^N, \dots, X_d^N), q \right\}$ indexed by monomials in non-commutative variables is tight. Any limit point $\{\tau(q), q\}$ can be extended by linearity to polynomials and then satisfies the Schwinger-Dyson equation

$$(11) \quad \tau_V(P(X_i - \mathcal{D}_i V)) = \tau_V \otimes \tau_V(\partial_i P)$$

with $\tau_V(I) = 1$. Moreover, for any monomial q , we deduce from (9) that

$$(12) \quad |\tau_V(q)| \leq C^{\deg(q)}.$$

When $V = \sum t_i q_i$ with the t_i small enough, there exists a unique solution to (11). Indeed, when $t_i = 0$, the moments are just defined inductively by (11). When the t_i are small enough, the equation still has a unique solution. Indeed, taking two solutions $\tau, \tilde{\tau}$ and denoting

$$\Delta_k := \sup_{q: \deg(q)=k} |\tau(q) - \tilde{\tau}(q)|$$

where the supremum is taken on monomials of degree smaller or equal to k , we have

$$\Delta_{k+1} = \max_i \sup_{q: \deg q=k} |\tau(X_i q) - \tilde{\tau}(X_i q)|$$

whereas by using (11), (12) and $\Delta_0 = 0$, if $D + 1 = \max \deg(q_i)$ we get

$$\begin{aligned} |\tau(X_i q) - \tilde{\tau}(X_i q)| &\leq |\tau \otimes \tau(\partial_i q) - \tilde{\tau} \otimes \tilde{\tau}(\partial_i q)| + D \sum |t_j| \sum_{\ell=1}^D \Delta_{k+\ell} \\ &\leq \sum_{l=1}^k \Delta_l C^{k-l} + D \sum_j |t_j| \sum_{\ell=1}^D \Delta_{k+\ell} \end{aligned}$$

Hence,

$$\Delta_{k+1} \leq \sum_{l=1}^k \Delta_l C^{k-l} + D \sum_j |t_j| \sum_{\ell=1}^D \Delta_{k+\ell}, \quad \Delta_k \leq 2C^k$$

so that for $\gamma < C^{-1} \wedge 1$, we deduce that

$$(13) \quad \Delta_\gamma = \sum_{k \geq 1} \gamma^k \Delta_k \leq \frac{\gamma}{1 - C\gamma} \Delta_\gamma + \frac{D \sum |t_j|}{\gamma^D} \Delta_\gamma$$

which entails $\Delta_\gamma = 0$ for $\gamma < C^{-1} \wedge 1$ so that

$$\frac{\gamma}{1 - C\gamma} + \frac{D^2 \sum |t_j|}{\gamma^D} < 1.$$

Such a $\gamma > 0$ exists for small enough t_j 's, which implies $\tau = \tilde{\tau}$.

We shall see that indeed this solution coincides with a generating function for the enumeration of maps.

THEOREM 5. *For t_i small enough, there exists $C < \infty$ so that*

$$\mathcal{M}_t(q) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_0((q, 1); (q_i, k_i), 1 \leq i \leq n)$$

is solution of equation (11) and (12) and therefore

$$\tau_V(q) = \mathcal{M}_t(q)$$

Let us remark that by definition of τ_V as a limit of trace of polynomials in matrices, for all polynomials P, Q ,

$$\tau_V(PP^*) \geq 0 \quad \tau_V(PQ) = \tau_V(QP)$$

where $(zX_{i_1} \cdots X_{i_n})^* = \bar{z}X_{i_n} \cdots X_{i_1}$. As a consequence, \mathcal{M}_t also satisfy these equations : for all P, Q

$$\mathcal{M}_t(PP^*) \geq 0 \quad \mathcal{M}_t(PQ) = \mathcal{M}_t(QP) \quad \mathcal{M}_t(1) = 1.$$

This means that \mathcal{M}_t is a tracial state. The traciality property can easily be derived by symmetry properties of the maps. However, the positivity property $\mathcal{M}_t(PP^*) \geq 0$ is not easy to prove combinatorially, and hence matrix models are an easy way to derive it. This property is sometimes useful to actually solve the combinatorial problem (i.e. find an explicit formula for \mathcal{M}_t).

Proof of Theorem 5. Let us denote in short, for $\mathbf{k} = (k_1, \dots, k_n)$ and a monomial q , $\mathcal{M}_\mathbf{k}(q) = \mathcal{M}_0((q, 1); (q_i, k_i), 1 \leq i \leq n)$ the number of planar maps with k_i stars of type q_i and one of type q . We generalize this definition to polynomials P by linearity. We let

$$\mathcal{M}_t(q) := \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_\mathbf{k}(q)$$

This series is a priori formal but we shall see below that in fact there exists a finite constant C so that for any monomials q_i

$$(14) \quad \mathcal{M}_\mathbf{k}(q) \leq \prod k_i! C^{\sum k_i \deg(q_i)}$$

converges absolutely for $|t_i| < 1/C$. \mathcal{M}_t then satisfies (11) if and only if for every \mathbf{k} and monomial P

$$(15) \quad \mathcal{M}_{\mathbf{k}}(X_i P) = \sum_{\substack{0 \leq p_j \leq k_j \\ 1 \leq j \leq n}} \prod_{j=1}^n C_{k_j}^{p_j} \mathcal{M}_{\mathbf{p}} \otimes \mathcal{M}_{\mathbf{k}-\mathbf{p}}(\partial_i P) + \sum_{1 \leq j \leq n} k_j \mathcal{M}_{\mathbf{k}-1_j}([D_i q_j] P)$$

where $1_j(i) = 1_{i=j}$ and $\mathcal{M}_{\mathbf{k}}(1) = 1_{\mathbf{k}=\mathbf{0}}$.

- We first check (15) for $\mathbf{k} = \mathbf{0} = (0, \dots, 0)$. By convention, $\mathcal{M}_{\mathbf{0}}(1) = 1$. We now check that for any monomial P

$$\mathcal{M}_{\mathbf{0}}(X_i P) = \mathcal{M}_{\mathbf{0}} \otimes \mathcal{M}_{\mathbf{0}}(\partial_i P) = \sum_{P=p_1 X_i p_2} \mathcal{M}_{\mathbf{0}}(p_1) \mathcal{M}_{\mathbf{0}}(p_2)$$

But in any planar map with only one star of type $X_i P$, the half-edge corresponding to X_i has to be glued with another half-edge of P . If X_i is glued with the half-edge X_i coming from the decomposition $P = p_1 X_i p_2$, the map is then split into two (independent) planar maps with stars p_1 and p_2 respectively (note here that p_1 and p_2 inherit the structure of stars since they inherit the orientation from P as well as a marked half-edge corresponding to the first neighbour of the glued X_i .) Hence the relation is satisfied. Moreover, we prove (14) by showing by induction that for any monomial P , $\mathcal{M}_{\mathbf{0}}(P)$ is bounded above by the Catalan number $C_{\deg P}$ based on the fact that

$$|\mathcal{M}(X_i P)| = \left| \sum_{P=p_1 X_i p_2} \mathcal{M}_{\mathbf{0}}(p_1) \mathcal{M}_{\mathbf{0}}(p_2) \right| \leq \sum_{k=0}^{\deg(P)-1} C_k C_{\deg(P)-1-k} = C_{\deg(P)}$$

- We now proceed by induction over \mathbf{k} and the degree of P ; we assume that (15) is true for $\sum k_i \leq M$ and all monomials, and for $\sum k_i = M + 1$ when $\deg(P) \leq L$. Note that $\mathcal{M}_{\mathbf{k}}(1) = 0$ for $|\mathbf{k}| \geq 1$ since we can not glue a vertex with no half-edges with any star. Hence, this induction can be started with $L = 0$. Now, consider $R = X_i P$ with P of degree less than L and the set of planar maps with a star of type $X_i P$ and k_j stars of type q_j , $1 \leq j \leq n$, with $|\mathbf{k}| = \sum k_i = M + 1$. Then,

◊ either the half-edge corresponding to X_i is glued with an half-edge of P , say to the half-edge corresponding to the decomposition $P = p_1 X_i p_2$; we then can use the argument as above; the map M is cut into two disjoint planar maps M_1 (containing the star p_1) and M_2 (resp. p_2), the stars of type q_i being distributed either in one or the other of these two planar maps; there will be $r_i \leq k_i$ stars of type q_i in M_1 , the rest in M_2 . Since all stars are labelled, there will be $\prod C_{k_i}^{r_i}$ ways to assign these stars in M_1 and M_2 .

Hence, the total number of planar maps with a star of type $X_i P$ and k_i stars of type q_i , such that the marked half-edge of $X_i P$ is glued with an half-edge of P is

$$(16) \quad \sum_{P=p_1 X_i p_2} \sum_{\substack{0 \leq r_i \leq k_i \\ 1 \leq i \leq n}} \prod_{i=1}^n C_{k_i}^{r_i} \mathcal{M}_{\mathbf{r}}(p_1) \mathcal{M}_{\mathbf{k}-\mathbf{r}}(p_2)$$

◇ Or the half-edge corresponding to X_i is glued with an half-edge of another star, say q_j ; let's say with the edge coming from the decomposition of q_j into $q_j = q_j^1 X_i q_j^2$. Then, once we are giving this gluing of the two edges, we can replace the two stars $X_i P$ and $q_j^1 X_i q_j^2$ glued by their X_i by the star $q_j^2 q_j^1 P$.

We have k_j ways to choose the star of type q_j and the total number of such maps is

$$\sum_{q_j = q_j^1 X_i q_j^2} k_j \mathcal{M}_{\mathbf{k}-1_j}(q_j^2 q_j^1 P)$$

Summing over j , we obtain by linearity of $\mathcal{M}_{\mathbf{k}}$

$$(17) \quad \sum_{j=1}^n k_j \mathcal{M}_{\mathbf{k}-1_j}([D_i q_j] P)$$

(16) and (17) give (15). Moreover, it is clear that (15) defines uniquely $\mathcal{M}_{\mathbf{k}}(P)$ by induction. In addition, we see that the solution to (15) satisfy (14) : Indeed we have seen that it is true for $\mathbf{k} = 0$ and otherwise we show by induction that $|\mathcal{M}_{\mathbf{k}}(P)| \leq \prod (k_i! C_{k_i}) C_{\deg(P)}$ as we did in the case $\mathbf{k} = 0$.

as free semi-circle variables are bounded by 2 and then follows for large \mathbf{k} by induction over $\sum k_i$.

It turns out that this strategy can be followed for each genera up to consider a family of loop equations which are obtained by differentiating the first one with respect to small additional potentials. The first point is to derive the second order Schwinger Dyson equation by varying V into $V + \varepsilon W$ and differentiating at $\varepsilon = 0$ the first order loop equation (7), hence getting equations for the cumulants. We refer the interested reader to [GMS07, Ma06] for full details, but outline the approach below. The first point is to prove an a priori rough estimate by showing that there exists a finite constant $C > 0$ so that for all t_i 's small enough, all monomials q of degree less than $N^{1/2-\varepsilon}$ for $\varepsilon > 0$, we have

$$|\mathbb{E}[\tau_{X_N}[q]] - \tau_V(q)| \leq \frac{C^{\deg(q)}}{N^2}.$$

The proof elaborates on the ideas developed around (13) to prove uniqueness of the solution to Schwinger-Dyson equation and the concentration inequalities (10) which give a fine control on the error term in the loop equation satisfied by $\mathbb{E}[\tau_{X_N}]$ with respect to the Schwinger-Dyson equation. In fact, putting

$$\Delta_k^N = \sup_{\deg(q)=k} |\mathbb{E}[\tau_{X_N}(q)] - \tau_V(q)|$$

the Schwinger-Dyson equation and (10), (14) imply that for $k \leq N^{\frac{1}{2}-\varepsilon}$

$$\Delta_k^N \leq \sum_{\ell=0}^{k-1} C^{k-1-\ell} \Delta_\ell^N + D \sum_{\ell=1}^D |t_i| \Delta_{k+\ell}^N + \frac{C^k}{N^2} + (\Delta_k^N)^2.$$

But we have seen that Δ_k^N goes to zero so that we conclude that for $\sum |t_i|$ small enough, we can find $\gamma > 0$, a positive constant c and a finite constant c' so that

$$c \sum_{k=1}^{N^{1/2-\varepsilon}} \gamma^k \Delta_k^N \leq \frac{1}{N^2} \sum_{k=1}^{N^{1/2-\varepsilon}} \gamma^k C^k + \sum_{k=N^{1/2-\varepsilon}+D}^{k=N^{1/2-\varepsilon}+D} \gamma^k C^k \leq \frac{c'}{N^2}$$

where the last estimate is based on $\gamma C < 1$. The desired estimate follows.

We next turn to the precise estimate of the asymptotics of $\bar{\delta}^N(P) = \mathbb{E}[\text{Tr}(P)] - N\tau_V(P)$. To this end, we shall introduce the following cumulants :

$$\begin{aligned} W_2^V(P, Q) &= \mathbb{E}[(\text{Tr}P - \mathbb{E}\text{Tr}P)(\text{Tr}Q - \mathbb{E}[\text{Tr}Q])] \\ &= \partial_\varepsilon \mathbb{P}^{V-\varepsilon N^{-1}Q}(\text{Tr}P)|_{\varepsilon=0} \\ W_3^V(P, Q, R) &= \partial_\varepsilon W_2^{V-\varepsilon N^{-1}R}(P, Q) \end{aligned}$$

Note that equation (7) can be written as

$$(18) \quad \mathbb{E}[\text{Tr}(\Xi_i P)] = \frac{1}{N} W_2(\partial_i P) + \frac{1}{N} \bar{\delta}^N \otimes \bar{\delta}^N(\partial_i P),$$

where

$$\Xi_i P = \partial_i V \# P - (\tau_V \otimes I + I \otimes \tau_V) \partial_i P.$$

By our a priori estimate on $\bar{\delta}_N$ the last term in the right hand side of (18) is at most of order N^{-3} . Hence, to estimate the first order correction, we would like to estimate the asymptotics of W_2 as well as “invert” Ξ_i . It turns out that even though Ξ_i is hardly invertible, a combination of the Ξ_i 's is. Namely, we let Ξ be the operator on $\mathcal{A}_0 = \{P \in \mathbb{C}\langle X_1, \dots, X_d \rangle : \tau_V(P) = 0\}$ given by

$$\Xi P = \sum_i \Xi_i \mathcal{D}_i P.$$

Then the image of Ξ lies in \mathcal{A}_0 by symmetry of Ξ in $L^2(\tau_V)$. Indeed, Ξ can be seen as a small perturbation of the infinitesimal generator of the free Brownian motion, see [GMS07].

To estimate W_2 , we obtain the second loop equation by changing $V \rightarrow V + \varepsilon N^{-1}W$ in (18) and identifying the linear terms in ε ; we find

$$\begin{aligned} W_2(\Xi_i P, W) &= \mathbb{E}\left[\frac{1}{N} \text{Tr}(P \mathcal{D}_i W)\right] \\ &\quad + N^{-1} W_3(\partial_i P, W) + (W_2 \otimes \bar{\delta}^N + \bar{\delta}^N \otimes W_2)(\partial_i P, W). \end{aligned}$$

It turns out that the terms in W_3 and W_2 are bounded by concentration inequalities (10) whereas $\bar{\delta}^N$ is of order N^{-1} by our previous rough estimate. Hence we conclude that

$$\lim_{N \rightarrow \infty} W_2(\Xi_i P, W) = \tau_V(P \mathcal{D}_i W)$$

for all i and P . Applying this convergence to $P = \mathcal{D}_i Q$ and summing we conclude that

$$\lim_{N \rightarrow \infty} W_2(P, W) = \tau_V \left(\sum_i \mathcal{D}_i \Xi^{-1} P \times \mathcal{D}_i W \right) =: w_2(P, W)$$

and therefore plugging this back into (18) we deduce the first order correction

$$\mathbb{E} \left[\frac{1}{N} \text{Tr}(P) \right] = \tau_V(P) + \frac{1}{N^2} w_2 \left[\sum_i \partial_i \mathcal{D}_i (\Xi^{-1} P) \right] + o(N^{-2}).$$

The next orders of the asymptotic expansion can be derived similarly.

It turns out that loop equations can be established for many other models which are not directly related with Gaussian random matrices. It seems that a large family of loop equations give rise to topological expansions. We describe below the case of the β -ensembles and the integration over the unitary group.

1.4. Other topological expansions.

1.4.1. *Topological expansion for β -matrix models.* The law of the eigenvalues of the GUE follows the distribution on \mathbb{R}^N

$$dP_N(\lambda) = \frac{1}{Z_N} \prod_{i < j} |\lambda_i - \lambda_j|^2 e^{-N \sum \lambda_i^2} \prod d\lambda_i$$

as can be checked by doing the change of variables associating to X its ordered eigenvalues and a parametrization of its eigenvectors. β -ensembles are the following generalization of this distribution :

$$dP_{N,\beta}^V(\lambda) = \frac{1}{Z_{N,\beta}^V} \prod_{i < j} |\lambda_i - \lambda_j|^\beta e^{-N \sum V(\lambda_i)} \prod d\lambda_i$$

It is related with invariant matrix ensembles only in the cases $\beta = 1, 2, 4$ and *a priori* has no relations with Gaussian entries otherwise. However, it was proved in [BG12], see [CE06] for a formal proof, that β -ensembles have a topological expansion. More precisely, assume that V is analytic in a neighborhood of the real line and such that the unique probability measure μ_V which minimizes

$$(19) \quad I(\mu) = \int V(x) d\mu(x) - \frac{\beta}{2} \int \int \log |x - y| d\mu(x) d\mu(y)$$

has a connected support, and V is off critical in the sense that $V'(x) - \beta \int (x - y)^{-1} d\mu_V(x)$ does not vanish in an open neighborhood of the support of μ_V , then for any $z \in \mathbb{C} \setminus \mathbb{R}$, and $K \geq 0$

$$(20) \quad \int \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \lambda_i} dP_{N,\beta}^V(\lambda) = \sum_{k=0}^K N^{-k} W^{V,k}(z) + o(N^{-K})$$

where $o(N^{-K})$ is uniform on compacts. Moreover, we have

$$W^{V,k}(z) = \sum_{g=0}^{\lfloor k/2 \rfloor} \left(\frac{\beta}{2} \right)^{-g} \left(1 - \frac{2}{\beta} \right)^{k-2g+1} \mathcal{W}^{V;(g;k-2g+1)}$$

and if V is a small perturbation of the quadratic potential, $\mathcal{W}^{V;(g;k-2g+1)}$ expands as a generating function of maps of genus g when ribbons can be “twisted”.

Note that the hypothesis that the support is connected is important since otherwise the result is not true in general. The proof of such expansion relies as well on the loop equations

$$(21) \quad \int \frac{\beta}{N^2} \sum_{i \neq j} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} dP_{N,\beta}^V(\lambda) = \int \left[\frac{1}{N} \sum f(\lambda_i) V'(\lambda_i) \right] dP_{N,\beta}^V(\lambda)$$

which can be proved by integration by parts. As a consequence, one sees that the equilibrium measure μ_V satisfies the limiting equation

$$(22) \quad \beta \int \int \frac{f(x) - f(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int f(x) V'(x) d\mu_V(x).$$

If V is a small perturbation of the quadratic potential one can develop arguments similar to those of the previous section to check that moments under μ_V are generating functions for planar maps.

In fact, the limiting equation (22) does not always have a unique solution as it is a weak characterization of the minimizers of (19), but it does as soon as V is strictly convex for instance. In any case, μ_V governs the first order of the expansion. To get the higher order terms in the expansion the idea is, as in the previous section, to write equations for all the cumulants

$$W_n^V(x_1, \dots, x_n) = \partial_{\epsilon_1} \cdots \partial_{\epsilon_n} \left(\ln Z_{N,\beta}^{V - \frac{2}{\beta N} \sum_i \frac{\epsilon_i}{x_i - \bullet}} \right) \Big|_{\epsilon_i=0}$$

by differentiating the loop equation (21) with respect to the potential.

1.4.2. *Loop equations for the unitary group.* In this section we shall consider the Haar measure dU on the unitary group, that is the unique measure on $U(N)$ which is invariant under left multiplication by unitary matrices and with mass one. We consider matrix integrals given by

$$I_N(V, A_i) = \int e^{N \text{Tr}(V(A_i, U_i, U_i^*, 1 \leq i \leq m))} dU_1 \cdots dU_m.$$

A well-known example is the Harich-Chandra-Itzykson-Zuber integral

$$HCIZ(A_1, A_2) = \int e^{N \text{Tr}(A_1 U A_2 U^*)} dU$$

where $(A_i, 1 \leq i \leq m)$ are $N \times N$ deterministic uniformly bounded matrices, and V is a polynomial function in the non-commutative variables $(U_i, U_i^*, A_i, 1 \leq i \leq m)$. We assume that the joint distribution of the $(A_i, 1 \leq i \leq m)$ converges; namely for all polynomial function P in m non-commutative indeterminates

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(P(A_i, 1 \leq i \leq m)) = \tau(P)$$

for some linear functional τ on the set of polynomials. For technical reasons, we assume that the polynomial V satisfies $\text{Tr}(V(U_i, U_i^*, A_i, 1 \leq i \leq m)) \in \mathbb{R}$ for all $U_i \in U(N)$ and all Hermitian matrices $A_i, 1 \leq i \leq m$ and $N \in \mathbb{N}$. Under those very general assumptions, the formal

convergence of the integrals could already be deduced from [C03]. The following theorem is a precise description of the results from [CGMS09] which gives an asymptotic convergence :

THEOREM 6. *Under the above hypotheses and if we further assume that the spectral radius of the matrices $(A_i, 1 \leq i \leq m, N \times N)$ is uniformly bounded (by say M), there exists $\epsilon = \epsilon(M, V) > 0$ so that for $z \in [-\epsilon, \epsilon]$, the limit*

$$F_{V,\tau}(z) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(zV, A_i)$$

exists. Moreover, $F_{V,\tau}(z)$ is an analytic function of $z \in \{z \in \mathbb{C} : |z| \leq \epsilon\}$. Furthermore, if we let

$$P_N(dU_1, \dots, dU_m) = \frac{1}{I_N(V, A_i)} e^{N \text{Tr}(V(U_i, U_i^*, A_i)_{1 \leq i \leq m})} dU_1 \dots dU_m$$

for all polynomial P in $(U_i, U_i^, A_i)_{1 \leq i \leq m}$ we have the convergence*

$$\tau_{V,\tau}(P) = \lim_{N \rightarrow \infty} \int \frac{1}{N} \text{Tr}(P((U_i, U_i^*, A_i)_{1 \leq i \leq m})) dP_N$$

The strategy of the proof of Theorem 6 is again to find and study the Schwinger-Dyson (or loop) equations under the associated Gibbs measure P_N . This equation is based on the invariance of the Haar measure, which somehow generalizes the Gaussian case where the loop equation was based on integration by parts, which can be seen as a consequence of the invariance by translation of Lebesgue measure.

To define this equation let us first define derivatives on polynomials in these matrices by the linear form such that for all $i, j \in \{1, \dots, m\}$

$$\partial_j A_i = 0 \quad \partial_j U_i = 1_{i=j} U_j \otimes 1 \quad \partial_j U_i^* = -1_{i=j} 1 \otimes U_j^*$$

and satisfying the Leibnitz rule, namely, for monomials P, Q ,

$$(23) \quad \partial_j(PQ) = \partial_j P \times (1 \otimes Q) + (P \otimes 1) \times \partial_j Q.$$

Here \times denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1 P_2 \otimes Q_1 Q_2$. We also let D_i be the corresponding cyclic derivatives such that if $m(A \otimes B) = BA$, then $D_j = m \circ \partial_j$. If q is a monomial, we more specifically have

$$(24) \quad \partial_j q = \sum_{q=q_1 U_j q_2} q_1 U_i \otimes q_2 - \sum_{q=q_1 U_j^* q_2} q_1 \otimes U_j^* q_2$$

$$(25) \quad D_j q = \sum_{q=q_1 U_j q_2} q_2 q_1 U_j - \sum_{q=q_1 U_j^* q_2} U_j^* q_2 q_1$$

Using the invariance by multiplication of the Haar measure one can prove the Schwinger-Dyson equation :

$$P_N \left(\frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}(\partial_j P) \right) + P_N \left(\frac{1}{N} \text{Tr}(P D_j V) \right) = 0$$

This is proved by noticing that if we set $U_j(t) = U_j e^{itB}$ and leave the other $U_k(t) = U_k$ unchanged for a Hermitian matrix B then for all $k, l \in \{1, \dots, N\}$

$$\partial_t \int P(U_p(t), 1 \leq p \leq m, A_i)(k, l) e^{N \text{Tr}(V(U_p(t), U_p(t)^*, A_i))} dU_1 \dots dU_m = 0$$

Taking $B = 1_{kl} + 1_{lk}$ and $B = i1_{kl} - i1_{lk}$ shows that we can choose by linearity $B = 1_{kl}$ even though this is not self-adjoint. This yields the result after summation over k and l . By using concentration of measure, we know that for all polynomial P , $N^{-1} \text{Tr}(P(U_i, U_i^*, A_i))$ is not far from its expectation and therefore we deduce that the limit points of these (bounded) quantities $\tau(P)$ satisfy the Schwinger-Dyson equation

$$\tau \otimes \tau(\partial_j P) + \tau(D_j V P) = 0$$

Uniqueness of the solution to such an equation in the perturbative regime is done as in the Gaussian case; when $V = 0$ it is clear as it defines all moments recursively from the knowledge of τ restricted to the A_i and a perturbation argument shows this is still true for small parameters. The uniqueness provides the convergence whereas the study of this solution shows that it expands as a generating series in the enumeration of some planar maps.

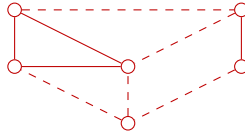
2. Loop models

In this last lecture we show how matrix integrals can also be used to enumerate loop models, which are in some cases equivalent to the famous Potts model (on random planar maps). We first discuss the relation with the Potts model and then show how to construct a matrix model to solve the related enumeration question.

2.1. The Potts model on random maps. The partition function of the Potts model on a graph $G = (V, E)$ is given by

$$\begin{aligned} Z_G &= \sum_{\sigma: V \rightarrow \{1, \dots, Q\}} \exp(K \sum_{\{i, j\} \in E} \delta_{\sigma_i, \sigma_j}) \\ &= \sum_{\sigma: V \rightarrow \{1, \dots, Q\}} \prod_{\{i, j\} \in E} (1 + v \delta_{\sigma_i, \sigma_j}) \\ &= \sum_{E' \subset E} v^{\# \text{ bonds}} Q^{\# \text{ clusters}} \end{aligned}$$

where $v = e^K - 1$, bonds are the edges in E' , a subset of E , the clusters are the connected components of the subgraph (V, E') . For instance the following graph where the edges in E' are bold whereas those in $E \setminus E'$ are dashed,



has weight $v^4 Q^3$.

We shall consider the Potts model on random planar maps. Recall (see section 1.1) that a map is a connected graph which is embedded into a surface in such a way that edges do not cross and faces (obtained by cutting the surface along the edges) are homeomorphic to a disk.

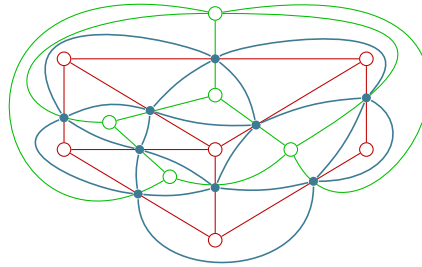
The genus of the map is the minimal genus of a surface in which it can be properly embedded. By Euler formula :

$$2 - 2g = \# \text{ vertices} + \# \text{ faces} - \# \text{ edges}.$$

We shall consider the Potts model on random planar maps. We assume these graphs are rooted, that is are given a distinguished oriented edge. It is given by the partition function

$$\begin{aligned} Z &= \sum_{G=(V,E)} x^{\#E} y^{\#V} Z_G \\ (26) \quad &= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{\sigma:V \rightarrow \{1,\dots,Q\}} \exp(K \sum_{\{i,j\} \in E} \delta_{\sigma_i, \sigma_j}) \\ &= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{E' \subset E} v^{\# \text{ bonds}} Q^{\# \text{ clusters}} \end{aligned}$$

If G is a planar map, there is a dual (green) and a medial (blue) planar graph G_m . We shall express the partition function Z as a generating function for (loop) configurations on the medial graph. The vertices of the dual graph are given by a point in each of the faces of the original graph and each of the edges of the dual graph intersect one (and only one) edge of the original graph. The vertices of the medial graph are the intersection of the edges of the dual graph and the original graph. The medial graph has an edge in each face of the graph obtained by taking both edges and vertices of the original and dual graph. The edges of the medial graph do not cut the edges of the original or the medial graph except at their intersections. Hence, the medial graph is four-valent.



By construction, the original graph and the dual graph are in bijection. Note that in each face of the medial graph there is either a vertex from the dual or from the original graph and that this choice is given by a checkerboard coloring of the medial graph corresponding to fixing

which type of vertex is inside a face. Hence, knowing the medial graph and the nature of the vertex in one of its face allows to reconstruct both the original and the dual graph.

We next describe the bijection between the configurations on the original, dual and medial graph. A configuration on the original graph just consists in coloring some edges and dashing the others. The configuration on the dual graph is obtained by dashing (resp. bolding) its edges that cut a bold (resp. dashed) edge of the original graph. The configuration on the medial graph is given by splitting the vertex so that it does not intersect a bold edge. Hence, there are two sorts of vertices according to the nature of the colored edge it does not intersect. The two sorts of vertices on the medial graph are described as follows :



Configurations are therefore described bijectively by the collection of loops of the medial graph as well as as a checkerboard coloring.



If G is a planar map, there is a bijection between the configuration on G and the set of loops and shaded vertices on the medial graph.

Moreover, writing Euler formula in each cluster gives the relation

$$\#\text{loops} = 2\#\text{clusters} + \#\text{bonds} - \#V$$

The equivalence to the loop model allows to state that if we take $y = Q^{-\frac{1}{2}}$ in (26)

$$\begin{aligned} Z &= \sum_{G=(V,E)} x^{\#E} Q^{-\frac{1}{2}\#V} \sum_{E' \subset E} v^{\#\text{ bonds}} Q^{\#\text{ clusters}} \\ &= \sum_{\Gamma} \delta^{\#\text{ loops}} \alpha^{\#} \beta^{\#} \end{aligned}$$

where the summation is restricted to 4-valent rooted planar maps, and

$$\delta = \sqrt{Q} \quad \frac{\alpha}{\beta} = \frac{v}{\sqrt{Q}} \quad \beta = x$$

δ is called the fugacity.

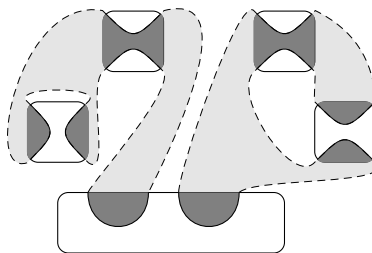
Hence, when $y = Q^{-1/2}$, the partition function Z of the Potts model on planar maps is a generating function for the number of possible matchings of the end points of n copies of the vertex



and m copies of the vertex



so that the resulting graph is planar, connected, has p loops, and is checkerboard shaded.

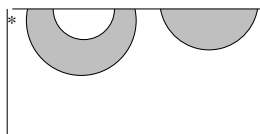


We shall consider generalizations of such enumeration questions in the following.

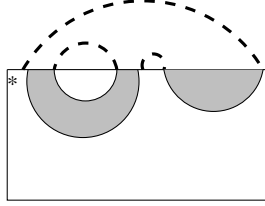
2.2. Loop models and Random matrices. We have already seen in the previous lecture that random matrices could be used to enumerate planar graphs. In this section we show how this point can be specified to enumerate loop models.

In the following we shall consider loop models with vertices given by Temperley-Lieb elements.

The Temperley-Lieb elements are boxes with boundary points connected by non-intersecting strings, equipped with a shading and a marked boundary point.



The easier loop models are those with only one vertex and the question one may ask is, being given a Temperley-Lieb element, to count the number of planar matching of the end points of the Temperley-Lieb element so that there are exactly n loops. The picture below shows the case of 2 loops :



This question has been connected with random matrices for a long time in the physics literature, see e.g. [EB99, EK95, KS92]. For a Temperley-Lieb element B , we denote $p \stackrel{B}{\sim} \ell$ if a string joins the p th boundary point with the ℓ th boundary point in B , then we associate to B with k strings the polynomial

$$q_B(X) = \sum_{\substack{i_j = i_p \text{ if } j \stackrel{B}{\sim} p \\ 1 \leq i_j \leq n}} X_{i_1} \cdots X_{i_{2k}}.$$

For instance, if B is given by



we have

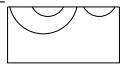
$$q_B(X) = \sum_{i,j,k=1}^n X_i X_j X_j X_i X_k X_k.$$

THEOREM 7. If ν^M denotes the law of n independent $M \times M$ GUE matrices,

$$\lim_{M \rightarrow \infty} \int \frac{1}{M} \text{Tr}(q_B(X)) \nu^M(dX) = \sum n^{\#\text{loops}}$$

where we sum over all planar maps that can be built on B .

Proof By Voiculescu's theorem, if $B =$



$$\begin{aligned} & \lim_{M \rightarrow \infty} \int \frac{1}{M} \text{Tr}(q_B(X)) \nu^M(dX) \\ &= \sum_{i,j,k=1}^n \lim_{M \rightarrow \infty} \int \frac{1}{M} \text{Tr}(X_i X_j X_j X_i X_k X_k) \nu^M(dX) \\ &= \sum_{i,j,k=1}^n \sum_{i,j,k} \text{Diagram} \\ &= \sum n^{\#\text{loops}} \end{aligned}$$

because the indices have to be constant along loops (since they have to correspond to the same matrix to give a non vanishing contribution).

The problem with the previous theorem is that moments of random matrices can only be used so far as generating function for the enumeration of loop configurations taken at integer values of the fugacity. This is enough to characterize polynomials but not the series we shall consider later.

In [J98], V. Jones proposed a construction of a planar algebra associated with a bipartite graph. It was used in [GJS10] to overcome this point. The idea is to take random matrices which are indexed by the edges of a bipartite graph instead of integer numbers and to modify the polynomial q_B in such a way that the fugacity is the Perron-Frobenius eigenvalue of the adjacency matrix of the graph.

To be more precise, let $\Gamma = (V = V_+ \cup V_-, E)$ be a bipartite graph with oriented edges so that if $e \in E$, its opposite e^o is also in E . Assume that the adjacency matrix of Γ has Perron-Frobenius eigenvalue δ . Note that this restricts the possible values of δ to $\{2 \cos(\frac{\pi}{n}), n \geq 3\} \cup [2, +\infty[$ which is however a set which contains limit points.

Now, let us define for a Temperley-Lieb element B the polynomial

$$q_B^v(X) = \sum_{e_j=e_p^o \text{ if } j \stackrel{B}{\sim} p} \sigma_B(w) X_{e_1} \cdots X_{e_{2k}}$$

where we recall that $j \stackrel{B}{\sim} p$ if a string joins the p th boundary point with the j th boundary point in the TL element B . The sum runs over loops $w = e_1 \cdots e_{2k}$ in Γ which starts at $v \in V$. $v \in V_+$ iff $*$ is in a white (i.e unshaded) region. σ_B is defined as follows. Denote $(\mu_v)_{v \in V}$ with $\mu_v \geq 0$ the eigenvector of Γ for the Perron-Frobenius eigenvalue δ and set, if $\sigma(e) := \sqrt{\frac{\mu_{t(e)}}{\mu_{s(e)}}}$, $e = (s(e), t(e))$,

$$\sigma_B(e_1 \cdots e_{2p}) = \prod_{\substack{i \stackrel{B}{\sim} j \\ i < j}} \sigma(e_i)$$

to be the products of $\sigma(e)$ so that each string of B brings $\sigma(e)$ with e the edge which labels the start of the string.

For $e \in E$, $e = (s(e), t(e))$, let X_e^M be independent (except for the symmetry constraint $X_{e^o}^M = (X_e^M)^*$) $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with i.i.d centered Gaussian entries with variance $1/(M\sqrt{\mu_{s(e)}\mu_{t(e)}})$. We denote ν_M their joint law.

THEOREM 8 (G-Jones-Shlyakhtenko [GJS10]). *Let Γ be a bipartite graph whose adjacency matrix has δ as Perron-Frobenius eigenvalue. Let B be a Temperley-Lieb element so that $*$ is in an unshaded region. Then, for all $v \in V^+$*

$$\tau_\delta(B) := \lim_{M \rightarrow \infty} \int \frac{1}{M\mu_v} \text{Tr}(q_B^v(X)) d\nu^M(X) = \sum \delta^{\#\text{loops}}$$

where the sum runs on all planar maps built on B .

Maybe the most illuminating “proof” is by trying examples.

If $B = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$, for all $v \in V^+$

$$\begin{aligned} \mathbb{E}\left[\frac{1}{M\mu_v} \text{Tr}\left(\sum_{e:s(e)=v} \sigma(e)X_e X_{e^0}\right)\right] &= \frac{1}{M\mu_v} \sum_{e:s(e)=v} \sum_{i=1}^{[M\mu_v]} \sum_{j=1}^{[M\mu_{t(e)}]} \sqrt{\frac{\mu_{t(e)}}{\mu_v}} \frac{1}{M\sqrt{\mu_{t(e)}\mu_{s(e)}}} \\ &= \frac{1}{M\mu_v} \sum_{e:s(e)=v} \sqrt{\frac{\mu_{t(e)}}{\mu_v}} \frac{M\mu_v M\mu_{t(e)}}{M\sqrt{\mu_{t(e)}\mu_{s(e)}}} \\ &= \frac{1}{\mu_v} \sum_{e:s(e)=v} \mu_{t(e)} = \delta \end{aligned}$$

If $B = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$, for all $v \in V^+$

$$\begin{aligned} &\lim_{M \rightarrow \infty} \mathbb{E}\left[\frac{1}{M\mu_v} \text{Tr}\left(\sum_{\substack{e:s(e)=v \\ s(f)=v}} \sigma(e)\sigma(f)X_e X_{e^0} X_f X_{f^0}\right)\right] \\ &= \delta^2 + \frac{1}{\mu_v} \sum_{e=f} \frac{\mu_{t(e)} \mu_v^2 \mu_{t(e)}}{\mu_v \mu_{t(e)} \mu_v} = \delta^2 + \delta \end{aligned}$$

More generally, the edges are constant along the loops and bring the contribution $\mu_{t(e)}/\mu_v$ hence leading after summation to δ .

As in the previous section we can make these enumeration questions more interesting by adding a potential, and in turn enumerating loop models with several Temperley-Lieb vertices. Let B_i be Temperley Lieb elements with $*$ with shading $\sigma_i = +$ (resp. $-$) in the unshaded (resp. shaded) region, $1 \leq i \leq p$. Let Γ be a bipartite graph whose adjacency matrix has eigenvalue δ as before. Let ν^M be the law of the previous independent rectangular Gaussian matrices and set $\nu_{(B_i)}^M(X)$ to be the law

$$d\nu_{(B_i, \beta_i)}^M(X) = \frac{\prod_{e \in E} \mathbb{1}_{\|X_e\|_\infty \leq L}}{Z_{(B_i, \beta_i)}^M} e^{M \text{Tr}(\sum_{i=1}^p \beta_i \sum_{v \in V_{\sigma_i}} \mu_v q_{B_i}^v(X))} d\nu^M(X).$$

Note at this point that Γ may need to be infinite for $\delta > 2$ and then $\nu_{(B_i, \beta_i)}^{M, \beta}$ may seem to be ill defined. However, at least for $\delta \in \{2 \cos(\pi/n), n \geq 3\}$, we can choose Γ to be finite whereas we show in [GJSZ12] that we can take $\Gamma = \mathbb{A}^\infty$ and then that the correlation between matrices decays very fast with the distance of the edges on the graph, hence allowing a construction of the measure on infinitely many matrices using Gibbs measures techniques. We shall not discuss this point hereafter.

THEOREM 9 (G-Jones-Shlyakhtenko-Zinn Justin [**GJSZ12**]). *For any $L > 2$, for β_i small enough real numbers, for any Temperley-Lieb element B with color σ , any $v \in V_\sigma$,*

$$\tau_{(B_i, \beta_i)_i}^\delta(B) := \lim_{M \rightarrow \infty} \int \frac{1}{M \mu_v} \text{Tr}(q_B^v(X)) d\nu_{(B_i, \beta_i)_i}^M(X) = \sum_{n_i \geq 0} \sum \delta^{\#\text{loops}} \prod_{i=1}^p \frac{\beta_i^{n_i}}{n_i!}$$

where we sum over the planar maps build on n_i TL elements B_i and one B .

The proof is based, as in the previous section, on Schwinger-Dyson's equation and concentration of measure. Note that the above theorem also provides the convergence of the partition function, at list in the case where $\sum \mu_v^2 < \infty$ since we have

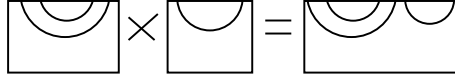
$$\frac{1}{M^2} \log \frac{Z_{(B_i, \beta_i)}^M}{Z_{(B_i, 0)}^M} = \int_0^1 \sum_{i=1}^p \beta_i \sum_{v \in V_{\sigma_i}} \mu_v^2 \int \frac{1}{M \mu_v} \text{Tr}(q_{B_i}^v(X)) d\nu_{(B_i, \alpha \beta_i)_i}^M(X) d\alpha$$

where the right hand side converges by the theorem, and remains bounded so that dominated convergence theorem applies. Taking the limit and performing the integral over α yields

$$\lim_{M \rightarrow \infty} \frac{1}{M^2 \sum \mu_v^2} \log \frac{Z_{(B_i, \beta_i)}^M}{Z_{(B_i, 0)}^M} = \sum_{\sum n_i \geq 1} \sum \delta^{\#\text{loops}} \prod_{i=1}^p \frac{\beta_i^{n_i}}{n_i!}$$

where we sum over the planar maps build on n_i TL elements B_i .

2.3. Loop models and subfactors. Another point of view on the previous section is subfactor theory. In fact, Temperley-Lieb algebra can be viewed as a special case of planar algebras and $\tau_{\delta, \beta}$ are tracial states on this planar algebra if they are equipped with the multiplication



and the involution which is given by taking the symmetric picture of the element. In other words, $\tau_{\delta, \beta}$ satisfies the following properties for any elements S, T of the Temperley-Lieb algebra :

$$\tau_{\delta, \beta}(SS^*) \geq 0, \tau_{\delta, \beta}(TS) = \tau_{\delta, \beta}(ST), \tau_{\delta, \beta}(1) = 1$$

where 1 represent the element with no string. To a tracial state on a Banach algebra we can associate a von Neumann algebra by the so-called GNS construction.

THEOREM 10 (G-Jones-Shlyakhtenko [**GJS10**]). *Take $\delta \in \{2 \cos(\pi/n), n \geq 3\} \cup [2, +\infty[$. Then*

- $\tau_{\delta, 0}$ is a tracial state on the Temperley Lieb algebra.
- The von Neumann algebra associated by the GNS construction to $\tau_{\delta, 0}$ is a factor, namely its center is trivial. A tower of subfactors with index δ^2 can be built.

The tower is build by changing the multiplication so that the nearest boundary points of both Temperley-Lieb elements are capped. The construction presented here can be generalized to any planar algebra. Hence, it shows that there is a canonical way to construct a tower of

subfactors from any subfactor planar algebra. It is still unknown whether the von Neumann algebras associated to $\tau_{\delta,\beta}$ are factors for $\beta \neq 0$.

2.4. Matrix model for the Potts model. Let $\delta \in \{2 \cos(\frac{\pi}{n})\}_{n \geq 3} \cup [2, \infty[$ and $\Gamma = (V_+ \cup V_-, E)$ be a bipartite graph with eigenvalue δ and Perron-Frobenius eigenvector μ . Assume that Γ is finite to simplify. This includes $\delta = 2 \cos(\pi/n), n \geq 3$, and therefore a set with an accumulation point. We set

$$\nu_{\beta_{\pm}}^M(dX_e) = \frac{1_{\|X_e\|_{\infty} \leq L}}{Z_{\beta_{\pm}}^M} e^{M \text{Tr} \left(\sum_{v \in V} \mu_v \sum_{\sigma = \pm} \beta_{\sigma} 1_{v \in V_{\sigma}} \left(\sum_{e: s(e)=v} \sigma(e) X_e X_e^* \right)^2 \right)}$$

$$\prod_e e^{-\frac{M}{2} (\mu_{s(e)} \mu_{t(e)})^{\frac{1}{2}} \text{Tr}(X_e X_e^*)} dX_e dX_e^*$$

THEOREM 11 (G-Jones-Shlyakhtenko-Zinn Justin [GJSZ12]). *Take $\delta \in \{2 \cos(\pi/n), n \geq 3\} \cup [2, +\infty[$. Then, for L large enough, β_{\pm} small enough, for all Temperley-Lieb element $B, v \in V_{\sigma_B}$,*

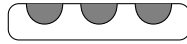
$$\lim_{M \rightarrow \infty} \frac{1}{M \mu_v} \int \text{Tr}(q_B^v(X)) \nu_{\beta_{\pm}}^M(dX)$$

$$= \sum_{n_+, n_- \geq 0} \sum \frac{\beta_+^{n_+} \beta_-^{n_-}}{n_+! n_-!} \delta^{\#\text{loops}} =: \text{Tr}_{\beta_{\pm}, \delta}(B)$$

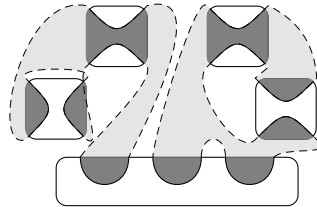
where we sum over all planar maps build by matching the end points of n_- (resp. n_+) vertices of type



and one of type B . If B is given by



we count the number of matchings of the following type :



2.5. Solving the Potts matrix model. By the previous construction, at least for $\delta \in \{2 \cos(\pi/n), n \geq 3\}$, we have equality between

$$(27) \quad \sum_{\Gamma} \delta^{\#\text{ loops}} \beta_{-}^{\#\text{ loops}} \beta_{+}^{\#\text{ loops}}$$

and

$$\lim_{M \rightarrow \infty} \frac{1}{M^2 \sum \mu(v)^2} \log Z_{\beta_{\pm}}^M$$

where, for L large enough, $Z_{\beta_{\pm}}^M$ equals to

$$\int_{\|X_e\| \leq L} e^{M \text{Tr} \left(\sum_{v \in V} \mu_v (\beta_{+} 1_{v \in V_{+}} + \beta_{-} 1_{v \in V_{-}}) (\sum_{e: s(e)=v} \sqrt{\mu_v \mu_{t(e)}} X_e X_e^* \right)^2 \right)} \nu^M(dX)$$

if ν^M is the law of X_e , $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with iid centered Gaussian entries with covariance $(M^2 \mu_{s(e)} \mu_{t(e)})^{-\frac{1}{2}}$

We can compute the matrix model and therefore solve the original combinatorial problem by using Hubbard-Stratonovich transformation. Namely, let G_v $[M\mu_v] \times [M\mu_v]$ independent matrices from the GUE and X_e be $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with covariance $(M^2 \mu_{s(e)} \mu_{t(e)})^{-\frac{1}{2}}$ under ν^M , with $\alpha_{\pm} = \sqrt{2\beta_{\pm}}$, $\Delta(\lambda) = \prod_{i \neq j} (\lambda_i - \lambda_j)$, we can write

$$\begin{aligned} Z_{\beta_{\pm}}^M &= \int_{\|X_e\| \leq L} e^{M \text{Tr} \left(\sum_{v \in V} \mu_v (\beta_{+} 1_{v \in V_{+}} + \beta_{-} 1_{v \in V_{-}}) (\sum_{e: s(e)=v} \sigma(e) X_e X_e^* \right)^2 \right)} \nu^M(dX) \\ &= \int_{\|X_e\| \leq L} e^{M \text{Tr} \left(\sum_{\sigma = \pm} \sum_{v \in V_{\sigma}} \alpha_{\sigma} G_v (\sum_{e: s(e)=v} \sqrt{\mu_v \mu_{t(e)}} X_e X_e^* \right)} \nu^M(dX, dG) \\ &\simeq \int_{\|G_v\| \leq L'} \prod_{e \in E_{+}} e^{-\text{Tr} \otimes \text{Tr} (\log(I + \alpha_{+} I \otimes G_{s(e)} + \alpha_{-} G_{t(e)} \otimes I))} \nu^M(dG) \\ &= Z^M \int_{|\lambda_i^e| \leq L'} \prod_{e \in E_{+}} e^{-\sum_{i=1}^{[M\mu_{s(e)}]} \sum_{j=1}^{[M\mu_{t(e)}]} \log(1 + \alpha_{+} \lambda_i^e + \alpha_{-} \eta_j^e)} \\ &\quad \Delta(\eta^e) \Delta(\lambda^e) e^{-\frac{[M\mu_{s(e)}]}{2} \sum_{i=1}^{[M\mu_{s(e)}]} (\lambda_i^e)^2 - \frac{[M\mu_{t(e)}]}{2} \sum_{j=1}^{[M\mu_{t(e)}]} (\eta_j^e)^2} d\lambda^e d\eta^e. \end{aligned}$$

where in the second line we used Hubbard-Stratonovich transformation and in the third took the expectation over the X_e (note here that the bound on the $\|X_e\|$ ensured that the log-density of the joint law in X, G was strictly concave; it thus keeps the G_v bounded with overwhelming probability by Brascamp-Lieb inequality which is the reason why the bound on $\|X_e\|$ transferred into a bound on $\|G\|_v$ in the third line). Finally in the last line we have diagonalized the matrices G_v and observed that the jacobian is given by the Vandermonde determinant Δ .

Hence, Hubbard-Stratonovich led us to define an auxiliary probability measure $P_\alpha^{M,L}$ on $\mathbb{R}^{M|V|}$ given by

$$\frac{1_{|\lambda^v| \leq L}}{Z_\alpha^{M,L}} \prod_{e \in E_+} \prod_{\substack{1 \leq i \leq [M\mu_s(e)] \\ 1 \leq j \leq [M\mu_t(e)]}} \frac{1}{1 + \alpha_+ \lambda_i^{s(e)} + \alpha_- \lambda_j^{t(e)}} \prod_{v \in V} \Delta(\lambda^v) e^{-\frac{M\mu_v}{2} \sum (\lambda_i^v)^2} \prod d\lambda_i^v$$

By large deviation analysis, see [BaG97, AGZ10], the asymptotics of this model can easily be studied and one finds that the law of $\{\hat{\mu}_v^M := \frac{1}{M\mu_v} \sum \delta_{\lambda_i^v}, v \in V\}$ under $P_\alpha^{M,L}$ satisfies a large deviation principle for the weak topology with speed M^2 and rate function given, up to a constant, by

$$\begin{aligned} I(\nu_v, v \in V) &= \sum_v \frac{\mu_v^2}{2} \left(\int x^2 d\nu_v(x) - 2\Sigma(\nu_v) \right) \\ &+ \sum_{e \in E_+} \mu_v \mu_{t(e)} \int \log |1 + \alpha_+ x + \alpha_- y| d\nu_v(x) d\nu_{t(e)}(y). \end{aligned}$$

if $\Sigma(\nu) = \int \log |x - y| d\nu(x) d\nu(y)$. Such result is easily guessed by seeing that approximately

$$dP_\alpha^{M,L} \sim \frac{1}{Z_{M,L}} e^{-M^2 I(\hat{\mu}_v^M)} \prod d\lambda_i^v$$

so that Laplace method yields the desired estimate. The only point is to deal with the logarithmic singularity to transform these heuristics in [BaG97] theorem.

Moreover, we then have

$$(28) \quad \lim_{M \rightarrow \infty} M^{-2} \log Z_\alpha^{M,L} = -\inf I$$

where the infimum is taken over the set of probability measures $\mu_v, v \in V$.

The minimizers of the rate function I can be studied and we get

THEOREM 12 ([GJSZ12]). • *I achieves its minimal value at a unique set of probability measures $\nu_v, v \in V$.*

- $\exists \nu_+, \nu_- \in P(\mathbb{R})$, so that $\nu_v = \nu_\pm$ if $v \in V_\pm$. (ν_+, ν_-) are the unique minimizers of

$$\begin{aligned} I_{\delta, \alpha_+, \alpha_-}(\mu_+, \mu_-) &= \sum_{\epsilon = \pm} \left(\frac{1}{2} \int x^2 d\mu_\epsilon(x) - \int \log |x - y| d\mu_\epsilon(x) d\mu_\epsilon(y) \right) \\ &+ \delta \int \log |1 + \alpha_+ x + \alpha_- y| d\mu_+(x) d\mu_-(y). \end{aligned}$$

- For all $L > 2$ not too large, α_+, α_- small enough

$$\lim_{M \rightarrow \infty} E \left[\frac{1}{M\mu_v} \sum_{i=1}^{[M\mu_v]} (\lambda_i^v)^p \right] = \int x^p d\nu_v(x) \quad \forall p \in \mathbb{N}, \quad v \in V.$$

We describe more precisely the measures ν_+, ν_- below. This already gives a formula for (27) according to (28). In fact, this gives us a slightly more precise information. Indeed, we can relate the asymptotic measures ν_+, ν_- with the original combinatorial problem we wished to solve, that is the enumeration for the loop model. Let $M(z) = \int \sum_{n \geq 0} z^n x^n d\nu_+(x)$. On the other hand, consider the generating function we are interested in, namely

$$C(z, \alpha_+, \alpha_-) = \sum_{n \geq 0} z^n \sum_{\ell} \delta^\ell \frac{\alpha_+^{n_+}}{n_+!} \frac{\alpha_-^{n_-}}{n_-!}$$

where we sum over the planar maps build over n_+ (resp. n_-) vertices with two strings and two black (resp. white) regions and one vertex with n strings with black inside so that there are exactly ℓ loops. Then, put

$$\gamma(z) = \alpha_+ z / (1 - z^2 M(z)).$$

We claim that for small z , we have

$$C(z, \alpha_+, \alpha_-) = \frac{\alpha_+}{z} \left[1 - \frac{\alpha_+ \gamma^{-1}(z)}{z} \right].$$

In fact this can be seen by giving a combinatorial interpretation to the matrix model $P_\alpha^{M,L}$; we refer the interested reader to [GJSZ12]. Hence, we see that the original combinatorial question encapsulated in $C(z, \alpha_+, \alpha_-)$ can be reduced to a variational problem, namely minimizing $I_{\delta, \alpha_+, \alpha_-}$. It turns out that this minimizing problem can be solved explicitly as follows. Let p_+ (resp. p_-) be the law of $1 + \alpha_+ x$ and $-\alpha_- y$ under ν_+ (resp. ν_-).

- For α_\pm small enough, p_\pm has a connected support $[a_\pm, b_\pm]$ around 1 (resp. 0) and $a_- < b_- < a_+ < b_+$.

- Set $G_\pm(z) = \int (z-x)^{-1} dp_\pm(x)$. Then

$$G_\pm(z+i0) + G_\pm(z-i0) = P_\pm(z) + \delta G_\mp(z) \quad z \in [a_\pm, b_\pm]$$

with $P_-(z) = z/\alpha_-$, $P_+(z) = (1-z)/\alpha_+$.

Introduce

$$u(z) = \frac{i}{2} \sqrt{(b_1 - a_1)(b_2 - a_2)} \int_{b_2}^z \frac{dz'}{\sqrt{(z' - a_1)(z' - a_2)(z' - b_1)(z' - b_2)}},$$

with inverse $z(u)$. Set

$$\omega_+(u) = \int \frac{1}{z(u) + \sqrt{8\beta_+ x}} d\nu_+(x)$$

Then, if q is such that $\delta = q + q^{-1}$, $q = e^{i\pi\nu}$, we have

$$\omega_+(u) = \frac{1}{q - q^{-1}} [(\phi_+(u) - \phi_+(-u)) + R(z(u))]$$

where, if Θ denotes the theta-function

$$\Theta(u) = 2 \sum_{k=0}^{\infty} e^{i\pi \frac{\omega_2}{\omega_1} (k+1/2)^2} \sin(2k+1) \frac{\pi u}{\omega_1},$$

we have

$$\phi_+(u) = c_+ \frac{\Theta(u - u_\infty - 2\nu K)}{\Theta(u - u_\infty)} + c_- \frac{\Theta(u + u_\infty - 2\nu K)}{\Theta(u + u_\infty)}$$

and

$$R(z) = \frac{2q}{1 - q^2} \frac{z}{\sqrt{8t_1}} + \frac{q^2 + 1}{1 - q^2} \frac{(z - 1)}{\sqrt{8t_2}}.$$

All the parameters can be computed as solutions of fixed point equations. This derivation is done in details in [GJSZ12].

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