Extreme value statistics for censored data with heavy tails under competing risks

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Abstract

This paper addresses the problem of estimating the extreme value index in presence of random censoring and competing risks for heavy tailed distributions. Asymptotic normality of the proposed estimator (which has the form of an Aalen-Johansen integral) is established. A small simulation study exhibits its performances for finite samples.

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1. Introduction

The study of duration data (lifetime, failure time, re-employment time...) subject to random censoring is a major topic of the domain of statistics, which finds applications in many areas (in the sequel we will, for convenience, talk about lifetimes to refer to these observed durations, but without restricting our scope to lifetime data analysis). In general, the interest lies in obtaining informations about the *central* characteristics of the underlying lifetime distribution (mean lifetime or survival probabilities for instance), often with the objective of comparing results between different conditions under which the lifetime data are acquired. In this work, we will address the problem of inferring about the (upper) *tail* of the lifetime distribution, for data subject both to random (right) censoring and competing risks.

Suppose indeed that we are interested in the lifetimes of n individuals or items, which are subject to K different causes of death or failure, and to random censorship (from the right) as well. We are particularly interested in one of these causes (this cause will be considered as cause number k thereafter, where $k \in \{1, \dots, K\}$), and we suppose that all causes are exclusive and are likely to be dependent on the others. The censoring time is assumed to be independent of the different causes of death or failure and of the observed lifetime itself. However, since the other causes (different from the k-th cause of interest) generally cannot be considered as independent of the main cause, in no way they can be included in the censoring mechanism. This prevents us from relying on the basic independent censoring statistical framework, and we are thus in the presence of what is called a competing risks framework (see Moeschberger and Klein (1995)).

For instance, if a patient is suffering from a very serious disease and starts some treatment, then the final outcome of the treatment can be death due to their main disease, or death due to other causes (nosocomial infection for instance). And censoring can occur due to loss of follow up or end of the clinical study. Another example, in a reliability experiment, is that the failure of some mechanical system can be due to the failure of a particular subpart, or component, of the system : since separating the different components for studying the reliability of only one of them is generally not possible, accounting for these different competing causes of failure is necessary. Another field where competing risks often arise are labor economics, for instance in re-employment studies (see Fermanian (2003) for practical examples).

One way of formalising this is to say that we observe a sample of n independent couples $(Z_i, \xi_i)_{1 \leqslant i \leqslant n}$ where

$$Z_i = \min(X_i, C_i), \quad \delta_i = \mathbb{I}_{X_i \leqslant C_i}, \quad \xi_i = \begin{cases} 0 & \text{if } \delta_i = 0, \\ \mathscr{C}_i & \text{if } \delta_i = 1. \end{cases}$$

The i.i.d. samples $(X_i)_{i \leqslant n}$ and $(C_i)_{i \leqslant n}$, of respective continuous distribution functions F and G, represent the lifetimes and censoring times of the individuals, and are supposed to be independent. For convenience, we will suppose in this work that their are non-negative. The variables $(\mathscr{C}_i)_{i \leqslant n}$ form a discrete sample with values in $\{1, \dots, K\}$, and represent the causes of failure or death of the n individuals or items. It is important to note that these causes are observed only when the data is uncensored (*i.e.* when $\delta_i = 1$), therefore we only observe the ξ_i's, not the complete $\mathscr{C}_i$'s.

One way of considering the failure times X_i is to write

$$X_i = \min(X_{i,1}, \dots, X_{i,K}),$$

where the variable $X_{i,k}$ is a (rather artificial) variable representing the imaginary latent lifetime of the i-th individual when the latter is only affected by the k-th cause (the other causes being absent). This viewpoint may be interesting in its own right, but we will not keep on considering it in the sequel, one reason being that such variables $X_{i,1}, \dots, X_{i,K}$ cannot be realistically considered as independent, and their respective distributions are of no practical use or interpretability (as explained and demonstrated in the competing risks literature, these distributions are in fact not statistically identifiable, see Tsiatis (1975) for example).

The object of interest is the probability that a subject dies or fails after some given time t, due to the k-th cause, for high values of t. This quantity, denoted by

$$\bar{F}^{(k)}(t) = \mathbb{P}[\, X > t \,, \mathscr{C} = k \,],$$

is related to the so-called *cumulative incidence function* $F^{(k)}$ defined by

$$F^{(k)}(t) = \mathbb{P}[\, X \leqslant t \,, \mathscr{C} = k \,].$$

Note that $\bar{F}^{(k)}(t)$ is not equal to $1-F^{(k)}(t)$, but to $\mathbb{P}(\mathscr{C}=k)-F^{(k)}(t)$, because $F^{(k)}$ is only a sub-distribution function. However we have $\bar{F}^{(k)}(t)=\int_t^\infty dF^{(k)}(u)$. In the sequel, the notation $\bar{S}(.)=S(\infty)-S(.)$ will be used, for any non-decreasing function S.

In this paper, we are interested in investigating the behaviour of $\bar{F}^{(k)}(t)$ for large values of t. This amounts to statistically study extreme values in a context of censored data under competing risks, and will lead us to consider some extreme value index γ_k related to $\bar{F}^{(k)}$, which will be defined in a few lines. Equivalently, the object of interest is the high quantile $x_p^{(k)}=(\bar{F}^{(k)})^-(p)=\inf\{\, x\in\mathbb{R}\,;\,\bar{F}^{(k)}(x)\geqslant p\,\}$ when p is close to 0, which can be interpreted as follows (in the context of lifetimes of individuals or failure times of systems) : in the presence of the other competing causes, a given individual (or item) will die (or fail), due to cause k after such a time $x_p^{(k)}$, only with small probability p. A nonparametric inference for quantiles of fixed (and therefore not extreme) order, in the competing risk setting, has been already proposed in Peng and Fine (2007).

One way of addressing this problem could be through a parametric point of view (see Crowder (2001) for further methods in the competing risk setting), however, the non-parametric approach is the most common choice of people faced with data presenting censorship or competing risks. Of course, the standard Kaplan-Meier method for survival analysis does not yield valid results for a particular risk if failures from other causes are treated as censored, because the other causes cannot always be considered independent of the particular cause of interest.

The commonly used nonparametric estimator of the cumulative incidence function $F^{(k)}$ is the so-called Aalen-Johansen estimator (see Aalen and Johansen (1978), or Geffray (2009) equation (7)) defined by

$$F_n^{(k)}(t)=\sum_{Z_i\leqslant t}\frac{\delta_i\mathbb{I}_{\mathscr{C}_i=k}}{n\bar{G}_n(Z_i^-)}$$

where $\bar{G}_n$ denotes the Kaplan-Meier estimator of G, so that we can introduce the following estimator for $\bar{F}^{(k)}$:

$$\bar{F}_n^{(k)}(t)=\sum_{Z_i>t}\frac{\delta_i\mathbb{I}_{\mathscr{C}_i=k}}{n\bar{G}_n(Z_i^-)}.$$

But if the value t considered is so high that only very few observations Z_i (such that $\mathscr{C}_i=k$) exceed t, then this purely nonparametric approach will lead to very unstable estimations $\bar{F}_n^{(k)}(t)$ of $\bar{F}^{(k)}(t)$. This is why a semiparametric approach is desirable, and the one we will consider here is the one inspired by classical extreme value theory.

First note that in this paper, we will only consider situations where the underlying distributions F and G of the variables X and C are supposed to present power-like tails (also commonly named heavy tails), and we will focus on the evaluation of the order of this tail. Our working hypothesis will be thus that the different functions $\bar{F}^{(k)}$ (for $k=1,\ldots,K$) as well as $\bar{G}=1-G$ belong to the Fréchet maximum domain of attraction. In other words, we assume that they are (see Definition 1) regularly varying at infinity, with respective negative indices $-1/\gamma_1,\ldots,-1/\gamma_K$ and $-1/\gamma_C$

$$\forall 1\leqslant k\leqslant K,\quad \forall x>0,\ \lim_{t\to+\infty}\bar{F}^{(k)}(tx)/\bar{F}^{(k)}(t)=x^{-1/\gamma_k}\quad\text{ and }\quad\lim_{t\to+\infty}\bar{G}(tx)/\bar{G}(t)=x^{-1/\gamma_C}. \tag{1}$$

Consequently, $\bar{F}=1-F=\sum_{k=1}^K\bar{F}^{(k)}$ and $\bar{H}=\bar{F}\bar{G}$ (the survival function of Z) are regularly varying (at $+\infty$) with respective indices $-1/\gamma_F$ and $-1/\gamma$, where $\gamma_F=\max(\gamma_1,\ldots,\gamma_K)$ and γ satisfies $\gamma^{-1}={\gamma_F}^{-1}+{\gamma_C}^{-1}$.

The estimation of γ_F has been already studied in the literature, as it corresponds to the random (right) censoring framework, without competing risks. We can cite Beirlant et al. (2007) and Einmahl et al. (2008), where the authors propose to use consistent estimators of γ divided by the proportion of non-censored observations in the tail, or Worms and Worms (2014), where two Hill-type estimators are proposed for γ_F, based on survival analysis techniques. However, our target is γ_k (for a fixed $k=1,\ldots,K$) and the point is that there seems to be no way to deduce an estimator of γ_k from an estimator of γ_F. Note that the useful trick used in Beirlant et al. (2007) and Einmahl et al. (2008) to construct an estimator of γ_F does not seem to be extendable to this competing risks setting. To the best of our knowledge, our present paper is the first one addressing the problem of estimating the cause-specific extreme value index γ_k .

Considering assumption (1), it is simple to check that, for a given k, we have

$$\lim_{t\to+\infty} \frac{1}{\bar{F}^{(k)}(t)} \int_t^{+\infty} \log(u/t)\, dF^{(k)}(t) \;=\; \gamma_k$$

It is therefore most natural to propose the following (Hill-type) estimator of γ_k , for some given threshold value t_n (assumptions on this threshold are detailed in the next section) :

$$\widehat{\gamma}_{n,k} \;=\; \int \widehat{\phi}_n(u) dF_n^{(k)}(u) \quad \text{where} \quad \widehat{\phi}_n(u) \;=\; \frac{1}{\bar{F}_n^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n}$$

which can be also written as

$$\widehat{\gamma}_{n,k} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{i=1}^{n} \frac{\log(Z_i/t_n)}{\bar{G}_n(Z_i^-)} \mathbb{I}_{\xi_i=k} \mathbb{I}_{Z_i>t_n} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{Z_{(i)}>t_n} \frac{\log(Z_{(i)}/t_n)}{\bar{G}_n(Z_{(i-1)})} \delta_{(i)} \mathbb{I}_{\mathscr{C}_{(i)}=k},$$

where $Z_{(1)} \leqslant \ldots \leqslant Z_{(n)}$ are the ordered random variables associated to $Z_1, \ldots, Z_n$, and $\delta_{(i)}$ and $\mathscr{C}_{(i)}$ are the censoring indicator and cause number which correspond to the order statistic $Z_{(i)}$. Note that the only observations which are used to estimate γ_k are those above the threshold t_n. It is clear that this estimator is a generalisation of one of the estimators proposed in Worms and Worms (2014), in which the situation $K = 1$ (with only one cause of failure/death) was considered. The asymptotic result we prove in the present work is then valid in the situation studied in Worms and Worms (2014), where only consistency was proved and a random threshold was used.

Our paper is organized as follows: in Section 2, we state the asymptotic normality result of the proposed estimator. Section 5 is devoted to its proof. In Section 3, we present some simulations in order to illustrate finite sample behaviour of our estimator. Some technical aspects of the proofs are postponed to the Appendix.

2. Assumptions and Statement of the results

The Central Limit Theorem which is going to be proved has the rate $\sqrt{v_n}$ where $v_n = n\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)$ and t_n is a threshold tending to ∞ with the following constraint

$$v_n \overset{n\to\infty}{\longrightarrow} +\infty \quad \text{such that} \quad n^{-\eta_0} v_n \overset{n\to\infty}{\longrightarrow} +\infty \quad \text{for some} \;\; \eta_0 > 0. \tag{2}$$

If we note l_k the slowly varying function associated to $\bar{F}^{(k)}$ (*i.e.* such that $\bar{F}^{(k)}(x) = x^{-1/\gamma_k} l_k(x)$ in condition (1)), the second order condition we consider is the classical $SR2$ condition for l_k (see Bingham, Goldie and Teugels (1987)),

$$\forall x > 0, \; \frac{l_k(tx)}{l_k(t)} - 1 \overset{t\to\infty}{\sim} h_{\rho_k}(x)\; g(t) \quad (\forall x > 1), \tag{3}$$

where g is a positive measurable function, slowly varying with index $\rho_k \leqslant 0$, and $h_{\rho_k}(x) = \frac{x^{\rho_k}-1}{\rho_k}$ when $\rho_k < 0$, or $h_{\rho_k}(x) = \log x$ when $\rho_k = 0$.

Theorem 1. *Under assumptions* (1), (2) *and* (3), *if there exists* $\lambda \geqslant 0$ *such that* $\sqrt{v_n} g(t_n) \overset{n\to\infty}{\longrightarrow} \lambda$, *and if* $\gamma_k < \gamma_C$ *then we have*

$$\sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k) \;\xrightarrow{d}\; \mathscr{N}(\lambda m, \sigma^2) \qquad \text{as } n \to \infty$$

where

$$m = \begin{cases} \frac{\gamma_k^2}{1-\gamma_k\rho_k} & \text{if } \rho_k < 0, \\ \gamma_k^2 & \text{if } \rho_k = 0, \end{cases} \quad \text{and} \;\; \sigma^2 = \frac{\gamma_k^2}{(1-q)^3}\left((1+q^2) - 2cq\right),$$

with $c = \lim_{x\to\infty} \bar{F}^{(k)}(x)/\bar{F}(x) \in [0,1]$ *and* $q = \gamma_k/\gamma_C$.

Remark 1. *Note that when* $\gamma_k < \gamma_F$, *then* $c = 0$, *and, when* $\gamma_k = \gamma_F$ *and* $c = 1$ *(for instance when there is only one cause), then* σ^2 *reduces to* $\gamma_F^2/(1-q)$.

Proposition 1. *Under assumptions* (1) *and* (2), *we have*

$$\widehat{\gamma}_{n,k} \;\xrightarrow{\mathbb{P}}\; \gamma_k \qquad \text{as } n \to \infty.$$

Remark 2. *The condition* $\gamma_k < \gamma_C$ *(weak censoring) is not necessary for the consistency of* $\widehat{\gamma}_{n,k}$.

3. Simulations

In this section, a small simulation study is conducted in order to illustrate the finite-sample behaviour of our new estimator in some simple cases, and discuss the main issues associated with the competing risks setting.

For simplicity, we focus on the situation with two competing risks ($K = 2$), also called causes below. Data are generated from one of the following two models : for c_1, c_2 non-negative constants satisfying $c_1 + c_2 = 1$, we consider the following (sub-)distribution for each cause-specific function $\bar{F}^{(k)}$ ($k \in \{1, 2\}$) :

– Fréchet : $\bar{F}^{(k)}(t) = c_k \ \exp(-t^{-1/\gamma_k})$, for $t \geqslant 0$;

– Pareto : $\bar{F}^{(k)}(t) = c_k \ t^{-1/\gamma_k}$, for $t \geqslant 1$.

The lifetime X, of survival function $\bar{F} = \bar{F}^{(1)} + \bar{F}^{(2)}$, is generated by acceptance-rejection algorithm. Censoring times are then generated from a Fréchet or a Pareto distribution :

$$\bar{G}(t) = \exp(-t^{-1/\gamma_C}) \ (t \geqslant 0) \qquad \text{or} \qquad \bar{G}(t) = t^{-1/\gamma_C} \ (t \geqslant 1).$$

In this section, we consider (as it is often done in simulation studies) that the threshold t_n used in the definition of our new estimator $\hat{\gamma}_{n,1}$ is taken equal to $Z_{(n-k_n)}$ (i.e. we consider it as random). One aim of this section is to show how our estimator (with random threshold)

$$\hat{\gamma}_1 = \frac{1}{n\bar{F}_n^{(1)}(Z_{(n-k_n)})} \sum_{i=1}^{k_n} \frac{\log(Z_{(n-i+1)}/Z_{(n-k_n)})}{\bar{G}_n(Z_{(n-i,n)})} \delta_{(n-i+1)} \mathbb{I}_{\mathscr{C}_{(n-i+1)}=1}$$

of γ_1 behaves when the proportion c_1 of cause 1 events varies : we consider $c_1 \in \{1, 0.9, 0.7, 0.5\}$, the case $c_1 = 1$ corresponding to the simple censoring framework, without competing risk.

Another aim is to illustrate the impact of dependency between the causes, when estimating the tail. The starting point is that, if cause 2 could be considered independent of cause 1, then we could (and would) include it in the censoring mechanism and we would be in the simple random censoring setting, without competing risk. In this case, it would be possible to estimate γ_1 by one of the following two estimators, the first one being proposed in Beirlant et al. (2007) (a Hill estimator weighted with a constant weight), and the second one in Worms and Worms (2014) (a Hill estimator weighted with varying Kaplan-Meier weights):

$$\hat{\gamma}_1^{(BDFG)} = \hat{\gamma}_{Z,Hill}/\hat{p}_1 \qquad \text{where} \qquad \hat{p}_1 = \tfrac{1}{k_n} \textstyle\sum_{i=1}^{k_n} \delta_{(n-i+1)} \mathbb{I}_{\mathscr{C}_{(n-i+1)}=1} \ , \tag{4}$$

$$\hat{\gamma}_1^{(KM)} = \frac{1}{n\bar{F}_{n,b}(Z_{(n-k_n)})} \sum_{i=1}^{k_n} \frac{\log(Z_{(n-i+1)}/Z_{(n-k_n)})}{\bar{G}_{n,b}(Z_{(n-i,n)})} \delta_{(n-i+1)} \mathbb{I}_{\mathscr{C}_{(n-i+1)}=1}, \tag{5}$$

where the Kaplan Meier estimators $\bar{F}_{n,b}$ and $\bar{G}_{n,b}$ are based on the $\tilde{\delta}_i = \delta_i \mathbb{I}_{\mathscr{C}_i=1}$. These two estimators consider the uncensored lifetimes associated to cause 2 as independent censoring times. Comparing our new estimator with these latter two estimators, when $c_1 < 1$, will empirically prove that considering cause 2 as a competing risk independent of cause 1 has a great (negative) impact on the estimation of γ_1. Note that when $c_1 = 1$, the new estimator $\hat{\gamma}_1$ and $\hat{\gamma}_1^{(KM)}$ are exactly the same.

We address these two aims for each set-up (Fréchet, or Pareto), by generating 2000 datasets of size 500, with three configurations of the triplet $(\gamma_1, \gamma_2, \gamma_C)$: $(0.1, 0.25, 0.3)$ ($\gamma_1 < \gamma_2$, moderate censoring $\gamma_C > \gamma_F$), $(0.1, 0.25, 0.2)$ ($\gamma_1 < \gamma_2$, heavy censoring $\gamma_C < \gamma_F$), or $(0.25, 0.1, 0.45)$ ($\gamma_1 > \gamma_2$, moderate censoring $\gamma_C < \gamma_F$). Median bias and mean squared error (MSE) of the different estimators are plotted against different values of k_n, the number of excesses used.

Figures 1, 2 and 3 illustrate the behaviour of our estimator when c_1 varies. In terms of bias and MSE, we can see that the first configuration is a little better than the second one, which is itself much better than the third one. We observed this phenomenon in many other cases, not reported here : our estimator behaves best when it is the smallest parameter γ_k which is estimated, and when the censoring is not too strong. Our simulations also show that the quality of our estimator (especially in terms of the MSE) diminishes with c_1.

Figures 4, 5, 6 and 7, 8, 9 present the comparison between our new estimator and the ones described in (4) and (5). A general conclusion (confirmed by other simulations not reported here) is that $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ behave worse in most cases, even for a value of c_1 of 0.9, which is only a slight modification of the situation without competing risk ($c_1 = 1$). Therefore, a contamination of the cause 1 distribution by another cause rapidly yield inadequate estimations of γ_1 if dependency between causes is ignored ; this conclusion is

true for both $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$, but to a greater extent for $\hat{\gamma}_1^{(BDFG)}$. In the atypical case $(0.25, 0.1, 0.45)$, the improvement provided by $\hat{\gamma}_1$ becomes notable when c_1 drops below 0.7.

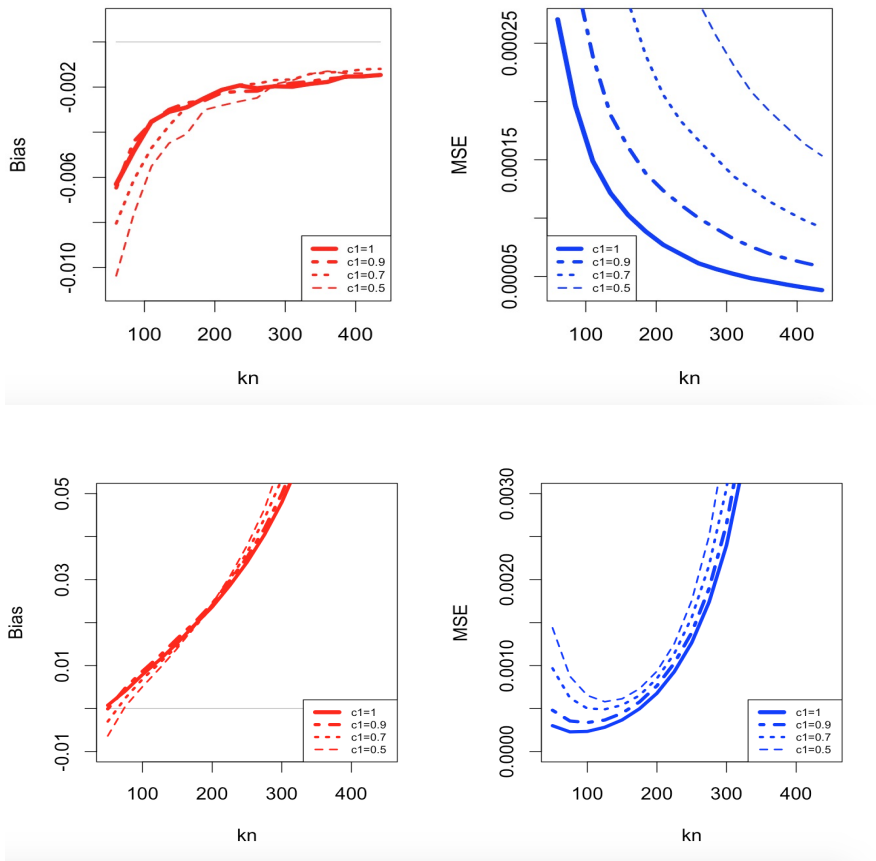

Figure 1: Bias and MSE of $\hat{\gamma}_1$ for different values of c_1, with $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_C = 0.3$: top : Pareto , bottom: Fréchet

4. Conclusion

In this paper, we consider heavy tailed lifetime data subject to random censoring and competing risks, and use the Aalen-Johansen estimator of the cumulative incidence function to construct an estimator for the extreme value index associated to the main cause of interest. Its asymptotic normality is proved and a small simulation study exhibiting its finite-sample performance shows that accounting for the dependency of the different causes is important, but that the bias can be particularly high. Estimating second order tail parameters would then be interesting in order to reduce this bias. A first step towards this aim is to study the following moments

$$M_n^{(\alpha)} = \frac{1}{n\bar{F}_n^{(k)}(t_n)} \sum_{i=1}^{n} \frac{\log^{\alpha}(Z_i/t_n)}{\bar{G}_n(Z_i^-)} \mathbb{I}_{\xi_i=k} \mathbb{I}_{Z_i>t_n},$$

which asymptotic behaviour can be derived following the same lines as in the proof of Theorem 1. The next direction of work would then be to exploit this methodology in order to estimate extreme quantiles of the cumulative incidence function.

5. Proofs

This Section is essentially devoted to the proof of the main Theorem 1. Some hints about the proof of the consistency result contained in Proposition 1 are given in the last Subsection 5.3.

We adopt a strategy developed by Stute in Stute (1995) in order to prove his Theorem 1.1, a well-known result which states that a Kaplan-Meier integral of the form $\int \phi \, dF_n$ can be approximated by a sum

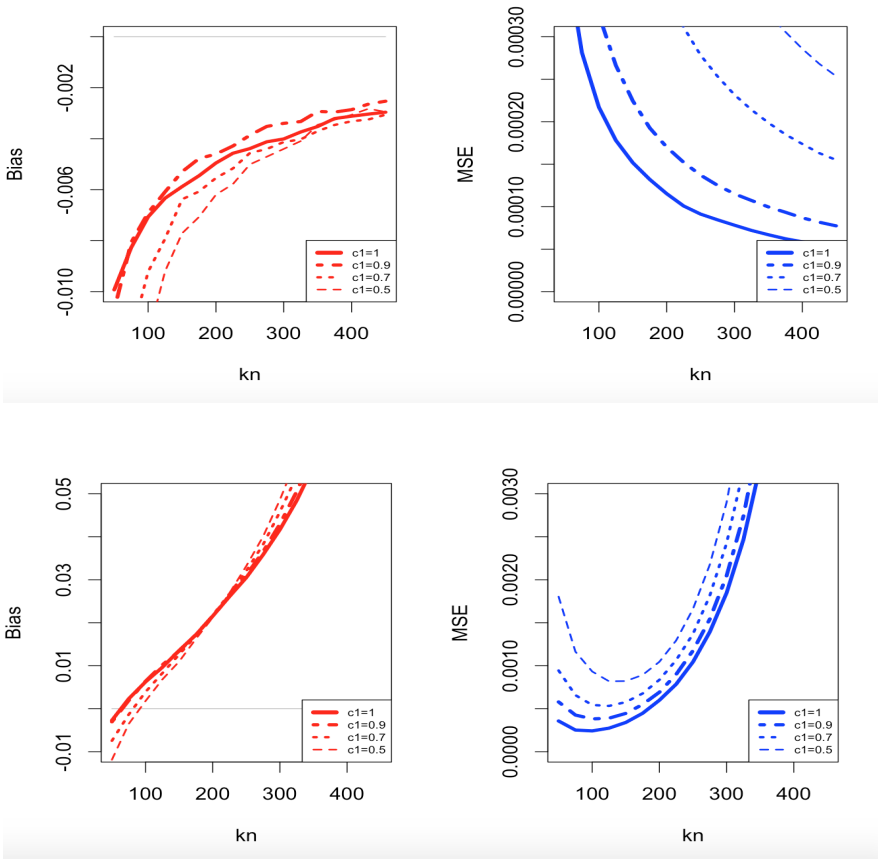

Figure 2: Bias ias and MSE of $\hat{\gamma}_1$ for different values of c_1, with $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_C = 0.2$: top : Pareto , bottom: Fréchet

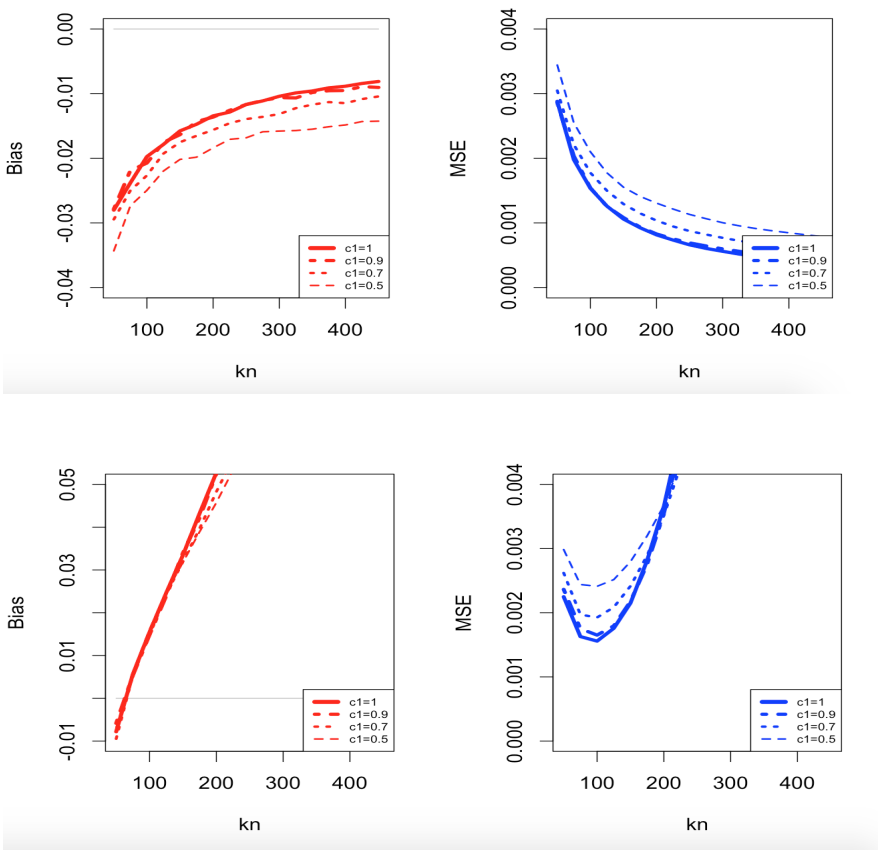

Figure 3: Bias ias and MSE of $\hat{\gamma}_1$ for different values of c_1, with $\gamma_1 = 0.25$, $\gamma_2 = 0.1$ and $\gamma_C = 0.45$: top : Pareto , bottom: Fréchet

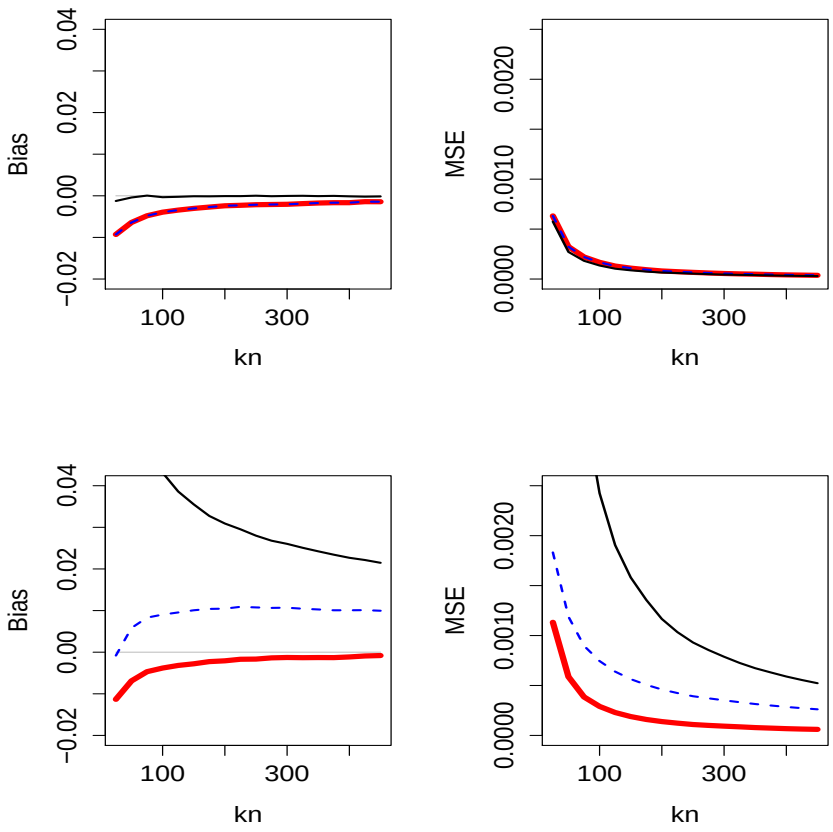

Figure 4: Comparison of bias and MSE for $\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dased) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Pareto distributions with $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_C = 0.30$

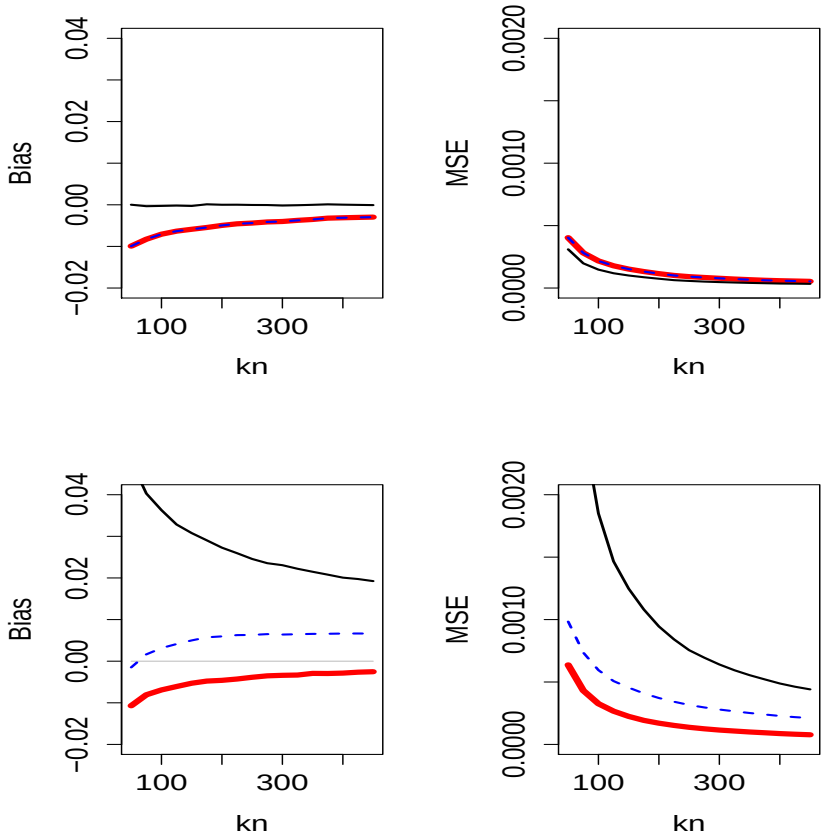

Figure 5: Comparison of bias and MSE for $\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dashed) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Pareto distributions $\gamma_1 = 0.10$, $\gamma_2 = 0.25$ and $\gamma_C = 0.2$

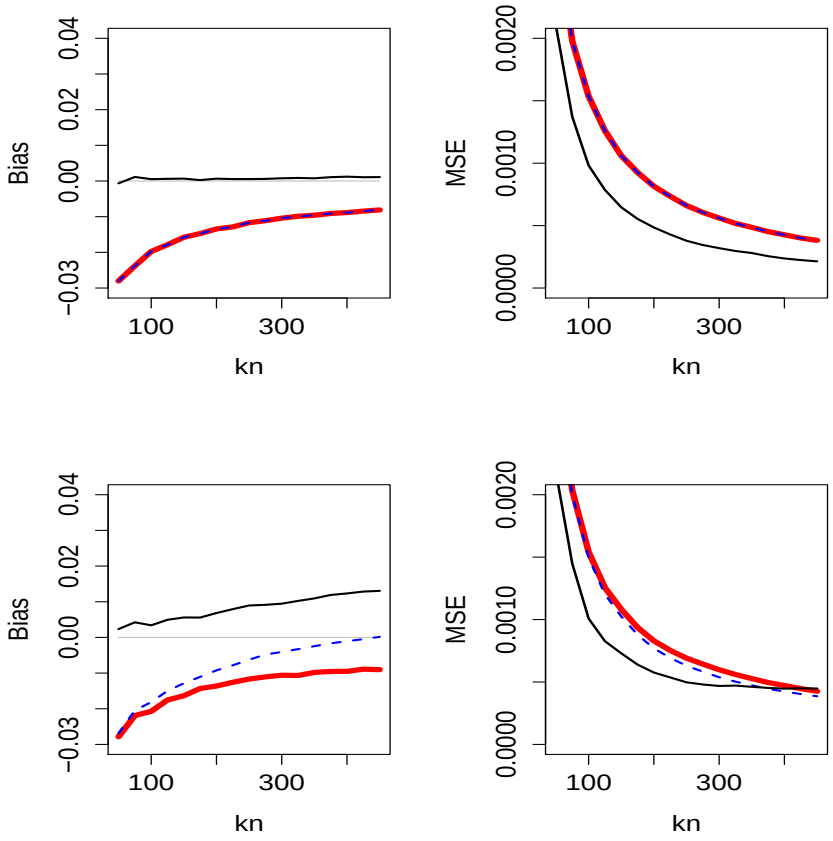

Figure 6: Comparison of bias and MSE for $\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dashed) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Pareto distributions $\gamma_1 = 0.25$, $\gamma_2 = 0.1$ and $\gamma_C = 0.45$

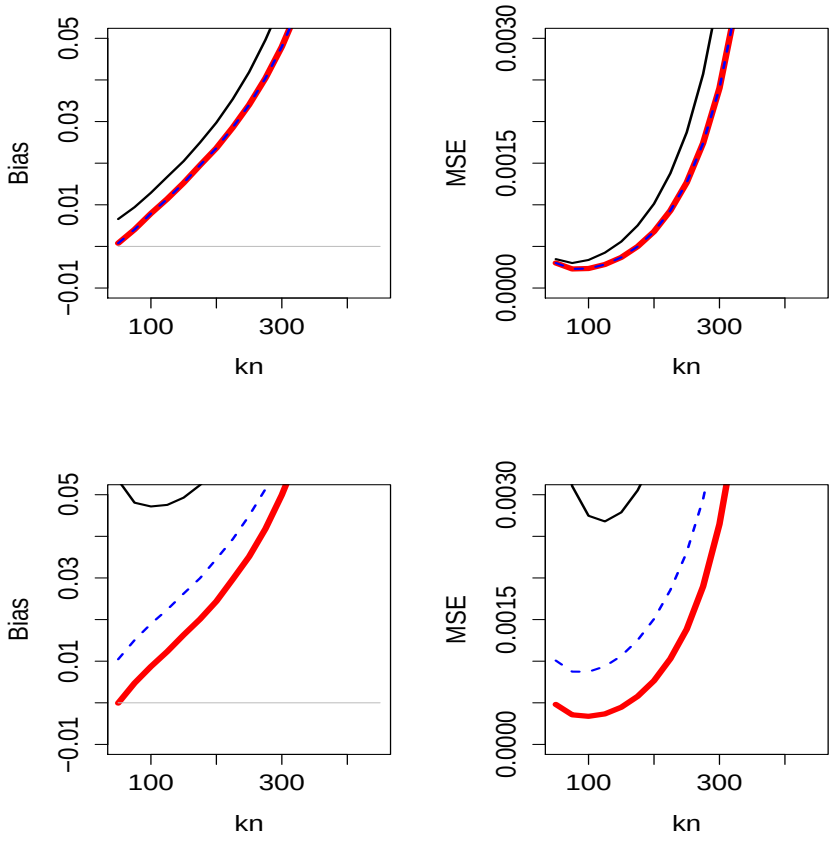

Figure 7: Comparison of bias and MSE for $\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dashed) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Fréchet distributions $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_C = 0.3$

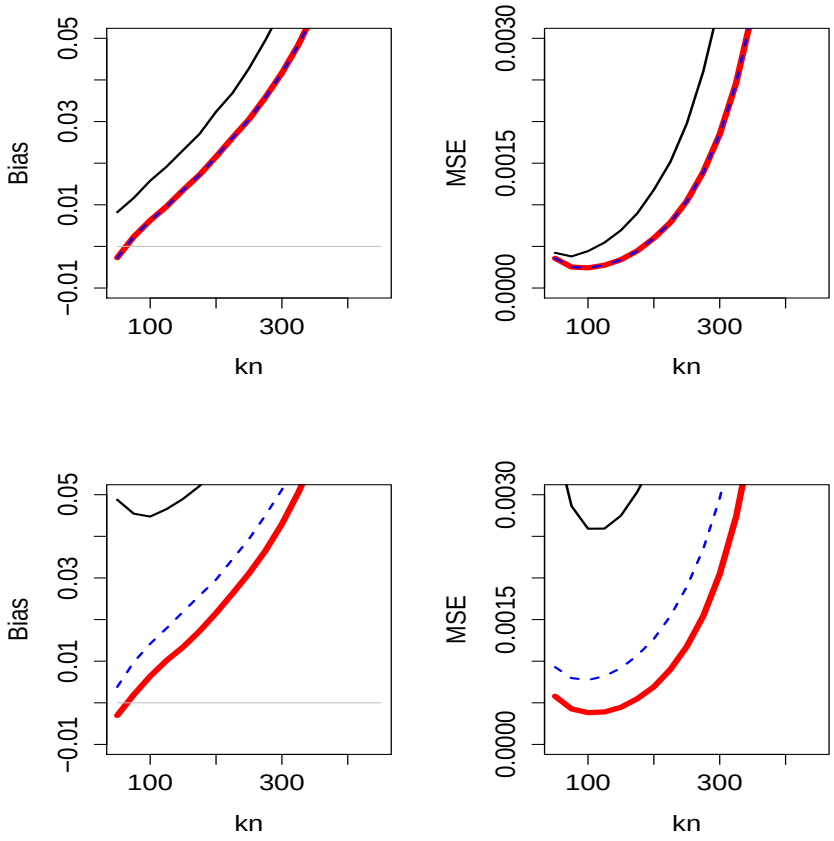

Figure 8: Comparison of bias and MSE for $\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dashed) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Fréchet distributions $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_C = 0.20$

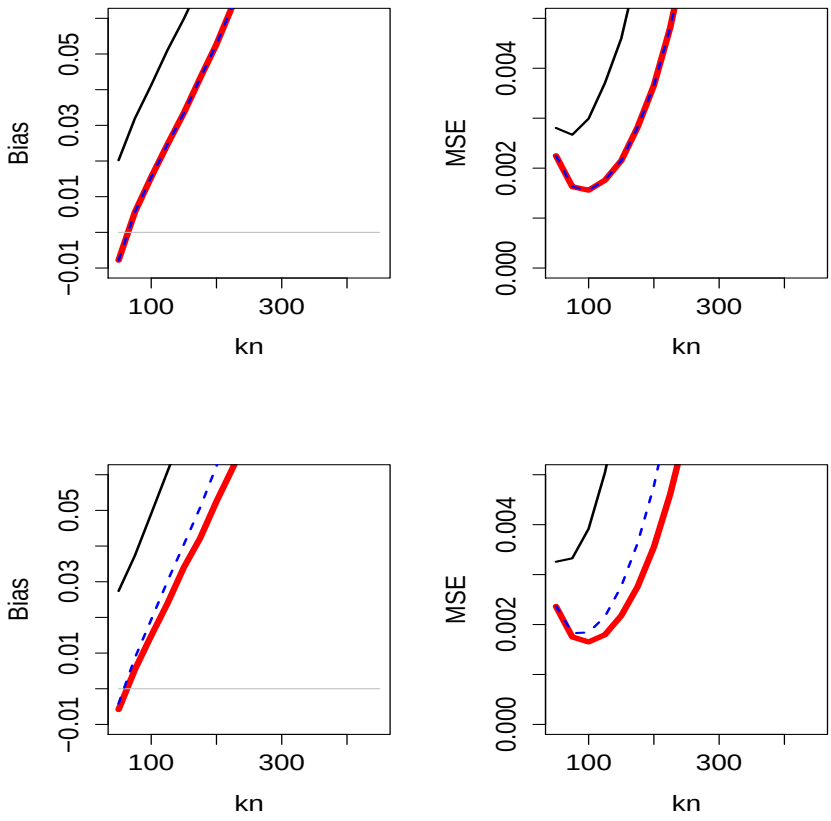

Figure 9: Comparison of bias and MSE for$\hat{\gamma}_1$ (thick), $\hat{\gamma}_1^{(BDFG)}$ and $\hat{\gamma}_1^{(KM)}$ (dashed) : top : $c_1 = 1$, bottom: $c_1 = 0.9$, for Fréchet distributions $\gamma_1 = 0.25$, $\gamma_2 = 0.1$ and $\gamma_C = 0.45$

of independent terms. This idea is used in Suzukawa (2002) in the context of competing risks. We thus intend to approximate $\hat{\gamma}_{n,k}$ by the integral $\int \phi_n \, dF_n^{(k)}$ of some deterministic function ϕ_n, with respect to the Aalen-Johansen estimator, and approximate this integral by the mean of independent variables $U_{i,n} + V_{i,n}$. However, a major difference with Stute (1995) or Suzukawa (2002) is that the function we integrate here, $\phi_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)} \log(u/t_n) \mathbb{I}_{u>t_n}$, is not only an unbounded function, depending on n, but it also has a "sliding" support $[t_n, +\infty[$, which therefore always include the endpoint of the distribution H. In Stute (1995), a crucial point of the proof consists in considering (at least temporarily) that the integrated function ϕ has a support which is bounded away from the endpoint of H (condition (2.3) there). Considering the kind of function ϕ_n we have to deal with here, we will not be able to make such an assumption. Dealing with the remainder terms is the most challenging part of this work. Finally, note that we will also have to consider simultaneously integrals (with respect to $F_n^{(k)}$) of another function g_n, defined below, which basically shares the same flaws as ϕ_n.

Let us first recall or define the following objects :

$$\begin{aligned}
\widehat{\phi}_n(u) &= \frac{1}{\bar{F}_n^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n} \\
\phi_n(u) &= \frac{1}{\bar{F}^{(k)}(t_n)} \log\left(\frac{u}{t_n}\right) \mathbb{I}_{u>t_n} \\
\gamma_{n,k} &= \int \phi_n(u) dF^{(k)}(u) \overset{n\to\infty}{\longrightarrow} \gamma_k \\
\widetilde{\gamma}_{n,k} &= \int \phi_n(u) dF_n^{(k)}(u) \\
\widehat{\gamma}_{n,k} &= \int \widehat{\phi}_n(u) dF_n^{(k)}(u)
\end{aligned}$$

We thus have $\widehat{\gamma}_{n,k} = \Delta_n^{-1} \widetilde{\gamma}_{n,k}$, where

$$\Delta_n = \bar{F}_n^{(k)}(t_n)/\bar{F}^{(k)}(t_n) = \int g_n(u) dF_n^{(k)}(u) \quad \text{and} \quad g_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)} \mathbb{I}_{u>t_n},$$

and we now introduce the following new quantities, related to the Stute-like decomposition of $\widetilde{\gamma}_{n,k}$ and Δ_n :

$$\begin{aligned}
U_{i,n}^{(1)} &= \frac{\phi_n(Z_i)}{\bar{G}(Z_i)} \delta_i \mathbb{I}_{\mathscr{C}_i=k} \quad \text{and} \quad V_{i,n}^{(1)} = \frac{g_n(Z_i)}{\bar{G}(Z_i)} \delta_i \mathbb{I}_{\mathscr{C}_i=k} \\
U_{i,n}^{(2)} &= \frac{1-\delta_i}{\bar{H}(Z_i)} \psi(\phi_n, Z_i) \quad \text{and} \quad V_{i,n}^{(2)} = \frac{1-\delta_i}{\bar{H}(Z_i)} \psi(g_n, Z_i) \\
U_{i,n}^{(3)} &= \int_0^{Z_i} \psi(\phi_n, u) \, dC(u) \quad \text{and} \quad V_{i,n}^{(3)} = \int_0^{Z_i} \psi(g_n, u) \, dC(u) \\
U_{i,n} &= U_{i,n}^{(1)} + U_{i,n}^{(2)} - U_{i,n}^{(3)} \quad \text{and} \quad V_{i,n} = V_{i,n}^{(1)} + V_{i,n}^{(2)} - V_{i,n}^{(3)}
\end{aligned}$$

where, for any function $f : \mathbb{R}_+ \to \mathbb{R}$, we note (for any given $z \geqslant 0$)

$$\psi(f, z) = \int_z^{+\infty} f(t) dF^{(k)}(t) \text{ and } C(z) = \int_0^z \frac{dG(t)}{\bar{H}(t)\bar{G}(t)}.$$

This enables us to finally define the important objects

$$\breve{\gamma}_{n,k} = \frac{1}{n} \sum_{i=1}^n U_{i,n} \quad \text{and} \quad \widehat{\Delta}_n = \frac{1}{n} \sum_{i=1}^n V_{i,n} \tag{6}$$

which are the triangular sums of independent terms which will respectively approximate $\widetilde{\gamma}_{n,k}$ and Δ_n. At the beginning of section 5.1, it will be proved that $\mathbb{E}(U_{i,n}^{(1)}) = \gamma_{n,k}$ and $\mathbb{E}(V_{i,n}^{(1)}) = 1$, while $\mathbb{E}(U_{i,n}^{(2)}) = \mathbb{E}(U_{i,n}^{(3)})$ and $\mathbb{E}(V_{i,n}^{(2)}) = \mathbb{E}(V_{i,n}^{(3)})$, yielding $\mathbb{E}(\breve{\gamma}_{n,k}) = \gamma_{n,k}$ and $\mathbb{E}(\widehat{\Delta}_n) = 1$, while the terms $U_{i,n}^{(2)}$, $U_{i,n}^{(3)}$, $V_{i,n}^{(2)}$ and $V_{i,n}^{(3)}$ only participate to the variance component of the estimator. The relation between all these quantities is made clearer in the following Lemma :

Lemma 1. *We have*

$$\sqrt{v_n}(\widehat{\gamma}_{n,k} - \gamma_k) = \Delta_n^{-1} \left(Z_n + \sqrt{v_n} R_n + \sqrt{v_n}(\gamma_{n,k} - \gamma_k) \right) \tag{7}$$

where

$$Z_n \;=\; \sqrt{v_n}\,\Big(\,(\breve{\gamma}_{n,k}-\gamma_{n,k})-\gamma_k(\widehat{\Delta}_n-1)\,\Big) \qquad \textit{and} \qquad R_n \;=\; (\widetilde{\gamma}_{n,k}-\breve{\gamma}_{n,k})-\gamma_k(\Delta_n-\widehat{\Delta}_n)$$

The proof of Lemma 1 is simple :

$$\begin{aligned}\sqrt{v_n}(\widehat{\gamma}_{n,k}-\gamma_k) &= \Delta_n^{-1}\sqrt{v_n}\,(\widetilde{\gamma}_{n,k}-\Delta_n\gamma_k)\\ &= \Delta_n^{-1}\sqrt{v_n}\,\{(\breve{\gamma}_{n,k}-\gamma_{n,k})+\gamma_k(1-\Delta_n)+(\widetilde{\gamma}_{n,k}-\breve{\gamma}_{n,k})+(\gamma_{n,k}-\gamma_k)\}\end{aligned}$$

which leads to the desired relation (7).

The main theorem thus becomes an immediate consequence of the following four results, the second one being the most difficult to establish.

Proposition 2. *Under condition* (1) *and assuming that*

$$v_n \overset{n\to\infty}{\longrightarrow} +\infty, \tag{8}$$

if $\gamma_k < \gamma_C$*, then*

$$Z_n \;\xrightarrow{\;d\;}\; \mathcal{N}(0,\sigma^2) \qquad \textit{as } n\to\infty$$

where σ^2 *is defined in Theorem 1.*

Proposition 3. *Under conditions* (1) *and* (2) *, if* $\gamma_k < \gamma_C$*, then*

$$\widetilde{\gamma}_{n,k} = \breve{\gamma}_{n,k} + R_{n,\phi} \qquad \textit{and} \qquad \Delta_n = \widehat{\Delta}_n + R_{n,g} \tag{9}$$

where $R_{n,\phi}$*,* $R_{n,g}$ *(and consequently* R_n *too) are* $o_{\mathbb{P}}(v_n^{-1/2})$*.*

Lemma 2. *Under conditions* (1)*,* (3) *and* $\sqrt{v_n}g(t_n)\to\lambda\geqslant 0$*, the bias term* $\sqrt{v_n}(\gamma_{n,k}-\gamma_k)$ *in (7) converges to* λm *as* $n\to\infty$*, where* m *is defined in Theorem 1.*

Lemma 3. *Under the conditions of Proposition 3,* Δ_n *converges to* 1 *in probability as* $n\to\infty$*.*

Propositions 2 and 3 will be proved in Sections 5.1, 5.2 respectively, sometimes with the help of other results stated and established in the Appendix. The proofs of Lemmas 2 and 3 are short, we state them below.

Concerning Lemma 2, recall that $\gamma_{n,k} = \int \phi_n(u)dF_n^{(k)}(u)$. An integration by parts and the fact that $\bar{F}^{(k)}(x) = x^{-1/\gamma_k}l_k(x)$ yields

$$\sqrt{v_n}(\gamma_{n,k}-\gamma_k) = \sqrt{v_n}\int_1^{+\infty} y^{-1/\gamma_k-1}\left(\frac{l_k(yt_n)}{l_k(t_n)}-1\right)dy,$$

and, using assumption (3) and Proposition 3.1 in de Haan and Ferreira (2006), we can write

$$\int_1^{+\infty} y^{-1/\gamma_k-1}\left(\frac{l_k(yt_n)}{l_k(t_n)}-1\right)dy = g(t_n)\int_1^{+\infty} y^{-1/\gamma_k-1}h_{\rho_k}(y)dy + o(g(t_n)).$$

The result then follows from assumption $\sqrt{v_n}g(t_n)\to\lambda\geqslant 0$ and the fact that $\int_1^{+\infty} y^{-1/\gamma_k-1}h_{\rho_k}(y)dy = m$.

Concerning Lemma 3, once the proof of Proposition 2 has been gone through, it will become clear to the reader that $\hat{\Delta}_n$ converges to 1 in probability, and therefore Δ_n as well (since Proposition 3 states that $\Delta_n = \hat{\Delta}_n + R_{n,g}$ where $R_{n,g}$ tends to 0 in probability).

In the rest of the paper, we will have a heavy use of the well-known sub-distibutions functions $H^{(0)}$ and $H^{(1,k)}$ defined, for all $t\geqslant 0$, by

$$H^{(0)}(t) = \mathbb{P}(Z\leqslant t, \delta = 0) \text{ and } H^{(1,k)}(t) = \mathbb{P}(Z\leqslant t, \xi = k).$$

Note that we have

$$dH^{(0)} = \bar{F}dG \text{ and } H^{(1,k)} = \bar{G}dF^{(k)}.$$

5.1. Proof of Proposition 2

We first write

$$Z_n \;=\; \sum_{i=1}^{n} W_{i,n} \quad \text{where} \quad W_{i,n} \;=\; \frac{\sqrt{v_n}}{n}\left(\tilde{U}_{i,n}-\gamma_k\tilde{V}_{i,n}\right) \qquad \text{and} \qquad \begin{cases}\tilde{U}_{i,n} = U_{i,n}-\gamma_{n,k}\\ \tilde{V}_{i,n} = V_{i,n}-1\end{cases}$$

where $W_{i,n}$, $\tilde{U}_{i,n}$ and $\tilde{V}_{i,n}$ are centred, because the random variables $U_{i,n}$ and $V_{i,n}$ have expectations respectively equal to $\gamma_{n,k}$ and 1. Indeed, we have

$$\mathbb{E}\left(U_{i,n}^{(1)}\right) = \mathbb{E}\left(\frac{\phi_n(Z_i)}{G(Z_i)}\delta_i \mathbb{I}_{\mathscr{C}_i=k}\right) = \int \frac{\phi_n(u)}{G(u)} dH^{(1,k)}(u) = \int \phi_n(u) dF^{(k)}(u) = \gamma_{n,k}$$

and

$$\mathbb{E}\left(U_{i,n}^{(2)}\right) = \mathbb{E}\left(\frac{1-\delta_i}{\bar{H}(Z_i)}\psi(\phi_n, Z_i)\right) = \int\int \frac{1}{\bar{H}(u)} \mathbb{I}_{t>u} \phi_n(t)\, dF^{(k)}(t)\, dH^{(0)}(u) = \int\int \frac{1}{\bar{G}(u)} \mathbb{I}_{t>u} \phi_n(t)\, dF^{(k)}(t)\, dG(u)$$

as well as

$$\mathbb{E}\left(U_{i,n}^{(3)}\right) = \mathbb{E}\left(\int_0^{Z_i} \psi(\phi_n, u)\, dC(u)\right) = \int\int\int \mathbb{I}_{z>u}\mathbb{I}_{t>u} \phi_n(t)\, dH(z)\, dF^{(k)}(t) \frac{dG(u)}{\bar{G}(u)\bar{H}(u)} = \mathbb{E}\left(U_{i,n}^{(2)}\right).$$

The proof for $\mathbb{E}(V_{i,n}) = 1$ is similar.

We will now prove the asymptotic normality of Z_n by using the Lyapunov criteria.

Lemma 4. *Under the conditions* (1) *and* (8) *, if* $\gamma_k < \gamma_C$ *:*

(*i*) *we have*

$$\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) = \mathbb{E}(U_{1,n}^2) + \gamma_k^2 \mathbb{E}(V_{1,n}^2) - 2\gamma_k \mathbb{E}(U_{1,n}V_{1,n}) + o(1) \tag{10}$$

(*ii*) *we have*

$$\mathbb{E}(U_{1,n}^2) = \int \frac{\phi_n^2(u)}{\bar{G}(u)} dF^{(k)}(u) - \int (\psi(\phi_n, u))^2\, dC(u) \tag{11}$$

$$\mathbb{E}(V_{1,n}^2) = \int \frac{g_n^2(u)}{\bar{G}(u)} dF^{(k)}(u) - \int (\psi(g_n, u))^2\, dC(u) \tag{12}$$

$$\mathbb{E}(U_{1,n}V_{1,n}) = \int \frac{\phi_n(u) g_n(u)}{\bar{G}(u)} dF^{(k)}(u) - \int \psi(\phi_n, u)\psi(g_n, u)\, dC(u) \tag{13}$$

(*iii*) *we have*

$$\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) = \frac{1}{\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)} \left(\frac{\gamma_k^2(1+q^2)}{(1-q)^3} + o(1)\right) - \frac{1-p}{\bar{H}(t_n)} \left(\frac{2\gamma_k^3}{\gamma(2-\gamma_k/\gamma)^3} + o(1)\right) + o(1) \tag{14}$$

Lemma 5. *Under the conditions* (1) *and* (8) *, if* $\gamma_k < \gamma_C$*, then*

$$\sum_{i=1}^n \mathbb{E}\,|W_{i,n}|^{2+\delta} \to 0, \text{ as } n \text{ tends to infinity, for some } \delta > 0.$$

We can then immediately prove Proposition 2. Indeed, since $Z_n = \sum_{i=1}^n W_{i,n}$, Lemma 4 yields

$$\mathbb{V}ar(Z_n) = n\mathbb{V}ar(W_{1,n}) = \frac{v_n}{n}\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}).$$

which, since $v_n = n\bar{F}_n^{(k)}(t_n)\bar{G}(t_n)$, becomes

$$\mathbb{V}ar(Z_n) = \gamma_k^2(1+q^2)(1-q)^{-3} - \left(2(1-p)\gamma_k^3\gamma^{-1}(2-\gamma_k/\gamma)^{-3}\right)\left(\bar{F}^{(k)}(t_n)/\bar{F}(t_n)\right) + o(1).$$

Therefore, depending on the limit c of the ratio $\bar{F}^{(k)}(t_n)/\bar{F}(t_n)$ when $n \to \infty$ (for instance, it converges to 0 when $\gamma_k < \gamma_F$), it is simple to check that the variance of Z_n converges to the value σ^2 described in the statement of Theorem 1. Thanks to Lemma 5, the Lyapunov CLT applies and Proposition 2 is proved.

The two subsections 5.1.1 and 5.1.2 are now respectively devoted to the proofs of Lemmas 4 and 5.

5.1.1. Proof of Lemma 4

Part (*i*) of the lemma is straightforward : since $\tilde{U}_{1,n}$ and $\tilde{V}_{1,n}$ are centred, we have indeed

$$\begin{aligned}\mathbb{V}ar(\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n}) &= \mathbb{E}((\tilde{U}_{1,n} - \gamma_k \tilde{V}_{1,n})^2) = \mathbb{E}(\tilde{U}_{1,n}^2) + \gamma_k^2 \mathbb{E}(\tilde{V}_{1,n}^2) - 2\gamma_k \mathbb{E}(\tilde{U}_{1,n}\tilde{V}_{1,n}) \\ &= (\mathbb{E}(U_{1,n}^2) - \gamma_{n,k}^2) + \gamma_k^2(\mathbb{E}(V_{1,n}^2) - 1) - 2\gamma_k(\mathbb{E}(U_{1,n}V_{1,n}) - \gamma_{n,k})\end{aligned}$$

and the result comes by using the fact that $\gamma_{n,k}$ converges to γ_k as $n \to \infty$.

Now we proceed to the proof of part (ii), and will only prove (11) because, by definition of ϕ_n and g_n, the proofs for (12) and (13) will be completely similar. First of all, we obviously have

$$(U_{1,n})^2 = (U_{1,n}^{(1)})^2 + (U_{1,n}^{(2)})^2 + (U_{1,n}^{(3)})^2 + 2U_{1,n}^{(1)}U_{1,n}^{(2)} - 2U_{1,n}^{(1)}U_{1,n}^{(3)} - 2U_{1,n}^{(2)}U_{1,n}^{(3)} \tag{15}$$

The first term in the right-hand side of (11) is equal to $\mathbb{E}((U_{1,n}^{(1)})^2)$, and the second one (without the minus sign) is equal to $\mathbb{E}((U_{1,n}^{(2)})^2)$ and to $\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(3)})$ because

$$\mathbb{E}((U_{1,n}^{(2)})^2) = \mathbb{E}\left(\frac{1-\delta_1}{\bar{H}^2(Z_1)}(\psi(\phi_n, Z_1))^2\right) = \int \frac{1}{\bar{H}^2(z)}(\psi(\phi_n, z))^2\, dH^{(0)}(z) = \int (\psi(\phi_n, z))^2\, dC(z)$$

and

$$\begin{aligned}\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(3)}) &= \int \frac{\phi_n(z)}{\bar{G}(z)}\left(\int_0^z \psi(\phi_n, u)dC(u)\right) dH^{(1,k)}(z) \\ &= \int \psi(\phi_n, u)\left(\int_u^\infty \phi_n(z)dF^{(k)}(z)\right) dC(u) \;=\; \int (\psi(\phi_n, z))^2\, dC(z)\end{aligned}$$

The expectation $\mathbb{E}(U_{1,n}^{(1)}U_{1,n}^{(2)})$ equals 0 because $\delta_1(1-\delta_1)$ is constantly 0, and we are now going to prove that $\mathbb{E}((U_{1,n}^{(3)})^2) = 2\mathbb{E}(U_{1,n}^{(2)}U_{1,n}^{(3)})$, which ends the proof of (11) in view of (15). Indeed, noting $h(z) = \int_0^z \psi(\phi_n, u)dC(u)$ and using the simple fact that $h(z) = h(y) + \int_y^z \psi(\phi_n, u)dC(u)$ for every $y < z$, we have

$$\begin{aligned}\mathbb{E}((U_{1,n}^{(3)})^2) &= \int_0^\infty \left(\int_0^z \psi(\phi_n, y)h(z)dC(y)\right) dH(z) \\ &= \int_0^\infty \left(\int_0^z \psi(\phi_n, y)h(y)dC(y)\right) dH(z) \;+\; \int_0^\infty \left\{\int_0^z \left(\int_y^z \psi(\phi_n, u)dC(u)\right)\psi(\phi_n, y)dC(y)\right\} dH(z) \\ &= \int_0^\infty \left(\int_0^z \psi(\phi_n, y)h(y)dC(y)\right) dH(z) \;+\; \int_0^\infty \left\{\int_0^z \left(\int_0^u \psi(\phi_n, y)dC(y)\right)\psi(\phi_n, u)dC(u)\right\} dH(z) \\ &= 2\int_0^\infty \left(\int_0^z \psi(\phi_n, y)h(y)dC(y)\right) dH(z)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(U_{1,n}^{(2)}U_{1,n}^{(3)}) &= \mathbb{E}\left(\frac{1-\delta_1}{\bar{H}(Z_1)}\psi(\phi_n, Z_1)\, h(Z_1)\right) \;=\; \int \bar{H}(y)h(y)\psi(\phi_n, y)\frac{dG(y)}{\bar{H}(y)\bar{G}(y)} \\ &= \int_0^\infty \left(\int_y^\infty dH(z)\right) h(y)\psi(\phi_n, y)\, dC(y) \\ &= \int_0^\infty \left(\int_0^z \psi(\phi_n, y)h(z)dC(y)\right) dH(z) \;=\; \frac{1}{2}\mathbb{E}((U_{1,n}^{(3)})^2)\end{aligned}$$

as announced.

We can now start proving part (iii) of the lemma, in which the exact nature of the function ϕ_n matters. let us defines the constant c_j and d_j $(j = 0, 1, 2)$ by

$$c_j = \frac{j!\gamma_k^j}{(1-q)^{j+1}} \quad d_j = \frac{j!\gamma_k^{j+1}}{\gamma(2-\gamma_k/\gamma)^{j+1}}.$$

According to Lemma 8 part (ii) ($a + b = 1/\gamma_C - 1/\gamma_k < 0$, for c_j and $a + b = -2/\gamma_k + 1/\gamma = (1/\gamma_C - 1/\gamma_k) + (1/\gamma_F - 1/\gamma_k) < 0$ for d_j) , we have

$$\int_{t_n}^\infty \log^j\left(\frac{u}{t_n}\right)\frac{dF^{(k)}(u)}{\bar{G}(u)} \;\sim\; c_j\frac{\bar{F}^{(k)}(t_n)}{\bar{G}(t_n)} \text{ and } \int_{t_n}^\infty \log^j\left(\frac{u}{t_n}\right)\left(\frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)}\right)^2\frac{dC(u)}{C(t_n)} \overset{n\to\infty}{\longrightarrow} d_j. \tag{16}$$

Hence, by definition of ϕ_n, g_n, the first terms of $\mathbb{E}(U_{1,n}^2)$, $\mathbb{E}(V_{1,n}^2)$ and $\mathbb{E}(U_{1,n}V_{1,n})$ in relations (11), (12) and (13) are respectively equivalent (as $n \to \infty$) to $c_2D(t_n)$, $c_0D(t_n)$ and $c_1D(t_n)$ where $D(t_n)$ denotes

$$D(t_n) = \frac{1}{\bar{F}^{(k)}(t_n)\bar{G}(t_n)}.$$

Since $c_2 + \gamma_k^2 c_0 - 2\gamma_k c_1$ is equal to $\gamma_k^2(1+q^2)/(1-q)^3$, in view of (10) this proves the first term in relation (14). We now need to obtain equivalent expressions for the quantities $\int(\psi(\phi_n, u))^2\, dC(u)$, $\int(\psi(g_n, u))^2\, dC(u)$

and $\int \psi(\phi_n,u)\psi(g_n,u)\,dC(u)$ in order to prove the second part of relation (14) and therefore finish the proof of Lemma 4.

For saving space, we will use temporarily the following notations :

$$l_n(u) = \log\left(u/t_n\right) \quad , \quad R_n(u) = \bar{F}^{(k)}(u)/\bar{F}^{(k)}(t_n)$$

According to the technical Lemma 10 of the Appendix and, after splitting the integral into $\int_0^{+\infty}$ and $\int_{t_n}^{+\infty}$, we can write

$$\begin{aligned}\int (\psi(\phi_n,u))^2\,dC(u) &= \gamma_{n,k}^2 \int_0^{t_n} dC(u) \\ &+ \int_{t_n}^{+\infty} \Big(l_n^2(u)R_n^2(u) + \gamma_k^2\left(u/t_n\right)^{-2/\gamma_k} + 2\gamma_k l_n(u)R_n(u)\left(u/t_n\right)^{-1/\gamma_k}\Big) dC(u) + o(C(t_n)), \quad (17)\end{aligned}$$

where $o(C(t_n))$ in (17) is due to part (ii) of Lemma 8 and to the fact that $\epsilon_n(u)$ in Lemma 10 converges to 0 uniformly in u. According to (16), we thus have

$$\int (\psi(\phi_n,u))^2\,dC(u) = (\gamma_k^2 + d_2 + \gamma_k^2 d_0 + 2\gamma_k d_1)C(t_n) + o(C(t_n)) \quad (18)$$

The other terms are treated similarly (using the fact that $\psi(g_n,u) = 1$ when $u \leqslant t_n$, and $= \bar{F}^{(k)}(u)/\bar{F}^{(k)}(t_n)$ when $u > t_n$) and we obtain

$$\int (\psi(g_n,u))^2\,dC(u) = (1+d_0)\,C(t_n) + o(C(t_n)), \quad (19)$$

$$\int \psi(\phi_n,u)\psi(g_n,u)\,dC(u) = (\gamma_k + d_1 + \gamma_k d_0)\,C(t_n) + o(C(t_n)). \quad (20)$$

In view of (10), combining (18), (19) and (20) and using Remark 3 to write that $C(t_n) \sim (1-p)/\bar{H}(t_n)$ (as $n \to \infty$) proves the second term in relation (14).

5.1.2. Proof of Lemma 5

We have to prove that, for some $\delta > 0$ small enough, $n\mathbb{E}\left|W_{1,n}\right|^{2+\delta}$ tends to 0, as $n \to \infty$. In the sequel, *cst* denotes an unspecified absolute positive constant. According to the definition of $W_{1,n}$, it is clear that

$$n\left|W_{1,n}\right|^{2+\delta} \leqslant cst\ \frac{v_n^{1+\delta/2}}{n^{1+\delta}} \left(\sum_{j=1}^{3} \left|U_{1,n}^{(j)}\right|^{2+\delta} + \gamma_k^{2+\delta} \sum_{j=1}^{3} \left|V_{1,n}^{(j)}\right|^{2+\delta} + \left|\gamma_{n,k} - \gamma_k\right|^{2+\delta} \right)$$

First, we clearly have $n^{-1-\delta}\ v_n^{1+\delta/2}\left|\gamma_{n,k} - \gamma_k\right|^{2+\delta} \longrightarrow 0$ as $n \to \infty$. Secondly, since $V_{1,n}^{(j)}$ has the same form as $U_{1,n}^{(j)}$, with g_n instead of ϕ_n (*i.e.* without the log factor), we will only prove that there exists some $\delta > 0$ such that, as $n \to \infty$,

$$n^{-1-\delta}\ v_n^{1+\delta/2}\mathbb{E}\left|U_{1,n}^{(j)}\right|^{2+\delta} = n^{-\delta/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1+\delta/2}\mathbb{E}\left|U_{1,n}^{(j)}\right|^{2+\delta} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for } j \in \{1,2,3\} \quad (21)$$

For $j = 1$, we have

$$\begin{aligned}\mathbb{E}\left|U_{1,n}^{(1)}\right|^{2+\delta} &= \int_0^{+\infty} \left|\frac{\phi_n(z)}{\bar{G}(z)}\right|^{2+\delta} dH^{(1,k)}(z) = (\bar{F}^{(k)}(t_n))^{-2-\delta} \int_{t_n}^{+\infty} (\bar{G}(z))^{-2-\delta} \left(\log(z/t_n)\right)^{2+\delta} \bar{G}(z) dF^{(k)}(z) \\ &= \left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{-1-\delta} \int_{t_n}^{+\infty} \left(\log(z/t_n)\right)^{2+\delta} \left(\frac{\bar{G}(t_n)}{\bar{G}(z)}\right)^{1+\delta} \frac{dF^{(k)}(z)}{\bar{F}^{(k)}(t_n)}.\end{aligned}$$

Applying part (ii) of Lemma 8 for $\alpha = 2+\delta$, $a = (1+\delta)/\gamma_C$ and $b = -1/\gamma_k$ (with δ sufficiently small so that $a + b = 1/\gamma_C - 1/\gamma_k + \delta/\gamma_C$ is kept < 0), and using the fact that $v_n = n\bar{F}^{(k)}(t_n)\bar{G}(t_n) \to \infty$, this ends the proof of (21) for $j = 1$.

For $j = 2$, we have

$$\mathbb{E}\left|U_{1,n}^{(2)}\right|^{2+\delta} = \int_0^{+\infty} \left|\frac{\psi(\phi_n,z)}{\bar{H}(z)}\right|^{2+\delta} \bar{F}(z) dG(z).$$

By definition of ψ, ϕ_n, and $\gamma_{n,k}$, we have $\psi(\phi_n,z) = \gamma_{n,k}$ when $z \leqslant t_n$. Therefore, splitting the integral

above into two integrals $\int_0^{t_n}$ and $\int_{t_n}^{+\infty}$ we obtain

$$\mathbb{E}\left|U_{1,n}^{(2)}\right|^{2+\delta} = I_1(t_n) + I_2(t_n),$$

where, on one hand,

$$I_1(t_n) \;=\; (\gamma_{n,k})^{2+\delta}\int_0^{t_n}\frac{\bar F(z)dG(y)}{(\bar H(z))^{2+\delta}} \;=\; (\gamma_{n,k})^{2+\delta}\int_0^{t_n}\frac{dC(z)}{(\bar H(z))^{\delta}} \;\leqslant\; (\gamma_{n,k})^{2+\delta}\frac{C(t_n)}{(\bar H(t_n))^{\delta}}$$

and, on the other hand, using the technical Lemma 10, for some $\delta' > 0$,

$$\begin{aligned}
I_2(t_n) \;&\leqslant\; \int_{t_n}^{+\infty}\left|\log\left(\frac{z}{t_n}\right)\frac{\bar F^{(k)}(z)}{\bar F^{(k)}(t_n)} + \gamma_k\left(\frac{z}{t_n}\right)^{-1/\gamma_k} + \epsilon_n(u)\left(\frac{z}{t_n}\right)^{-1/\gamma_k+\delta'}\right|^{2+\delta}\frac{\bar F(z)dG(z)}{(\bar H(z))^{2+\delta}}\\
&\leqslant\; cst\left\{\int_{t_n}^{+\infty}\log^{2+\delta}\left(\frac{z}{t_n}\right)\left(\frac{\bar F^{(k)}(z)}{\bar F^{(k)}(t_n)}\right)^{2+\delta}\frac{dC(z)}{(\bar H(z))^{\delta}}\right.\\
&\qquad + \;\gamma_k^{2+\delta}\int_{t_n}^{+\infty}\left(\frac{z}{t_n}\right)^{-(2+\delta)/\gamma_k}\frac{dC(z)}{(\bar H(z))^{\delta}} \;+\; \sup_{u>t_n}|\epsilon_n(u)|^{2+\delta}\int_{t_n}^{+\infty}\left(\frac{z}{t_n}\right)^{(-1/\gamma_k+\delta')(2+\delta)}\left.\frac{dC(z)}{(\bar H(z))^{\delta}}\right\}\\
&=\; cst\frac{C(t_n)}{(\bar H(t_n))^{\delta}}\left\{\int_{t_n}^{+\infty}\log^{2+\delta}\left(\frac{z}{t_n}\right)\left(\frac{\bar F^{(k)}(z)}{\bar F^{(k)}(t_n)}\right)^{2+\delta}\left(\frac{\bar H(t_n)}{\bar H(z)}\right)^{\delta}\frac{dC(z)}{C(t_n)}\right.\\
&\qquad + \gamma_k^{2+\delta}\int_{t_n}^{+\infty}\left(\frac{z}{t_n}\right)^{-(2+\delta)/\gamma_k}\left(\frac{\bar H(t_n)}{\bar H(z)}\right)^{\delta}\frac{dC(z)}{C(t_n)} + o(1)\int_{t_n}^{+\infty}\left(\frac{z}{t_n}\right)^{(-1/\gamma_k+\delta')(2+\delta)}\left(\frac{\bar H(t_n)}{\bar H(z)}\right)^{\delta}\left.\frac{dC(z)}{C(t_n)}\right\}
\end{aligned}$$

Applying Lemma 9 to $\delta = 1$, we have $C(t_n) = O\left(1/\bar H(t_n)\right)$, therefore $I_1(t_n) = O\left((\bar H(t_n))^{-1-\delta}\right)$. It is then easy to check that $n^{-\delta/2}\left(\bar F^{(k)}(t_n)\bar G(t_n)\right)^{1+\delta/2}\; I_1(t_n)$ tends to 0, because $\bar F^{(k)} \leqslant \bar F$ and $n\bar H(t_n) \to \infty$, since $\bar H(t_n) \geqslant \bar F^{(k)}(t_n)\bar G(t_n)$.

For $I_2(t_n)$, since by Lemma 9 the function C is regularly varying with index $1/\gamma$, the application of part (ii) of Lemma 8 to $\alpha = 0$ or $2+\delta$ and to various couples of values of a and b finally yields $I_2(t_n) = O\left((\bar H(t_n))^{-1-\delta}\right)$, and consequently $n^{-\delta/2}\left(\bar F^{(k)}(t_n)\bar G(t_n)\right)^{1+\delta/2}\; I_2(t_n)$ tends to 0.

We now come to the study of relation (21) for $j = 3$. We have

$$\mathbb{E}\left|U_{1,n}^{(3)}\right|^{2+\delta} = \int_0^{+\infty}\left(\int_0^z \psi(\phi_n,u)dC(u)\right)^{2+\delta}dH(z).$$

Proceeding as above by splitting the integral into two integrals $\int_0^{t_n}$ and $\int_{t_n}^{+\infty}$, we obtain

$$\mathbb{E}\left|U_{1,n}^{(3)}\right|^{2+\delta} = J_1(t_n) + J_2(t_n),$$

where

$$J_1(t_n) \;=\; (\gamma_{n,k})^{2+\delta}\int_0^{t_n}(C(z))^{2+\delta}dH(z) \;=\; -(\gamma_{n,k})^{2+\delta}\bar H(t_n)\,(C(t_n))^{2+\delta}\int_0^{t_n}\left(\frac{C(z)}{C(t_n)}\right)^{2+\delta}\frac{d\bar H(z)}{\bar H(t_n)}$$

and

$$J_2(t_n) \;\leqslant\; cst(J_2^{(1)}(t_n) + J_2^{(2)}(t_n)),$$

where

$$J_2^{(1)}(t_n) \;=\; \int_{t_n}^{+\infty}\left(\int_0^{t_n}\psi(\phi_n,u)dC(u)\right)^{2+\delta}dH(z) \;=\; \gamma_{n,k}^{2+\delta}\int_{t_n}^{+\infty}(C(t_n))^{2+\delta}dH(z) \;=\; \gamma_{n,k}^{2+\delta}\,(C(t_n))^{2+\delta}\,\bar H(t_n)$$

and, using the technical Lemma 10 as we did some lines above,

$$
\begin{aligned}
J_2^{(2)}(t_n) &= \int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\psi(\phi_n,u)dC(u)\right)^{2+\delta}dH(z)\\
&\leqslant cst\left(\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\log\left(\frac{u}{t_n}\right)\frac{\bar F^{(k)}(u)}{\bar F^{(k)}(t_n)}dC(u)\right)^{2+\delta}dH(z)+\gamma_k\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\left(\frac{u}{t_n}\right)^{-1/\gamma_k}dC(u)\right)^{2+\delta}dH(z)\right.\\
&\quad\left.+\sup_{u>t_n}|\epsilon_n(u)|^{2+\delta}\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\left(\frac{u}{t_n}\right)^{-1/\gamma_k+\delta'}dC(u)\right)^{2+\delta}dH(z)\right). \qquad (22)
\end{aligned}
$$

Using Lemma 9 and part (iii) of Lemma 8, we find that both $J_1(t_n)$ and $J_2^{(1)}(t_n)$ are $O\left((\bar H(t_n))^{-1-\delta}\right)$ and, though the term $J_2^{(2)}(t_n)$ is more involved, we are also going to prove below that the same property holds for $J_2^{(2)}(t_n)$: this will finish the proof of Lemma 5 because $n^{-\delta/2}\left(\bar F^{(k)}(t_n)\bar G(t_n)\right)^{1+\delta/2}(\bar H(t_n))^{-1-\delta}$ tends to 0, as already seen in the proof for $j=2$.

We only treat the first integral in the right-hand side of (22), since the two others are very similar, *i.e.* we need to prove that

$$
\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\log\left(\frac{u}{t_n}\right)\frac{\bar F^{(k)}(u)}{\bar F^{(k)}(t_n)}dC(u)\right)^{2+\delta}dH(z)=O\left(\frac{1}{(\bar H(t_n))^{1+\delta}}\right). \qquad (23)
$$

Now,

$$
\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\log\left(\frac{u}{t_n}\right)\frac{\bar F^{(k)}(u)}{\bar F^{(k)}(t_n)}dC(u)\right)^{2+\delta}dH(z) = (C(t_n))^{2+\delta}\int_{t_n}^{+\infty}\left(\int_{1}^{z/t_n}\log(y)\frac{\bar F^{(k)}(yt_n)}{\bar F^{(k)}(t_n)}\frac{dC(yt_n)}{C(t_n)}\right)^{2+\delta}dH(z)
$$

Using Potter-bounds (40) for $\bar F^{(k)}\in RV_{-1/\gamma_k}$, integration by parts and then Potter-bounds (40) for $C\in RV_{1/\gamma}$, it is easy to see that for n sufficiently large and $\epsilon>0$, there exists some positive constants c, c', c'' such that

$$
\left(\int_1^{z/t_n}\log(y)\frac{\bar F^{(k)}(yt_n)}{\bar F^{(k)}(t_n)}\frac{dC(yt_n)}{C(t_n)}\right)^{2+\delta} \leqslant c\log^{2+\delta}\left(\frac{z}{t_n}\right)\left(\frac{z}{t_n}\right)^{a}+c'\left(\frac{z}{t_n}\right)^{a}+c''.
$$

where $a=(2+\delta)(\frac{1}{\gamma}-\frac{1}{\gamma_k}+2\epsilon)$. Consequently

$$
\begin{aligned}
&\int_{t_n}^{+\infty}\left(\int_{t_n}^{z}\log\left(\frac{u}{t_n}\right)\frac{\bar F^{(k)}(u)}{\bar F^{(k)}(t_n)}dC(u)\right)^{2+\delta}dH(z)\\
&\leqslant (C(t_n))^{2+\delta}\bar H(t_n)\left(-c\int_{t_n}^{+\infty}\log^{2+\delta}\left(\frac{z}{t_n}\right)\left(\frac{z}{t_n}\right)^{a}\frac{d\bar H(z)}{\bar H(t_n)}-c'\int_{t_n}^{+\infty}\left(\frac{z}{t_n}\right)^{a}\frac{d\bar H(z)}{\bar H(t_n)}-c''\right).
\end{aligned}
$$

This yields (23), by using part (ii) of Lemma 8 to this value of a, to $b=-1/\gamma$ (and to $\alpha=2+\delta$ or $\alpha=0$), as well as Lemma 9.

5.2. Proof of Proposition 3

Let us start with an important note. In Proposition 3, the main result is that the remainder terms $R_{n,\phi}$ and $R_{n,g}$ are $o_{\mathbb{P}}(v_n^{-1/2})$. Proving this will be conducted in a similar way as proving that R_n is $o_{\mathbb{P}}(n^{-1/2})$ in Theorem 1.1 of Stute (1995). But, recall that in our situation, the function that we integrate here is ϕ_n, which is depending on n, with a "sliding" support $[t_n,+\infty[$. We will need to be particularly cautious with integrability issues, especially when dealing with U-statistics for the terms $R_{n,2}$ and $R_{n,3}$ in the remainder $R_{n,C}$, defined below.

Before we proceed with the proof, let us define the following empirical (sub)-distribution functions : for $t\geqslant 0$,

$$
H_n(t)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{Z_i\leqslant t},\quad H_n^{(0)}(t)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{Z_i\leqslant t}\ \delta_i,\quad H_n^{(1,k)}(t)=\sum_{i=1}^{n}\mathbb{I}_{Z_i\leqslant t}\ \delta_i\ \mathbb{I}_{\mathscr{C}_i=k}.
$$

First note that, since g_n is the function ϕ_n without the log factor, it should be clear to the reader that proving that $\Delta_n=\hat\Delta_n+R_{n,g}$ and $\sqrt{v_n}R_{n,g}=o_{\mathbb{P}}(1)$ will be simpler than proving that $\widetilde\gamma_{n,k}=\breve\gamma_{n,k}+R_{n,\phi}$ and $\sqrt{v_n}R_{n,\phi}=o_{\mathbb{P}}(1)$. We will thus only prove the latter two relations.

Let us start with the first one, in other words let us define the remainder term $R_{n,\phi}$. Remind that the definitions of $\widetilde{\gamma}_{n,k}$ and $\breve{\gamma}_{n,k}$ are $\widetilde{\gamma}_{n,k} = \int \phi_n(u) dF_n^{(k)}(u)$ and $\breve{\gamma}_{n,k} = \bar{U}_n^{(1)} + \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$, where $\bar{U}_n^{(j)}$ denotes the mean of the n variables $U_{i,n}^{(j)}$. We need to decompose the integral of ϕ_n with respect to $F_n^{(k)}$, which is a stepwise subdistribution function which jumps at the (ordered) observations $Z_{(i)}$ are equal to $\mathbb{I}_{\xi_{(i)}=k}/(n\bar{G}_n(Z_{(i-1)}))$. But it is known that (see Lemma 2.1 in Stute (1995))

$$\frac{1}{\bar{G}_n(Z_{(i-1)})} = \exp\left\{ n \int_0^{Z_{(i)}^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \right\}$$

Therefore, using the fact that $\bar{G}(z) = \exp\left(-\int_0^z \bar{H}^{-1} dH^{(0)}\right)$, we have

$$\begin{aligned}
\widetilde{\gamma}_{n,k} &= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_{(i)})}{\bar{G}(Z_{(i)}^-)} \mathbb{I}_{\xi_{(i)}=k} \frac{\bar{G}(Z_{(i)}^-)}{\bar{G}_n(Z_{(i-1)})} \\
&= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} \exp\left(n \int_0^{Z_i^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \; - \; \int_0^{Z_i^-} \frac{dH^{(0)}}{\bar{H}} \right).
\end{aligned}$$

Consequently, using the mean value theorem for exp, and introducing the important notations

$$\begin{aligned}
B_{i,n} &= n \int_0^{Z_i^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \; - \; \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n} \\
C_{i,n} &= \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n} \; - \; \int_0^{Z_i^-} \frac{dH^{(0)}}{\bar{H}},
\end{aligned}$$

it is easy to see that

$$\begin{aligned}
\widetilde{\gamma}_{n,k} &= \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} \; + \; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} B_{i,n} \; + \; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \\
&\qquad + \; \frac{1}{2n}\sum_{i=1}^n \phi_n(Z_i) \mathbb{I}_{\xi_i=k} (B_{i,n} + C_{i,n})^2 e^{\Delta_{i,n}} \\
&= \bar{U}_n^{(1)} \; + \; R_{n,B} \; + \; \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \; + \; R_{n,\Delta}, \qquad (24)
\end{aligned}$$

where $\bar{U}_n^{(1)}$ is the first term in the definition of $\breve{\gamma}_{n,k}$, and $\Delta_{i,n}$ is a random quantity lying between $\int_0^{Z_i^-} \bar{H}^{-1} dH^{(0)}$ and $n\int_0^{Z_i^-} \log(1 + (n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x)$.

What we now need to do is to show that the term involving the quantity $C_{i,n}$ in relation (24) above can be written as $\bar{U}_n^{(2)} - \bar{U}_n^{(3)}$ plus a remainder term $R_{n,C}$, and therefore we have $\widetilde{\gamma}_{n,k} = \breve{\gamma}_{n,k} + R_{n,\phi}$, where

$$R_{n,\phi} \; = \; R_{n,B} + R_{n,C} + R_{n,\Delta}. \qquad (25)$$

The rest of the proof will, afterwards, be devoted to showing that each term of $R_{n,\phi}$ is $o_{\mathbb{P}}(v_n^{-1/2})$.

Proceeding as in Stute (1995) or Suzukawa (2002), and using the fact that for any given function f we have $\int f\, dH_n^{(1,k)} = \frac{1}{n}\sum_{i=1}^n f(Z_i)\mathbb{I}_{\xi_i=k}$, we can write

$$\frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} \; = \; -C_n^{(1)} + 2C_n^{(2)} - C_n^{(3)} + R_{n,1} \qquad (26)$$

where

$$
\begin{aligned}
C_n^{(1)} &= \iiint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}^2(v)} \mathbb{I}_{z>v}\mathbb{I}_{u>v}\, dH_n(u) dH_n^{(0)}(v) dH_n^{(1,k)}(z), \\
C_n^{(2)} &= \iint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}(v)} \mathbb{I}_{z>v}\, dH_n^{(0)}(v) dH_n^{(1,k)}(z), \\
C_n^{(3)} &= \iint \frac{\phi_n(z)}{\bar{G}(z)\bar{H}(v)} \mathbb{I}_{z>v}\, dH^{(0)}(v) dH_n^{(1,k)}(z), \\
R_{n,1} &= \iint \frac{\phi_n(z)}{\bar{G}(z)} \mathbb{I}_{z>v} \frac{(\bar{H}_n - \bar{H})^2(v)}{\bar{H}^2(v)\bar{H}_n(v)} dH_n^{(0)}(v) dH_n^{(1,k)}(z).
\end{aligned}
$$

Note that $C_n^{(1)}$ and $C_n^{(2)}$ are a kind of U-statistics, which need to be approximated by sums of independent variables called Hoeffding decompositions : more precisely, if we introduce the functions (important in the sequel)

$$
h(v,w) = \frac{\phi_n(w)}{\bar{G}(w^-)\bar{H}(v)} \mathbb{I}_{w>v}\mathbb{I}_{v<\infty}\mathbb{I}_{w<\infty} \quad \text{and} \quad \underline{h}(u,v,w) = h(v,w)\frac{\mathbb{I}_{u>v}}{\bar{H}(v)} \tag{27}
$$

for $u \in \mathbb{R}$, $v \in \mathbb{R} \cup \{+\infty\}$ and $w \in \mathbb{R} \cup \{+\infty\}$, then these decompositions are defined by

$$
\begin{aligned}
\widehat{C}_n^{(1)} = {} & \iiint \underline{h}(u,v,w) dH_n(u) dH^{(0)}(v) dH^{(1,k)}(w) + \\
& \iiint \underline{h}(u,v,w) dH(u) dH_n^{(0)}(v) dH^{(1,k)}(w) + \\
& \iiint \underline{h}(u,v,w) dH(u) dH^{(0)}(v) dH_n^{(1,k)}(w) \\
& - 2 \iiint \underline{h}(u,v,w) dH(u) dH^{(0)}(v) dH^{(1,k)}(w)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\widehat{C}_n^{(2)} = {} & \iint h(v,w) dH_n^{(0)}(v) dH^{(1,k)}(w) + \iint h(v,w) dH^{(0)}(v) dH_n^{(1,k)}(w) \\
& - \iint h(v,w) dH^{(0)}(v) dH^{(1,k)}(w).
\end{aligned} \tag{29}
$$

Therefore, if we introduce the remainder terms

$$
R_{n,2} = C_n^{(1)} - \widehat{C}_n^{(1)} \quad \text{and} \quad R_{n,3} = C_n^{(2)} - \widehat{C}_n^{(2)} \tag{30}
$$

then (26) becomes

$$
\frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)} \mathbb{I}_{\xi_i=k} C_{i,n} = -\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} + R_{n,C} \quad \text{where} \quad R_{n,C} = R_{n,1} - R_{n,2} + 2R_{n,3}.
$$

We are thus left to prove that $-\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} = \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$. This is indeed the case because, if we note

$$
\theta_n = \iint h(v,w) dH^{(0)}(v) dH^{(1,k)}(w), \tag{31}
$$

then, by definition of $\underline{h}$, the last (fourth) term in $\widehat{C}_n^{(1)}$ equals $-2\theta_n$, the third one equals $C_n^{(3)}$, the second one is (because $dH^{(1,k)}(w) = \bar{G}(w^-) dF^{(k)}(w)$)

$$
\begin{aligned}
\iiint \underline{h}(u,v,w) dH(u) dH_n^{(0)}(v) dH^{(1,k)}(w) &= \iint h(v,w) dH_n^{(0)}(v) dH^{(1,k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n (1-\delta_i) \int h(Z_i,w) \bar{G}(w^-)\, dF^{(k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \frac{1-\delta_i}{\bar{H}(Z_i)} \int_{Z_i}^\infty \phi_n(w)\, dF^{(k)}(w) \\
&= \frac{1}{n}\sum_{i=1}^n \frac{1-\delta_i}{\bar{H}(Z_i)} \psi(\phi_n, Z_i) = \bar{U}_n^{(2)}
\end{aligned}
$$

and the first one is (because $dH^{(1,k)}(w) = \bar{G}(w^-)dF^{(k)}(w)$ and $dH^{(0)}(v) = \bar{F}(v)dG(v)$)

$$\begin{aligned}\iiint \underline{h}(u,v,w)dH_n(u)dH^{(0)}(v)dH^{(1,k)}(w) &= \frac{1}{n}\sum_{i=1}^n \iint \underline{h}(Z_i,v,w)dH^{(0)}(v)dH^{(1,k)}(w)\\ &= \frac{1}{n}\sum_{i=1}^n \iint \frac{\phi_n(w)}{\bar{G}(v)\bar{H}(v)}\mathbb{I}_{v<w}\mathbb{I}_{v<Z_i}\, dG(v)\, dF^{(k)}(w)\\ &= \frac{1}{n}\sum_{i=1}^n \int_0^{Z_i} \psi(\phi_n,v)\, dC(v) \;=\; \bar{U}_n^{(3)}.\end{aligned}$$

Likewise, the first term of $\widehat{C}_n^{(2)}$ equals $\bar{U}_n^{(2)}$, the second one equals $C_n^{(3)}$, and the last one equals $-\theta_n$. After straightforward simplifications, we obtain the desired equality $-\widehat{C}_n^{(1)} + 2\widehat{C}_n^{(2)} - C_n^{(3)} = \bar{U}_n^{(2)} - \bar{U}_n^{(3)}$, and the proof of $\widetilde{\gamma}_{n,k} = \breve{\gamma}_{n,k} + R_{n,\phi}$ is over.

The proof of Proposition 3 is now based on the following two lemmas : Lemma 6 is proved in subsection 5.2.1, and Lemma 7 is the longest to establish, its proof will be split across subsections 5.2.2 to 5.2.5.

Lemma 6. *If conditions* (1) *and (2) hold with* $\gamma_k < \gamma_C$*, then we have*

$$\sqrt{v_n}R_{n,B} = o_{\mathbb{P}}(1),\ \sqrt{v_n}R_{n,1} = o_{\mathbb{P}}(1),\ \textit{and}\ \sqrt{v_n}R_{n,\Delta} = o_{\mathbb{P}}(1).$$

Lemma 7. *If conditions* (1) *and (2) hold with* $\gamma_k < \gamma_C$*, then we have*

$$\sqrt{v_n}R_{n,j} = o_{\mathbb{P}}(1) \qquad \textit{for } j = 2 \textit{ and for } j = 3.$$

5.2.1. Proof of Lemma 6

- We start with the remainder term $R_{n,B}$, which is defined as

$$R_{n,B} = \frac{1}{n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}B_{i,n},$$

where $B_{i,n} = n\int_0^{Z_i^-} \log(1+(n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x) \;-\; \int_0^{Z_i^-} \frac{dH_n^{(0)}}{\bar{H}_n}$. Since, for all $x \geqslant 0,\ x - \frac{x^2}{2} \leqslant \log(1+x) \leqslant x$, we obtain

$$-\frac{1}{2n}\int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}_n(x))^2} \leqslant B_{i,n} \leqslant 0$$

and then

$$\begin{aligned}|R_{n,B}| &\leqslant \frac{1}{n}\sum_{i=1}^n \left\{ \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\frac{1}{2n}\left(\int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}_n(x))^2}\right)\right\}\\ &\leqslant \sup_{0\leqslant x<Z_{(n)}}\left(\frac{\bar{H}(x)}{\bar{H}_n(x)}\right)^2 \frac{1}{2n^2}\sum_{i=1}^n \left\{ \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\left(\int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}(x))^2}\right)\right\}. \qquad (32)\end{aligned}$$

But

$$\int_0^{Z_i^-} \frac{dH_n^{(0)}(x)}{(\bar{H}(x))^2} = \int_0^{Z_i^-} \frac{dH^{(0)}(x)}{(\bar{H}(x))^2} + \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2},$$

so, if we define

$$T_n^{(1)} = \frac{1}{n}\sum_{i=1}^n T_{i,n}^{(1)} \text{ and } T_n^{(2)} = \frac{1}{n}\sum_{i=1}^n T_{i,n}^{(2)}$$

where

$$\begin{aligned}T_{i,n}^{(1)} &= \frac{1}{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\int_0^{Z_i^-} \frac{dH^{(0)}(x)}{(\bar{H}(x))^2}\\ T_{i,n}^{(2)} &= \frac{1}{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2},\end{aligned}$$

then it remains to prove (thanks to part (i) of Lemma 11) that $\sqrt{v_n}T_n^{(1)} = o_{\mathbb{P}}(1)$ and $\sqrt{v_n}T_n^{(2)} = o_{\mathbb{P}}(1)$.

Concerning $T_n^{(1)}$, since $\bar{H} \geqslant \bar{H}^{(0)}$ implies that $T_{i,n}^{(1)} \leqslant \frac{1}{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)\bar{H}^{(0)}(Z_i^-)}\mathbb{I}_{\xi_i=k}$, then $\sqrt{v_n}T_n^{(1)} = o_{\mathbb{P}}(1)$ is a consequence of Lemma 12, used with $\alpha = 0$ and $d = 1$.

Concerning $T_n^{(2)}$, an integration by parts yields

$$\begin{aligned}\left|\int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2}\right| &\leqslant \frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(Z_i^-)}{\bar{H}^2(Z_i^-)} + 2\int_0^{Z_i^-} \frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(x)}{\bar{H}^3(x)}\, dH(x) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)| \\ &\leqslant \sup_{0\leqslant x<Z_{(n)}} \sqrt{n}\frac{|\bar{H}_n^{(0)} - \bar{H}^{(0)}|(x)}{(\bar{H}^{(0)}(x))^{\frac{1}{2}-\alpha}}\left(\frac{\bar{H}^{(0)}(x)}{\bar{H}(x)}\right)^{\frac{1}{2}-\alpha} \times \\ &\quad \left(\frac{1}{\sqrt{n}\left(\bar{H}(Z_i^-)\right)^{\frac{3}{2}+\alpha}} + \frac{2}{\sqrt{n}}\int_0^{Z_i^-} \frac{dH(x)}{\left(\bar{H}(x)\right)^{\frac{5}{2}+\alpha}}\right) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|,\end{aligned}$$

for any given $0 < \alpha < \frac{1}{2}$. Lemma 11 (applied with $a = 1/2 - \alpha < 1/2$) and the fact $\bar{H}^{(0)} \leqslant \bar{H}$ thus imply that

$$\left|\int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})(x)}{(\bar{H}(x))^2}\right| \leqslant O_{\mathbb{P}}(1)\frac{1}{\sqrt{n}\left(\bar{H}(Z_i^-)\right)^{\frac{3}{2}+\alpha}} + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|,$$

so that, by definition of $T_{i,n}^{(2)}$, the desired statement $\sqrt{v_n}T_n^{(2)} = o_{\mathbb{P}}(1)$ is a consequence of Lemma 12, applied with $\alpha > 0$ sufficiently small and $d = \frac{3}{2}$, and of

$$\frac{\sqrt{v_n}}{n}\sum_{i=1}^n \frac{1}{n}\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}|\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)| = \sqrt{n}|\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)| \times \frac{\sqrt{\bar{F}^{(k)}(t_n)\bar{G}(t_n)}}{n} \times \bar{U}_n^{(1)} = o_{\mathbb{P}}(1).$$

Indeed $\bar{U}_n^{(1)}$ converges to γ_k and $\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)$ equals $\frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\delta_i=0} - \mathbb{P}(\delta = 0)$, which is $O_{\mathbb{P}}(n^{-1/2})$ by the standard central limit theorem.

- Let us now turn to the remainder term $R_{n,1}$, which is defined as

$$R_{n,1} = \iint \frac{\phi_n(z)}{\bar{G}(z)}\mathbb{I}_{z>v}\frac{(\bar{H}_n - \bar{H})^2(v)}{\bar{H}^2(v)\bar{H}_n(v)} dH_n^{(0)}(v) dH_n^{(1,k)}(z).$$

A simple calculation leads to

$$R_{n,1} \leqslant \sup_{0\leqslant x<Z_{(n)}}\left(\sqrt{n}\frac{|H_n - H|(x)}{(\bar{H}(x))^{\frac{1}{2}-\alpha}}\right)^2 \sup_{0\leqslant x<Z_{(n)}}\frac{\bar{H}(x)}{\bar{H}_n(x)} \times \frac{1}{n^2}\sum_{i=1}^n\left\{\frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}\left(\int_0^{Z_i^-}\frac{dH_n^{(0)}(v)}{(\bar{H}(v))^{2+2\alpha}}\right)\right\}$$

for any $0 < \alpha < \frac{1}{2}$. Taking α sufficiently small, the rest of the proof is very similar to the one for $R_{n,B}$ (compare to (32)) and relies on Lemma 11 and Lemma 12.

- We can finally deal with the last remainder term $R_{n,\Delta}$, defined as

$$R_{n,\Delta} = \frac{1}{2n}\sum_{i=1}^n \phi_n(Z_i)\mathbb{I}_{\xi_i=k}(B_{i,n} + C_{i,n})^2 e^{\Delta_{i,n}},$$

where $\Delta_{i,n}$ is a random quantity lying between $a_n := \int_0^{Z_i^-} \bar{H}^{-1}dH^{(0)}$ and $b_n := \int_0^{Z_i^-} \log(1+(n\bar{H}_n(x))^{-1})\, dH_n^{(0)}(x)$. Since $e^a = 1/\bar{G}(Z_i^-)$, we have

$$R_{n,\Delta} = \frac{1}{2n}\sum_{i=1}^n \frac{\phi_n(Z_i)}{\bar{G}(Z_i^-)}\mathbb{I}_{\xi_i=k}(B_{i,n} + C_{i,n})^2 e^{\Delta_{i,n}-a_n}.$$

Since $b_n - a_n = B_{i,n} + C_{i,n}$, where $B_{i,n} < 0$ and $C_{i,n} = \int_0^{Z_i^-}\frac{dH_n^{(0)}}{H_n} - \int_0^{Z_i^-}\frac{dH^{(0)}}{H}$, we clearly have

$$e^{(\Delta_{i,n}-a)} \leqslant \max(1, e^{C_{i,n}}) \leqslant e^{|C_{i,n}|}.$$

But $C_{i,n} = \hat{\Lambda}_{n,G}(Z_i) - \Lambda_G(Z_i)$, where Λ_G is the cumulative hazard function assiciated to G, and $\hat{\Lambda}_{n,G}$ its Nelson-Alen estimator. Relying on Zhou (1991) Theorem 2.1, we can deduce that $\sup_{1\leqslant i\leqslant n}|C_{i,n}| = O_{\mathbb{P}}(1)$. Hence, $e^{(\Delta_{i,n}-a)} = O_{\mathbb{P}}(1)$.

Now,

$$C_{i,n} - \frac{1}{2n}\int_0^{Z_i^-} \frac{dH_n^{(0)}}{(\bar{H}_n)^2} \leqslant B_{i,n} + C_{i,n} \leqslant C_{i,n}.$$

By writing

$$C_{i,n} = \int_0^{Z_i^-} \frac{d(H_n^{(0)} - H^{(0)})}{\bar{H}_n} + \int_0^{Z_i^-} \left(\frac{1}{\bar{H}_n} - \frac{1}{\bar{H}}\right) dH^{(0)},$$

we prove (using Lemma 11 and simple integrations as for the previous treatment of $T_n^{(2)}$ above) that $|C_{i,n}| \leqslant O_{\mathbb{P}}(1)\ 1/(\sqrt{n}(\bar{H}(Z_i))^{1/2+\alpha}) + |\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|$ for $0 < \alpha < 1/2$.

Hence, on one hand $(C_{i,n})^2 \leqslant O_{\mathbb{P}}(1)(n(\bar{H}(Z_i))^{1+2\alpha})^{-1} + O_{\mathbb{P}}(n^{-1}) \leqslant O_{\mathbb{P}}(1)(n(\bar{H}^{(0)}(Z_i))^{1+2\alpha})^{-1} + O_{\mathbb{P}}(n^{-1})$, and on the other hand

$$\left(C_{i,n} - \frac{1}{2n}\int_0^{Z_i^-} \frac{dH_n^{(0)}}{(\bar{H}_n)^2}\right)^2 \leqslant O_{\mathbb{P}}(1)\left(\frac{1}{n(\bar{H}^{(0)}(Z_i))^{1+2\alpha}} + \frac{1}{n^2(\bar{H}^{(0)}(Z_i))^2}\right) + O_{\mathbb{P}}(n^{-1}),$$

for any given $0 < \alpha < 1/2$ (where the $O_{\mathbb{P}}(n^{-1})$ comes from $|\bar{H}_n^{(0)}(0) - \bar{H}^{(0)}(0)|^2$, which does not depend on i). Therefore, it is sufficient to prove that

$$\frac{1}{n}\sum_{i=1}^n \frac{\sqrt{v_n}\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)n(\bar{H}^{(0)}(Z_i))^{1+2\alpha}} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^n \frac{\sqrt{v_n}\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)n^2(\bar{H}^{(0)}(Z_i))^2}$$

are $o_{\mathbb{P}}(1)$, and that

$$n^{-1/2}\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1/2} \times \frac{1}{n}\sum_{i=1}^n \frac{\log(Z_i/t_n)\mathbb{I}_{\xi_i=k}\mathbb{I}_{Z_i>t_n}}{\bar{F}^{(k)}(t_n)\bar{G}(Z_i^-)}$$

is $o_{\mathbb{P}}(1)$ as well. But the first two statements are consequences of Lemma 12 with $\alpha > 0$ sufficiently close to 0 and, respectively, $d = 1$ and $d = 2$. And for the third statement, the expectation of the expression turns out (thanks to Lemma 8 part (ii)) to be equivalent to a constant times $n^{-1/2}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2}$, which tends to 0.

5.2.2. Preliminaries to the proof of Lemma 7

We start this section by introducing important objects, issued from an idea appearing (to the best of our knowledge) in Stute (1994). We define the improper variables $(V_i)_{1\leqslant i\leqslant n}$ and $(W_j)_{1\leqslant j\leqslant n}$ by

$$V_i = \begin{cases} Z_i & \text{if } \delta_i = 0 \\ +\infty & \text{if } \delta_i = 1 \end{cases} \quad \text{and} \quad W_j = \begin{cases} +\infty & \text{if } \delta_j = 0 \text{ or } \mathscr{C}_j \neq k \\ Z_j & \text{if } \delta_j = 1 \text{ and } \mathscr{C}_j = k \end{cases}$$

which have $H^{(0)}$ and $H^{(1,k)}$ for respective subdistribution functions. We thus have $1 - \delta_i = \mathbb{I}_{V_i<\infty}$ and $\mathbb{I}_{\mathscr{C}_j=k} = \mathbb{I}_{W_j<\infty}$, which, according to the definitions of $C_n^{(1)}$ and $C_n^{(2)}$ on one hand, and of functions h and $\underline{h}$ (in (27)) on the other hand, leads to

$$C_n^{(2)} = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{\phi_n(Z_j)}{\bar{G}(Z_j^-)\bar{H}(Z_i)}\mathbb{I}_{Z_j>Z_i}(1-\delta_i)\delta_j\mathbb{I}_{\mathscr{C}_j=k} = \frac{1}{n^2}\sum\sum_{i\neq j} h(V_i, W_j)$$

and

$$C_n^{(1)} = \frac{1}{n^3}\sum_{l=1}^n\sum_{i=1}^n\sum_{j=1}^n \frac{\phi_n(Z_j)}{\bar{G}(Z_j^-)\bar{H}^2(Z_i)}\mathbb{I}_{Z_j>Z_i}\mathbb{I}_{Z_l>Z_i}(1-\delta_i)\delta_j\mathbb{I}_{\mathscr{C}_j=k} = \frac{1}{n^3}\sum\sum\sum_{i\neq j, i\neq l} \underline{h}(Z_l, V_i, W_j).$$

Since the latter triple sum is not convenient, we also define

$$C_n^{(1)} = \widetilde{C}_n^{(1)} + \widetilde{\widetilde{C}}_n^{(1)} \quad \text{where} \quad \widetilde{C}_n^{(1)} = \frac{1}{n^3}\sum_{i,j,l}\sum\sum_{distincts} \underline{h}(Z_l, V_i, W_j) \quad \text{and} \quad \widetilde{\widetilde{C}}_n^{(1)} = \frac{1}{n^3}\sum\sum_{i\neq j} h(V_i, W_j)/\bar{H}(V_i),$$

where $\widetilde{C}_n^{(1)}$ will be the quantity approximated by $\widehat{C}_n^{(1)}$, and $\widetilde{\widetilde{C}}_n^{(1)}$ will be a remainder. We can indeed rewrite (30) as

$$R_{n,3} = \frac{n(n-1)}{n^2}\left(\frac{n^2}{n(n-1)}C_n^{(2)} - \widehat{C}_n^{(2)}\right) - \widehat{C}_n^{(2)}/n, \tag{33}$$

$$R_{n,2} = \frac{n(n-1)(n-2)}{n^3}\left(\frac{n^3}{n(n-1)(n-2)}\widetilde{C}_n^{(1)} - \widehat{C}_n^{(1)}\right) - \frac{3n-2}{n^2}\widehat{C}_n^{(1)} + \widetilde{\widetilde{C}}_n^{(1)}. \tag{34}$$

The terms in parentheses in (33) and (34) turn out to be genuine U-statistics of 2 and 3 variables, denoted by

$$\mathscr{U}_n = \tfrac{1}{n(n-1)} \sum\sum_{i\neq j} \mathscr{H}(V_i, W_j) \quad \text{and} \quad \mathscr{V}_n = \tfrac{1}{n(n-1)(n-2)} \sum\sum\sum_{i,j,l \ distincts} \underline{\mathscr{H}}(Z_l, V_i, W_j) \tag{35}$$

where functions $\mathscr{H}$ and $\underline{\mathscr{H}}$ will be defined in a few lines (relation (36)) after some preliminaries, certainly well-known in the U-statistics literature, but which we include here to make our proof self-contained (and since we are dealing with improper variables).

If V and W denote *independent* improper random variables with subdistribution functions $H^{(0)}$ and $H^{(1,k)}$ (*i.e.* $V = Z\mathbb{I}_{\delta=0} + \infty\mathbb{I}_{\delta=1}$ and $W = Z'\delta'\mathbb{I}_{\mathscr{C}'=k} + \infty(1-\delta'+\mathbb{I}_{\mathscr{C}'\neq k})$ where $(Z,\delta,\mathscr{C})$ and $(Z',\delta',\mathscr{C}')$ are independent copies of $(Z_1,\delta_1,\mathscr{C}_1)$), we introduce the following notations : for any function $g : [0,\infty]\times[0,\infty] \to \mathbb{R}$,

$$g_{1\bullet}(v) = \mathbb{E}(g(v,W)) \quad \text{and} \quad g_{\bullet 1}(w) = \mathbb{E}(g(V,w)) \ ,$$

as well as, for any function $g : [0,\infty[\times[0,\infty]\times[0,\infty] \to \mathbb{R}$, with Z (of distribution function H) independent of V and W,

$$g_{1\bullet\bullet}(u) = \mathbb{E}(g(u,V,W)) \quad , \quad g_{\bullet 1\bullet}(v) = \mathbb{E}(g(Z,v,W)) \quad \text{and} \quad g_{\bullet\bullet 1}(w) = \mathbb{E}(g(Z,V,w)).$$

Since $h(v,w) = 0$ whenever v or w equals ∞, we then have (the proof is simple)

$$\theta_n = \iint h(v,w) dH^{(0)}(v) dH^{(1,k)}(w) = \mathbb{E}(h(V,W)) = \mathbb{E}(\underline{h}(Z,V,W)).$$

Therefore, setting (for z in $[0,\infty[$ and v and w in $[0,\infty]$)

$$\begin{array}{rcl} \mathscr{H}(v,w) & = & h(v,w) - h_{1\bullet}(v) - h_{\bullet 1}(w) + \theta_n \\ \underline{\mathscr{H}}(z,v,w) & = & h(z,v,w) - \underline{h}_{1\bullet\bullet}(z) - \underline{h}_{\bullet 1\bullet}(v) - \underline{h}_{\bullet\bullet 1}(w) + 2\theta_n \end{array} \tag{36}$$

it is then not difficult to check (using (28) and (29)) that $\mathscr{U}_n$ and $\mathscr{V}_n$ in relation (35) are indeed equal to the differences in parentheses in relations (33) and (34), respectively. Lemma 7 thus becomes a consequence of the following facts : $\sqrt{v_n}\mathscr{U}_n = o_{\mathbb{P}}(1)$, $\sqrt{v_n}\mathscr{V}_n = o_{\mathbb{P}}(1)$, and

$$\text{the three sequences } \ \widetilde{\widetilde{C}}_n^{(1)} \ , \ \widehat{C}_n^{(1)}/n \ \text{ and } \ \widehat{C}_n^{(2)}/n \ \text{ are } \ o_{\mathbb{P}}(v_n^{-1/2}). \tag{37}$$

We will prove these statements in the next 3 subsections.

5.2.3. Proof of $\sqrt{v_n}\mathscr{U}_n = o_{\mathbb{P}}(1)$

We note $\mathscr{I} = \{I = (i,j)\,;\, 1 \leqslant i < j \leqslant n\}$, $\mathscr{H}_I = \mathscr{H}(V_i, W_j)$ when $I = (i,j) \in \mathscr{I}$, and $N = n(n-1)/2$. It is clear that it suffices to prove that

$$S_N = o_{\mathbb{P}}(1) \quad \text{where} \quad S_N = \sum_{I\in\mathscr{I}} \tfrac{\sqrt{v_n}}{N} \mathscr{H}_I.$$

The good point is that S_N turns out to be a sum of identically distributed centred and uncorrelated random variables $\mathscr{H}_I$, but unfortunately these variables $\mathscr{H}_I$ are not square-integrable and potentially only have a moment of order slightly larger than $4/3$ when $\gamma_k < \gamma_C$. In order to deal with this difficulty, since we cannot handle directly the L^p norm of S_N of order $p = 4/3$, we will follow a strategy similar to that found in Csorgo, Szyszkowicz and Wang (2008), based on truncation. We set

$$\mathscr{H}^*(v,w) \ = \ \mathscr{H}(v,w)\mathbb{I}_{|\mathscr{H}(v,w)|\leqslant M_n} - \mathbb{E}(\mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n}) \ \text{ where } \ \left\{ \begin{array}{rcl} \mathscr{H}_1 & = & \mathscr{H}(V_1,W_2) \\ M_n & = & n^2/\sqrt{v_n} \end{array} \right. \tag{38}$$

The variables $\mathscr{H}_I^* = \mathscr{H}^*(V_i,W_j)$ ($I \in \mathscr{I}$) are centred and bounded, but they lose the non-correlation property of the variables $\mathscr{H}_I$. This is why we define now

$$\mathscr{H}_I^{**} \ = \ \mathscr{H}^{**}(V_i,W_j) \ \text{ where } \ \mathscr{H}^{**}(v,w) = \mathscr{H}^*(v,w) - \mathscr{H}_{1\bullet}^*(v) - \mathscr{H}_{\bullet 1}^*(w)$$

which are centred and bounded but are also uncorrelated (see part (i) of Lemma 14), and we write

$$S_N \ = \ S_N^{(1)} + S_N^{(2)} \ = \ \tfrac{\sqrt{v_n}}{N} \textstyle\sum_{I\in\mathscr{I}} \mathscr{H}_I^{**} \ + \ \tfrac{\sqrt{v_n}}{N} \textstyle\sum_{I\in\mathscr{I}} (\mathscr{H}_I - \mathscr{H}_I^{**}). \tag{39}$$

We thus need to prove that $S_N^{(1)}$ and $S_N^{(2)}$ both converge to 0 in probability.

Concerning $S_N^{(1)}$, since the $\mathscr{H}_I^{**}$ are centred and uncorrelated, we have

$$\mathbb{E}\left((S_N^{(1)})^2\right) \;=\; (v_n/N^2)\mathbb{E}\left((\textstyle\sum_{I\in\mathscr{I}}\mathscr{H}_I^{**})^2\right) \;=\; (v_n/N)\mathbb{E}\left((\mathscr{H}^{**}(V_1,W_2))^2\right) \;\leqslant\; cst(v_n/n^2)\mathbb{E}(\mathscr{H}_1^2\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n})$$

where $\mathscr{H}_1$ was defined in (38) (the justification of the last inequality is postponed to part (ii) of Lemma 14). Remind that $\mathscr{H}_1$ is not square-integrable and $M_n = n^2/\sqrt{v_n} = n^{3/2}/(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2}$, and introduce $m_n = n^{3/2}/(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1/2-\epsilon} = o(M_n)$ for some given small $\epsilon > 0$. We then write

$$\mathbb{E}\left((S_N^{(1)})^2\right) \;\leqslant\; cst\frac{v_n}{n^2}m_n^{2/3}\mathbb{E}\left(|\mathscr{H}_1|^{4/3}\right) \;+\; cst\frac{v_n}{n^2}M_n^{2/3}\mathbb{E}\left(|\mathscr{H}_1|^{4/3}\mathbb{I}_{|\mathscr{H}_1|>m_n}\right) \;=\; cst(A_n+B_n).$$

Thanks to Lemma 13 (parts (i) and (ii)) and to the definition of m_n, the term A_n is bounded by a quantity which is equivalent (as $n\to\infty$) to $\frac{v_n}{n^2}m_n^{2/3}\left(\frac{v_n}{n}\right)^{-2/3} = (\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{2\epsilon/3} = o(1)$. We now rely on Hölder's inequality for dealing with the term B_n . Let $p>1$ and $q>1$ such that $1/p+1/q=1$. Since $\theta_n = \mathbb{E}(\mathscr{H}_1)$, again thanks to Lemma 13 ((i), (ii) and (v)), for p sufficiently close to 1 so that $4p/3 < 1+(1+2\gamma_k/\gamma_C)^{-1}$, we have

$$\begin{aligned}
B_n &\leqslant \left(\frac{v_n}{n}\right)^{2/3}\left(\mathbb{E}|\mathscr{H}_1|^{4p/3}\right)^{1/p}\left(\mathbb{P}(|\mathscr{H}_1|>m_n)\right)^{1/q}\\
&\leqslant \left(\frac{v_n}{n}\right)^{2/3}\left(O\left(\left(\frac{v_n}{n}\right)^{2(1-4p/3)}\right)\right)^{1/p} m_n^{-1/q}\,(4\theta_n)^{1/q}\\
&\leqslant O(1)\left(\frac{v_n}{n}\right)^{2/3+2(1-1/q)-8/3} M_n^{-1/q}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-\epsilon/q}(-\log\bar{G}(t_n))^{1/q}\\
&\leqslant O(1)\,v_n^{-3/2q}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-2\epsilon/q}\left((\bar{G}(t_n))^{\epsilon}(-\log\bar{G}(t_n))\right)^{1/q} \;=\; o(1)\left(n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1+4\epsilon/3}\right)^{-3/2q}
\end{aligned}$$

which converges to 0 thanks to assumption (2), for $\epsilon>0$ small enough (we used part (v) of Lemma 13 in the third upper bound).

We are thus left to prove that $S_N^{(2)}$ also converges to 0, but this time in L^1. We start by writing that

$$\begin{aligned}
\mathbb{E}\left(|S_N^{(2)}|\right) &\leqslant \tfrac{\sqrt{v_n}}{N}\textstyle\sum_{I\in\mathscr{I}}\mathbb{E}\left(|\mathscr{H}_I-\mathscr{H}_I^{**}|\right) \;=\; \sqrt{v_n}\,\mathbb{E}\left(\,|\mathscr{H}_1-\mathscr{H}^*(V_1,W_2)+\mathscr{H}_{1\bullet}^*(V_1)+\mathscr{H}_{\bullet 1}^*(W_2)|\,\right)\\
&\leqslant 4\sqrt{v_n}\,\mathbb{E}\left(|\mathscr{H}_1|\mathbb{I}_{|\mathscr{H}_1|>M_n}\right),
\end{aligned}$$

the last inequality being proved in the appendix (part (iii) of Lemma 14). The follow-up is a bit similar to the treatment of B_n above, relying on Lemma 13 (parts (i), (ii) and (v)) and on Hölder's inequality : for $p>1$ close to 1 and a large q such that $1/p+1/q=1$, we can write

$$\begin{aligned}
\mathbb{E}\left(|S_N^{(2)}|\right) &\leqslant 4\sqrt{v_n}M_n^{-1/3}\left(\mathbb{E}|\mathscr{H}_1|^{4p/3}\right)^{1/p}\left(\mathbb{P}(|\mathscr{H}_1|>M_n)\right)^{1/q}\\
&\leqslant O(1)\,v_n^{-3/2q}(-\log\bar{G}(t_n))^{1/q} \;\leqslant\; O(1)\left\{\left(n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1+\epsilon}\right)^{-3/2}(\bar{G}(t_n))^{3\epsilon/2}(-\log\bar{G}(t_n))\right\}^{1/q}
\end{aligned}$$

which, for $\epsilon>0$ small enough, is $o(1)$ thanks to assumption (2).

5.2.4. Proof of $\sqrt{v_n}\mathscr{V}_n = o_{\mathbb{P}}(1)$

The proof is very similar to the one contained in the previous subsection. We nonetheless provide a few details to convince the reader of the validity of the result. We note now $\mathscr{I} = \{I=(i,j,l)\,;\,1\leqslant i<j<l\leqslant n\}$ and $\underline{\mathscr{H}}_I = \underline{\mathscr{H}}(Z_l,V_i,W_j)$ when $I=(i,j,l)\in\mathscr{I}$, with $N=n(n-1)(n-2)/6$ denoting the cardinal of the index set $\mathscr{I}$. Since the observations $(Z_i)_{i\leqslant n}$ are i.i.d., it should be clear to the reader that it suffices to prove that

$$S_N = o_{\mathbb{P}}(1) \quad \text{where} \quad S_N = \sum_{I\in\mathscr{I}}\tfrac{\sqrt{v_n}}{N}\underline{\mathscr{H}}_I.$$

As previously, the problem lies with the moments of the centred and uncorrelated variables $\underline{\mathscr{H}}_I$, and now we only have a guaranteed moment of order slightly more than 6/5 instead of 4/3 in the previous situation. Fortunately, the cardinal N is now of order n^3, which turns out to be the right compensation.

We thus define, for $(u,v,w)\in[0,\infty[\times[0,\infty]\times[0,\infty]$,

$$\underline{\mathscr{H}}^*(u,v,w) \;=\; \underline{\mathscr{H}}(u,v,w)\mathbb{I}_{|\underline{\mathscr{H}}(u,v,w)|\leqslant M_n} \;-\; \mathbb{E}(\underline{\mathscr{H}}_1\mathbb{I}_{|\underline{\mathscr{H}}_1|\leqslant M_n}) \quad \text{where} \quad \left\{\begin{array}{lcl}\underline{\mathscr{H}}_1 &=& \underline{\mathscr{H}}(Z_3,V_1,W_2)\\ M_n &=& n^3/\sqrt{v_n}\end{array}\right.$$

as well as

$$\underline{\mathscr{H}}_I^{**} \;=\; \underline{\mathscr{H}}^{**}(Z_l, V_i, W_j) \;\text{ where }\; \underline{\mathscr{H}}^{**}(u,v,w) = \underline{\mathscr{H}}^{*}(u,v,w) - \underline{\mathscr{H}}^{*}_{1\bullet\bullet}(u) - \underline{\mathscr{H}}^{*}_{\bullet 1\bullet}(v) - \underline{\mathscr{H}}^{*}_{\bullet\bullet 1}(w)$$

which are centred and bounded but are also uncorrelated (see part (i) of Lemma 14 in the Appendix), and we write

$$S_N \;=\; S_N^{(1)} + S_N^{(2)} \;=\; \tfrac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}} \underline{\mathscr{H}}_I^{**} \;+\; \tfrac{\sqrt{v_n}}{N}\sum_{I\in\mathscr{I}} (\underline{\mathscr{H}}_I - \underline{\mathscr{H}}_I^{**}).$$

Introducing $m_n = M_n(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{\epsilon}$ and skipping details, we assess that

$$\mathbb{E}\left((S_N^{(1)})^2\right) \leqslant cst\frac{v_n}{n^3} m_n^{4/5}\mathbb{E}\left(|\underline{\mathscr{H}}_1|^{6/5}\right) \;+\; cst\frac{v_n}{n^3} M_n^{4/5}\mathbb{E}\left(|\underline{\mathscr{H}}_1|^{6/5}\mathbb{I}_{|\mathscr{H}_1|>m_n}\right)$$

and that this quantity converges to 0, as $n\to\infty$, thanks to parts (i) and (iii) of Lemma 13. The same argument is used to prove that $\mathbb{E}(|S_N^{(2)}|) \overset{n\to\infty}{\longrightarrow} 0$.

5.2.5. Proof of relation (37)

Let us first prove that, for some $d\in]4/5,1[$, $\mathbb{E}(|\sqrt{v_n}\widetilde{\widetilde{C}}_n^{(1)}|^d)$ tends to 0, as n tends to infinity. Recall that $\widetilde{\widetilde{C}}_n^{(1)} = \frac{1}{n^3}\sum\sum_{i\neq j} h(V_i,W_j)/\bar{H}(V_i)$. Since $d<1$, we have

$$\mathbb{E}(|\sqrt{v_n}\widetilde{\widetilde{C}}_n^{(1)}|^d) \;\leqslant\; n^{d/2-3d}\left(\bar{G}(t_n)\bar{F}^{(k)}(t_n)\right)^{d/2} n(n-1)\mathbb{E}\left(|h(V_1,W_2)/\bar{H}(V_1)|^d\right).$$

According to part (iv) of Lemma 13, the right-hand side of the inequality above is $O(1)\; v_n^{2-5d/2}$, which tends to 0, since $d>4/5$, and so we are done.

Let us now prove that $\mathbb{E}(|\sqrt{v_n}\widehat{C}_n^{(1)}/n|)$ tends to 0, as n tends to infinity. $\widehat{C}_n^{(1)}$ is defined in (28), where the expectation of each of the four integrals is θ_n : therefore, we only need to prove that $\frac{\sqrt{v_n}}{n}\theta_n$ tends to 0. This is straightforward using part (v) of Lemma 13.

We can prove in a very similar way that $\mathbb{E}(|\sqrt{v_n}\widehat{C}_n^{(2)}/n|)$ tends to 0, as n tends to infinity.

5.3. Proof of Proposition 1

Using the same notations as in the begining of Section 5, we have,

$$\widehat{\gamma}_{n,k} - \gamma_k \;=\; \Delta_n^{-1}\left(\frac{Z_n}{\sqrt{v_n}} + R_n + (\gamma_{n,k}-\gamma_k)\right).$$

The fact that $\frac{Z_n}{\sqrt{v_n}} \overset{\mathbb{P}}{\to} 0$ is due to the application of a triangular weak law of large numbers (see Chow and Teicher (1997) for example) to $\frac{1}{n}\sum\tilde{U}_{i,n}$ and to $\frac{1}{n}\sum\tilde{V}_{i,n}$. By carrefully following the proof of proposition 3 in Section 5.2, we can see that $R_n = o_{\mathbb{P}}(1)$. The condition $\gamma_k<\gamma_C$ is not used, neither in the treatment of $\frac{Z_n}{\sqrt{v_n}}$ nor in that of R_n. Details are omited.

6. Appendix

This appendix contains various results : some of them are used repeatedly in the proof of the main result (in particular Proposition 4, Lemmas 8, and 11, and to a lesser extent Lemmas 10 and 9), the other ones concern parts of the main proof which are postponed to the appendix for better clarity of the main flow of the proof (Lemmas 12, 13 and 14).

Definition 1. *An ultimately positive function $f : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying (at infinity) with index $\alpha\in\mathbb{R}$, if*

$$\lim_{t\to+\infty}\frac{f(tx)}{f(t)} = x^{\alpha}\;\;(\forall x>0).$$

This is noted $f\in RV_\alpha$. If $\alpha=0$, f is said to be slowly varying.

Proposition 4. *(See de Haan and Ferreira (2006) Proposition B.1.9)*
Suppose $f\in RV_\alpha$. If $x<1$ and $\epsilon>0$, then there exists $t_0=t_0(\epsilon)$ such that for every $t\geqslant t_0$,

$$(1-\epsilon)x^{\alpha+\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha-\epsilon}$$

and if $x \geqslant 1$,

$$(1-\epsilon)x^{\alpha-\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha+\epsilon}. \tag{40}$$

Lemma 8. *Let* $x \in \mathbb{R}_+^*$, $\alpha \in \mathbb{R}_+$, $\beta > -1$, *and for* a *and* b *real numbers,* f *and* g *are two regular varying functions at infinity, with index, respectively,* a *and* b. *Then, as* $t \to +\infty$,

(i) $J_\beta(x) = \displaystyle\int_1^{+\infty} \log^\beta(y)\ y^{-x-1}dy = \frac{\Gamma(\beta+1)}{x^{\beta+1}}.$

(ii) $I_{\alpha,a,b} = \displaystyle\int_1^{+\infty} \log^\alpha(y)\ \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b\Gamma(\alpha+1)}{(-a-b)^{\alpha+1}},$ *if* $a+b<0$

(iii) $J_{a,b} = \displaystyle\int_0^1 \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b}{a+b},$ *if* $a+b>0$

Proof :

(i) A simple change of variable and the definition of the Γ function yields the result.

(ii) For the sake of simplicity, we are going to treat the case $a<0$ and $b<0$. The only difference for the other cases is the sign in front of the ϵ or ϵ' appearing below (coming from the application of (40) several times), which can depend on the sign of a, b or another constant, but does not affect the result. Using Potter-bounds (40) for f yields, for n sufficiently large and $\epsilon > 0$,

$$(1+\epsilon)\int_1^{+\infty} \log^\alpha(y)\ y^{a+\epsilon}\ \frac{dg(yt_n)}{g(t_n)} \leqslant I_{\alpha,a,b} \leqslant (1-\epsilon)\int_1^{+\infty} \log^\alpha(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)}.$$

Let us treat only the upper bound and the case $\alpha \neq 0$ (the other cases being similar). By integration by parts, with $a+b<0$, we have

$$\int_1^{+\infty} \log^\alpha(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)} = -\alpha\int_1^{+\infty} \log^{\alpha-1}(y)\ y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy - (a-\epsilon)\int_1^{+\infty} \log^\alpha(y)\ y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy.$$

Using Potter-bounds (40) for g yields, for n sufficiently large and $\epsilon' > 0$

$$\int_1^{+\infty} \log^\alpha(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)} \leqslant -\alpha(1-\epsilon')J_{\alpha-1}(-a-b+\epsilon+\epsilon') - (a-\epsilon)(1+\epsilon')J_\alpha(-a-b+\epsilon-\epsilon').$$

Doing the same with the lower bound and making ϵ and ϵ' tend to 0, yields the result after simplifications.

(iii) As in (ii), using Potter-bounds (40) for f, integration by parts and then again (40) for g yields the result.

Lemma 9. *For any* $\delta > 0$, *let* C_δ *denote the function*

$$C_\delta(t) = \int_0^t \frac{dG(v)}{\bar{G}(v)\bar{H}^\delta(v)}.$$

Under condition (1), *this function is regularly varying of order* δ/γ *and we have* $C_\delta(t) \sim (\gamma/\gamma_C)/(\delta\bar{H}^\delta(t))$, *as* $t \to +\infty$.

Proof : by writing $\bar{H}^\delta(t)C_\delta(t) = -\int_0^1 \frac{\bar{H}^\delta(t)}{H^\delta(tu)}\frac{\bar{G}(t)}{\bar{G}(tu)}\frac{d\bar{G}(tu)}{\bar{G}(t)}$, the lemma is an immediate consequence of part (iii) of Lemma 8, with $a+b = (\delta/\gamma + 1/\gamma_C) + (-1/\gamma_C) = \delta/\gamma > 0$ and $-b/(a+b) = (\gamma/\gamma_C)/\delta$.

Remark 3. *In the Lemma above,* C_1 *is the important function* C *introduced at the beginning of Section 5, and thus* $C(t) \sim (\gamma/\gamma_C)/\bar{H}(t)$, *as* $t \to +\infty$. *Hence,* C *is regularly varying at infinity with index* $1/\gamma$, *a property which proves useful several times in the main proofs.*

Lemma 10. *Let* $\psi(\phi_n, u) = \int_u^{+\infty} \phi_n(s)dF^{(k)}(x)$, *for* $u \geqslant 0$ *and* $\phi_n(u) = \frac{1}{\bar{F}^{(k)}(t_n)}\log(u/t_n)\mathbb{I}_{u>t_n}$. *Under condition* (1), *we have*

$$\begin{aligned}\psi(\phi_n,u) &= \gamma_{n,k}, \text{ if } u \leqslant t_n \\ &= \log\left(\frac{u}{t_n}\right)\frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} + \gamma_k\left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \epsilon_n(u)\left(\frac{u}{t_n}\right)^{-1/\gamma_k+\delta} \quad \text{if } u > t_n,\end{aligned}$$

where $\epsilon_n(u)$ is a sequence tending to 0 uniformly in u, as $n \to \infty$, and δ a positive real number such that $-\frac{1}{\gamma_k} + \delta < 0$.

Proof : We only consider the second situation where $u > t_n$ (the first one is straightforward) :

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = -\int_{\frac{u}{t_n}}^{+\infty} \log(y) \frac{d\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)}$$

An integration by part and the fact that $\bar{F}^{(k)}$ is regularly varying at infinity with index $-1/\gamma_k$, yields

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = \log\left(\frac{u}{t_n}\right) \frac{\bar{F}^{(k)}(u)}{\bar{F}^{(k)}(t_n)} + \gamma_k \left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \Delta_n(u),$$

where

$$\Delta_n(u) = \int_{\frac{u}{t_n}}^{+\infty} \left(\frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-1/\gamma_k}\right) \frac{dy}{y}$$

Let δ be a positive real number. Then

$$\begin{aligned} |\Delta_n(u)| &= \left| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} \left(y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right) dy \right| \\ &\leqslant \sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy, \end{aligned}$$

where the function $y \to y^{1/\gamma_k - \delta} \bar{F}^{(k)}(y)$ is regularly varying with index $-\delta$. Then since

$$\sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\bar{F}^{(k)}(yt_n)}{\bar{F}^{(k)}(t_n)} - y^{-\delta} \right| \overset{n \to \infty}{\longrightarrow} 0$$

and, when $-\frac{1}{\gamma_k} + \delta < 0$, we have $\int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy = cst\,(u/t_n)^{-1/\gamma_k + \delta}$, this concludes the proof.

Lemma 11. *Recalling that H is a distribution function with infinite right endpoint, we have :*

(i) $\sup_{0 \leqslant x < Z^{(n)}} \bar{H}(x)/\bar{H}_n(x) = O_{\mathbb{P}}(1)$

(ii) for any $a < 1/2$,

$$\sqrt{n} \sup_{t \geqslant 0} \frac{|\bar{H}_n(t) - \bar{H}(t)|}{(\bar{H}(t))^a} = O_{\mathbb{P}}(1) \quad \text{and} \quad \sqrt{n} \sup_{t \geqslant 0} \frac{|\bar{H}_n^{(0)}(t) - \bar{H}^{(0)}(t)|}{(\bar{H}^{(0)}(t))^a} = O_{\mathbb{P}}(1).$$

Proof : part (i) is well known (see for instance section 3 of chapter 10 of Shorack and Wellner (1986)), while the two statements in (ii) are proved by usual empirical processes techniques, showing that the family of functions $(f_t)_{t<\infty}$ defined in one case by $f_t(z) = \mathbb{I}_{z>t}/(\bar{H}(t))^a$, and in the other case by $f_t(\delta, z) = (1-\delta)\mathbb{I}_{z>t}/(\bar{H}^{(0)}(t))^a$ are Donsker whenever $a < 1/2$ (using respective square integrable envelope functions $f^*(z) = 1/(\bar{H}(z))^a$ and $f^*(\delta, z) = (1-\delta)/(\bar{H}^{(0)}(z))^a$, which bound from above the functions f_t uniformly in t) .

Lemma 12. *Under conditions (1) and (2), suppose that $\alpha \geqslant 0$ and $d \geqslant 1$ are real numbers. If $\gamma_k < \gamma_C$ and*

$$X_{i,n} = \frac{\sqrt{v_n}}{n^{1+d}} \frac{\phi(Z_i)}{\bar{G}(Z_i)(\bar{H}^{(0)}(Z_i))^{d+\alpha}} \mathbb{I}_{\xi_i = k},$$

then we have $\sum_{i=1}^n X_{i,n} \xrightarrow{\mathbb{P}} 0$, as n tends to infinity, if α is 0 or sufficiently close to it.

Proof :

According to the LLN for triangular arrays, we need to prove the following three statements :

$$\begin{aligned} &(i) \quad \forall \epsilon > 0, \ \sum_{i=1}^n \mathbb{P}(|X_{i,n}| > \epsilon) \overset{n \to \infty}{\longrightarrow} 0 \\ &(ii) \quad \sum_{i=1}^n \mathbb{E}((X_{i,n})^2 \mathbb{I}_{|X_{i,n}| \leqslant 1}) \overset{n \to \infty}{\longrightarrow} 0 \\ &(iii) \quad \sum_{i=1}^n \mathbb{E}(X_{i,n} \mathbb{I}_{|X_{i,n}| \leqslant 1}) \overset{n \to \infty}{\longrightarrow} 0 \end{aligned}$$

But, $X_{i,n}$ being positive, (iii) clearly implies (ii). We thus need to prove that (i) and (iii) hold.

Let us start with assertion (i). If $\epsilon > 0$ is given, then

$$X_{i,n} \;=\; \frac{v_n^{1/2}}{n^{1+d}} \frac{\log(Z_i/t_n)}{\bar F^{(k)}(t_n)\bar G(t_n)(\bar H^{(0)}(t_n))^{d+\alpha}} \mathbb{I}_{Z_i>t_n}\mathbb{I}_{\xi_i=k} \frac{\bar G(t_n)}{\bar G(Z_i)} \left(\frac{\bar H^{(0)}(t_n)}{\bar H^{(0)}(Z_i)}\right)^{d+\alpha}.$$

Now, put $a = \frac{1}{\gamma_C} + \frac{d+\alpha}{\gamma}$ (> 0); since, for a given $\epsilon' > 0$, there exists $c > 0$ such that $\forall x \geqslant 1$, $\log(x) \leqslant cx^{\epsilon'}$, and using Potter-bounds (40) for $\bar G^{-1}(\bar H^{(0)})^{-(d+\alpha)} \in RV_{-a}$, we can write (using the definition of v_n)

$$\begin{aligned}\{\,|X_{i,n}| > \epsilon\,\} \;&\subset\; \left\{ v_n^{-1/2} n^{-d} \left(\bar H^{(0)}(t_n)\right)^{-(d+\alpha)} c(1+\epsilon') \left(\frac{Z_i}{t_n}\right)^{a+2\epsilon'} > \epsilon \right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\} \\ &\subset\; \left\{\, Z_i > c(\epsilon,\epsilon')\, t_n w_n \,\right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\},\end{aligned}$$

where $w_n = \left(v_n^{1/2} n^d \left(\bar H^{(0)}(t_n)\right)^{d+\alpha}\right)^{1/(a+2\epsilon')}$ and $c(\epsilon,\epsilon')$ is a constant depending on ϵ and ϵ' only. Consequently, if w_n tends to infinity,

$$\begin{aligned}\sum_{i=1}^n \mathbb{P}(|X_{i,n}| > \epsilon) \;&\leqslant\; n\mathbb{E}(\mathbb{I}_{Z_i > c(\epsilon,\epsilon')\, t_n w_n}\mathbb{I}_{\xi_i=k}) = n\int_{c(\epsilon,\epsilon')\, t_n w_n}^{\infty} \bar G(x) dF^k(x) \\ &\leqslant\; v_n \frac{(\bar F^{(k)}\bar G)(c(\epsilon,\epsilon')\, t_n w_n)}{(\bar F^{(k)}\bar G)(t_n)} \\ &\leqslant\; cst\; v_n w_n^{-\beta},\end{aligned}$$

where $\beta = \frac{1}{\gamma_C} + \frac{1}{\gamma_k} - \epsilon'$ and the last inequality is due to Potter-bounds (40) applied to $\bar F^{(k)}\bar G \in RV_{-\frac{1}{\gamma_C}-\frac{1}{\gamma_k}}$. Then, assertion (i) above will be true as soon as we prove that $w_n \to \infty$ and $v_n w_n^{-\beta} \to 0$, as $n \to \infty$. Since $\bar H^{(0)}(t)$ is equivalent to a positive constant times $\bar H(t)$ when $t \to +\infty$, and $\bar H(t_n) \geqslant v_v/n$, then $w_n^{a+2\epsilon'} \geqslant cst\ (n^{-\eta} v_n)^r$, for $r = \frac{1}{2} + d + \alpha > 0$ and $\eta = \frac{\alpha}{r} \geqslant 0$. Assumption (2) finally yields that w_n tends to $+\infty$, since $0 \leqslant \eta \leqslant \eta_0$ for α sufficiently close to 0.

Now, proving that $v_n w_n^{-\beta}$ tends to 0 is equivalent to proving that $v_n^{-(a+2\epsilon')/\beta} v_n^{1/2} n^d \left(\bar H^{(0)}(t_n)\right)^{d+\alpha}$ tends to $+\infty$. The same arguments as in the previous paragraph yield that it is sufficient to prove that $v_n^A n^{-\alpha} = (n^{-\eta} v_n)^A$ tends to $+\infty$, for $A = -(a+2\epsilon')/\beta + 1/2 + d + \alpha$ and $\eta = \frac{\alpha}{A}$. This is a consequence of hypothesis (2), since $A > 0$ and $\alpha \leqslant \eta_0 A$, for α sufficiently close to 0. This ends the proof of (i).

Let us now start the proof of assertion (iii). If $\epsilon > 0$ is given, using Potter-Bounds (40) for $\bar G^{-1}(\bar H^{(0)})^{-(d+\alpha)}$ which belongs to RV_{-a}, and introducing $h(x) = \log(x) x^{a-\epsilon}$, we find that (for some positive constant c)

$$\mathbb{I}_{|X_{i,n}|\leqslant 1}\mathbb{I}_{Z_i>t_n} \leqslant \mathbb{I}_{h(Z_i/t_n)\leqslant c w_n}\mathbb{I}_{Z_i>t_n}$$

where we set $w_n = v_n^{1/2} n^d \left(\bar H^{(0)}(t_n)\right)^{d+\alpha}$. Hence, denoting by h^{-1} the inverse function of h,

$$\mathbb{I}_{|X_{i,n}|\leqslant 1}\mathbb{I}_{Z_i>t_n}\mathbb{I}_{\xi_i=k} \leqslant \mathbb{I}_{t_n<Z_i<t_n h^{-1}(cw_n)}\mathbb{I}_{\xi_i=k}.$$

Consequently, using once again Potter-Bounds (40) and bounding the log with a constant times a power of z/t_n, we get

$$\begin{aligned} n\mathbb{E}(X_{1,n}\mathbb{I}_{|X_{1,n}|\leqslant 1}) \;&\leqslant\; \frac{v_n^{1/2}}{n^d} \int_{t_n}^{t_n h^{-1}(cw_n)} \frac{\log(z/t_n)}{\bar F^{(k)}(t_n)\bar G(z)(\bar H^{(0)}(z))^{d+\alpha}} dH^{(1,k)}(z) \\ &\leqslant\; cst\ \frac{v_n}{w_n} \int_1^{h^{-1}(cw_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(st_n)}{\bar F^{(k)}(t_n)},\end{aligned}$$

where $b = \frac{d+\alpha}{\gamma}$ and $\epsilon' > 0$ is some given positive value (the inequality $\log(s) \leqslant cst\, s^{\epsilon'}$, $\forall s \geqslant 1$, was used). But, by integration by parts and (40) applied to $\bar F^{(k)}$, setting $h_n = h^{-1}(cw_n)$, we have

$$\frac{v_n}{w_n} \int_1^{h^{-1}(cw_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(st_n)}{\bar F^{(k)}(t_n)} \leqslant cst \frac{v_n}{w_n}\left(1 + h_n^{b-1/\gamma_k+3\epsilon'}\right).$$

Proceeding similarly as in the previous paragraphs, we find that $w_n/v_n \to \infty$ (and thus w_n and h_n as well) thanks to assumption (2), for α close to 0. We are thus left to prove that $(v_n/w_n) \times h_n^{b'}$ tends to 0, where $b' = b - 1/\gamma_k + 3\epsilon'$. If $b - 1/\gamma_k$ is negative, this is immediate. We thus suppose that $b - 1/\gamma_k \geqslant 0$ and, after some simple computations, we find out that $(v_n/w_n)h_n^{b'}$ tends to 0 if $v_n^{-a+\epsilon'} w_n^{a-b'-\epsilon'}$ tends to ∞, a property

which holds true thanks to assumption (2), for α close to 0 (we omit the details).

Lemma 13. *Suppose that V_1 and W_2 are independent improper random variables of respective subdistribution functions $H^{(0)}$ and $H^{(1,k)}$, and Z_3 is independent of V_1 and W_2 and has distribution H. Consider h, $\underline{h}$, $\mathscr{H}$ and $\underline{\mathscr{H}}$ the functions defined in (27) and (36).*

(*i*) *For any $d \geqslant 1$, there exist some positive constants c and c' such that*

$$\mathbb{E}\left(\,|\mathscr{H}^d(V_1,W_2)|\,\right) \leqslant c\,\mathbb{E}\left(\,h^d(V_1,W_2)\,\right) \quad \text{and} \quad \mathbb{E}\left(\,|\underline{\mathscr{H}}^d(Z_3,V_1,W_2)|\,\right) \leqslant c'\,\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right).$$

(*ii*) *For any $d \in]1, 1+(1+2\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,h^d(V_1,W_2)\,\right) = O\left((\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{2(1-d)}\right).$$

In particular, if $\gamma_k < \gamma_C$, then $\mathbb{E}(h^{4/3}(V_1,W_2))$ is of the order of $(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-2/3}$ and $\mathbb{E}(h^d(V_1,W_2))$ is finite whenever d is (greater than but) sufficiently close to $4/3$.

(*iii*) *For any $d \in]1, 1+(1+3\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right) = O\left((\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{3(1-d)}\right).$$

In particular, if $\gamma_k < \gamma_C$, then $\mathbb{E}(\underline{h}^{6/5}(Z_3,V_1,W_2))$ is of the order of $(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{-3/5}$ and $\mathbb{E}(\underline{h}^d(Z_3,V_1,W_2))$ is finite whenever d is (greater than but) sufficiently close to $6/5$.

(*iv*) *For any $d \in]1/2, (2\gamma_C^{-1}+\gamma_F^{-1}+\gamma_k^{-1})/(3\gamma_C^{-1}+2\gamma_F^{-1})[$, we have $\mathbb{E}\left(h^d(V_1,W_2)/\bar{H}^d(V_1)\right) = O\left((\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{2-3d}\right)$. In particular, if $\gamma_k < \gamma_C$ then taking δ (greater than but) sufficiently close to $4/5$ is permitted, otherwise it is $2/3$ instead of $4/5$.*

(*v*) *The integral $\theta_n = \iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w)$ is equivalent, as $n \to \infty$, to $\gamma_k(-\log \bar{G}(t_n))$.*

Proof :

(*i*) Let $d \geqslant 1$, and remind that h is a non-negative function. Using several times the inequality $|a+b|^d \leqslant 2^{d-1}(|a|^d + |b|^d)$, we can write

$$\mathbb{E}(|\mathscr{H}^d(V_1,W_2)|) \leqslant cst\left\{\mathbb{E}(h^d(V_1,W_2)) + \mathbb{E}[(h_{1\bullet}(V_1))^d] + \mathbb{E}[(h_{\bullet 1}(W_2))^d] + (\mathbb{E}(h(V_1,W_2)))^d\right\}.$$

But using the fact that the L^1 norm is bounded by the L^d norm whenever $d \geqslant 1$, we have $(\mathbb{E}(h(V_1,W_2)))^d \leqslant \mathbb{E}(h^d(V_1,W_2))$ and it is quite simple to prove (by independency of V_1 and W_2) that it is also the case of $\mathbb{E}[(h_{1\bullet}(V_1))^d] = \mathbb{E}[(\mathbb{E}(h(V_1,W_2)|V_1))^d] \leqslant \mathbb{E}[\mathbb{E}(h^d(V_1,W_2)|V_1)] = \mathbb{E}(h^d(V_1,W_2))$, as well as for $\mathbb{E}[(h_{\bullet 1}(W_2))^d]$. The inequality is thus proved. The other one (concerning $\underline{\mathscr{H}}$ and $\underline{h}$) is proved similarly.

(*ii*) Let $d > 1$. Since $h(v,\infty) = h(\infty,w) = 0$ $(\forall v,w)$, we have

$$\begin{aligned}
\mathbb{E}(h^d(V_1,W_2)) &= (\bar{F}^{(k)}(t_n))^{-d} \iint \log^d(w/t_n)(\bar{H}(v)\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w) \\
&= (\bar{F}^{(k)}(t_n))^{1-d}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\int_0^w \frac{dG(v)}{\bar{G}(v)\bar{H}^{d-1}(v)}\right)\bar{G}^{1-d}(w)\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)} \\
&= \frac{C_{d-1}(t_n)}{(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{d-1}}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\frac{\bar{G}(t_n)}{\bar{G}(w)}\right)^{d-1}\frac{C_{d-1}(w)}{C_{d-1}(t_n)}\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)}
\end{aligned}$$

where the function C_{d-1} was defined in the statement of Lemma 9. This lemma and Lemma 8, applied with $\alpha = d$, $a = (d-1)/\gamma_C + (d-1)/\gamma$ and $b = -1/\gamma_k$ (the constraint specified on d certifies that $a+b<0$), imply that the integral in the previous line converges to a constant. And Lemma 9 also implies that the ratio in front of this integral is equivalent, as $n \to \infty$, to a positive constant times $\left(\bar{H}(t_n)\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{1-d}$, which is itself lower than $\left(\bar{F}^{(k)}(t_n)\bar{G}(t_n)\right)^{2(1-d)}$, as desired.

(*iii*) Let $d > 1$. By definition of $\underline{h}$ in (27), and proceeding as in the previous item, $\mathbb{E}(\underline{h}^d(Z_3,V_1,W_2))$ equals

$$\begin{aligned}
&(\bar{F}^{(k)}(t_n))^{-d}\iiint \log^d\left(\tfrac{w}{t_n}\right)(\bar{H}(v))^{-2d}(\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}\mathbb{I}_{u>v}dH(u)dH^{(0)}(v)dH^{(1,k)}(w) \\
= \;& \frac{C_{2d-2}(t_n)}{(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{d-1}}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\frac{\bar{G}(t_n)}{\bar{G}(w)}\right)^{d-1}\frac{C_{2d-2}(w)}{C_{2d-2}(t_n)}\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)},
\end{aligned}$$

which is equivalent to $O\left((\bar{H}(t_n))^{2-2d}(\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{1-d}\right) = O\left((\bar{F}^{(k)}(t_n)\bar{G}(t_n))^{3(1-d)}\right)$ as soon as, thanks to Lemma 8, the sum $((d-1)/\gamma_C + (2d-2)/\gamma) - 1/\gamma_k$ is negative, which turns out to be true whenever $d < 1 + (2 + 3\gamma_k/\gamma_C)^{-1}$, as specified.

(*iv*) The proof is very similar to the previous ones, starting from

$$\mathbb{E}\left(h^d(V_1,W_2)/\bar{H}^d(V_1)\right) = (\bar{F}^{(k)}(t_n))^{-d}\iint \log^d(w/t_n)(\bar{H}(v))^{-2d}(\bar{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w)$$

so we omit the details.

(*v*) Noting that $-\log\bar{G}$ is slowly varying at infinity null at 0, we have

$$\theta_n = \int_{t_n}^{\infty}\log(w/t_n)\left(\int_0^w dG(v)/\bar{G}(v)\right)\frac{dF^{(k)}(w)}{\bar{F}^{(k)}(t_n)} = (-\log\bar{G}(t_n))\left(-\int_1^{\infty}\log(u)\frac{-\log\bar{G}(ut_n)}{-\log\bar{G}(t_n)}\frac{d\bar{F}^{(k)}(ut_n)}{\bar{F}^{(k)}(t_n)}\right)$$

which can be dealt with using part (*ii*) of Lemma 8 with $\alpha = 1$, $a = 0$ and $b = -1/\gamma_k$: the obtained constant is indeed equal to γ_k.

Lemma 14. *In this Lemma, various notations defined in sections 5.2.2 to 5.2.4 are used.*

(*i*) *The variables $\mathscr{H}_I^{**}$ for $I \in \{(i,j)\,;\,1 \leqslant i < j \leqslant n\}$ are centred and uncorrelated . This is also true for the variables $\underline{\mathscr{H}}_I^{**}$ for $I \in \{(i,j,l)\,;\,1 \leqslant i < j < l \leqslant n\}$.*

(*ii*) *We have $\mathbb{E}\left[(\mathscr{H}^{**}(V_1,W_2))^2\right] \leqslant 48\mathbb{E}[\mathscr{H}_1^2\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n}]$.*

(*iii*) *We have $\mathbb{E}\left(\,|\mathscr{H}_1 - \mathscr{H}^*(V_1,W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\,\right) \leqslant 4\mathbb{E}\left(|\mathscr{H}_1|\mathbb{I}_{|\mathscr{H}_1|>M_n}\right)$*

Proof :

(*i*) Let us consider the first situation, where $\mathscr{I} = \{(i,j)\,;\,1 \leqslant i < j \leqslant n\}$. First, if $I = (i,j) \in \mathscr{I}$, then $\mathbb{E}(\mathscr{H}_I^{**}) = 0 - \mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) - \mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j))$; but, by definition of $\mathscr{H}_{1\bullet}^*$ and independency of V_i and W_j, we have $\mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) = \mathbb{E}(\mathscr{H}^*(V_i,W_j)) = 0$, and $\mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j)) = 0$ is obtained similarly, so we proved that $\mathbb{E}(\mathscr{H}_I^{**}) = 0$. Note that we can prove (with similar arguments) that $\mathscr{H}_{1\bullet}^{**}(v) = \mathscr{H}_{\bullet 1}^{**}(w) = 0$ for every v, w in $[0,\infty]$, a property which is repeatedly used below . Let us now deal with the non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$, by considering the various cases where $I \neq I'$ with $I = (i,j)$ and $I' = (k,l)$ are in $\mathscr{I}$.

If all four indices i,j,k,l are distinct, then non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$ is immediate by mutual independence of the variables $Z_1,\ldots,Z_n$.

If $i = k$ but $j \neq l$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i))$ where $\psi(v) = \mathbb{E}(\mathscr{H}^{**}(v,W_j)\mathscr{H}^{**}(v,W_l)) = (\mathscr{H}_{1\bullet}^{**}(v))^2 = 0$, by independence of V_i with (W_j,W_l), and of W_j and W_l.

The case $i \neq k$ and $j = l$ is similar using $\mathscr{H}_{\bullet 1}^{**}(\cdot) \equiv 0$.

If $i = l$ but $j \neq k$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i,W_i))$ where $\psi(v,w) = \mathbb{E}(\mathscr{H}^{**}(v,W_j)\mathscr{H}^{**}(V_k,w)) = \mathscr{H}_{1\bullet}^{**}(v)\mathscr{H}_{\bullet 1}^{**}(w) = 0 \times 0 = 0$; the case $j = k$ and $i \neq l$ is treated similarly.

Note that the case $i = l$ and $j = k$ (*i.e.* $\mathscr{H}_I^{**} = \mathscr{H}^{**}(V_i,W_j)$, $\mathscr{H}_{I'} = \mathscr{H}^{**}(V_j,W_i)$) is not permitted (it would lead to dependency) since we cannot have simultaneously $i < j$ and $j < i$; this is the reason why, in the beginning of section 5.2.3, we restricted the study of the sum $\mathscr{U}_n$ to that of the sum S_N having terms $\mathscr{H}(V_i,W_j)$ satisfying $i < j$.

The second situation, for $\underline{\mathscr{H}}_I^{**}$ and $\underline{\mathscr{H}}_{I'}^{**}$ with $I \neq I'$ in $\mathscr{I} = \{I = (i,j,l)\,;\,1 \leqslant i < j < l \leqslant n\}$, is a bit more tedious (with more cases to detail) but very similar, so we omit its proof.

(*ii*) We start by the trivial bound

$$\mathbb{E}[(\mathscr{H}^{**}(V_1,W_2))^2] \leqslant 4\left\{\mathbb{E}[(\mathscr{H}^*(V_1,W_2))^2] + \mathbb{E}[(\mathscr{H}_{1\bullet}^*(V_1))^2] + \mathbb{E}[(\mathscr{H}_{\bullet 1}^*(W_2))^2]\right\}.$$

Noting $\mathscr{H}_1^- = \mathscr{H}_1\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n}$, we can write, on one hand, by definition of $\mathscr{H}^*$, $\mathbb{E}[(\mathscr{H}^*(V_1,W_2))^2] \leqslant 2\left\{\mathbb{E}[(\mathscr{H}_1^-)^2] + (\mathbb{E}[\mathscr{H}_1^-])^2\right\} \leqslant 4\mathbb{E}[(\mathscr{H}_1^-)^2]$. On the other hand, if W is independent of V_1, we have $\mathbb{E}[(\mathscr{H}_{1\bullet}^*(V_1))^2] = \mathbb{E}[(\mathbb{E}[\mathscr{H}^*(V_1,W)|V_1])^2] \leqslant \mathbb{E}[\mathbb{E}[(\mathscr{H}^*(V_1,W))^2|V_1]] = \mathbb{E}[(\mathscr{H}^*(V_1,W_2))^2]$, which is the same term as the first one, and is thus lower than $4\mathbb{E}[(\mathscr{H}_1^-)^2]$. The same is true of $\mathbb{E}[(\mathscr{H}_{\bullet 1}^*(W_2))^2]$, so the desired inequality is proved.

(*iii*) First recall that $\mathscr{H}_1$ denotes $\mathscr{H}(V_1, W_2)$. Now, since $\mathscr{H}_1$ is centred and we trivially have $\mathscr{H}_1 = \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| \leqslant M_n} + \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| > M_n}$, noting $\mathscr{H}_1^+ = \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| > M_n}$ yields

$$\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) = \mathscr{H}_1^+ - \mathbb{E}(\mathscr{H}_1^+).$$

Secondly, using the fact that $\mathscr{H}_{1\bullet}(\cdot) \equiv 0$ (simple to prove), we can write

$$\mathscr{H}_{1\bullet}^*(v) = \mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| \leqslant M_n}) - \mathbb{E}(\mathscr{H}_1 \mathbb{I}_{|\mathscr{H}1| \leqslant M_n}) = -\mathscr{H}_{1\bullet}^+(v) + \mathbb{E}(\mathscr{H}_1^+),$$

where $\mathscr{H}_{1\bullet}^+(v)$ denotes $\mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| > M_n})$ and satisfies $\mathbb{E}(\mathscr{H}_{1\bullet}^+(V_1)) = \mathbb{E}(\mathscr{H}_1^+)$, and similarly

$$\mathscr{H}_{\bullet 1}^*(w) = -\mathscr{H}_{\bullet 1}^+(w) + \mathbb{E}(\mathscr{H}_1^+)$$

with $\mathscr{H}_{\bullet 1}^+(w) = \mathbb{E}(\mathscr{H}(V, w)\mathbb{I}_{|\mathscr{H}(V,w)| > M_n})$ and $\mathbb{E}(\mathscr{H}_{\bullet 1}^+(W_2)) = \mathbb{E}(\mathscr{H}_1^+)$. Summing these three terms finally leads to

$$\mathbb{E}\left(\,|\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\,\right) \quad = \quad \mathbb{E}\left(\,|\mathscr{H}_1^+ - \mathscr{H}_{1\bullet}^+(V_1) - \mathscr{H}_{\bullet 1}^+(W_2) + \mathbb{E}(\mathscr{H}_1^+)|\,\right)$$

which is lower than $4\mathbb{E}(|\mathscr{H}_1^+|)$, as announced.

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