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Stable recovery of the factors from a deep matrix product

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Abstract—We study a deep matrix factorization problem. It takes as input the matrix X obtained by multiplying K matrices (called factors) and aims at recovering the factors. When $K = 1$, this is the usual compressed sensing framework; $K = 2$: Examples of applications are dictionary learning, blind deconvolution, self-calibration; $K \geq 3$: includes convolution network problems as well as many fast transforms.

Using a Lifting, we provide : a necessary and sufficient conditions for the identifiability of the factors (up to a scale rearrangement); - an analogue of the Null-Space-Property, called the Deep-Null-Space-Property which is necessary and sufficient to guarantee the stable recovery of the factors.

I. INTRODUCTION

Let $K \in \mathbb{N}^*$, $m_1 \dots m_{K+1} \in \mathbb{N}$, write $m_1 = m$, $m_{K+1} = n$. We impose the factors to be structured matrices defined by a (typically small) number S of unknown parameters. More precisely, for $k = 1 \dots K$, let

$$\begin{aligned} M_k : \mathbb{R}^S &\longrightarrow \mathbb{R}^{m_k \times m_{k+1}}, \\ h &\longmapsto M_k(h) \end{aligned}$$

be a linear map.

We assume that we know the matrix $X \in \mathbb{R}^{m \times n}$ which is provided by

$$X = M_1(\mathbf{h}_1) \cdots M_K(\mathbf{h}_K) + e, \quad (1)$$

for an unknown error term e and parameters $\mathbf{h} = (\mathbf{h}_k)_{1 \leq k \leq K} \in \mathcal{M}^{\bar{L}} \subset \mathbb{R}^{S \times K}$ for some \bar{L} , where we assume that we know a collection of models $\mathcal{M} = (\mathcal{M}^L)_{L \in \mathbb{N}}$ such that, for every L , $\mathcal{M}^L \subset \mathbb{R}^{S \times K}$.

This work investigates models/constraints imposed on (1) for which we can (up to obvious scale rearrangement) identify or stably recover the parameters \mathbf{h} from X . A preliminary version of this work is presented in [1].

Set $\mathbb{N}_K = \{1, \dots, K\}$ and

$$\mathbb{R}_*^{S \times K} = \{\mathbf{h} \in \mathbb{R}^{S \times K}, \forall k \in \mathbb{N}_K, \|\mathbf{h}_k\| \neq 0\}.$$

Define an equivalence relation in $\mathbb{R}_*^{S \times K}$: for any $\mathbf{h}, \mathbf{g} \in \mathbb{R}_*^{S \times K}$, $\mathbf{h} \sim \mathbf{g}$ if and only if there exists $(\lambda_k)_{k \in \mathbb{N}_K} \in \mathbb{R}^K$ such that

$$\prod_{k=1}^K \lambda_k = 1 \quad \text{and} \quad \forall k \in \mathbb{N}_K, \mathbf{h}_k = \lambda_k \mathbf{g}_k.$$

Denote the equivalence class of $\mathbf{h} \in \mathbb{R}_*^{S \times K}$ by $[\mathbf{h}]$. We consider a metric denoted d_p on $\mathbb{R}_*^{S \times K} / \sim$. It is based on the l^p norm.

We say that a tensor $T \in \mathbb{R}_*^{S \times K}$ is of rank 1 if and only if there exists a collection of vectors $\mathbf{h} \in \mathbb{R}_*^{S \times K}$ such that T is the outer product of the vectors \mathbf{h}_k , for $k \in \mathbb{N}_K$, that is, for any $\mathbf{i} \in \mathbb{N}_S^K$,

$$T_{\mathbf{i}} = \mathbf{h}_{1,i_1} \cdots \mathbf{h}_{K,i_K}.$$

The set of all the tensors of rank 1 is denoted by Σ_1 .

Moreover, we parametrize $\Sigma_1 \subset \mathbb{R}^{S^K}$ by the Segre embedding

$$\begin{aligned} P : \mathbb{R}^{S \times K} &\longrightarrow \Sigma_1 \subset \mathbb{R}^{S^K} \\ \mathbf{h} &\longmapsto (\mathbf{h}_{1,i_1} \mathbf{h}_{2,i_2} \cdots \mathbf{h}_{K,i_K})_{\mathbf{i} \in \mathbb{N}_S^K} \end{aligned}$$

Following [2], [3], [4], [5], [6], [7] where problems such that $K = 2$ are studied, we can lift the problem and show that the map

$$(\mathbf{h}_1, \dots, \mathbf{h}_K) \longmapsto M_1(\mathbf{h}_1)M_2(\mathbf{h}_2) \cdots M_K(\mathbf{h}_K),$$

uniquely determines a linear map

$$\mathcal{A} : \mathbb{R}^{S^K} \longrightarrow \mathbb{R}^{m \times n},$$

such that for all $\mathbf{h} \in \mathbb{R}^{S \times K}$

$$M_1(\mathbf{h}_1)M_2(\mathbf{h}_2) \cdots M_K(\mathbf{h}_K) = \mathcal{A}P(\mathbf{h}).$$

When $\|e\| = 0$, we can prove that every element of $\mathbf{h} \in \mathcal{M}$ is identifiable (i.e. the elements of $[\mathbf{h}]$ are the only solutions of (1)) if and only if for any L and $L' \in \mathbb{N}$

$$\text{Ker}(\mathcal{A}) \cap (P(\mathcal{M}^L) - P(\mathcal{M}^{L'})) = \{0\}.$$

When $\|e\| \leq \delta$, we further assume that we have a way to find L^* and $\mathbf{h}^* \in \mathcal{M}^{L^*}$ such that, for some parameter $\eta > 0$,

$$\|\mathcal{A}P(\mathbf{h}^*) - X\|^2 \leq \eta. \quad (2)$$

Using the following definition we can prove Theorem 1.

Definition 1. Deep-Null Space Property

Let $\gamma > 0$, we say that $\text{Ker}(\mathcal{A})$ satisfies the deep-Null Space Property (deep-NSP) with respect to the model collection \mathcal{M} with constant γ if there exists $\varepsilon > 0$ such that for any L and $L' \in \mathbb{N}$, any $T \in P(\mathcal{M}^L) - P(\mathcal{M}^{L'})$ satisfying $\|AT\| \leq \varepsilon$ and any $T' \in \text{Ker}(\mathcal{A})$, we have

$$\|T\| \leq \gamma \|T - T'\|.$$

Theorem 1. Sufficient condition for stable recovery

Assume $\text{Ker}(\mathcal{A})$ satisfies the deep-NSP with respect to the collection of models \mathcal{M} and with the constant $\gamma > 0$. For any \mathbf{h}^* as in (2) with η and δ sufficiently small, we have

$$\|P(\mathbf{h}^*) - P(\bar{\mathbf{h}})\| \leq \frac{\gamma}{\sigma_{\min}} (\delta + \eta),$$

where σ_{\min} is the smallest non-zero singular value of \mathcal{A} . Moreover, if $\bar{\mathbf{h}} \in \mathbb{R}_*^{S \times K}$

$$d_p([\mathbf{h}^*], [\bar{\mathbf{h}}]) \leq \frac{7(KS)^{\frac{1}{p}} \gamma}{\sigma_{\min}} \min \left(\|P(\bar{\mathbf{h}})\|_{\infty}^{\frac{1}{K}-1}, \|P(\mathbf{h}^*)\|_{\infty}^{\frac{1}{K}-1} \right) (\delta + \eta).$$

We also prove that the deep-NSP condition is **necessary** for the stable recovery of the factors. We study a variante of the deep-NSP, dedicated to models \mathcal{M} enforcing sparsity, and discuss the applications of these results for the stable recovery of convolution network.

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