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ABSTRACT. Spectral representations uniquely define the covariance functions associated to random fields defined over spheres or spheres cross time. Covariance functions on spheres cross time are usually modelled under the assumptions of either spatial isotropy or axial symmetry, and the assumption of temporal stationarity. This paper goes beyond these assumptions. In particular, we consider the problem of spatially anisotropic covariance functions on spheres. The crux of our criterion is to escape from the addition theorem for spherical harmonics. We also challenge the problem of temporal nonstationarity in nonseparable space-time covariance functions, where space is the n-dimensional sphere.

Keywords: Gegenbauer Polynomial, Positive Definite, Space-time Random Field, Spectral Representation, Spherical Harmonic.

1. INTRODUCTION

The literature on positive definite kernels on spheres has become ubiquitous, and we refer the reader to the essay by [6], with the list of references given there. Both mathematical and statistical communities have been interested in the construction of positive definite functions defined over product spaces where the sphere is involved. The recent tours de force in [2, 7, 9] in concert with the works by [6, 3, 18] and the recent review in [12] are apparent indications that there are some important branches of the mathematical and statistical communities devoted to studying such constructions.

Recently, [17] have proposed an overview of statistical approaches for global data, showing that positive definite functions have a crucial role for modeling temporally evolving phenomena defined over the sphere representing planet Earth. Recent statistical works are concerned with the

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construction of space-time covariances where the space is the spherical shell (see [12, 18, 4] for instance). Characterization theorems for positive definite kernels on n-dimensional spheres have been provided in seminal papers by [19] and [16]. Then, [3] and [18] extended Schoenberg's representation theorem to space-time. In particular, [18] focus on construction principles that allow for algebraically tractable closed forms, and then analyze the discrepancy between using the correct metric on the sphere and other metrics. The literature on spatial anisotropy on spheres has instead been sparse, with the work of [11] being a notable exception. Specifically, the authors show spectral representations for anisotropic processes defined over $\mathbb{R}^3$ and then restricted to the sphere. Under such a scheme, the covariance function depends on the chordal distance, being the segment *below* the arc joining any pair of points located over the spherical shell. The natural metric on the sphere is the great circle or geodesic distance, and for constructive criticism about the use of chordal distance the reader is referred to [6] as well as to [17].

This paper goes beyond spatial isotropy and temporal stationarity by avoiding the traditionally used addition theorem for spherical harmonics (see [19] as a seminal reference and [14, 3] for instance as more recent works). A similar strategy is adopted by [13] and by [20] to obtain spectral representations for axially symmetric processes.

In particular, we provide spectral representations for positive definite kernels on $(\mathbb{S}^n \times T)^2$, with $\mathbb{S}^n$ being the n-dimensional sphere of $\mathbb{R}^{n+1}$ with unit radius, and where T denotes time, which might be the whole real line or a compact set. On the basis of such spectral representations, we illustrate how to obtain anisotropy with respect to space, and nonstationarity with respect to time. The paper is organized as follows. In Section 2, we give necessary notation and background. The main results of the paper are provided in Section 3.

2. Background

This section is largely expository and contains basic material that will be useful for the exposition of our results. Although most of our work is related to the product space $\mathbb{S}^n \times T$, it will be convenient to present some basic facts by making reference to a metric space, denoted M throughout. We call kernel on M any mapping $k : M \times M \to \mathbb{C}$, such that $\overline{k}(q,p) = k(p,q)$ for $p, q \in M$, where $\overline{a}$ denotes the complex conjugate of the complex number a. The kernel k is positive definite if, for every positive integer m and every set of points $p_1, \dots, p_m \in M$, the $m \times m$ matrix with entries $[k(p_i, p_{i'})]_{1 \le i,i' \le m}$ satisfies, for any $c_1, \dots, c_m \in \mathbb{C}$,

$$\sum_{i,i'=1}^{m} c_i \, \overline{c_{i'}} \, k(p_i, p_{i'}) \ge 0.$$

Kolmogorov's existence theorem implies that a real-k is positive definite if and only if there exists a zero mean real-valued Gaussian random field X defined on M such that k is the covariance function of X. Namely, we have

$$k(p_1, p_2) := \operatorname{cov}(X(p_1), X(p_2)) = \mathbb{E}[X(p_1)X(p_2)], \qquad p_1, p_2 \in M.$$

The class of positive definite kernels on M is a convex cone being stable by multiplication and closed under the topology of pointwise convergence. Furthermore, if k_1 and k_2 are positive definite kernels on M_1 and M_2 respectively, then their tensor product $k_1 \otimes k_2$ is a positive definite kernel on $M_1 \times M_2$.

Next lemma provides an integral characterization of continuous positive definite kernels on $M = \mathbb{S}^n \times \mathbb{R}$. It extends the special case given by Lemma 4.3 in [3]. We do not give a proof because it is obtained by following similar arguments.

Lemma 2.1. *Let k be a continuous kernel on $\mathbb{S}^n \times \mathbb{R}$.*

Then, k is positive definite if and only if, for any continuous function $c : \mathbb{S}^n \times \mathbb{R} \to \mathbb{C}$ with compact support,

$$\int_{\mathbb{S}^n \times \mathbb{R}} \int_{\mathbb{S}^n \times \mathbb{R}} k((p_1, t_1), (p_2, t_2)) \, c(p_1, t_1) \, \overline{c}(p_2, t_2) \, \mathrm{d}\sigma_n(p_1) \mathrm{d}t_1 \mathrm{d}\sigma_n(p_2) \mathrm{d}t_2 \ \ge 0, \tag{2.1}$$

where σ_n is the surface measure on $\mathbb{S}^n$.

Through the manuscript we consider the geodesic (or great circle) distance as the mapping $\theta : \mathbb{S}^n \times \mathbb{S}^n \to [0, \pi]$ defined by $\theta(p_1, p_2) = \arccos(\langle p_1, p_2 \rangle)$, $p_1, p_2 \in \mathbb{S}^n$, where $\langle \cdot, \cdot \rangle$ denotes the classical inner product. A kernel $k : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ is called isotropic if there exists a function $\psi : [0, \pi] \to \mathbb{R}$ such that

$$k(p_1, p_2) = \psi(\theta(p_1, p_2)), \qquad p_1, p_2 \in \mathbb{S}^n. \tag{2.2}$$

Isotropic covariance functions have a well established literature: [19] has shown that all continuous covariance functions on n-dimensional spheres, being additionally isotropic, admit series expansions depending on spherical harmonics. We now recall some basic facts on these topics. We refer to the recent monographs [21] and [5] for more details.

Let $d_{n,0} = 1$ and for $\ell \in \mathbb{N}$,

$$d_{n,\ell} = \frac{(2\ell + n - 1)(\ell + n - 2)!}{\ell!(n-1)!}, \tag{2.3}$$

and let us consider $\{Y_{\ell,j} : \ell \in \mathbb{N}_0,\ j = 1, \ldots, d_{n,\ell}\}$, the family of spherical harmonics on the sphere $\mathbb{S}^n$ as defined in [16] (see also [13] for instance in the case $n = 2$). These are complex-valued functions and they provide a complete orthonormal system in the space of the Lebesgue square integrable functions $\mathcal{L}^2(\mathbb{S}^n, \mathrm{d}\sigma_n)$. They moreover satisfy the next relation

$$\overline{Y_{\ell,j}} = Y_{\ell,1-j+d_{n,\ell}} \qquad \ell \in \mathbb{N}_0,\ j = 1, \ldots, d_{n,\ell}.$$

A classical result provides an explicit relation between Gaussian random fields with a continuous covariance and its related expansion in terms of spherical harmonics. Formally, it states that any zero mean, real-valued and squared integrable Gaussian random field X on $\mathbb{S}^n$ with covariance function k admits the expansion

$$X(p) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} b_{\ell,j} Y_{\ell,j}(p), \qquad p \in \mathbb{S}^n, \tag{2.4}$$

with the coefficients $\{b_{\ell,j}\}_{\ell,j}$ being complex-valued random variables with zero mean and satisfying $\overline{b_{\ell,j}} = b_{\ell,1-j+d_{n,\ell}}$ for any (ℓ, j). Specifically, we have

$$b_{\ell,j} = \int_{\mathbb{S}^n} X(p)\overline{Y_{\ell,j}}(p)\mathrm{d}\sigma_n(p), \qquad \ell \in \mathbb{N}_0,\ j = 1, \ldots, d_{n,\ell}.$$

When the field is isotropic, the variance only depends on ℓ through

$$\mathbb{E}[b_{\ell,j}\overline{b_{\ell',j'}}] = a_\ell\, \delta_{\ell,\ell'}\delta_{j,j'}, \tag{2.5}$$

where δ is the Kronecker delta function. Then, the associated covariance kernel admits the expression

$$\mathbb{E}[X(p_1)X(p_2)] = \sum_{\ell=0}^{\infty} a_\ell \sum_{j=1}^{d_{n,\ell}} Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2), \qquad p_1, p_2 \in \mathbb{S}^n, \tag{2.6}$$

where the sequence $\{a_\ell\}_{\ell\in\mathbb{N}_0}$, identified through the relation (2.5), is called angular power spectrum.

Representation (2.6) naturally yields the family of Gegenbauer polynomials $\{C_\ell^\nu : \ell \in \mathbb{N}_0\}$ for $\nu \geq 0$, which constitutes an orthogonal basis for the space $\mathcal{L}^2([-1,1],(1-z^2)^{\nu-1/2}\mathrm{d}z)$. Actually, when $n = 2, 3, \ldots$, the $(n-1)/2$-Gegenbauer polynomial of degree ℓ can be expressed in terms of the spherical harmonics through the addition Theorem (see [16] Equation 4.5 or [21] Lemma 17.3):

$$C_\ell^{(n-1)/2}(\langle p_1, p_2\rangle) = \omega_n \frac{(n-1)}{(2\ell+n-1)} \sum_{j=1}^{d_{n,\ell}} Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2), \qquad p_1, p_2 \in \mathbb{S}^n, \tag{2.7}$$

where $\omega_n = 2\pi^{(n+1)/2}/\Gamma\left((n+1)/2\right)$ is the total mass of the surface measure σ_n on $\mathbb{S}^n$, being $\Gamma(\cdot)$ the Gamma function. Note that the Gegenbauer polynomials are real-valued. Besides, for all $\ell \in \mathbb{N}_0$,

$$|C_\ell^{(n-1)/2}(z)| \leq C_\ell^{(n-1)/2}(1) = \binom{\ell+n-2}{\ell}, \qquad \forall\, z \in [-1,1], \tag{2.8}$$

and

$$\begin{aligned} \left\| C_\ell^{(n-1)/2} \right\|^2_{\mathcal{L}^2([-1,1],(1-z^2)^{(n-2)/2}\mathrm{d}z)} &= \int_0^\pi \left(C_\ell^{(n-1)/2}(\cos\theta)\right)^2 \sin\theta^{n-1}\mathrm{d}\theta \\ &= \frac{\omega_n}{\omega_{n-1}} \frac{\binom{\ell+n-2}{\ell}}{(1+\frac{2\ell}{n-1})}. \end{aligned} \tag{2.9}$$

In the case $n = 2$, the Gegenbauer polynomials coincide with the Legendre polynomials [5]. When $n = 1$, the addition Theorem simplifies to

$$C_\ell^0(\langle p_1, p_2\rangle) = \pi \sum_{j=1}^{2} Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2), \qquad p_1, p_2 \in \mathbb{S}^1,$$

and according to Equation 22.3.14 in [1], $C_\ell^0(\cos\theta) = 2\,\ell^{-1}\cos(\ell\,\theta)$ for $\ell \in \mathbb{N}$, and $C_0^0(\cos\theta) = 1$.

The ingredients in (2.6) and (2.5) sum up nicely to provide the following statement, known as Schoenberg's Theorem [19]. Let n be a positive integer. Then, any kernel $k : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ is positive definite and isotropic if and only if

$$k(p_1, p_2) = \psi(\theta(p_1, p_2)) = \sum_{\ell=0}^{\infty} \alpha_\ell^n C_\ell^{(n-1)/2}(\cos\theta), \qquad p_1, p_2 \in \mathbb{S}^n, \tag{2.10}$$

with all $\alpha_\ell^n \geq 0$ and $\sum_{\ell=0}^{\infty} \alpha_\ell^n \binom{\ell+n-2}{\ell} < \infty$ when $n = 2, 3, \ldots$ or $\sum_{\ell=1}^{\infty} \alpha_\ell^1 \, \ell^{-1} < \infty$ when $n = 1$. Note that we use the upper index n in α_ℓ^n to emphasize the dependence of the Schoenberg's coefficients with respect to the dimension of the sphere $\mathbb{S}^n$.

When $n = 2$ and $T = \mathbb{R}$, a direct way to extend representation (2.4) to processes defined over $\mathbb{S}^2 \times \mathbb{R}$ is proposed by [13]:

$$X(p, t) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{2,\ell}} b_{\ell,j}(t) Y_{\ell,j}(p), \qquad p \in \mathbb{S}^2, t \in \mathbb{R}, \tag{2.11}$$

where $\{b_{\ell,j}(\cdot)\}$ is a uniquely determined sequence of stochastic processes defined on $\mathbb{R}$. Again, the hypothesis on such sequence are crucial to determine the properties of the associated covariance. If $\mathbb{E}[b_{\ell,j}(\cdot)] = 0$ and $\mathbb{E}[b_{\ell,j}(t_1)\overline{b_{\ell',j'}}(t_2)] = a_\ell(t_1 - t_2)\delta_{\ell,\ell'}\delta_{j,j'}$, then the associated covariance is isotropic in space and stationary in time. In the same vein, it is shown in [3] that all covariance functions on $\mathbb{S}^n \times \mathbb{R}$ being isotropic in space and stationary in time are uniquely defined as

$$k(p_1, t_1, p_2, t_2) = \sum_{\ell=0}^{\infty} a_\ell(t_1 - t_2) C_\ell^{(n-1)/2}(\cos\theta), \quad (p_i, t_i) \in \mathbb{S}^n \times \mathbb{R}, \; i = 1, 2,$$

for a uniquely determined sequence $\{a_\ell(\cdot)\}$ of positive definite functions with the requirement that $\sum_\ell a_\ell(0) < \infty$ (otherwise X in (2.11) would have infinite variance).

In view of [13]'s result, it becomes apparent that the spectral representation (2.11) is the building block for considering alternatives to isotropy. The following results show that this spectral representation actually permits spatial anisotropy and temporal nonstationarity.

3. Beyond Spatial Isotropy and Temporal Stationarity

Recall that T denotes either a compact interval in $\mathbb{R}$ or $\mathbb{R}$ itself. We are concerned with the representation of real-valued positive definite kernels on $\mathbb{S}^n \times T$. It should be remarked that isotropy and stationarity occur in a separate way in each respective space.

Definition 1. *Let k be a continuous positive definite kernel on $\mathbb{S}^n \times T$.*

i) k is called isotropic with respect to space when there exists a function $\widetilde{k}_S : [0,\pi]\times T\times T \to \mathbb{R}$ such that $k(p_1,t_1,p_2,t_2) = \widetilde{k}_S(\theta(p_1,p_2),t_1,t_2)$, for $p_1,p_2 \in \mathbb{S}^n$, $t_1,t_2 \in T$.

ii) k is called stationary with respect to time when there exists a function $\widetilde{k}_T : \mathbb{S}^n\times\mathbb{S}^n\times\mathbb{R} \to \mathbb{R}$ such that $k(p_1,t_1,p_2,t_2) = \widetilde{k}_T(p_1,p_2,t_2-t_1)$, for $p_1,p_2 \in \mathbb{S}^n$, $t_1,t_2 \in T$.

We first focus on a class of kernels defined on the product space $\mathbb{S}^n \times T$. We call $\mathcal{E}(\mathbb{S}^n,T)$ the set of continuous symmetric maps $k : (\mathbb{S}^n \times T)^2 \to \mathbb{R}$ that satisfy the following:

for any $t_1,t_2 \in T$ fixed, there exists a sequence $\{\varphi^n_{\ell,j}(t_1,t_2)\}_{\ell,j}$ of complex numbers such that

$$k(p_1,t_1,p_2,t_2) = \sum_{\ell=0}^{\infty}\sum_{j=1}^{d_{n,\ell}} \varphi^n_{\ell,j}(t_1,t_2) Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2), \qquad p_1,p_2 \in \mathbb{S}^n, \tag{3.1}$$

where the convergence holds uniformly with respect to (p_1,p_2) in $\mathbb{S}^n \times \mathbb{S}^n$ and the sequence $\{\varphi^n_{\ell,j}(t_1,t_2)\}_{\ell,j}$ satisfies

$$\overline{\varphi^n_{\ell,j}}(t_1,t_2) = \varphi^n_{\ell,1-j+d_{n,\ell}}(t_1,t_2), \qquad \ell \in \mathbb{N}_0,\ j = 1,\ldots,d_{n,\ell}.$$

An example of a kernel k belonging to $\mathcal{E}(\mathbb{S}^n,T)$ is given by the covariance function of a real-valued random field X on $\mathbb{S}^n \times T$ with a spectral representation similar to (2.11),

$$X(p,t) = \sum_{\ell=0}^{\infty}\sum_{j=1}^{d_{n,\ell}} b_{\ell,j}(t) Y_{\ell,j}(p), \qquad p \in \mathbb{S}^n, t \in \mathbb{R}, \tag{3.2}$$

where $\{b_{\ell,j}(t) : \ell \in \mathbb{N}_0,\ j = 1,\ldots,d_{n,\ell}\}$ is a family of complex-valued centered random processes on T such that, for all $t,t' \in T$, $\overline{b_{\ell,j}}(t) = b_{\ell,1-j+d_{n,\ell}}(t)$, $\mathbb{E}\left[\sum_{\ell=0}^{\infty}\sum_{j=1}^{d_{n,\ell}} |b_{\ell,j}(t)|^2\right] < \infty$ and

$\mathbb{E}[b_{\ell,j}(t)\overline{b_{\ell',j'}}(t')] = 0$ for $(\ell,j) \neq (\ell',j')$. In that case, the covariance function associated with X satisfies Equation (3.1) with

$$\varphi^n_{\ell,j}(t_1,t_2) = \mathbb{E}[b_{\ell,j}(t_1)\overline{b_{\ell,j}}(t_2)], \qquad t_1,t_2 \in T.$$

Theorem 3.1. *Let k belong to $\mathcal{E}(\mathbb{S}^n, T)$. Then, the following holds.*

a) *For every $t_1,t_2 \in T$, the series $\sum_{\ell=0}^{\infty}\sum_{j=1}^{d_{n,\ell}} |\varphi^n_{\ell,j}(t_1,t_2)|^2$ converges and for any $\ell \in \mathbb{N}_0$ and any $j \in \{1,\dots,d_{n,\ell}\}$,*

$$\varphi^n_{\ell,j}(t_1,t_2) = \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} k(p_1,t_1,p_2,t_2)\overline{Y_{\ell,j}}(p_1)Y_{\ell,j}(p_2)\mathrm{d}\sigma_n(p_1)\mathrm{d}\sigma_n(p_2). \tag{3.3}$$

Additionally, $\varphi^n_{\ell,j}$ is continuous on $T\times T$ for every $\ell \in \mathbb{N}_0$ and $j \in \{1,\dots,d_{n,\ell}\}$.

b) *The kernel k is a positive definite kernel on $\mathbb{S}^n \times T$ if and only if, for any $\ell \in \mathbb{N}_0$ and any $j \in \{1,\dots,d_{n,\ell}\}$, $\varphi^n_{\ell,j}$ is a positive definite kernel on T.*

Proof of Theorem 3.1.

a) For any positive integer N, let us define

$$k_N(p_1,t_1,p_2,t_2) := \sum_{\ell=0}^{N}\sum_{j=1}^{d_{n,\ell}} \varphi^n_{\ell,j}(t_1,t_2)Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2). \tag{3.4}$$

For every t_1 and t_2 in T, $k_N(\cdot,t_1,\cdot,t_2)$ converges to $k(\cdot,t_1,\cdot,t_2)$ as $N\to\infty$ uniformly on $\mathbb{S}^n\times\mathbb{S}^n$, and hence in $\mathcal{L}^2(\mathbb{S}^n\times\mathbb{S}^n, \mathrm{d}\sigma_n\otimes\mathrm{d}\sigma_n)$, *i.e.*

$$\|k(\cdot,t_1,\cdot,t_2) - k_N(\cdot,t_1,\cdot,t_2)\|^2_{\mathcal{L}^2(\mathbb{S}^n\times\mathbb{S}^n)} \underset{N\to\infty}{\longrightarrow} 0. \tag{3.5}$$

The left-hand side in (3.5) is equal to

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\left|\sum_{\ell=N+1}^{\infty}\sum_{j=1}^{d_{n,\ell}}\varphi^n_{\ell,j}(t_1,t_2)Y_{\ell,j}(p_1)\overline{Y_{\ell,j}}(p_2)\right|^2\mathrm{d}\sigma_n(p_1)\mathrm{d}\sigma_n(p_2)\\
&=\sum_{\ell=N+1}^{\infty}\sum_{j=1}^{d_{n,\ell}}\sum_{\ell'=N+1}^{\infty}\sum_{j'=1}^{d_{n,\ell}}\varphi^n_{\ell,j}(t_1,t_2)\overline{\varphi^n_{\ell',j'}}(t_1,t_2)\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}Y_{\ell,j}(p_1)\overline{Y_{\ell',j'}}(p_1)\,\overline{Y_{\ell,j}}(p_2)Y_{\ell',j'}(p_2)\mathrm{d}\sigma_n(p_1)\mathrm{d}\sigma_n(p_2)\\
&=\sum_{\ell=N+1}^{\infty}\sum_{j=1}^{d_{n,\ell}}|\varphi^n_{\ell,j}(t_1,t_2)|^2\ .
\end{aligned}$$

Hence, $\sum_{\ell=0}^{\infty}\sum_{j=1}^{d_{n,\ell}}|\varphi_{\ell,j}^n(t_1,t_2)|^2<\infty$. By the orthonormality of the spherical harmonics, we can thus prove (3.3). For any ℓ and j, application of Lebesgue's Theorem to the integral in (3.3) shows continuity of $\varphi_{\ell,j}^n$ on $T\times T$.

b) Suppose that k is a positive definite kernel on $\mathbb{S}^n\times T$. For any fixed ℓ and j, and any compactly supported function q on T, we apply Lemma 2.1 with $c(p,t)=Y_{\ell,j}(p)\,q(t)$ for $(p,t)\in\mathbb{S}^n\times T$, and a Direct application of Fubini's Theorem yields

$$\int_T\int_T q(t_1)\,\overline{q(t_2)}\left(\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}k(p_1,t_1,p_2,t_2)\,Y_{\ell,j}(p_1)\,\overline{Y_{\ell,j}}(p_2)\,\mathrm{d}\sigma_n(p_1)\mathrm{d}\sigma_n(p_2)\right)\mathrm{d}t_1\mathrm{d}t_2\ \geq 0.$$

The inner integral being equal to $\varphi_{\ell,j}^n(t_1,t_2)$, it proves that $\varphi_{\ell,j}^n$ is a positive definite kernel on T.

Conversely, suppose that $\varphi_{\ell,j}^n$ is a positive definite kernel on T for all ℓ and j. Then, for $a_1,\dots,a_m$ in $\mathbb{C}$ and $(p_1,t_1),\dots,(p_m,t_m)$ in $\mathbb{S}^n\times T$, we have

$$\begin{aligned}\sum_{i,i'=1}^{m}a_i\overline{a_{i'}}k(p_i,t_i,p_{i'},t_{i'})&=\sum_{i,i'=1}^{m}a_i\overline{a_{i'}}\lim_{N\to\infty}\sum_{\ell=0}^{N}\sum_{j=1}^{d_{n,\ell}}\varphi_{\ell,j}^n(t_i,t_{i'})Y_{\ell,j}(p_i)\overline{Y_{\ell,j}}(p_{i'})\\&=\lim_{N\to\infty}\sum_{\ell=0}^{N}\sum_{j=1}^{d_{n,\ell}}\sum_{i,i'=1}^{m}a_i\overline{a_{i'}}\varphi_{\ell,j}^n(t_i,t_{i'})Y_{\ell,j}(p_i)\overline{Y_{\ell,j}}(p_{i'})\geq 0,\end{aligned}$$

since $\varphi_{\ell,j}^n(t_i,t_{i'})Y_{\ell,j}(p_i)\overline{Y_{\ell,j}}(p_{i'})$ is a positive definite kernel on $\mathbb{S}^n\times T$, being the tensor product of positive definite kernels on T and $\mathbb{S}^n$ respectively, for every ℓ and j. ■

Next result provides a spectral representation for the positive definite kernels that are isotropic with respect to the space $\mathbb{S}^n$ and not necessarily stationary with respect to time. On the one hand, it is a generalization of the main result given in [3], which provides a characterization for the stationary case. On the other hand, it is a corollary of Theorem 3.1.

Theorem 3.2. *Let k be a continuous kernel on $\mathbb{S}^n\times T$.*

Then, k is positive definite and isotropic with respect to space if and only if we have,

$$k(p_1,t_1,p_2,t_2)=\sum_{\ell=0}^{\infty}\alpha_\ell^n(t_1,t_2)\,C_\ell^{(n-1)/2}(\cos\theta),\quad (p_i,t_i)\in\mathbb{S}^n\times T,\quad i=1,2, \tag{3.6}$$

where,

i) *for every* $\ell \in \mathbb{N}_0$, α_ℓ^n *is a real-valued continuous positive definite kernel on* T*;*

ii) *for every* t *in* T, $\sum_{\ell=1}^{\infty} \alpha_\ell^n(t,t)\ell^{n-2} < \infty$*;*

iii) *let* $\widetilde{k}_S(\theta, t_1, t_2) = k(p_1, t_1, p_2, t_2)$. *Then, for every* $t_1, t_2 \in T$, *the convergence in* (3.6) *is uniform with respect to* $\theta \in [0, \pi]$.

Moreover, for every $t_1, t_2 \in T$ *and* $\ell \in \mathbb{N}_0$,

$$\alpha_\ell^n(t_1,t_2) = \frac{\omega_{n-1}}{\omega_n} \frac{(1+\frac{2\ell}{n-1})}{\binom{\ell+n-2}{\ell}} \int_0^\pi \widetilde{k}_S(\theta,t_1,t_2)\, C_\ell^{(n-1)/2}(\cos\theta)\, \sin^{n-1}\theta \,\mathrm{d}\theta, \quad \text{for } n = 2,3,\ldots \ , \tag{3.7}$$

and

$$\alpha_\ell^1(t_1,t_2) = \begin{cases} \frac{\ell}{\pi}\int_0^\pi \widetilde{k}_S(\theta,t_1,t_2)\, \cos(\ell\theta)\,\mathrm{d}\theta, & \text{for} \quad \ell = 1,2,\ldots \\ \frac{1}{\pi}\int_0^\pi \widetilde{k}_S(\theta,t_1,t_2)\,\mathrm{d}\theta, & \text{for} \quad \ell = 0. \end{cases} \tag{3.8}$$

Proof of Theorem 3.2.

Suppose that k is a positive definite kernel on $\mathbb{S}^n \times T$ and additionnaly spatially isotropic.

First, by Definition 1, there exists a function $\widetilde{k}_S : [0,\pi] \times T \times T \to \mathbb{R}$ such that for every $t_1, t_2 \in T$ and $p_1, p_2 \in \mathbb{S}^n$, $k(p_1,t_1,p_2,t_2) = \widetilde{k}_S(\theta(p_1,p_2),t_1,t_2)$.

Let us fix t_1 and t_2 in T. Since the map $\widetilde{k}_S(\arccos(\cdot),t_1,t_2)$ belongs to $\mathcal{L}^2([-1,1],(1-z^2)^{(n-2)/2}\mathrm{d}z)$, it admits an expansion in terms of the Gegenbauer polynomials, *i.e.* there exists a sequence of real numbers $\{\alpha_\ell^n(t_1,t_2)\}_{\ell\in\mathbb{N}_0}$ such that

$$\widetilde{k}_S(\arccos(\cdot),t_1,t_2) = \sum_{\ell=0}^{\infty} \alpha_\ell^n(t_1,t_2)\, C_\ell^{(n-1)/2}(\cdot), \tag{3.9}$$

where the convergence holds in $\mathcal{L}^2([-1,1],(1-z^2)^{(n-2)/2}\mathrm{d}z)$. Using (2.7), we see that k satisfies Equation (3.1) with, for $n = 2,3,\ldots$,

$$\varphi_{\ell,j}^n(t_1,t_2) = \frac{\omega_n\,(n-1)}{2\ell+n-1}\,\alpha_\ell^n(t_1,t_2), \qquad \ell\in\mathbb{N}_0,\ j = 1,\ldots,d_{n,\ell},$$

and $\varphi_{\ell,j}^1(t_1,t_2) = \pi\,\alpha_\ell^1(t_1,t_2)$ for $\ell\in\mathbb{N}_0$, $j = 1,2$. Hence, k belongs to $\mathcal{E}(\mathbb{S}^n,T)$. Then, Theorem 3.1 applies and shows that $\{\alpha_\ell^n\}_{\ell\in\mathbb{N}_0}$ is a sequence of continuous positive definite kernels on T. Point $i)$ is thus established.

We now focus on point *ii*). Let $t \in T$ be fixed and let us recall that Expansion (3.9) holds in $\mathcal{L}^2([-1,1],(1-z^2)^{(n-2)/2}\mathrm{d}z)$ for $t_1 = t_2 = t$. Moreover, since $\widetilde{k}_S(\arccos(\cdot),t,t)$ is continuous on $[-1,1]$, we have that the series in (3.9) is Abel summable on $[0,1)$ for any $z \in [-1,1]$ (see Theorem 9 in [15]), *i.e.* the limit $\lim_{r\to 1^-}\sum_{\ell=0}^{\infty}(\alpha_\ell^n(t,t)\, C_\ell^{(n-1)/2}(z)\, r^\ell)$ exists. For $z=1$, since $\alpha_\ell^n(t,t)\, C_\ell^{(n-1)/2}(1) \geq 0$ for all $\ell \in \mathbb{N}_0$, it implies that the series $\sum_{\ell=0}^{\infty}(\alpha_\ell^n(t,t)\, C_\ell^{(n-1)/2}(1))$ is finite. Noting that, for $n = 2,3,\ldots$, $C_\ell^{(n-1)/2}(1) = \binom{\ell+n-2}{\ell} \sim \frac{\ell^{n-2}}{(n-2)!}$ and $C_\ell^0(1) \sim 2\ell^{-1}$ when $\ell \to \infty$, the convergence of the series is equivalent to $\sum_{\ell=1}^{\infty}\alpha_\ell^n(t,t)\ell^{n-2} < \infty$. Hence, we get *ii*).

Finally, it remains to establish assertion *iii*). We consider Expansion (3.9) for fixed $t_1, t_2 \in T$. By Cauchy-Schwarz inequality and by (2.8), we can see that for $z \in [-1,1]$ and for all $\ell \in \mathbb{N}_0$,

$$|\alpha_\ell^n(t_1,t_2)C_\ell^{(n-1)/2}(z)| \leq \frac{1}{2}\left(\alpha_\ell^n(t_1,t_1)+\alpha_\ell^n(t_2,t_2)\right)\binom{\ell+n-2}{\ell},$$

with $\sum_{\ell=1}^{\infty}\alpha_\ell^n(t_i,t_i)\ell^{n-2} < \infty$ for $i = 1,2$. Then, by the M-test of Weierstrass, the series in (3.9) uniformly converges to a continuous function, being precisely the function $\widetilde{k}_S(\arccos(\cdot),t_1,t_2)$. This proves the uniform convergence of the series in (3.6) for fixed $(t_1,t_2) \in T \times T$.

At last, Equation (3.7) follows directly from the orthogonality of the Gegenbauer polynomials and (2.9).

Let us prove now the converse part of Theorem 3.2. If k is a kernel that satisfies Expansion (3.9) with *i*), *ii*), *iii*) then, using the addition formula (2.7), it is easy to see that k belongs to $\mathcal{E}(\mathbb{S}^n,T)$. The converse part of Theorem 3.1 allows us to state that k is a positive definite kernel on $\mathbb{S}^n \times T$. The isotropy property is clear from (3.9). ∎

Next result comes from Theorems 3.1 and 3.2.

Corollary 3.3. *Let $k : (\mathbb{S}^n \times T)^2 \to \mathbb{R}$ be a kernel in $\mathcal{E}(\mathbb{S}^n,T)$ that is positive definite.*

Then, k is isotropic with respect to space if and only if the functions $\{\varphi_{\ell,j}^n\}_{\ell,j}$ appearing in Expansion (3.1) do not depend on the j index, i.e. they are such that

$$\varphi_{\ell,j}^n \equiv \varphi_{\ell,j'}^n\ , \quad \textit{for all}\ \ j,j' \in \{1,\ldots,d_{n,\ell}\}\ ,\ \ \ell \in \mathbb{N}_0.$$

Furthermore, the functions $\{\varphi^n_{\ell,j}\}_{\ell,j}$ *are linked with the functions* $\{\alpha^n_\ell\}_\ell$ *in Expansion* (3.6) *by the following. For* $n \geq 2$*,*

$$\varphi^n_{\ell,j}(t_1,t_2) = \frac{\omega_n\,(n-1)}{2\ell+n-1}\,\alpha^n_\ell(t_1,t_2)\ ,\ t_1,t_2 \in T, \tag{3.10}$$

and $\varphi^1_{\ell,j}(t_1,t_2) = \pi\,\alpha^1_\ell(t_1,t_2)$ *for* $n = 1$*.*

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