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Characterization Theorems for Covariance Functions on the n -Dimensional Sphere Across Time

ANNE ESTRADE, ALESSANDRA FARIÑAS, EMILIO PORCU

Abstract

This paper provides representation theorems for classes of nonstationary covariance functions on spheres across time. Isotropy on spheres comes out naturally in virtue of the addition theorem, and we show how to escape from this assumption, in order to obtain kernels that can be anisotropic. The same criterion is used in order to escape from stationarity in the temporal component. Our findings are then illustrated through a simulation study.

Keywords: Positive definite kernel, Spherical harmonics, Gegenbauer polynomials, Spectral representation, Space-time random fields

1 Introduction

The literature on positive definite kernels on spheres has become ubiquitous, and we refer the reader to the essays [Gneiting, T. \(2013\)](#); [Narcowich, F. J. \(1995\)](#), with the impressive list of references given there. Both mathematical and statistical communities have been interested in the construction of positive definite functions defined over product spaces where the sphere is involved. The recent tours de force in [Barbosa, V. S. and Menegatto, V. A. \(2016\)](#); [Guella, J. C. and Menegatto, V. A. \(2016\)](#); [Guella, J. C., Menegatto, V. A. and Peron, A. P. \(2016\)](#); [Guella, J. C. and Menegatto, V. A. \(2016\)](#) are apparent indications that there are some important branches of the mathematical community devoted to studying such constructions.

The statistical theory has been recently interested in the analysis of global data, where typically a process is observed on the whole sphere representing planet Earth, and where measurements are repeated across time. Statistical inferences, as well as best linear unbiased predictions, of Gaussian fields on spheres across time are completely determined by their covariance functions, also called positive definite kernels in numerical analysis literature.

Recent statistical works are concerning with the construction of space-time covariances where the space is the spherical shell. In particular, the works [Jun, M. and Stein, M. L. \(2007\)](#); [Berg, C. and Porcu, E. \(2016\)](#); [Porcu, E., Bevilacqua, M. and Genton, M. G. \(2016\)](#); [Genton, M. G. and Kleiber, W. \(2015\)](#) put mathematical theory for the characterization of such covariances. The work [Porcu, E., Bevilacqua, M. and Genton, M. G. \(2016\)](#) is focused on construction principles that allow for algebraically tractable closed forms, and then analyze the discrepancy between using the correct metric and other metrics. The classical representation was provided by [Schoenberg, I. J. \(1942\)](#), which establishes the fundamental theoretical results for the characterization of isotropic positive definite kernels on $\mathbb{S}^n \times \mathbb{S}^n$, for $n \geq 1$. Other, analogous and also classical was given by the Bochner's Theorem, stated in the Euclidean case to characterizes stationary covariances [Adler, R. and Taylor, J. \(2007\)](#), both classical constructions are based on generalized Fourier theory [Narcowich, F. J. \(1995\)](#).

The works [Jun, M. and Stein, M. L. \(2007, 2008\)](#) put emphasis on the fact that global data are affected by strong nonstationarities. This idea is pursued, albeit with different approaches by [Castruccio, S. and Guinness, J. \(2016\)](#). They note that statistical models on a spherical domain could be nonstationary for different latitudes, but stationary at the same latitude.

Our paper considers the problem of anisotropy with respect to space and nonstationarity with respect to time from the point of view of these mathematical objects. Our basis can be found in celebrated Schoenberg's theorem [Schoenberg, I. J. \(1942\)](#), followed by the characterization given in [Berg, C. and Porcu, E. \(2016\)](#) for isotropic and stationary positive definite kernels on $(\mathbb{S}^n \times G)^2$, where G is a locally compact group. In particular, Schoenberg's representation, in concert with the addition theorem for spherical harmonics [Marinucci, D. and Peccati, G. \(2011\)](#), imply that the spherical isotropy is a natural assumption for the construction of covariances over spheres. Thus, we put some effort on providing representation theorems that allow to treat general cases.

In summary, we provide spectral representations for positive definite kernels on $(\mathbb{S}^n \times T)^2$, being T the parameter set for the time variable. Each representation is characterized by a sequence of positive definite kernels on $T \times T$. Certain conditions allow us to obtain anisotropy with respect to space and nonstationarity with respect to time. The paper is organized as follows. In Section 2, we give a background with the necessary terminology or notations, and with fundamental theory. Then, the main results of the paper can be found in Section 3. In Section 4, we first construct a covariance model with isotropy with respect to space and nonstationarity with respect to time, and we simulate approximations of the associated Gaussian random field. Finally, we construct a covariance model that is anisotropic and nonstationary, followed by the simulation of its associated approximated Gaussian random field.

2 Background

Let M be a metric space. We call kernel any mapping $k : M \times M \rightarrow \mathbb{C}$ (or $k : M \times M \rightarrow \mathbb{R}$). The kernel k is called Hermitian if for any $p_1, p_2 \in M$, we have $k(p_1, p_2) = \overline{k(p_2, p_1)}$ (or symmetric if k is real-valued).

Definition 1. Let $k : M \times M \rightarrow \mathbb{C}$ be a kernel. Then, k is positive definite if, for every set of points $p_1, \dots, p_m \in M$, the $m \times m$ matrix with entries $[k(p_i, p_{i'})]_{1 \leq i, i' \leq m}$, called Gram matrix, satisfies, for any $c_1, \dots, c_m \in \mathbb{C}$,

$$\sum_{i, i'=1}^m c_i \overline{c_{i'}} k(p_i, p_{i'}) \geq 0. \quad (2.1)$$

Every Gram matrix of a positive definite kernel k on $M \times M$ is a positive semi-definite matrix, with its diagonal elements being non-negative. Additionally, for every pair $p_i, p_{i'} \in M$, Cauchy-Schwarz inequality is satisfied *i.e.*,

$$|k(p_i, p_{i'})|^2 \leq k(p_i, p_i)k(p_{i'}, p_{i'}), \quad p_i, p_{i'} \in M.$$

This in turn implies that

$$|k(p_i, p_{i'})| \leq (k(p_i, p_i) + k(p_{i'}, p_{i'}))/2.$$

An important class of positive definite kernels is given by the next procedure. Let X be a real-valued zero mean random field on M with finite second moments. The covariance function

$$\text{cov}(p_1, p_2) := \mathbb{E}[X(p_1)\overline{X(p_2)}], \quad p_1, p_2 \in M,$$

defines a positive definite kernel, usually known as covariance kernel. Notice here that we use the conjugate by convenience, in order to simplify later formulas.

The class of positive definite kernels on $M \times M$ is a convex cone being closed under the topology of pointwise convergence. Furthermore, it is stable by multiplication.

One defines the tensor product between two kernels k_1 and k_2 on $M_1 \times M_1$ and $M_2 \times M_2$ respectively, as $k_1 \otimes k_2(p_1, t_1, p_2, t_2) := k_1(p_1, t_1)k_2(p_2, t_2)$, where $p_i, t_i \in M_i$ for $i = 1, 2$. The following result, proved in [Berg, C., Reus, J. P. and Ressel, P. \(1984\)](#), holds.

Proposition 1. Let M_1 and M_2 be metric spaces and let $k_1 : M_1 \times M_1 \rightarrow \mathbb{R}$ and $k_2 : M_2 \times M_2 \rightarrow \mathbb{R}$ be positive definite kernels, then their tensor product $k_1 \otimes k_2$ is positive definite on $(M_1 \times M_2)^2$.

Arguments in [Narcowich, F. J. \(1995\)](#) show that any continuous kernel on a compact metric space $M \times M$ is positive definite if and only if

$$\int_M \int_M k(p_1, p_2) c(p_1) \overline{c(p_2)} d\mu(p_1) d\mu(p_2) \geq 0, \quad p_1, p_2 \in M, \quad (2.2)$$

for all $c \in \mathcal{L}^2(M, d\mu)$, and for μ a finite measure on M . In particular, this criterion holds in the case $M = \mathbb{S}^n \times [0, 1]$, where \mathbb{S}^n is the n -dimensional sphere and $\mu = \sigma_n \otimes \lambda$, being σ_n the surface measure on the spherical shell of \mathbb{R}^{n+1} and λ the Lebesgue measure on $[0, 1]$.

Lemma 1. Let $k : (\mathbb{S}^n \times [0, 1])^2 \rightarrow \mathbb{R}$ be a symmetric and continuous kernel.

Then, k is positive definite if and only if, for any continuous function $c : \mathbb{S}^n \times [0, 1] \rightarrow \mathbb{C}$,

$$\int \int_{(\mathbb{S}^n \times [0, 1])^2} k(p_1, t_1, p_2, t_2) c(p_1, t_1) \overline{c(p_2, t_2)} d\sigma_n(p_1) dt_1 d\sigma_n(p_2) dt_2 \geq 0.$$

Henceforth, we use the abuse of notation dt for $d\lambda(t)$, $t \in [0, 1]$. An extended version of the previous lemma is stated over \mathbb{R} . It extends the particular cases given by Lemma 4.3 of [Berg, C. and Porcu, E. \(2016\)](#) and Proposition 6.4 in [Wendland, H. \(2005\)](#).

Definition 2. A continuous function $c : \mathbb{S}^n \times \mathbb{R} \rightarrow \mathbb{C}$ has compact support in \mathbb{R} if there exists a compact subset F in \mathbb{R} such that $c(p, t) = 0$ for all $p \in \mathbb{S}^n$, $t \in F^c$.

Lemma 2. Let $k : (\mathbb{S}^n \times \mathbb{R})^2 \rightarrow \mathbb{R}$ be a symmetric and continuous kernel.

Then, k is positive definite if and only if, for any continuous function $c : \mathbb{S}^n \times \mathbb{R} \rightarrow \mathbb{C}$ with compact support in \mathbb{R} ,

$$\int \int_{(\mathbb{S}^n \times \mathbb{R})^2} k(p_1, t_1, p_2, t_2) c(p_1, t_1) \overline{c(p_2, t_2)} d\sigma_n(p_1) dt_1 d\sigma_n(p_2) dt_2 \geq 0. \quad (2.3)$$

Proof. Suppose that k is a positive definite kernel. First we note that the integral in (2.3) converges since the function c has compact support in \mathbb{R} and k is bounded. Let us call

$$I(\mathbb{S}^n, \mathbb{R}) := \int \int_{(\mathbb{S}^n \times \mathbb{R})^2} k(p_1, t_1, p_2, t_2) c(p_1, t_1) \overline{c(p_2, t_2)} d\sigma_n(p_1) dt_1 d\sigma_n(p_2) dt_2.$$

If $\text{Supp}(c) = \mathbb{S}^n \times F$ is the support of c , then

$$I(\mathbb{S}^n, \mathbb{R}) = \int \int_{(\mathbb{S}^n \times F)^2} k(p_1, t_1, p_2, t_2) c(p_1, t_1) \overline{c(p_2, t_2)} d\sigma_n(p_1) dt_1 d\sigma_n(p_2) dt_2 := I(\mathbb{S}^n, F).$$

By the compactness of $\mathbb{S}^n \times F$, it is possible to approximate the integral $I(\mathbb{S}^n, F)$ in the sense of Riemann. Thus, for every $\epsilon > 0$, there exists a positive integer N , $\{(p_1, t_1), \dots, (p_N, t_N)\}$ in $\mathbb{S}^n \times F$ and weights ρ_1, \dots, ρ_N in \mathbb{R} such that

$$\left| I(\mathbb{S}^n, F) - \sum_{i=1}^N \sum_{j=1}^N k(p_i, t_i, p_j, t_j) c(p_i, t_i) \rho_i \overline{c(p_j, t_j)} \rho_j \right| < \epsilon.$$

Then,

$$I(\mathbb{S}^n, \mathbb{R}) \geq \sum_{i=1}^N \sum_{j=1}^N k(p_i, t_i, p_j, t_j) c(p_i, t_i) \rho_i \overline{c(p_j, t_j)} \rho_j - \epsilon \geq -\epsilon,$$

for every ϵ . Therefore, we conclude $I(\mathbb{S}^n, \mathbb{R}) \geq 0$.

Suppose now that inequality (2.3) is true for any continuous function $c : \mathbb{S}^n \times \mathbb{R} \rightarrow \mathbb{C}$ with compact support in \mathbb{R} . Let $\{\vartheta_N\}_{N \geq 0}$ be a sequence of continuous functions from \mathbb{R} to \mathbb{R}^+ such that, for all $N \geq 0$, ϑ_N is supported by $[-N-1, N+1]$ and $\vartheta_N(x) = 1$ for all $x \in [-N, N]$.

For all $N \geq 0$, let us define the kernel k_N by

$$k_N(p_1, t_1, p_2, t_2) = k(p_1, t_1, p_2, t_2) \vartheta_N(t_1) \vartheta_N(t_2), \quad p_1, p_2 \in \mathbb{S}^n, t_1, t_2 \in \mathbb{R},$$

which is clearly continuous and symmetric on $(\mathbb{S}^n \times \mathbb{R})^2$. By hypothesis, inequality (2.3) remains true for all continuous functions c with support in $\mathbb{S}^n \times [-N, N]$ when k is replaced by k_N . Hence, using Lemma 1 with $[-N, N]$ instead of $[0, 1]$, we get that k_N is positive definite. Since k_N tends to k as N tends to infinity, we conclude by using Proposition 1 *iii*). \square

Definition 3. A kernel $k : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ is isotropic if there exists a function $\psi : [0, \pi] \rightarrow \mathbb{R}$ such that

$$k(p_1, p_2) = \psi(\theta(p_1, p_2)), \quad p_1, p_2 \in \mathbb{S}^n,$$

where $\theta(p_1, p_2) = \arccos(\langle p_1, p_2 \rangle)$ is the geodesic distance, denoted θ for short.

We call a random field X on \mathbb{S}^n isotropic if its covariance kernel is isotropic. Spectral representations of isotropic random fields and isotropic kernels on \mathbb{S}^n have been provided in [Schoenberg, I. J. \(1942\)](#), [Adler, R. and Taylor, J. \(2007\)](#), and are given by series expansions in terms of spherical harmonics. These are the eigenfunctions of the Laplace-Beltrami operator on \mathbb{S}^n , where the spectrum forms the sequence $\lambda_\ell = \ell(\ell + n - 1)$ with $\ell \geq 0$. For each eigenvalue λ_ℓ , the associated eigenspace has dimension $d_{n,\ell}$, where

$$d_{n,\ell} = \begin{cases} \frac{(2\ell+n-1)(\ell+n-2)!}{\ell!(n-1)!} & \text{if } \ell \geq 1 \\ 1 & \text{if } \ell = 0. \end{cases} \quad (2.4)$$

The eigenspaces are mutually orthogonal, and the direct sum of them is $\mathcal{L}^2(\mathbb{S}^n, d\sigma_n)$. Thus, the set of functions $\{Y_{\ell,j} : \ell \geq 0, j = 1, \dots, d_{n,\ell}\}$ that satisfy the eigenproblem

$$\Delta_{\mathbb{S}^n} Y_{\ell,j} - \ell(\ell + n - 1)Y_{\ell,j} = 0,$$

is termed set of spherical harmonics with order ℓ , and constitutes a complete orthonormal system for $\mathcal{L}^2(\mathbb{S}^n, d\sigma_n)$, where in the above formula $\Delta_{\mathbb{S}^n}$ denotes the Laplace-Beltrami operator on \mathbb{S}^n . Consequently, for any $f \in \mathcal{L}^2(\mathbb{S}^n, d\sigma_n)$, introducing

$$f_N(p) := \sum_{\ell=0}^N \sum_{j=1}^{d_{n,\ell}} b_{\ell,j} Y_{\ell,j}(p), \quad p \in \mathbb{S}^n, \quad \text{with} \quad b_{\ell,j} = \int_{\mathbb{S}^n} f(p) \overline{Y_{\ell,j}}(p) d\sigma_n(p),$$

one can prove (see Equation 3.3 in [Narcowich, F. J. \(1995\)](#)) that

$$\int_{\mathbb{S}^n} |f(p) - f_N(p)| d\sigma_n(p) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

In the same vein, any isotropic random field on \mathbb{S}^n admits a series expansion in terms of the spherical harmonics (see page 121 in [Adler, R. and Taylor, J. \(2007\)](#)).

Theorem 2.1 (Spectral representation of isotropic random fields). *Let X be a centered, real-valued and squared integrable random field on \mathbb{S}^n with isotropic covariance. Then X admits the expansion*

$$X(p) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} b_{\ell,j} Y_{\ell,j}(p), \quad p \in \mathbb{S}^n, \quad (2.5)$$

with $\{b_{\ell,j}\}_{\ell,j}$ given by

$$b_{\ell,j} = \int_{\mathbb{S}^n} X(p) \overline{Y_{\ell,j}}(p) d\sigma_n(p),$$

being real-valued uncorrelated centered random variables with variance depending only on ℓ .

The representation in Equation (2.5) is also known as Karhunen-Loève expansion and the associated covariance kernel admits the expression

$$\mathbb{E}[X(p_1) \overline{X(p_2)}] = \sum_{\ell=0}^{\infty} a_{\ell} \sum_{j=1}^{d_{n,\ell}} Y_{\ell,j}(p_1) \overline{Y_{\ell,j}}(p_2), \quad p_1, p_2 \in \mathbb{S}^n, \quad (2.6)$$

where the sequence $\{a_{\ell}\}_{\ell \geq 0}$ is called angular power spectrum and is identified through the relation

$$\mathbb{E}[b_{\ell_1 j_1} \overline{b_{\ell_2 j_2}}] = a_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{j_1 j_2}, \quad (2.7)$$

where δ is the Kronecker delta function. The representation of isotropic kernels is related to the Gegenbauer polynomials $\{C_{\ell}^{\nu}(z) : z \in [-1, 1], \ell \geq 0\}$ for $\nu \geq 0$ [Gneiting, T. \(2013\)](#). They constitute an orthogonal basis for $\mathcal{L}^2([-1, 1], (1 - z^2)^{\nu-1/2} dz)$.

Considering $n \geq 2$, it is important to note that the $(n-1)/2$ -Gegenbauer polynomial of degree ℓ can be expressed in terms of the spherical harmonics through the addition Theorem (See Equation 4.5 in [Narcowich, F. J. \(1995\)](#))

$$C_\ell^{(n-1)/2}(\langle p_1, p_2 \rangle) = \binom{\ell + n - 2}{\ell} \frac{\omega_n}{d_{n,\ell}} \sum_{j=1}^{d_{n,\ell}} Y_{\ell j}(p_1) \overline{Y_{\ell j}(p_2)}, \quad p_1, p_2 \in \mathbb{S}^n, \quad (2.8)$$

where $\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the total mass of the surface measure σ_n on \mathbb{S}^n . Besides, for all $\ell \geq 0$,

$$|C_\ell^{(n-1)/2}(z)| \leq C_\ell^{(n-1)/2}(1) = \binom{\ell + n - 2}{\ell}, \quad \forall z \in [-1, 1], \quad (2.9)$$

and

$$\|C_\ell^{(n-1)/2}\|_{\mathcal{L}^2([-1,1], (1-z^2)^{(n-2)/2} d\sigma_n)}^2 = \int_0^\pi \left(C_\ell^{(n-1)/2}(\cos \theta) \right)^2 \sin \theta^{n-1} d\theta = \frac{\omega_n \binom{\ell+n-2}{\ell}}{\omega_{n-1} \left(1 + \frac{2\ell}{n-1}\right)}. \quad (2.10)$$

One can also note that when $n = 2$, the Gegenbauer polynomials simplify to Legendre polynomials [Narcowich, F. J. \(1995\)](#).

When $n = 1$, the addition Theorem takes the form

$$C_\ell^0(\langle p_1, p_2 \rangle) = \pi \sum_{j=1}^2 Y_{\ell j}(p_1) \overline{Y_{\ell j}(p_2)}, \quad p_1, p_2 \in \mathbb{S}^1,$$

and according to Equation 22.3.14 in [Abramowitz, M. and Stegun, I. \(1972\)](#), $C_\ell^0(\cos(\theta)) = 2\ell^{-1} \cos(\ell\theta)$ for $\ell \geq 1$, and $C_\ell^0(\cos(\theta)) = 1$ for $\ell = 0$.

The sum of the ingredients in (2.6) and (2.7) offers Schoenberg's representation theorem, which gives a representation for the continuous, isotropic and positive definite kernels on \mathbb{S}^n through the Gegenbauer polynomials [Schoenberg, I. J. \(1942\)](#).

Theorem 2.2 (Schoenberg, 1942). *Any isotropic positive definite kernel k on $\mathbb{S}^n \times \mathbb{S}^n$ can be represented by the following series expansion*

$$k(p_1, p_2) = \psi(\theta(p_1, p_2)) = \sum_{\ell=0}^{\infty} \alpha_\ell^n C_\ell^{(n-1)/2}(\cos(\theta(p_1, p_2))), \quad p_1, p_2 \in \mathbb{S}^n, \quad (2.11)$$

with all $\alpha_\ell^n \geq 0$ and $\sum_{\ell=0}^{\infty} \alpha_\ell^n \binom{\ell+n-2}{\ell} < \infty$ when $n \geq 2$ or $\sum_{\ell=1}^{\infty} \alpha_\ell^1 \ell^{-1} < \infty$ when $n = 1$.

Note that we use the upper index n in α_ℓ^n to emphasize the dependence of the Schoenberg's coefficients with respect to the dimension of the sphere \mathbb{S}^n .

3 Results

From now on, T denotes either the closed interval $[0, 1]$ or \mathbb{R} . Let us quote that one can replace $[0, 1]$ by any compact interval in \mathbb{R} . We now make emphasis on the representation of positive definite kernels on $(\mathbb{S}^n \times T)^2$. At this point, it should be remarked that isotropy or stationarity occur in a separate way in each respective space.

Definition 4. Let $M = \mathbb{S}^n \times T$ and $k : M \times M \rightarrow \mathbb{R}$ be a positive definite kernel.

- i.- k is called isotropic with respect to space when there exists a function $\widetilde{k}_S : [0, \pi] \times T \times T \rightarrow \mathbb{R}$ such that

$$k(p_1, t_1, p_2, t_2) = \widetilde{k}_S(\arccos(\langle p_1, p_2 \rangle), t_1, t_2), \quad p_1, p_2 \in \mathbb{S}^n, t_1, t_2 \in T.$$

- ii.- k is called stationary with respect to time when there exists a function $\widetilde{k}_T : \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$k(p_1, t_1, p_2, t_2) = \widetilde{k}_T(p_1, p_2, t_2 - t_1), \quad p_1, p_2 \in \mathbb{S}^n, t_1, t_2 \in T.$$

In the next theorem, we consider the class of kernels that admit an expansion of the form (3.1). Note that Narkowich also considered the same class (see Equation (3.8) in [Narcowich, F. J. \(1995\)](#)).

Theorem 3.1. *Let $k : (\mathbb{S}^n \times T)^2 \rightarrow \mathbb{R}$ be a continuous and symmetric kernel admitting the expansion*

$$k(p_1, t_1, p_2, t_2) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2) Y_{\ell j}(p_1) \overline{Y_{\ell j}(p_2)}, \quad (3.1)$$

where, for any $(t_1, t_2) \in T \times T$, the convergence in (3.1) holds uniformly with respect to $(p_1, p_2) \in \mathbb{S}^n \times \mathbb{S}^n$. Then, the following results hold.

- a) For every $t_1, t_2 \in T$, the series $\sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2)^2$ converges and, for any $\ell \geq 0$ and any $j \in \{1, \dots, d_{n,\ell}\}$,

$$\varphi_{\ell,j}^n(t_1, t_2) = \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} k(p_1, t_1, p_2, t_2) \overline{Y_{\ell j}(p_1)} Y_{\ell j}(p_2) d\sigma_n(p_1) d\sigma_n(p_2). \quad (3.2)$$

Moreover, $\varphi_{\ell,j}^n$ is continuous and symmetric on $T \times T$ for every ℓ and j .

- b) The kernel k is positive definite on $(\mathbb{S}^n \times T)^2$ if and only if, for any $\ell \geq 0$ and any $j \in \{1, \dots, d_{n,\ell}\}$, $\varphi_{\ell,j}^n$ is a positive definite kernel on $T \times T$.

Proof. a) For any positive integer N , let us define

$$k_N(p_1, t_1, p_2, t_2) := \sum_{\ell=0}^N \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2) Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)}. \quad (3.3)$$

For every t_1 and t_2 in T , $k_N(\cdot, t_1, \cdot, t_2)$ converges to $k(\cdot, t_1, \cdot, t_2)$ as $N \rightarrow \infty$ uniformly on $\mathbb{S}^n \times \mathbb{S}^n$, and hence in $\mathcal{L}^2(\mathbb{S}^n \times \mathbb{S}^n, d\sigma_n \otimes d\sigma_n)$, i.e.

$$\|k(\cdot, t_1, \cdot, t_2) - k_N(\cdot, t_1, \cdot, t_2)\|_{\mathcal{L}^2(\mathbb{S}^n \times \mathbb{S}^n)}^2 \xrightarrow{N \rightarrow \infty} 0.$$

This last quantity is equal to

$$\begin{aligned}
& \left\| \sum_{\ell=N+1}^{\infty} \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2) (Y_{\ell,j} \otimes \overline{Y_{\ell,j}}) \right\|_{\mathcal{L}^2(\mathbb{S}^n \times \mathbb{S}^n)}^2 \\
&= \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \lim_{M \rightarrow \infty} \left| \sum_{\ell=N+1}^M \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} \right|^2 d\sigma_n(p_1) d\sigma_n(p_2) \\
&= \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \lim_{M \rightarrow \infty} \sum_{\ell=N+1}^M \sum_{j=1}^{d_{n,\ell}} \sum_{\ell'=N+1}^M \sum_{j'=1}^{d_{n,\ell'}} \varphi_{\ell,j}^n(t_1, t_2) Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} \varphi_{\ell',j'}^n(t_1, t_2) \overline{Y_{\ell',j'}(p_1)} Y_{\ell',j'}(p_2) d\sigma_n(p_1) d\sigma_n(p_2) \\
&= \sum_{\ell=N+1}^{\infty} \sum_{j=1}^{d_{n,\ell}} \sum_{\ell'=N+1}^{\infty} \sum_{j'=1}^{d_{n,\ell'}} \varphi_{\ell,j}^n(t_1, t_2) \varphi_{\ell',j'}^n(t_1, t_2) \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} Y_{\ell,j}(p_1) \overline{Y_{\ell',j'}(p_1)} \overline{Y_{\ell,j}(p_2)} Y_{\ell',j'}(p_2) d\sigma_n(p_1) d\sigma_n(p_2) \\
&= \sum_{\ell=N+1}^{\infty} \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2)^2.
\end{aligned}$$

Hence $\sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_1, t_2)^2 < \infty$. By the orthonormal property of the spherical harmonics, for any $t_1, t_2 \in T$, we can thus prove that

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} k(p_1, t_1, p_2, t_2) Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} d\sigma_n(p_1) d\sigma_n(p_2) = \varphi_{\ell,j}^n(t_1, t_2).$$

Application of Lebesgue's Theorem to the above integral shows continuity of $\varphi_{\ell,j}^n$ on $T \times T$ for all ℓ and j .

b) Suppose that k is a positive definite kernel on $(\mathbb{S}^n \times T)^2$. We need to prove that for any ℓ and j , $\varphi_{\ell,j}^n$ is a positive definite kernel on $T \times T$. Let $q : T \rightarrow \mathbb{C}$ be any continuous map, having compact support in the case $T = \mathbb{R}$. We apply either Lemma 1 in the case $T = [0, 1]$ or Lemma 2 in the case $T = \mathbb{R}$, with $c(p, t) = Y_{\ell,j}(p) q(t)$ for $(p, t) \in \mathbb{S}^n \times T$, and obtain

$$\int \int_{(\mathbb{S}^n \times T)^2} k(p_1, t_1, p_2, t_2) Y_{\ell,j}(p_1) q(t_1) \overline{Y_{\ell,j}(p_2)} \overline{q(t_2)} d\sigma_n(p_1) d\sigma_n(p_2) dt_1 dt_2 \geq 0.$$

Direct application of Fubini's Theorem yields

$$\int \int_{(T)^2} q(t_1) \overline{q(t_2)} \left(\int \int_{(\mathbb{S}^n)^2} k(p_1, t_1, p_2, t_2) Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} d\sigma_n(p_1) d\sigma_n(p_2) \right) dt_1 dt_2 \geq 0.$$

The inner integral is exactly the function $\varphi_{\ell,j}^n(\cdot, \cdot)$ as defined through Equation (3.2). Thus,

$$\int \int_{(T)^2} q(t_1) \overline{q(t_2)} \varphi_{\ell,j}^n(t_1, t_2) dt_1 dt_2 \geq 0,$$

for any q as asserted. This proves that $\varphi_{\ell,j}^n$ is a positive definite kernel on $T \times T$.

Conversely, suppose that $\varphi_{\ell,j}^n$ is a positive definite kernel on $T \times T$ for all ℓ and j . Then, for $\{a_1, \dots, a_m\} \in \mathbb{C}$ and $\{(p_1, t_1), \dots, (p_m, t_m)\} \in \mathbb{S}^n \times T$, we have

$$\begin{aligned} \sum_{i,i'=1}^m a_i \overline{a_{i'}} k(p_i, t_i, p_{i'}, t_{i'}) &= \sum_{i,i'=1}^m a_i \overline{a_{i'}} \lim_{N \rightarrow \infty} \sum_{\ell=0}^N \sum_{j=1}^{d_{n,\ell}} \varphi_{\ell,j}^n(t_i, t_{i'}) Y_{\ell j}(p_i) \overline{Y_{\ell j}(p_{i'})} \\ &= \lim_{N \rightarrow \infty} \sum_{\ell=0}^N \sum_{j=1}^{d_{n,\ell}} \sum_{i,i'=1}^m a_i \overline{a_{i'}} \varphi_{\ell,j}^n(t_i, t_{i'}) Y_{\ell j}(p_i) \overline{Y_{\ell j}(p_{i'})} \geq 0, \end{aligned}$$

since $\varphi_{\ell,j}^n(t_i, t_{i'}) Y_{\ell j}(p_i) \overline{Y_{\ell j}(p_{i'})}$ is positive definite on $(\mathbb{S}^n \times T)^2$, because it is the tensor product of positive definite kernels on $T \times T$ and $\mathbb{S}^n \times \mathbb{S}^n$ respectively, for every ℓ and j . Then, the limit of a sequence of positive definite kernels on $(\mathbb{S}^n \times T)^2$ is also a positive definite kernel on $(\mathbb{S}^n \times T)^2$. \square

Next result provides a spectral representation for the positive definite kernels that are isotropic with respect to the space \mathbb{S}^n and nonstationary with respect to time. Therefore, it can be treated as a generalization of the main result given in [Berg, C. and Porcu, E. \(2016\)](#) as Theorem 3.3, which provides a similar characterization in the stationary case. Notice that for convenience, we often use the short notation θ instead of $\theta(p_1, p_2)$ for $p_1, p_2 \in \mathbb{S}^n$.

Theorem 3.2. *A continuous symmetric kernel $k : (\mathbb{S}^n \times T)^2 \rightarrow \mathbb{R}$ is positive definite and isotropic with respect to space if and only if for every t_1, t_2 in T , p_1, p_2 in \mathbb{S}^n ,*

$$k(p_1, t_1, p_2, t_2) = \sum_{\ell=0}^{\infty} \alpha_{\ell}^n(t_1, t_2) C_{\ell}^{(n-1)/2}(\cos(\theta(p_1, p_2))), \quad (3.4)$$

where,

- i.- for every $\ell \geq 0$, α_{ℓ}^n is a real-valued continuous symmetric positive definite kernel on $T \times T$;
- ii.- for every t in T , $\sum_{\ell=0}^{\infty} \alpha_{\ell}^n(t, t) \ell^{n-2} < \infty$.

Moreover, for any t_1, t_2 in T , we can write $k(\cdot, t_1, \cdot, t_2) = \widetilde{k}_S(\theta, t_1, t_2)$, the convergence in (3.4) is uniform with respect to $\theta \in [0, \pi]$, and

$$\alpha_{\ell}^n(t_1, t_2) = \frac{\omega_{n-1} \left(1 + \frac{2\ell}{n-1}\right)}{\omega_n \binom{\ell+n-2}{\ell}} \int_0^{\pi} \widetilde{k}_S(\theta, t_1, t_2) C_{\ell}^{(n-1)/2}(\cos \theta) (\sin \theta)^{n-1} d\theta. \quad (3.5)$$

Proof. Suppose that k is a positive definite kernel on $(\mathbb{S}^n \times T)^2$ and is isotropic with respect to space.

First note that by Definition 4, there exists a function $\widetilde{k}_S : [0, \pi] \times T \times T \rightarrow \mathbb{R}$ such that for every $t_1, t_2 \in T$ and $p_1, p_2 \in \mathbb{S}^n$,

$$k(p_1, t_1, p_2, t_2) = \widetilde{k}_S(\theta(p_1, p_2), t_1, t_2).$$

Let us fix t_1 and t_2 in T . Since the map $\widetilde{k}_S(\arccos(\cdot), t_1, t_2)$ belongs to $\mathcal{L}^2([-1, 1], (1-z^2)^{(n-2)/2} dz)$, it admits an expansion in terms of the Gegenbauer polynomials, *i.e.* there exists a sequence of real numbers $\{\alpha_{\ell}^n\}_{\ell \geq 0}$ such that

$$\widetilde{k}_S(\arccos(\cdot), t_1, t_2) = \sum_{\ell=0}^{\infty} \alpha_{\ell}^n(t_1, t_2) C_{\ell}^{(n-1)/2}(\cdot), \quad (3.6)$$

where the convergence holds in $\mathcal{L}^2([-1, 1], (1 - z^2)^{(n-2)/2} dz)$. It implies that, for any $\ell \geq 0$,

$$\begin{aligned} & \int_{[-1, 1]} \widetilde{k}_S(\arccos(z), t_1, t_2) C_{\ell}^{(n-1)/2}(z) (1 - z^2)^{(n-2)/2} dz \\ &= \sum_{\ell'=0}^{\infty} \alpha_{\ell'}^n(t_1, t_2) \int_{[-1, 1]} C_{\ell'}^{(n-1)/2}(z) C_{\ell}^{(n-1)/2}(z) (1 - z^2)^{(n-2)/2} dz. \end{aligned}$$

Using the orthogonality of the Gegenbauer polynomials, as well as (2.10), one gets

$$\int_{[-1, 1]} \widetilde{k}_S(\arccos(z), t_1, t_2) C_{\ell}^{(n-1)/2}(z) (1 - z^2)^{(n-2)/2} dz = \alpha_{\ell}^n(t_1, t_2) \frac{\omega_n \binom{\ell+n-2}{\ell}}{\omega_{n-1} (1 + \frac{2\ell}{n-1})}.$$

Thus, after the change of variable $z = \cos(\theta)$ in the above integral, the coefficients $\{\alpha_{\ell}^n\}_{\ell \geq 0}$ introduced in (3.6) are proved to satisfy Equation (3.5).

We now establish *i*). Let $\ell \geq 0$ be fixed and let us prove that α_{ℓ}^n is a positive definite kernel on $T \times T$.

For any compactly supported continuous function $q : T \rightarrow \mathbb{C}$, we apply Lemma 2 with $c(p, t) = Y_{\ell j}(p) q(t)$ for $(p, t) \in \mathbb{S}^n \times T$ and some $j \in \{1, \dots, d_{n, \ell}\}$. By Fubini's Theorem, we get

$$\int \int_{T^2} q(t_1) \overline{q(t_2)} \left(\int \int_{(\mathbb{S}^n)^2} k(p_1, t_1, p_2, t_2) Y_{\ell j}(p_1) \overline{Y_{\ell j}(p_2)} d\sigma_n(p_1) d\sigma_n(p_2) \right) dt_1 dt_2 \geq 0.$$

This proves that the function defined by the inner double integral constitutes a positive definite kernel on $T \times T$. Taking the sum on all j in $\{1, \dots, d_{n, \ell}\}$ proves that the kernel β_{ℓ}^n , defined through

$$\beta_{\ell}^n : (t_1, t_2) \mapsto \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} k(p_1, t_1, p_2, t_2) \sum_{j=1}^{d_{n, \ell}} Y_{\ell j}(p_1) \overline{Y_{\ell j}(p_2)} d\sigma_n(p_1) d\sigma_n(p_2),$$

is positive definite on $T \times T$. It is moreover continuous and symmetric thanks to the properties of k . Let us compute β_{ℓ}^n and establish that it is proportional to α_{ℓ}^n .

Applying the addition Theorem (2.8), one gets for every t_1, t_2 in T ,

$$\beta_{\ell}^n(t_1, t_2) = \frac{d_{n, \ell}}{\omega_n \binom{\ell+n-2}{\ell}} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \widetilde{k}_S(\arccos(\langle p_1, p_2 \rangle), t_1, t_2) C_{\ell}^{(n-1)/2}(\langle p_1, p_2 \rangle) d\sigma_n(p_1) d\sigma_n(p_2).$$

Following the change of variables as in page 479 in [Vilenkin, N. J. \(1968\)](#), we define the unit vector $\zeta = (1, \dots, 0) \in \mathbb{R}^{n+1}$ and $z := \langle \zeta, p_1 \rangle$, where $z \in [-1, 1]$, so that we have

$$p_1 = z\zeta + \sqrt{1 - z^2} \eta,$$

where $\eta \in \mathbb{S}^{n-1}$ is generated by x_1, \dots, x_n in \mathbb{R}^n . Thus, the Lebesgue measure on \mathbb{S}^n is

$$d\sigma_n(p_1) = (1 - z^2)^{(n-2)/2} dz d\sigma_{n-1}(\eta).$$

It follows that

$$\begin{aligned} \beta_\ell^n(t_1, t_2) &= \frac{d_{n,\ell}}{\omega_n \binom{\ell+n-2}{\ell}} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} k(p_1, t_1, p_2, t_2) C_\ell^{(n-1)/2}(\langle p_1, p_2 \rangle) d\sigma_n(p_1) d\sigma_n(p_2) \\ &= \frac{d_{n,\ell}}{\omega_n \binom{\ell+n-2}{\ell}} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \widetilde{k}_S(\arccos(\langle p_1, p_2 \rangle), t_1, t_2) C_\ell^{(n-1)/2}(\langle p_1, p_2 \rangle) d\sigma_n(p_1) d\sigma_n(p_2) \\ &= \frac{d_{n,\ell}}{\omega_n \binom{\ell+n-2}{\ell}} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \widetilde{k}_S(\arccos(\langle \zeta, p_1 \rangle), t_1, t_2) C_\ell^{(n-1)/2}(\langle \zeta, p_1 \rangle) d\sigma_n(p_1) d\sigma_n(p_2) \\ &= \frac{d_{n,\ell} \omega_{n-1}}{\binom{\ell+n-2}{\ell}} \int_{[-1,1]} \widetilde{k}_S(\arccos(z), t_1, t_2) C_\ell^{(n-1)/2}(z) (1 - z^2)^{(n-2)/2} dz. \end{aligned}$$

Moreover, if $\cos \theta = z$ with $\theta \in [0, \pi]$, we have

$$\beta_\ell^n(t_1, t_2) = \frac{d_{n,\ell} \omega_{n-1}}{\binom{\ell+n-2}{\ell}} \int_{[0,\pi]} \widetilde{k}_S(\theta, t_1, t_2) C_\ell^{(n-1)/2}(\cos \theta) (\sin \theta)^{n-1} d\theta,$$

which is clearly proportional to $\alpha_\ell^n(t_1, t_2)$ thanks to (3.5). Point *i*) is proved.

We now focus on point *ii*). Let $t \in T$ be fixed and let us recall that Expansion (3.6) holds in $\mathcal{L}^2([-1, 1], (1 - z^2)^{(n-2)/2} dz)$ for $t_1 = t_2 = t$. Moreover, since $\widetilde{k}_S(\arccos(\cdot), t, t)$ is continuous on $[-1, 1]$, one knows that the series in (3.6) is Abel summable on $[0, 1]$ for any $z \in [-1, 1]$ (see Theorem 9 in Müller, C. (1966)), *i.e.* the limit $\lim_{r \rightarrow 1^-} \sum_{\ell=0}^{\infty} (\alpha_\ell^n(t, t) C_\ell^{(n-1)/2}(z) r^\ell)$ exists. For $z = 1$, since $\alpha_\ell^n(t, t) C_\ell^{(n-1)/2}(1) \geq 0$ for all $\ell \geq 0$, it implies that the series $\sum_{\ell=0}^{\infty} (\alpha_\ell^n(t, t) C_\ell^{(n-1)/2}(1))$ is finite. Noting that $\binom{\ell+n-2}{\ell}$ is equivalent to $\frac{\ell^{n-2}}{(n-2)!}$ when $\ell \rightarrow \infty$, the convergence of the series is equivalent to

$$\sum_{\ell=0}^{\infty} \alpha_\ell^n(t, t) \ell^{n-2} < \infty,$$

and hence we get *ii*).

Finally, let us come back to Expansion (3.6) for fixed $t_1, t_2 \in T$. By Cauchy-Schwarz inequality and by (2.9), we can see that for $z \in [-1, 1]$ and for all $\ell \geq 0$,

$$|\alpha_\ell^n(t_1, t_2) C_\ell^{(n-1)/2}(z)| \leq \left(\frac{\alpha_\ell^n(t_1, t_1) + \alpha_\ell^n(t_2, t_2)}{2} \right) \binom{\ell + n - 2}{\ell},$$

with $\sum_{\ell=0}^{\infty} \alpha_\ell^n(t_i, t_i) \ell^{n-2} < \infty$ for $i = 1, 2$. Then, by the M-test of Weierstrass, the series in (3.6) converges uniformly to a continuous function, being precisely the function $\widetilde{k}_S(\arccos(\cdot), t_1, t_2)$. This proves that Expansion (3.4) holds pointwise, as well as the uniform convergence of the series for fixed $(t_1, t_2) \in T \times T$.

Conversely, we argue that (3.4) defines a positive definite kernel k on $(\mathbb{S}^n \times T)^2$ using the same arguments as in the converse implication of part *b*) of Theorem 3.1. Moreover k is isotropic with respect to space, since the right-hand side in (3.4) only depends on the scalar product of p_1 and p_2 in \mathbb{S}^n . \square

Remark 3.1. In general, when k is not stationary with respect to time, uniform convergence of the expansion (3.4) does not hold on $(\mathbb{S}^n \times T)^2$. Nevertheless, in the stationary case it does (see Theorem 3.3 in Berg, C. and Porcu, E. (2016)).

Remark 3.2. When $n = 1$, Expansion (3.4) takes the following form

$$k(p_1, t_1, p_2, t_2) = \sum_{\ell=0}^{\infty} \alpha_{\ell}^1(t_1, t_2) \cos(\ell \theta(p_1, p_2)), \quad p_1, p_2 \in \mathbb{S}^1, t_1, t_2 \in T, \quad (3.7)$$

where for every t in T , $\sum_{\ell=0}^{\infty} \alpha_{\ell}^1(t, t) \ell^{-1} < \infty$. Moreover, for any t_1, t_2 in T ,

$$\alpha_{\ell}^1(t_1, t_2) = \frac{2}{\pi} \int_0^{\pi} \widetilde{k}_S(\theta, t_1, t_2) \cos(\ell \theta) d\theta.$$

In the next corollary, we sum up the statements of Theorems 3.1 and 3.2.

Corollary 3.3. Let $k : (\mathbb{S}^n \times T)^2 \rightarrow \mathbb{R}$ be a kernel satisfying the same assumption as in Theorem 3.1. Assume moreover that k is positive definite.

Then, k is isotropic with respect to space if and only if the functions $\{\varphi_{\ell j}^n\}_{\ell, j}$ appearing in expansion (3.1) do not depend on the j index, *i.e.* they are such that

$$\varphi_{\ell j}^n \equiv \varphi_{\ell j'}^n, \quad \text{for all } j, j' \text{ in } \{1, \dots, d_{n, \ell}\}, \ell \geq 0.$$

Furthermore, the functions $\{\varphi_{\ell j}^n\}_{\ell, j}$ are linked with the functions $\{\alpha_{\ell}^n\}_{\ell}$ appearing in expansion (3.4) by the following

$$\varphi_{\ell, j}^n(t_1, t_2) = \frac{\omega_n \binom{\ell+n-2}{\ell}}{d_{n, \ell}} \alpha_{\ell}^n(t_1, t_2), \quad t_1, t_2 \in T. \quad (3.8)$$

Proof. Suppose that k is isotropic with respect to space. Then, by Definition 4, k can be expressed as $k(p_1, t_1, p_2, t_2) = \widetilde{k}_S(\theta(p_1, p_2), t_1, t_2)$. By Theorem 3.2, for every t_1, t_2 in T , $\widetilde{k}_S(\theta, t_1, t_2)$ admits the expansion

$$\widetilde{k}_S(\theta, t_1, t_2) = \sum_{\ell=0}^{\infty} \alpha_{\ell}^n(t_1, t_2) C_{\ell}^{(n-1)/2}(\cos \theta), \quad \theta \in [0, \pi], \quad (3.9)$$

which converges uniformly on $[0, \pi]$.

Replacing the expression (3.9) in (3.2) and using (2.8) with $\langle p_2, p_1 \rangle$ instead of $\langle p_1, p_2 \rangle$ yield

$$\begin{aligned}
\varphi_{\ell,j}^n(t_1, t_2) &= \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \sum_{\ell'=0}^{\infty} \alpha_{\ell'}^n(t_1, t_2) \frac{\omega_n(\ell'+n-2)}{d_{n,\ell'}} \sum_{j'=1}^{d_{n,\ell'}} \overline{Y_{\ell',j'}(p_1)} Y_{\ell',j'}(p_2) Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} d\sigma_n(p_1) d\sigma_n(p_2) \\
&= \sum_{\ell'=0}^{\infty} \alpha_{\ell'}^n(t_1, t_2) \frac{\omega_n(\ell'+n-2)}{d_{n,\ell'}} \sum_{j'=1}^{d_{n,\ell'}} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \overline{Y_{\ell',j'}(p_1)} Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} Y_{\ell',j'}(p_2) d\sigma_n(p_1) d\sigma_n(p_2) \\
&= \alpha_{\ell}^n(t_1, t_2) \frac{\omega_n(\ell+n-2)}{d_{n,\ell}}.
\end{aligned}$$

Thus $\varphi_{\ell,j}^n(t_1, t_2)$ does not depend on the j index for all $\ell \geq 0$ and (3.8) is proved.

To prove the converse implication, we suppose that the functions $\varphi_{\ell,j}^n$ in (3.1) do not depend on the j index for all $j = 1, \dots, d_{n,\ell}$. Then, using (2.8), Expansion (3.1) takes the form

$$\begin{aligned}
k(p_1, t_1, p_2, t_2) &= \sum_{\ell=0}^{\infty} \varphi_{\ell,j}^n(t_1, t_2) \sum_{j=1}^{d_{n,\ell}} Y_{\ell,j}(p_1) \overline{Y_{\ell,j}(p_2)} \\
&= \sum_{\ell=0}^{\infty} \alpha_{\ell}^n(t_1, t_2) C_{\ell}^{(n-1)/2}(\langle p_1, p_2 \rangle),
\end{aligned}$$

with $\alpha_{\ell}^n(t_1, t_2)$ as in (3.8). Hence k isotropic with respect to space. \square

4 Examples of nonstationary models on $\mathbb{S}^n \times \mathbb{R}$

4.1 An isotropic nonstationary model on $\mathbb{S}^n \times \mathbb{R}$

We propose the kernel $k(p_1, t_1, p_2, t_2) = \tilde{k}_S(\arccos(\langle p_1, p_2 \rangle), t_1, t_2)$, such that

$$\tilde{k}_S(\theta, t_1, t_2) = \exp(\lambda(g(t_1, t_2) \cos \theta - 1)), \quad (4.1)$$

where, $\lambda > 0$ and

$$g(t_1, t_2) = \frac{1}{t_1^2 + t_2^2 + 1}, \quad t_1, t_2 \in \mathbb{R}$$

defines a nonstationary positive definite kernel on $\mathbb{R} \times \mathbb{R}$. We note that the kernel (4.1) is non-separable, isotropic with respect to space and nonstationary with respect to time. The stationary analogue case is showed in Table 1 in [Porcu, E., Bevilacqua, M. and Genton, M. G. \(2016\)](#), called Poisson model. The Schoenberg's functions $\{\alpha_{\ell}^n\}_{\ell \geq 0}$ that appear in (3.4) when expanding the kernel are given by (3.5). If $n = 2$ for every $t_1, t_2 \in \mathbb{R}$,

$$\alpha_{\ell}^2(t_1, t_2) = \frac{\exp(-\lambda) 2\pi(1 + 2\ell)}{\omega_2} \int_{[0,\pi]} \exp(\lambda g(t_1, t_2) \cos \theta) C_{\ell}^{1/2}(\cos \theta) \sin \theta d\theta, \quad \ell \geq 0. \quad (4.2)$$

Recalling that when $n = 2$, the Gegenbauer polynomials coincide with the Legendre polynomials, hence series and integral representations are allowed. (see Equations 20 and 16 in Chapter 3 in [Vilenkin, N. J. \(1968\)](#))

4.2 Simulation of an isotropic nonstationary Gaussian field on $\mathbb{S}^2 \times \mathbb{R}$

We simulate a Gaussian random field on $\mathbb{S}^2 \times \mathbb{R}$ whose covariance function is approximately given by (4.1). The implementation was made in R software. First, we estimate the α_ℓ^2 following (4.2) using the Package *orthopolynom* in the computation of the Gegenbauer polynomials. Then we take a couple of pairs of time instants, for example $t_1 = 0.1$, $t_2 = 0.2$ and $t_3 = 0.4$, $t_4 = 1$, in order to observe the changes in the distribution of the field across the time. Thus, it is required to simulate a realization of the space-time random field over the sphere at four temporal instants. We use the spectral representation of the fields given by the spherical harmonics series expansion in Theorem 2.1, where the truncation of the series expansion was put at $N = 10$, since in this case the decay of α_ℓ^2 is fast enough as ℓ increases. Additionally, to compute the spherical harmonics functions, we apply the Remark 3.24 in [Marinucci, D. and Peccati, G. \(2011\)](#) implemented through the Package *gsl*.

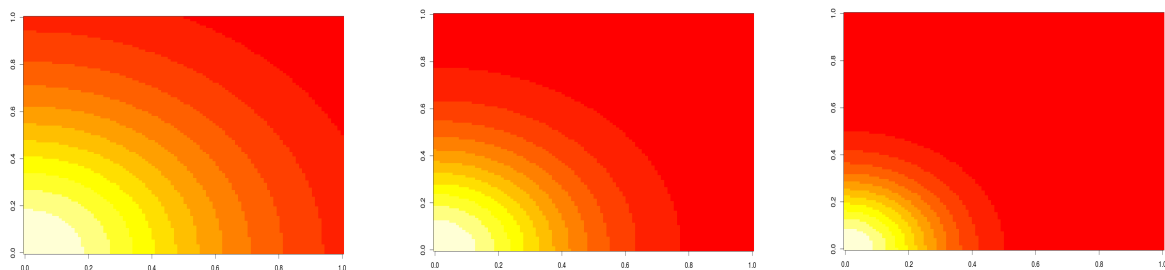


Figure 1. $\alpha_\ell^2(\cdot, \cdot)$ for $\ell = 1, 6$ and 12 , with $\lambda = 2$.

The set of variables $\{b_{\ell_j}(t_i) : \ell \geq 0, j = 1, \dots, d_{2,\ell}, i = 1, 2, 3, 4\}$ in Theorem 2.1, are simulated as centered Gaussian random variables following the time dependence given by α_ℓ^2 (see Figure 1), where we take into account that for every ℓ and t_i fixed $\{b_{\ell_j}(t_i) : j = 1, \dots, d_{2,\ell}\}$ are not correlated. This fact induces isotropy with respect to space. On the other hand, with respect to time we have that for every ℓ fixed and for $t_1, t_2 \in \mathbb{R}$, $b_{\ell_j}(t_2)$ follows a conditional Gaussian distribution given $b_{\ell_j}(t_1)$, we do subsequently the same recurrence for $b_{\ell_j}(t_3), b_{\ell_j}(t_4)$, with $j = 1, \dots, d_{2,\ell}$.

At the end, it is possible to evaluate the error of the approximation applying Proposition 5.2 of Lang, A. and Schwab, C. (2013), since for every $t_i \in \mathbb{R}$, $\alpha_\ell^2(t_i, t_i)$ decays algebraically with order $\alpha > 2$ i.e., for $i = 1, \dots, 4$, there exist constants $C_i > 0$ and $\ell_0 \geq 0$ such that $\alpha_\ell^2(t_i, t_i) \leq C_i \ell^{-\alpha}$ for all $\ell \geq \ell_0$. Thus, it holds

$$\|X(\cdot, t_i) - X_N(\cdot, t_i)\|_{\mathcal{L}^2(\mathbb{S}^2)} \leq \hat{C}_i N^{-(\alpha-2)/2},$$

for $N \geq \ell_0$, where $\hat{C}_i^2 = C_i \left(\frac{2}{\alpha-2} + \frac{1}{\alpha-1} \right)$. As an example, the decay of $\alpha_\ell^2(t_i, t_i)$ for $i = 1, 2$, is compared with ℓ^{-3} in Figure 2.

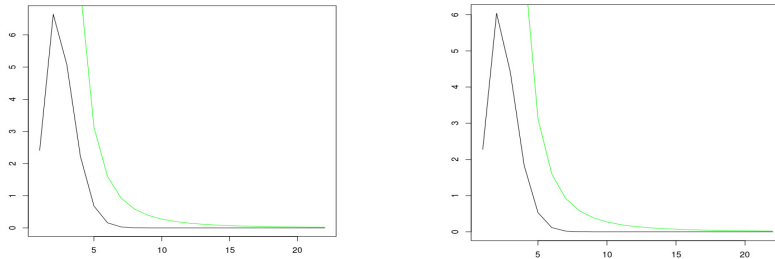


Figure 2. $\alpha_\ell^2(t_1, t_1)$ and $\alpha_\ell^2(t_2, t_2)$ respectively in color black with curves in $\mathcal{O}(\ell^{-3})$ in green, identifying the x-axis with the index ℓ .

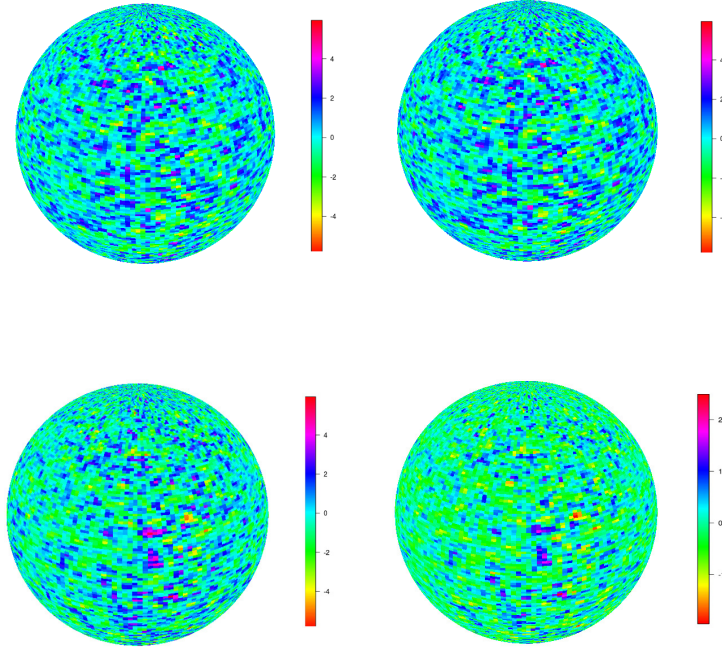


Figure 3. Isotropic nonstationary Gaussian random field simulated at time instants $t_1 = 0.1$, $t_2 = 0.2$ and $t_3 = 0.4$, $t_4 = 1$ respectively, with $N = 10$ over a mesh in \mathbb{S}^2 with 40000 points.

4.3 An anisotropic nonstationary model on $\mathbb{S}^n \times \mathbb{R}$

In this section, we give an anisotropic nonstationary covariance model on $\mathbb{S}^n \times \mathbb{R}$, modifying the model given in the above section. Let us apply Theorem 2.1, then for every $t \in \mathbb{R}$ we define a squared integrable random field on \mathbb{S}^n , which admits the next series representation

$$X(p, t) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{n,\ell}} b_{\ell,j}(t) Y_{\ell,j}(p), \quad p \in \mathbb{S}^n.$$

Considering two instants $t_1, t_2 \in \mathbb{R}$, the set of variables $\{b_{\ell,j}(t_i) : \ell \geq 0, j = 1, \dots, d_{2,\ell}, i = 1, 2\}$ satisfies

$$\mathbb{E}[b_{\ell,j}(t_1) b_{\ell,j}(t_2)] := \varphi_{\ell,j}^n(t_1, t_2),$$

where we impose that $\varphi_{\ell,j}^n$ has dependence on the j index, obtaining in this way anisotropy with respect to space. In particular, when $n = 2$ we propose to use

$$\varphi_{\ell,j}^2(t_1, t_2) = \frac{2j+1}{d_{2,\ell}} \alpha_{\ell}^2(t_1, t_2), \quad (4.3)$$

with α_ℓ^2 being defined as in (4.2).

4.4 Simulation of an anisotropic nonstationary Gaussian field on $\mathbb{S}^2 \times \mathbb{R}$

We approximate a Gaussian random field on $\mathbb{S}^2 \times \mathbb{R}$. The procedure to apply is analogous to the one explained in the last section, simulating in this case the $\varphi_{\ell,j}^2$ through formula (4.3).

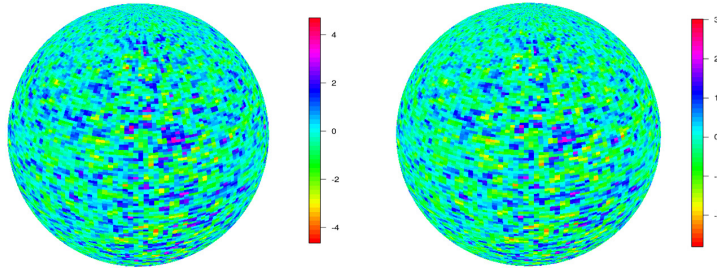


Figure 4. Anisotropic nonstationary Gaussian random field simulated at time instants $t_1 = 0.2$, $t_2 = 0.6$ respectively, with $N = 7$ over a mesh in \mathbb{S}^2 with 40000 points.

In order to appreciate better the anisotropy, we plot the field at $t_1 = 0.2$ from different viewing directions, which are obtained varying the azimuthal angle or rotating the sphere.

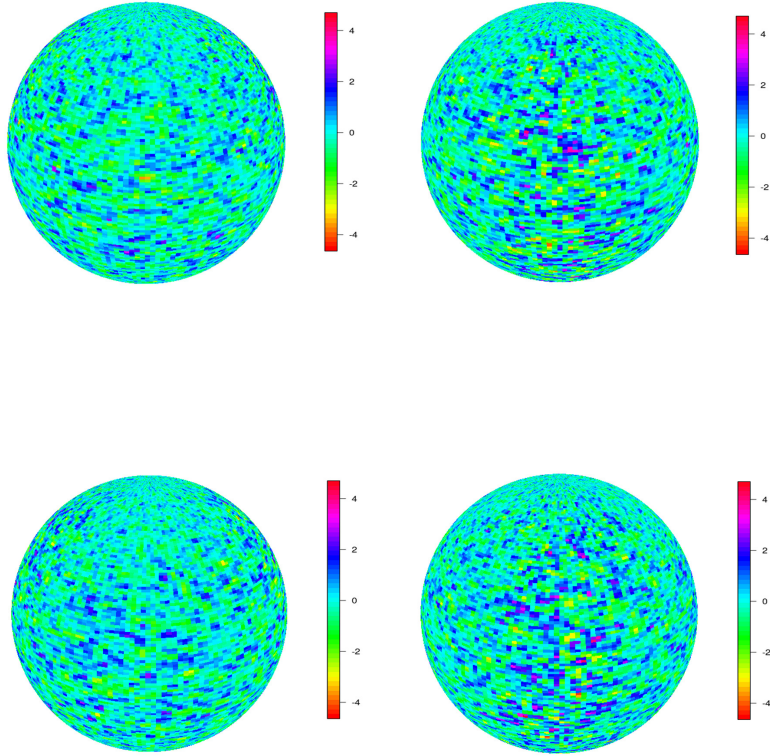


Figure 3. Anisotropic nonstationary Gaussian random field simulated at $t_1 = 0.2$ with viewing directions given by the azimuthal angles 0° , 90° , 180° and 270° respectively.

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