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HARNACK PARTS OF ρ -CONTRACTIONS

GILLES CASSIER^{1*}, MOHAMMED BENHARRAT² AND SOUMIA BELMOUHOUB³

ABSTRACT. The purpose of this paper is to describe the Harnack parts for the operators of class C_ρ ($\rho > 0$) on Hilbert spaces which were introduced by B. Sz. Nagy and C. Foias in [25]. More precisely, we study Harnack parts of operators with ρ -numerical radius one. The case of operators with ρ -numerical radius strictly less than 1 was described in [10]. We obtain a general criterion for compact ρ -contractions to be in the same Harnack part. We give a useful equivalent form of this criterion for usual contractions. Operators with numerical radius one received also a particular attention. Moreover, we study many properties of Harnack equivalence in the general case.

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space and $B(H)$ the set of all bounded linear operators on H . For $\rho > 0$, we say that an operator $T \in B(H)$ admits a *unitary ρ -dilation* if there is a Hilbert space \mathcal{H} containing H as a closed subspace and a unitary operator $U \in B(\mathcal{H})$ such that

$$T^n = \rho P_H U^n|_H, \quad n \in \mathbb{N}^*, \quad (1.1)$$

where P_H denotes the orthogonal projection onto the subspace H in \mathcal{H} .

In the sequel, we denote by $C_\rho(H)$, $\rho > 0$, the set of all operators in $B(H)$ which admit unitary ρ -dilations. A famous theorem due to B. Sz.-Nagy [22] asserts that $C_1(H)$ is exactly the class of all contractions, i.e., operators T such that $\|T\| \leq 1$. C. A. Berger [5] showed that the class $C_2(H)$ is precisely the class of all operators $T \in B(H)$ whose the *numerical radius*

$$w(T) = \sup\{|\langle Tx, x \rangle| : x \in H, \|x\| = 1\}$$

is less or equal to one. In particular, the classes $C_\rho(H)$, $\rho > 0$, provide a framework for simultaneous investigation of these two important classes of operators. Any operator $T \in C_\rho(H)$ is *power-bounded*:

$$\|T^n\| \leq \rho, \quad n \in \mathbb{N}, \quad (1.2)$$

moreover, its *spectral radius*

$$r(T) = \lim_{n \rightarrow +\infty} \|T^n\|^{\frac{1}{n}} \quad (1.3)$$

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is at most one. In [23], an example of a power-bounded operator which is not contained in any of the classes $C_\rho(H)$, $\rho > 0$, is given. However, J. A. R. Holbrook [17] and J. P. Williams [26], independently, introduced the ρ -numerical radius (or the *operator radii*) of an operator $T \in B(H)$ by setting

$$w_\rho(T) := \inf\{\gamma > 0 : \frac{1}{\gamma}T \in C_\rho(H)\}. \quad (1.4)$$

Note that $w_1(T) = \|T\|$, $w_2(T) = w(T)$ and $\lim_{\rho \rightarrow \infty} w_\rho(T) = r(T)$. Also, $T \in C_\rho(H)$ if and only if $w_\rho(T) \leq 1$, hence operators in $C_\rho(H)$ are contractions with respect to the ρ -numerical radius, and the elements of $C_\rho(H)$ are called ρ -contractions.

Some properties of the classes $C_\rho(H)$ become more clear (see for instance, [8],[9], [10] and [7]) due to the use of the following operatorial ρ -kernel for a bounded operator T having its spectrum in the closed unit disc $\overline{\mathbb{D}}$, harmonic method in operator analysis introduced and first systematically developed in [6, 8, 9]:

$$K_z^\rho(T) = (I - \bar{z}T)^{-1} + (I - zT^*)^{-1} + (\rho - 2)I, \quad (z \in \mathbb{D}). \quad (1.5)$$

The ρ -kernels are related to ρ -contraction by the next result. An operator T is in the class $C_\rho(H)$ if and only if, $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $K_z^\rho(T) \geq 0$ for any $z \in \mathbb{D}$ (see [9]).

We say that T_1 is Harnack dominated by T_0 , if T_0 and T_1 satisfy one of the following equivalent conditions of the following theorem:

Theorem 1.1. [10, Theorem 3.1] *For $T_0, T_1 \in C_\rho(H)$ and a constant $c \geq 1$, the following statements are equivalent:*

- (i) $Rep(T_1) \leq c^2 Rep(T_0) + (c^2 - 1)(\rho - 1)Rep(O_H)$, fo any polynomial p with $Rep \geq 0$ on $\overline{\mathbb{D}}$.
- (ii) $Rep(rT_1) \leq c^2 Rep(rT_0) + (c^2 - 1)(\rho - 1)Rep(O_H)$, fo any $r \in]0, 1[$ and each polynomial p with $Rep \geq 0$ on $\overline{\mathbb{D}}$.
- (iii) $K_z^\rho(T_1) \leq c^2 K_z^\rho(T_0)$, for all $z \in \mathbb{D}$.
- (iv) $\varphi_{T_1}(g) \leq c^2 \varphi_{T_0}(g)$ for any function $g \in C(\mathbb{T})$ such that $g \geq 0$ on $\mathbb{T} = \overline{\mathbb{D}} \setminus \mathbb{D}$.
- (v) If V_i acting on $K_i \supseteq H$ is the minimal isometric ρ -dilation of T_i ($i = 0, 1$), then there is an operator $S \in B(K_0, K_1)$ such that $S(H) \subset H$, $S|_H = I$, $SV_0 = V_1S$ and $\|S\| \leq c$.

When T_1 is Harnack dominated by T_0 in $C_\rho(H)$ for some constant $c \geq 1$, we write $T_1 \overset{H}{\prec}_c T_0$, or also $T_1 \overset{H}{\prec} T_0$. The relation $\overset{H}{\prec}$ is a preorder relation in $C_\rho(H)$. The induced equivalent relation is called Harnack equivalence, and the associated classes are called the Harnack parts of $C_\rho(H)$. So, we say that T_1 and T_0 are Harnack equivalent if they belong to the same Harnack parts. In this later case, we write $T_1 \overset{H}{\sim} T_0$.

We say that an operator $T \in C_\rho(H)$ is a strict ρ -contraction if $w_\rho(T) < 1$. In [16] C. Foiaş proved that the Harnack parts of contractions containing the null operators O_H consists of all strict contractions. More recently, G. Cassier and N. Suciú proved in [10, Theorem 4.4] that the Harnack parts of $C_\rho(H)$ containing

the null operators O_H consists of all strict ρ -contractions. According to this fact the following natural question arises:

If T an operator with ρ -numerical radius one, what can be said about the Harnack part of T ?

Recall that a ρ -contractions is similar to a contraction [24] but many properties are not preserved under similarity (and an operator similar to a contraction is not necessarily a ρ -contraction!), in particular it is true for the numerical range properties. Thus, the study of Harnack parts for ρ -contractions cannot be deduced from the contractions case. Notice also that some properties are of different nature (see for instance Theorem 2.1 and Remark 2.7). We find a few answers in the literature of the previous question, essentially in the class of contractions with norm one. In [2], the authors have proved that if T is either isometry or coisometry contraction then the Harnack part of T is trivial (i.e. equal to $\{T\}$), and if T is compact or $r(T) < 1$, or normal and nonunitary, then its Harnack part is not trivial in general. The authors have asked that it seems interesting to give necessary and/ or sufficient condition for a contractions to have a trivial Harnack part. It was proved in [20] that the Harnack part of a contraction T is trivial if and only if T is an isometry or a coisometry (the adjoint of an isometry), this a response of the question posed by T. Ando and al. in the class of contractions. Recently the authors of [4] proved that maximal elements for the Harnack domination in $C_1(H)$ are precisely the singular unitary operators and the minimal elements are isometries and coisometries.

This paper is a continuation and refinement of the research treatment of the Harnack domination in the general case of the ρ -contractions. Note that this treatment yields certain useful properties and new techniques for studies of the Harnack parts of an operator with ρ -numerical radius one. More precisely, we show that two ρ -contractions belong to the same Harnack parts have the same spectral values in \mathbb{T} . This property has several consequences and applications. In particular, it will be shown that if T is a compact (i.e. $T \in \mathcal{K}(H)$) with $w_\rho(T) = 1$ and whose spectral radius is strictly less than one, then a ρ -contraction $S \in \mathcal{K}(H)$ is Harnack equivalent to T if and only if $K_z^\rho(S)$ and $K_z^\rho(T)$ have the same kernel for all $z \in \mathbb{T}$. As a corollary, in the case of contractions, we show that if T is a compact contraction with $\|T\| = 1$ and $r(T) < 1$, then a contraction $S \in \mathcal{K}(H)$ is Harnack equivalent to T if and only if $I - S^*S$ and $I - T^*T$ have the same kernel and S and T restricted to the kernel of $I - T^*T$ coincide. A nice application is the description of the Harnack part of the (nilpotent) Jordan block of size n . We also obtain precise results about the relationships between the closure of the numerical range and the Harnack domination for every $\rho \in [1, 2]$. The case of $\rho = 2$ plays a crucial role. We characterizes the weak stability of a ρ -contraction in terms of its minimal isometric ρ -dilation. The details of these basic facts are explained in Section 2. In the last section we apply the results in the foregoing section to describe the Harnack part of some nilpotent matrices with numerical radius one, in three cases: a nilpotent matrix of order two in the two dimensional case, a nilpotent matrix of order two in \mathbb{C}^n and a nilpotent matrix of order three in the three dimensional case. In particular, we show that in the first case the Harnack

part is trivial, while in the third case the Harnack part is an orbit associated with the action of a group of unitary diagonal matrices.

2. MAIN RESULTS

2.1. Spectral properties and Harnack domination. We denote by $\Gamma(T)$ the set of complex numbers defined by $\Gamma(T) = \sigma(T) \cap \mathbb{T}$, where $\mathbb{T} = \overline{\mathbb{D}} \setminus \mathbb{D}$ is the unidimensional torus. In the following results, we prove that ρ -contractions belonging to the same Harnack parts have the same spectral values in the torus.

Theorem 2.1. *Let $T_0, T_1 \in C_\rho(H)$, ($\rho \geq 1$), if $T_1 \overset{H}{\prec} T_0$ then $\Gamma(T_1) \subseteq \Gamma(T_0)$.*

Proof. Let $T_0, T_1 \in C_\rho(H)$ be such that $T_1 \overset{H}{\prec} T_0$. Then there exists $c \geq 1$ such that

$$K_z^\rho(T_1) \leq c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D}, \quad (2.1)$$

so,

$$\begin{aligned} K_z^\rho(T_1) &= (I - zT_1^*)^{-1} [\rho I + 2(1 - \rho) \operatorname{Re}(\bar{z}T_1) + (\rho - 2) |z|^2 T_1^* T_1] (I - \bar{z}T_1)^{-1} \\ &\leq c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D}. \end{aligned}$$

Hence

$$\rho I + 2(1 - \rho) \operatorname{Re}(\bar{z}T_1) + (\rho - 2) |z|^2 T_1^* T_1 \leq c^2 (I - zT_1^*) K_z^\rho(T_0) (I - \bar{z}T_1), \quad (2.2)$$

for all $z \in \mathbb{D}$. Now, let $\lambda = e^{i\omega} \in \Gamma(T_1) \subseteq \sigma_{ap}(T_1)$, then there exists a sequence $(x_n)_{n \geq 0}$ of unit vectors such that $T_1 x_n - e^{i\omega} x_n = y_n$ converge to 0. From the inequality (2.2), we derive

$$\begin{aligned} &\rho I + 2(1 - \rho) \operatorname{Re}(\bar{z} \langle T_1 x_n, x_n \rangle) + (\rho - 2) |z|^2 \|T_1 x_n\|^2 \\ &\leq c^2 \langle K_z^\rho(T_0) (I - \bar{z}T_1) x_n, (I - \bar{z}T_1) x_n \rangle \\ &= c^2 \langle K_z^\rho(T_0) [(1 - \bar{z}e^{i\omega}) x_n - \bar{z}y_n], (1 - \bar{z}e^{i\omega}) x_n - \bar{z}y_n \rangle \\ &= c^2 |1 - \bar{z}e^{i\omega}|^2 \langle K_z^\rho(T_0) x_n, x_n \rangle - c^2 z(1 - \bar{z}e^{i\omega}) \langle K_z^\rho(T_0) x_n, y_n \rangle \\ &\quad - c^2 \bar{z}(1 - ze^{-i\omega}) \langle K_z^\rho(T_0) y_n, x_n \rangle + c^2 |z|^2 \langle K_z^\rho(T_0) y_n, y_n \rangle, \end{aligned}$$

for any $z \in \mathbb{D}$ and all $n \geq 0$. Note that

$$\left| \|T_1 x_n - e^{i\omega} x_n\| - \|x_n\| \right| \leq \|T_1 x_n\| \leq \|T_1 x_n - e^{i\omega} x_n\| + 1.$$

Letting $n \rightarrow +\infty$, from the two previous inequalities we obtain

$$\rho + 2(1 - \rho) \operatorname{Re}(\bar{z}e^{i\omega}) + (\rho - 2) |z|^2 \leq c^2 |1 - \bar{z}e^{i\omega}|^2 \limsup_{n \rightarrow +\infty} \langle K_z^\rho(T_0) x_n, x_n \rangle,$$

for any $z \in \mathbb{D}$. Then, if we take $z = (1 - t)e^{i\omega}$ with $0 < t < 1$, we get

$$\rho + 2(1 - \rho)(1 - t) + (\rho - 2)(1 - t)^2 \leq c^2 t^2 \limsup_{n \rightarrow +\infty} \left\langle K_{(1-t)e^{i\omega}}^\rho(T_0) x_n, x_n \right\rangle.$$

Assume that $e^{i\omega} \notin \Gamma(T_0)$, then $K_{(1-t)e^{i\omega}}^\rho(T_0)$ is uniformly bounded in $]0, 1[$, then there exists $\gamma > 0$ such that

$$\rho + 2(1 - \rho)(1 - t) + (\rho - 2)(1 - t)^2 \leq \gamma c^2 t^2,$$

which implies

$$2t \leq (\gamma c^2 + 2 - \rho)t^2,$$

for all $t > 0$, and hence

$$2 \leq (\gamma c^2 + 2 - \rho)t.$$

Now, we get a contradiction by letting $t \rightarrow 0$. Hence $e^{i\omega} \in \Gamma(T_0)$. \square

From Theorem 2.1, we also obtain the following result

Corollary 2.2. *If T_1 and T_0 are Harnack equivalent in $C_\rho(H)$ then $\Gamma(T_1) = \Gamma(T_0)$.*

Let $T \in B(H)$ and E be a closed invariant subspace of T , ($T(E) \subset E$). Then $T \in B(E \oplus E^\perp)$, has the following form:

$$T = \begin{pmatrix} T_1 & R \\ 0 & T_2 \end{pmatrix},$$

with $T_1 \in B(E)$, $T_2 \in B(E^\perp)$ and R is a bounded operator from E^\perp to E . We denote by $\Gamma_p(T) = \sigma_p(T) \cap \mathbb{T}$ the point spectrum of $T \in C_\rho(H)$ in the unidimensional torus.

Theorem 2.3. *Let $T_0, T_1 \in C_\rho(H)$ ($\rho \geq 1$), if $T_1 \overset{H}{\prec} T_0$ then $\Gamma_p(T_1) \subseteq \Gamma_p(T_0)$ and $\text{Ker}(T_1 - \lambda I) \subseteq \text{Ker}(T_0 - \lambda I)$ for all $\lambda \in \Gamma_p(T_1)$.*

For the proof of this theorem we need the following lemma.

Lemma 2.4. *Let $T \in C_\rho(H)$. Then*

$$\|(I - \bar{\lambda}T)K_z^\rho(T)(I - \lambda T^*)\| \leq \rho(1 + 2|1 - \rho| + |\rho - 2|\rho)(1 + \rho \frac{|z - \lambda|}{1 - |z|})^2,$$

for all $z \in \mathbb{D}$ and $\lambda \in \bar{\mathbb{D}}$.

Proof. Let $z \in \mathbb{D}$ and $\lambda \in \bar{\mathbb{D}}$, we have

$$\begin{aligned} (I - zT^*)^{-1}(I - \lambda T^*) &= (I - zT^*)^{-1}[I - zT^* + (z - \lambda)T^*] \\ &= I + (z - \lambda) \sum_{n=0}^{+\infty} z^n T^{*n+1}. \end{aligned}$$

Then by (1.2),

$$\|(I - zT^*)^{-1}(I - \lambda T^*)\| \leq 1 + \rho \frac{|z - \lambda|}{1 - |z|}.$$

Taking into account this inequality and the fact that

$$K_z^\rho(T_1) = (I - \bar{z}T_1)^{-1}[\rho I + 2(1 - \rho)\text{Re}(\bar{z}T_1) + (\rho - 2)|z|^2 T_1 T_1^*](I - zT_1^*)^{-1},$$

we obtain the desired inequality. \square

Proof of Theorem 2.3. Let $\lambda \in \Gamma_p(T_1)$. Then the operator $T_1 \in C_\rho(H)$ on $\text{Ker}(T_1 - \lambda I) \oplus \text{Ker}(T_1 - \lambda I)^\perp$ takes the form

$$T_1 = \begin{pmatrix} \lambda & C \\ 0 & \tilde{T}_1 \end{pmatrix}.$$

Since $|\lambda| = 1$, by using Proposition [11, Proposition 3.] we can see that $C = 0$. Thus, we have

$$K_z^\rho(T_1) = \begin{pmatrix} \frac{\rho+2(1-\rho)Re(\bar{\lambda}z)+(\rho-2)|\lambda|^2|z|^2}{|1-\bar{\lambda}z|^2}I & 0 \\ 0 & K_z^\rho(\tilde{T}_1) \end{pmatrix},$$

Now, if $T_0 \in C_\rho(H)$ be such that $T_1 \overset{H}{\prec} T_0$, then there exists $c \geq 1$ such that

$$K_z^\rho(T_1) \leq c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D},$$

Let $x \in Ker(T_1 - \lambda I)$ and $y \in Im(T_0^* - \lambda I)$. The Cauchy-Schwarz inequality yields

$$|\langle K_z^\rho(T_1)x, y \rangle|^2 \leq c^2 \langle K_z^\rho(T_1)x, x \rangle \langle K_z^\rho(T_0)y, y \rangle.$$

We derive

$$\frac{\rho I + 2(1-\rho)Re(\bar{\lambda}z) + (\rho-2)|\lambda|^2|z|^2}{|1-\bar{\lambda}z|^2} |\langle x, y \rangle|^2 \leq c^2 \langle K_z^\rho(T_0)y, y \rangle \|x\|^2.$$

Since $y \in Im(T_0^* - \lambda I)$, there exists $u \in H$ such that $y = (I - \lambda T_0^*)u$. By Lemma 2.4, we have

$$\begin{aligned} \langle K_z^\rho(T_0)y, y \rangle &= \langle (I - \bar{\lambda}T_0)K_z^\rho(T_0)(I - \lambda T_0^*)u, (I - \lambda T_0^*)u \rangle \\ &\leq \rho(1 + 2|1 - \rho| + |\rho - 2|\rho)(1 + \rho \frac{|z - \lambda|}{1 - |z|})^2 \|u\|^2. \end{aligned}$$

Let $z = r\lambda$, with $0 < r < 1$. Then

$$\frac{|\rho + 2(1-\rho)r + (\rho-2)r^2|}{(1-r)^2} |\langle x, y \rangle|^2 \leq c^2 \rho(1 + 2|1 - \rho| + |\rho - 2|\rho)(1 + \rho)^2 \|u\|^2 \|x\|^2.$$

This implies

$$|\rho + 2(1-\rho)r + (\rho-2)r^2| |\langle x, y \rangle|^2 \leq c^2 (1-r)^2 \rho(1 + 2|1 - \rho| + |\rho - 2|\rho)(1 + \rho)^2 \|u\|^2 \|x\|^2.$$

By letting r to 1, it follows that $\langle x, y \rangle = 0$, and hence $x \in Im(T_0^* - \lambda I)^\perp = Ker(T_0 - \lambda I)$. So, $\Gamma_p(T_1) \subseteq \Gamma_p(T_0)$ and $Ker(T_1 - \lambda I) \subseteq Ker(T_0 - \lambda I)$. \square

Remark 2.5. By Theorem 2.3, if $I_H \overset{H}{\prec} T$ on $C_\rho(H)$, ($\rho \geq 1$) then $T = I_H$. This means that I_H is a maximal element for the Harnack domination on $C_\rho(H)$ and its Harnack part is trivial, for all $\rho \geq 1$.

From Theorem 2.3, we also obtain the following result

Corollary 2.6. *If T_1 and T_0 are Harnack equivalent in $C_\rho(H)$ then $\Gamma_p(T_1) = \Gamma_p(T_0)$ and $Ker(T_1 - \lambda I) = Ker(T_0 - \lambda I)$ for all $\lambda \in \Gamma_p(T_0)$.*

Remark 2.7. After the authors have obtained Theorem 2.1, they have learned that C. Badea, D. Timotin and L. Suciuc [4] have proved using an other method that, in the case of contractions ($\rho = 1$), the domination suffices for the equality of the point spectrum in the torus. But in the case of $\rho > 1$ the inclusion in Theorem 2.3 may be strict, for instance, we have

- For $\rho > 1$, we have $0_H \overset{H}{\prec}_c I$ in $C_\rho(H)$ with $c = \sqrt{\frac{\rho}{\rho-1}}$.

- For $1 < \rho$, the operator T defined on \mathbb{C}^2 by $T = \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix}$ satisfies $T \stackrel{H}{\prec} I$ in $C_\rho(H)$ with $c = \sqrt{\frac{2\rho}{\rho-1}}$.

Corollary 2.8. *Let $T_0, T_1 \in C_\rho(H)$ ($\rho \geq 1$) such that $\Gamma(T_0) = \Gamma_\rho(T_0)$. Then T_0 and T_1 are Harnack equivalent in $C_\rho(H)$ if and only if $T_0 = U \oplus \tilde{T}_0$ and $T_1 = U \oplus \tilde{T}_1$ on $H = E \oplus E^\perp$, where $E = \bigoplus_{\lambda \in \Gamma_\rho(T_0)} \text{Ker}(T_0 - \lambda I) = \bigoplus_{\lambda \in \Gamma_\rho(T_1)} \text{Ker}(T_1 - \lambda I)$, U is an unitary diagonal operator on E and \tilde{T}_0 and \tilde{T}_1 are Harnack equivalent in $C_\rho(E^\perp)$.*

Proof. First we prove that if $\lambda, \mu \in \Gamma_\rho(T_0)$, then $\text{Ker}(T_0 - \lambda I) \perp \text{Ker}(T_0 - \mu I)$ for $\lambda \neq \mu$. Let $x \in \text{Ker}(T_1 - \lambda I)$ and $y \in \text{Ker}(T_0 - \mu I)$. Then

$$\begin{aligned} \langle K_z^\rho(T_0)x, y \rangle &= \langle ((I - \bar{z}T)^{-1} + (I - zT^*)^{-1} + (2 - \rho)I)x, y \rangle \\ &= \frac{1}{1 - \bar{z}\lambda} \langle x, y \rangle + \frac{1}{1 - z\bar{\mu}} \langle x, y \rangle + (2 - \rho) \langle x, y \rangle. \end{aligned}$$

By Cauchy-Schwarz inequality

$$|\langle K_z^\rho(T_0)x, y \rangle|^2 \leq \langle K_z^\rho(T_0)x, x \rangle \langle K_z^\rho(T_0)y, y \rangle.$$

Thus

$$\begin{aligned} & \left| \frac{1}{1 - \bar{z}\lambda} + \frac{1}{1 - z\bar{\mu}} + (2 - \rho) \right|^2 |\langle x, y \rangle|^2 \leq \\ & \frac{(\rho + 2(1 - \rho)\text{Re}(\bar{\lambda}z) + (\rho - 2)|z|^2)(\rho + 2(1 - \rho)\text{Re}(\bar{\mu}z) + (\rho - 2)|z|^2)}{|1 - \bar{z}\lambda|^2 |1 - \bar{\mu}z|^2} \|x\|^2 \|y\|^2. \end{aligned}$$

So

$$\begin{aligned} & \left| 1 + \frac{1 - \bar{z}\lambda}{1 - z\bar{\mu}} + (2 - \rho)(1 - \bar{z}\lambda)|z|^2 \right|^2 |\langle x, y \rangle|^2 \leq \\ & \frac{(\rho + 2(1 - \rho)\text{Re}(\bar{\lambda}z) + (\rho - 2)|z|^2)(\rho + 2(1 - \rho)\text{Re}(\bar{\mu}z) + (\rho - 2)|z|^2)}{|1 - \bar{\mu}z|^2} \|x\|^2 \|y\|^2. \end{aligned}$$

By taking z to λ , we get $\langle x, y \rangle = 0$. By [11, Corollary 4.] the subspace E reduces T_0 and T_1 . \square

Example 2.9. Recall that an operator $T \in B(H)$ is called to be quasi-compact operator (or quasi-strongly completely continuous in the terminology of [27]) if there exists a compact operator K and an integer m such that $\|T^m - K\| < 1$. Since every operator $T \in C_\rho(H)$ ($\rho \geq 1$) is power-bounded, by [27, Theorem 4]; if $T \in C_\rho(H)$ ($\rho \geq 1$) is a quasi-compact operator then $\Gamma(T) = \Gamma_\rho(T)$ and contains a finite number of eigenvalues and each of them is of finite multiplicity. Now if we assume that T, S are two quasi-compact operators which are Harnack equivalent in $C_\rho(H)$, ($\rho \geq 1$), then $S = U \oplus \tilde{S}$ where U is an unitary diagonal operator on $E = \bigoplus_{i=1}^k \text{Ker}(T - \lambda_i I)$ and \tilde{S} is Harnack equivalent to 0 in $C_\rho(E^\perp)$.

Corollary 2.10. *Let $T \in C_\rho(H)$ ($\rho \geq 1$) be a compact normal operator with $w_\rho(T) = 1$. If the operator $S \in C_\rho(H)$ is Harnack equivalent to T , then for all $\lambda \in \Gamma_\rho(T)$, $S|_E = T|_E$ where $E = \bigoplus_{\lambda \in \Gamma_\rho(T)} \text{Ker}(T - \lambda I)$, E is a reducing subspace for S and S_{E^\perp} is Harnack equivalent to 0, i.e. $w_\rho(S_{E^\perp}) < 1$.*

Proof. By Corollary 2.8, for all $\lambda \in \Gamma_p(T)$, we have $T = U \oplus \tilde{T}$ and $S = U \oplus \tilde{S}$ on $E \oplus E^\perp$, where $E = \bigoplus_{\lambda \in \Gamma_p(T)} \text{Ker}(T - \lambda I)$ and \tilde{T} and \tilde{S} are Harnack equivalent in $C_\rho(E^\perp)$. Since $T \in C_\rho(H)$ is a compact normal operator we also have

$$w_\rho(\tilde{T}) = \sup\{|\lambda|, \lambda \in \sigma(T) \setminus \Gamma_p(T)\} < 1.$$

This means that \tilde{T} and \tilde{S} are Harnack equivalent to 0. \square

In the following proposition, we prove that the ρ -contractions belong to the same Harnack parts have the same kernel for their operatorial ρ -kernels.

Proposition 2.11. *Let $T_0, T_1 \in C_\rho(H)$. If T_1 and T_2 are Harnack equivalent in $C_\rho(H)$ then $\text{Ker}K_z^\rho(T_1) = \text{Ker}K_z^\rho(T_2)$ for all $z \in \mathbb{D}$.*

Proof. Since $T_1 \stackrel{H}{\sim} T_2$, then by Theorem 1.1, there exist $\alpha, \beta > 0$ ($\alpha \leq 1, \beta \geq 1$) such that

$$\alpha K_z^\rho(T_1) \leq K_z^\rho(T_2) \leq \beta K_z^\rho(T_1), \quad \text{for all } z \in \mathbb{D}. \quad (2.3)$$

If $x \in \text{Ker}K_z^\rho(T_1)$, then by the right side of the inequality (2.3), we also have,

$$0 \leq \langle K_z^\rho(T_2)x, x \rangle \leq \beta \langle K_z^\rho(T_1)x, x \rangle = 0,$$

this implies that $\left\| \sqrt{K_z^\rho(T_2)}x \right\| = 0$, so $K_z^\rho(T_2)x = 0$, hence $\text{Ker}K_z^\rho(T_1) \subseteq \text{Ker}K_z^\rho(T_2)$ for all $z \in \mathbb{D}$. The converse inclusion holds from the left-side of the inequality (2.3). \square

Proposition 2.12. *If $w(T) = 1$ and $\Gamma(T)$ is empty then there exists $z_0 \in \mathbb{T}$ such that $K_{z_0}^2(T)$ is not invertible.*

Proof. Since $w(T) = 1$, there exists a sequence $(x_n)_{n \geq 0}$ of unit vectors such that $\langle Tx_n, x_n \rangle$ converge to $z_0 = e^{i\omega} \in \mathbb{T}$. Set $y_n = (I - e^{-i\omega}T)x_n$, then $\|y_n\|$ not converge to 0. If not we have $e^{i\omega} \in \sigma(T)$, this contradicts that $\Gamma(T)$ is empty. Thus, we may suppose that $\|y_n\| \rightarrow l > 0$ and we have

$$\begin{aligned} \langle K_{e^{i\omega}}^2(T)y_n, y_n \rangle &= 2 \langle (I - \text{Re}(e^{-i\omega}T))x_n, x_n \rangle \\ &= 2(1 - \text{Re}(e^{-i\omega} \langle Tx_n, x_n \rangle)), \end{aligned}$$

hence

$$\left\langle K_{e^{i\omega}}^2(T) \frac{y_n}{\|y_n\|}, \frac{y_n}{\|y_n\|} \right\rangle = 2 \frac{(1 - \text{Re}(e^{-i\omega} \langle Tx_n, x_n \rangle))}{\|y_n\|^2} \rightarrow 0.$$

This implies that $0 \in \sigma_{ap}(\sqrt{K_{e^{i\omega}}^2(T)})$ and hence $0 \in \sigma_{ap}(K_{e^{i\omega}}^2(T))$. \square

2.2. Numerical range properties and Harnack domination. Firstly, we give a proposition which is useful in this subsection.

Proposition 2.13. *Let $T_0, T_1 \in C_{\rho_1}(H)$ and $\rho_2 \geq \rho_1$. Then we have*

- (i) *If $T_1 \stackrel{H}{\underset{c}{\sim}} T_0$ in $C_{\rho_1}(H)$, then $T_1 \stackrel{H}{\underset{c}{\sim}} T_0$ in $C_{\rho_2}(H)$.*
- (ii) *If $T_1 \stackrel{H}{\overset{c}{\sim}} T_0$ in $C_{\rho_1}(H)$, then $T_1 \stackrel{H}{\overset{c}{\sim}} T_0$ in $C_{\rho_2}(H)$.*

Proof. (i) Since the C_ρ classes increase with ρ , the two operators T_0 and T_1 belong to $C_{\rho_2}(H)$. From Theorem 1.1, we know that there exists $c \geq 1$ such that $K_z^{\rho_1}(T_1) \leq c^2 K_z^{\rho_1}(T_0)$ for all $z \in \mathbb{D}$. As $c \geq 1$, it yields to

$$K_z^{\rho_2}(T_1) = K_z^{\rho_1}(T_1) + (\rho_2 - \rho_1)I \leq c^2 [K_z^{\rho_1}(T_0) + (\rho_2 - \rho_1)I] = c^2 K_z^{\rho_2}(T_0).$$

Using again Theorem 1.1, we obtained the desired conclusion.

The assertion (ii) is a direct consequence of (i). \square

Let $T \in B(H)$, we denote by $W(T)$ the numerical range of T which is the set given by

$$W(T) = \{\langle Tx, x \rangle; x \in H, \|x\| = 1\}.$$

The following result give relationships between numerical range and Harnack domination.

Theorem 2.14. *Let $T_0, T_1 \in C_\rho(H)$ with $1 \leq \rho \leq 2$, then we have:*

- (i) *Assume that $\rho = 1$ and $T_1 \overset{H}{\prec} T_0$, then $\overline{W(T_0)} \cap \mathbb{T} = \overline{W(T_1)} \cap \mathbb{T}$.*
- (ii) *Suppose that $1 < \rho \leq 2$, $T_1 \overset{H}{\prec} T_0$ and $\Gamma(T_0) = \emptyset$, then $\overline{W(T_0)} \cap \mathbb{T} \subseteq \overline{W(T_1)} \cap \mathbb{T}$.*
- (iii) *If $T_1 \overset{H}{\sim} T_0$, then $\overline{W(T_0)} \cap \mathbb{T} = \overline{W(T_1)} \cap \mathbb{T}$.*

Proof. (i) Let $\lambda = e^{i\omega} \in \overline{W(T_0)} \cap \mathbb{T}$, then there exists a sequence (x_n) of unit vectors such that $\langle T_0 x_n, x_n \rangle \rightarrow \lambda$. We have for some $c \geq 1$, $0 \leq K_{r,\theta}(T_1) \leq c^2 K_{r,\theta}(T_0)$ for all $z \in \mathbb{D}$. Multiplying these inequalities by the nonnegative function $1 - \operatorname{Re}(\bar{\lambda} e^{i\theta})$, integrating with respect to the Haar measure m and letting r to 1, we get $0 \leq I - \operatorname{Re}(\bar{\lambda} T_1) \leq c^2 [I - \operatorname{Re}(\bar{\lambda} T_0)]$. We deduce that $1 - \operatorname{Re}(\bar{\lambda} \langle T_1 x_n, x_n \rangle) \rightarrow 0$. Since $\langle T_1 x_n, x_n \rangle$ belongs to the closed unit disc, it forces $\langle T_1 x_n, x_n \rangle \rightarrow \lambda$. Hence $\overline{W(T_0)} \cap \mathbb{T} \subseteq \overline{W(T_1)} \cap \mathbb{T}$. Now, let $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$, then there exists a sequence (y_n) of unit vectors such that $\langle T_1 y_n, y_n \rangle \rightarrow \lambda$. As T_1 is a contraction, it follows that $1 = \lim |\langle T_1 y_n, y_n \rangle| \leq \liminf \|T_1 y_n\| \leq \limsup \|T_1 y_n\| \leq 1$, thus $\|T_1 y_n\| \rightarrow 1$. It implies $\|T_1 y_n - \lambda y_n\|^2 = \|T_1 y_n\|^2 - 2\operatorname{Re}(\bar{\lambda} \langle T_1 y_n, y_n \rangle) + 1 \rightarrow 0$. Consequently, we have $\lambda \in \Gamma(T_1)$, by using Theorem 2.1 we see that $\lambda \in \Gamma(T_0) \subseteq \overline{W(T_0)} \cap \mathbb{T}$. So we get the desired equality.

(ii) Tacking into account Proposition 2.13, it suffices to treat the case where $\rho = 2$. Let $\lambda = e^{i\omega} \in \overline{W(T_0)} \cap \mathbb{T}$, then there exists a sequence (x_n) of unit vectors such that $\langle T x_n, x_n \rangle \rightarrow \lambda$. Set $y_n = (I - e^{-i\omega} T_0)x_n$, since $\Gamma(T_0) = \emptyset$ we necessarily have $\gamma = \inf\{\|y_n\|; n \geq 0\} > 0$. Tacking $u_n = y_n/\|y_n\|$, we can see that

$$\begin{aligned} \langle K_{e^{i\omega}}^2(T_0)u_n, u_n \rangle &= \frac{2}{\|y_n\|^2} \langle (I - \operatorname{Re}(e^{-i\omega} T_0))x_n, x_n \rangle \\ &\leq \frac{2}{\gamma^2} \langle (I - \operatorname{Re}(e^{-i\omega} T_0))x_n, x_n \rangle \rightarrow 0. \end{aligned}$$

Since $T_1 \overset{H}{\prec} T_0$, there exists $c \geq 1$ such that

$$K_z^2(T_1) \leq c^2 K_z^2(T_0), \quad \text{for all } z \in \mathbb{D}. \quad (2.4)$$

On the one hand, if $\lambda \in \Gamma(T_1)$ we have obviously $\lambda \in \overline{W(T_1)}$. On the other hand, if $\lambda \notin \Gamma(T_1)$ we can extend (2.4) at $z = \lambda$ and we get

$$0 \leq \langle K_{e^{i\omega}}^2(T_1)u_n, u_n \rangle \leq c^2 \langle K_{e^{i\omega}}^2(T_0)u_n, u_n \rangle \longrightarrow 0,$$

hence $\langle K_{e^{i\omega}}^2(T_1)u_n, u_n \rangle \longrightarrow 0$. Observe that $\inf\{\|(I - \operatorname{Re}(e^{-i\omega}T_1))^{-1}u_n\|; n \geq 0\} \geq \frac{1}{3}$. Set $v_n = (1/\|(I - e^{-i\omega}T_1)^{-1}u_n\|)(I - \operatorname{Re}(e^{-i\omega}T_1))^{-1}u_n$, we obtain

$$\langle (I - \operatorname{Re}(e^{-i\omega}T_1))v_n, v_n \rangle \leq \frac{9}{2} \langle K_{e^{i\omega}}^2(T_1)u_n, u_n \rangle \longrightarrow 0.$$

We deduce that $\langle \operatorname{Re}(e^{-i\omega}T_1)v_n, v_n \rangle \longrightarrow 1$. As $T_1 \in C_2(H)$, it yields to:

$$1 \geq |\langle T_1 v_n, v_n \rangle|^2 = |\langle \operatorname{Re}(e^{-i\omega}T_1)v_n, v_n \rangle|^2 + |\langle \operatorname{Im}(e^{-i\omega}T_1)v_n, v_n \rangle|^2,$$

and we derive successively that $\langle \operatorname{Im}(e^{-i\omega}T_1)v_n, v_n \rangle \longrightarrow 0$ and $\langle T_1 v_n, v_n \rangle \longrightarrow \lambda$. Thus $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$ and it ends the proof of (i).

(iii) As before, we may suppose that $\rho = 2$. Assume that $T_1 \stackrel{H}{\sim} T_0$ and $\lambda \in \overline{W(T_0)} \cap \mathbb{T}$. By Corollary (2.2), we have $\Gamma(T_0) = \Gamma(T_1)$. So, if $\lambda \in \Gamma(T_0)$ then $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$. Now, if $\lambda \notin \Gamma(T_0)$, we proceed as in the second item (ii) to prove that $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$. Interchanging the roles of T_0 and T_1 gives the desired equality. \square

Remark 2.15. (1) The condition $\Gamma(T_0) = \emptyset$, in (ii), cannot be relaxed. In fact, we have $T_1 = 0_H \stackrel{H}{\prec}_c I = T_0$ in $C_\rho(H)$ ($1 < \rho \leq 2$) with $c = \sqrt{\frac{\rho}{\rho-1}}$ but $\overline{W(T_0)} \cap \mathbb{T} = \{1\}$ and $\overline{W(T_1)} \cap \mathbb{T} = \emptyset$.

(2) When T is a contraction, we have $\overline{W(T)} \cap \mathbb{T} = \Gamma(T)$ (see for instance the end of the proof of (i)). So, the assertion (i) of Theorem 2.14 restore, in the case of domination, the equality of the point spectrum in the torus obtained by C. Badea, D. Timotin and L. Suciuc in [4] by another way.

Corollary 2.16. *Let $T_0 \in C_\rho(H)$ with $1 \leq \rho \leq 2$. If $\overline{W(T_0)} = \overline{\mathbb{D}}$, and satisfies $\Gamma(T_0) = \emptyset$ when $\rho \neq 1$, then $\overline{W(T_1)} = \overline{\mathbb{D}}$ for every $T_1 \in C_\rho(H)$ such that $T_1 \stackrel{H}{\prec} T_0$. Furthermore, in the case of Harnack equivalence, we have $\overline{W(T_1)} = \overline{\mathbb{D}}$ as soon as $\overline{W(T_0)} = \overline{\mathbb{D}}$.*

Proof. By Theorem 2.14, Proposition 2.13 and the convexity theorem of Toeplitz-Hausdorff, we obtain the desired conclusions. \square

2.3. Harnack parts in the space of compact operators. Denote by $\mathcal{K}(H)$ the set of all compact operators. We have

Theorem 2.17. *Let $T \in C_\rho(H) \cap \mathcal{K}(H)$ with $w_\rho(T) = 1$ and $\Gamma_p(T)$ is empty. Then $S \in C_\rho(H) \cap \mathcal{K}(H)$ is Harnack equivalent to T if and only if $\ker(K_z^\rho(S)) = \ker(K_z^\rho(T))$ for all $z \in \mathbb{T}$.*

For the proof of this theorem we need the following lemma.

Lemma 2.18. *Let $T \in C_\rho(H) \cap \mathcal{K}(H)$ as in the previous theorem. If $\|K_z^\rho(T)\| = \lambda_1(z) \geq \lambda_2(z) \geq \dots \geq \lambda_n(z) \geq \dots$ are the eigenvalues of $K_z^\rho(T)$, arranged in decreasing order, then the mapping $z \mapsto \lambda_n(z)$ is continuous on $\overline{\mathbb{D}}$, for all n .*

Proof. Let R such that $\text{rang}(R) < n$. We have

$$\lambda_n(z) \leq \|K_z^\rho(T) - R\| \leq \|K_z^\rho(T) - K_{z'}^\rho(T)\| + \|K_{z'}^\rho(T) - R\|.$$

Hence

$$\lambda_n(z) \leq \|K_z^\rho(T) - K_{z'}^\rho(T)\| + \lambda_n(z').$$

By interchanging z by z' , we get

$$|\lambda_n(z) - \lambda_n(z')| \leq \|K_z^\rho(T) - K_{z'}^\rho(T)\|. \quad (2.5)$$

□

Proof of Theorem 2.17. Let $T, S \in C_\rho(H) \cap \mathcal{K}(H)$ such that $T \stackrel{H}{\sim} S$. Since $\Gamma_p(T)$ is empty, by Corollary 2.2, the operators T and S not admits eigenvalues in \mathbb{T} . Hence, $K_z^\rho(T)$ and $K_z^\rho(S)$ are uniformly bounded in \mathbb{D} and may be extended to a positive operators on \mathbb{D} . Furthermore, if we proceed as in the proof of Proposition 2.11, we deduce that $\text{Ker}K_z^\rho(T) = \text{Ker}K_z^\rho(S)$ for all $z \in \mathbb{T}$.

Conversely, Let $E_T(z) = \text{ker}(K_z^\rho(T))$ and $E_S(z) = \text{ker}(K_z^\rho(S))$, both $K_z^\rho(T)$ and $K_z^\rho(S)$ on $H = E_T(z) \oplus E_T(z)^\perp$, take the following forms

$$K_z^\rho(T) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{K}_z^\rho(T) \end{pmatrix} \quad \text{and} \quad K_z^\rho(S) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{K}_z^\rho(S) \end{pmatrix}.$$

Denote by $\lambda_1^T(z), \lambda_2^T(z), \dots, \lambda_n^T(z), \dots$ are the eigenvalues of $K_z^\rho(T)$, arranged in decreasing order $\lambda_1^T(z) = \|K_z^\rho(T)\| \geq \lambda_2^T(z) \geq \dots \geq \lambda_n^T(z) \geq \dots$. We put $\lambda^T(z) = \inf_{n \geq 1} \lambda_n^T(z)$. We claim that $\lambda^T(z) > 0$. Indeed, if we assume that there exist z_0 such that $\lambda^T(z_0) = 0$, then there exists a sequence $(x_n)_n$ in $E_T(z_0)^\perp$ with $\|x_n\| = 1$ such that

$$\lambda_n^T(z) = \langle K_{z_0}^\rho(T)x_n, x_n \rangle = \langle \tilde{K}_{z_0}^\rho(T)x_n, x_n \rangle = \rho + \langle R_{z_0}^\rho(T)x_n, x_n \rangle,$$

with $R_{z_0}^\rho(T) = \sum_{n=1}^{+\infty} \bar{z}_0^n T^n + \sum_{n=1}^{+\infty} z_0^n T^{*n}$. Since T is compact operator with $r(T) < 1$, both of the series $\sum_{n=1}^{+\infty} \bar{z}_0^n T^n$ and $\sum_{n=1}^{+\infty} z_0^n T^{*n}$ are converge to a compact operator in the operator norm, so $R_{z_0}^\rho(T)$ is compact. There exist a subsequence $(x_{j(n)})$ of $(x_n)_n$ such that $(x_{j(n)})$ converges to some $x \in E_T(z_0)^\perp$ in the weak star topology, this implies that $R_{z_0}^\rho(T)x_{j(n)} \longrightarrow R_{z_0}^\rho(T)x$ strongly and

$$0 = \rho + \langle R_{z_0}^\rho(T)x, x \rangle \geq \|x\|^2 + \langle R_{z_0}^\rho(T)x, x \rangle = \langle K_{z_0}^\rho(T)x, x \rangle \geq 0,$$

so $x \in E_T(z_0)$. If $x = 0$, then $R_{z_0}^\rho(T)x_k \longrightarrow 0$ strongly and $\lambda_k^T(z_0) \rightarrow 1$, which is a contradiction with $\lambda_k^T(z_0) \rightarrow 0$. Then $x \neq 0$ but $x \in E_T(z_0) \cap E_T(z_0)^\perp$ this is again a contradiction and $\lambda^T(z_0)$ must be strictly positive. On the other hand $(\lambda_n^T(z))_n$ is a decreasing bounded below sequence so it converge to $\lambda^T(z)$, furthermore, since, by Lemma 2.18, the mapping $z \mapsto \lambda_n^T(z)$ is continuous, then by letting n to $+\infty$ in (2.5), we deduce that the mapping $z \mapsto \lambda^T(z)$ is also continuous and has a minimum in \mathbb{T} denoted by $m(T) = \inf_{\lambda \in \mathbb{T}} \lambda^T(z) > 0$. The same arguments holds for the compact operator S .

Let $P(z)$ denote the orthogonal projection on $E_T(z) = E_S(z)$ and $Q(z) = I - P(z)$. We put $M(T) = \sup_{\lambda \in \mathbb{T}} \|K_z^\rho(T)\|$, for all $z \in \mathbb{T}$, we also have

$$m(T)Q(z) \leq \tilde{K}_z^\rho(T) \leq M(T)Q(z)$$

and

$$m(S)Q(z) \leq \tilde{K}_z^\rho(S) \leq M(S)Q(z).$$

This two inequalities gives

$$\frac{m(S)}{M(T)}\tilde{K}_z^\rho(T) \leq \tilde{K}_z^\rho(S) \leq \frac{M(S)}{m(T)}\tilde{K}_z^\rho(T).$$

Hence

$$\frac{m(S)}{M(T)}K_z^\rho(T) \leq K_z^\rho(S) \leq \frac{M(S)}{m(T)}K_z^\rho(T),$$

for all $z \in \mathbb{T}$. Now, by the uniqueness of harmonic extension we also have

$$\frac{m(S)}{M(T)}K_z^\rho(T) \leq K_z^\rho(S) \leq \frac{M(S)}{m(T)}K_z^\rho(T),$$

for all $z \in \mathbb{D}$. This complete the proof of the theorem. \square

Remark 2.19. In the previous theorem the hypothesis $\Gamma_p(T)$ is empty can be relaxed. In this case we can use the Corollary 2.8 and we applied the Theorem 2.17 for \tilde{T} and \tilde{S} as in the decomposition of T and S respectively, given by the Corollary 2.8.

Corollary 2.20. *Let $T \in C_\rho(H) \cap \mathcal{K}(H)$ with $w_\rho(T) = 1$ and $\Gamma_p(T)$ is empty. If there exists an unitary operator U such that $U(\ker(K_z^\rho(T))) \subseteq \ker(K_z^\rho(T))$ for all $z \in \mathbb{T}$, then U^*TU is Harnack equivalent to T .*

Proof. We have $K_z^\rho(U^*TU) = U^*K_z^\rho U$ and $\text{Ker}(K_z^\rho(U^*TU)) = \text{Ker}(K_z^\rho U) = U^* \text{Ker}(K_z^\rho)$. As we see in the proof the preceding theorem that $K_z^\rho(T) = \rho I + R_z^\rho(T)$ with $R_z^\rho(T)$ is compact. Hence $\text{Ker}(K_z^\rho)$ is finite dimensional, so the restriction of U to $\text{Ker}(K_z^\rho)$ is injective, equivalently to the restriction of U^* to $\text{Ker}(K_z^\rho)$ is surjective. Thus $U^* \text{Ker}(K_z^\rho) = \text{Ker}(K_z^\rho(T))$ and $\text{Ker}(K_z^\rho(U^*TU)) = \text{Ker}(K_z^\rho(T))$ for all $z \in \mathbb{T}$. By Theorem 2.17 we conclude that $T \stackrel{H}{\sim} S$. \square

Corollary 2.21. *Let $T \in C_1(H) \cap \mathcal{K}(H)$ with $\|T\| = 1$ and $\Gamma_p(T)$ is empty. Then $S \in C_1(H) \cap \mathcal{K}(H)$ is Harnack equivalent to T if and only if $E = \ker(I - T^*T) = \ker(I - S^*S)$ and $T|_E = S|_E$.*

Proof. Let $T, S \in C_1(H) \cap \mathcal{K}(H)$ such that $T \stackrel{H}{\sim} S$. By Theorem 2.17, $\text{Ker}K_z(T) = \text{Ker}K_z(S)$ for all $z \in \mathbb{T}$. On the other hand, the fact that

$$K_z(T) = (I - zT^*)^{-1}[I - |z|^2 T^*T](I - \bar{z}T)^{-1},$$

we can easily deduce that

$$\text{ker}K_z(T) = (I - \bar{z}T)(\text{ker}(I - T^*T)) \quad \text{for all } z \in \mathbb{T},$$

and similarly

$$\text{ker}K_z(S) = (I - \bar{z}S)(\text{ker}(I - S^*S)) \quad \text{for all } z \in \mathbb{T},$$

Then we also have,

$$(I - \bar{z}T)(\text{ker}(I - T^*T)) = (I - \bar{z}S)(\text{ker}(I - S^*S)) \quad \text{for all } z \in \mathbb{T}.$$

We put $E = \ker(I - T^*T)$. Let $x \in \ker(I - S^*S)$ and $z \in \mathbb{T}$. Then $(I - zS)x = (I - zT)y(z)$ with $y(z) \in E$, hence $y(z) = (I - zT)^{-1}(I - zS)x$ have an analytic

extensions in a neighbourhood of $\overline{\mathbb{D}}$. It follows that $x = y(0) = \int_0^{2\pi} y(e^{i\theta}) dm(\theta) \in E$, since E is closed. This proves $\ker(I - S^*S) \subseteq E$. Now the equality holds by interchanging the roles of T and S . Furthermore, we can see that for all $x \in E$, we have

$$y(z) = (I - zT)^{-1}(I - zS)x \in E \quad \text{for all } z \in \overline{\mathbb{D}}.$$

But

$$\begin{aligned} y(z) &= (I - zT)^{-1}(I - zT + z(T - S))x \\ &= x + \sum_{n=1}^{+\infty} z^n T^{n-1}(T - S)x \end{aligned}$$

On the other hand, we have

$$T^{n-1}(T - S)x = \int_0^{2\pi} e^{-in\theta} y(e^{i\theta}) dm(\theta) \in E \quad \text{for all } n \geq 1,$$

and

$$\langle (I - T^*T)T^{n-1}x, T^{n-1}(T - S)x \rangle = 0 \quad \text{for all } n \geq 1.$$

thus

$$\|(I - T^*T)T^{n-1}(T - S)x\|^2 - \|T^{n-1}(T - S)x\|^2 = 0 \quad \text{for all } n \geq 1,$$

so

$$\|(T - S)x\| = \|T^n(T - S)x\|^2 \longrightarrow 0,$$

because $r(T) < 1$. This implies that $Tx = Sx$ for all $x \in E$.

Conversely, if $E = \ker(I - T^*T) = \ker(I - S^*S)$ and $T|_E = S|_E$, then for all $z \in \mathbb{T}$, we have

$$\text{Ker}K_z(T) = (I - zT)(\ker(I - T^*T)) = (I - zS)(\ker(I - S^*S)) = \text{Ker}K_z(S).$$

Thus, by Theorem 2.17, $T \stackrel{H}{\sim} S$. □

For each $n \geq 1$, let

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

denote the (nilpotent) Jordan block of size n . By Corollary 2.21 and the fact that $\ker(I - J_n^*J_n) = \text{span}\{e_2, \dots, e_n\}$, the Harnack parts of J_n is given by

Corollary 2.22. *The Harnack parts of J_n is precisely the set of all matrices of $C_2(\mathbb{C}^n)$ of the form*

$$M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \\ z & 0 & \dots & \dots & 0 \end{pmatrix},$$

where z is in the open unit disc.

In the case of compact operators, we deduce from Theorem 2.14 the next result.

Proposition 2.23. *Let $T_0, T_1 \in C_\rho(H) \cap \mathcal{K}(H)$ with $1 \leq \rho \leq 2$, then we have:*

- (i) *Assume that $\rho = 1$ and $T_1 \overset{H}{\prec} T_0$, then $W(T_0) \cap \mathbb{T} = W(T_1) \cap \mathbb{T}$.*
- (ii) *Suppose that $1 < \rho \leq 2$, $T_1 \overset{H}{\prec} T_0$ and $\Gamma(T_0) = \emptyset$, then $W(T_0) \cap \mathbb{T} \subseteq W(T_1) \cap \mathbb{T}$.*
- (iii) *If $T_1 \overset{H}{\sim} T_0$, then $W(T_0) \cap \mathbb{T} = W(T_1) \cap \mathbb{T}$.*

Proof. By using the Theorem 2.14 and Proposition 2.13, it suffices to prove that $\overline{W(T)} \cap \mathbb{T} = W(T) \cap \mathbb{T}$ for each $T \in C_2(H) \cap \mathcal{K}(H)$. Indeed, let $\lambda \in \overline{W(T)} \cap \mathbb{T}$, then λ is a limit of $\langle Tx_n, x_n \rangle$ for some sequence (x_n) of unit vectors. Therefore, there exist a subsequence $(x_{j(n)})$ of (x_n) such that $x_{j(n)}$ converge to some x in the weak star topology. Since T is a compact operator then $Tx_{j(n)} \rightarrow Tx$ in the norm topology, this implies that $\lambda = \langle Tx, x \rangle$, and hence $x \neq 0$. Consequently, $\frac{\lambda}{\|x\|^2} \in W(T) \subseteq \overline{\mathbb{D}}$. So $\frac{1}{\|x\|^2} \leq 1$ and hence $\|x\|^2 \geq 1$, but we also have $\|x\|^2 \leq 1$, this means that $\|x\| = 1$ and $\lambda \in W(T)$. \square

2.4. Weak stability and Harnack domination. One says that an operator is weakly stable if $\lim_{n \rightarrow +\infty} T^n = 0$ in the weak topology of $B(H)$. Also we have that this is equivalent to T^* is weakly stable.

We give the following proposition which is useful to study this property.

Proposition 2.24. *Let H be a separable Hilbert space. Then, we have*

- (i) *Let $T \in C_\rho(H)$ and denote by V its minimal isometric ρ -dilation. Then, for every $m \geq 1$, we have*

$$\left\| \sum_{k=1}^m V^{*k+1} x_k \right\| \leq \left\| \sum_{k=1}^m T^{*k} x_k \right\| \leq \rho \left\| \sum_{k=1}^m V^{*k} x_k \right\|$$

for any m -tuple (x_1, \dots, x_m) of vectors of H .

- (ii) *Assume that T_1 be Harnack dominated by T_0 in $C_\rho(H)$ for a constant $c \geq 1$. If V_i acting on $K_i \supseteq H$ is the minimal isometric ρ -dilation of T_i ($i = 0, 1$), then we have*

$$\left\| \sum_{k=1}^m V_1^k x_k \right\| \leq c \left\| \sum_{k=1}^m V_0^k x_k \right\|$$

for any m -tuple (x_1, \dots, x_m) of vectors of H .

Proof. (i) Let $h = \sum_{i=0}^n V^i h_i$ with $h_i \in H$, then we have

$$\begin{aligned} \left\langle \sum_{k=1}^m T^{*k} x_k, Vh \right\rangle &= \sum_{k=1}^m \sum_{i=0}^n \langle T^{*k} x_k, V^{i+1} h_i \rangle = \frac{1}{\rho} \sum_{k=1}^m \sum_{i=0}^n \langle T^{*k} x_k, T^{i+1} h_i \rangle \\ &= \sum_{k=1}^m \sum_{i=0}^n \langle V^{*k+i+1} x_k, h_i \rangle = \left\langle \sum_{k=1}^m V^{*k+1} x_k, h \right\rangle \end{aligned}$$

Since the subset of all elements h having the above form is dense in K , we get

$$\begin{aligned} \left\| \sum_{k=1}^m T^{*k} x_k \right\| &= \sup_{\|h\|=1} \left| \left\langle \sum_{k=1}^m T^{*k} x_k, h \right\rangle \right| \geq \sup_{\|h\|=1} \left| \left\langle \sum_{k=1}^m T^{*k} x_k, Vh \right\rangle \right| \\ &\geq \sup_{\|h\|=1} \left| \left\langle \sum_{k=1}^m V^{*k+1} x_k, h \right\rangle \right| = \left\| \sum_{k=1}^m V^{*k+1} x_k \right\| \end{aligned}$$

and the left-hand side inequality is obtained. The right-hand side inequality is obvious.

(ii) Now, suppose that $T_1 \stackrel{H}{\prec} T_0$ in $C_\rho(H)$ and V_i acting on $K_i \supseteq H$ is the minimal isometric ρ -dilation of T_i ($i = 0, 1$). Using Theorem 1.1, we know that there exists an operator $S \in B(K_0, K_1)$ such that $S(H) \subset H$, $S|_H = I$, $SV_0 = V_1S$ and $\|S\| \leq c$. Let (x_1, \dots, x_m) be a m -tuple of vectors of H . Observe that $SV_0^k = V_1^k S$ for any positive integer k , thus we get

$$\left\| \sum_{k=1}^m V_1^k x_k \right\| = \left\| \sum_{k=1}^m V_1^k S x_k \right\| = \|S\| \left\| \sum_{k=1}^m V_0^k x_k \right\| \leq c \left\| \sum_{k=1}^m V_0^k x_k \right\|.$$

□

Lemma 2.25. *A ρ -contraction T is weakly stable if and only if the minimal isometric ρ -dilation of T is weakly stable.*

Proof. Let us assume that T is weakly stable and $[V, K]$ is the minimal isometric ρ -dilation of T . Hence T^* is also weakly stable, i.e. $T^{*n}h \rightarrow 0$ in the weak topology. Since T^* has the Blum-Hanson property, for each $h \in H$ and every increasing sequence $(k_n)_{n \geq 0}$ of positive integers, we have

$$\frac{1}{N} \sum_{n=0}^N T^{*k_n} h \rightarrow 0$$

in the norm topology. For each N , set $x_k = h/N$ if there exists an integer n such that $k = k_n$ and $x_k = 0$ otherwise, and use Proposition 2.24 (i). We derive

$$\frac{1}{N} \sum_{n=1}^N V^{*k_{n+1}} h \rightarrow 0.$$

It is enough to ensure that

$$\frac{1}{N} \sum_{n=0}^N V^{*l_n} x \rightarrow 0. \quad (2.6)$$

for any increasing sequence $(l_n)_{n \geq 0}$ of positive integers and any $x \in H$. Now, let $x = \sum_{i=1}^m V^i x_i$ with $x_i \in H$, we easily deduce from (2.6) that

$$\frac{1}{N} \sum_{n=1}^N V^{*l_n} x \rightarrow 0.$$

Since the subset of all elements x having the above form is dense in K and that the sequence of operators $1/N \left[\sum_{n=1}^N V^{*l_n} \right]$ is a sequence of contractions, we derive that V^* has the Blum-Hanson property. Thus, the sequence $(V^{*n}x)$ weakly converge to 0 for any $x \in K$. Hence V is weakly stable.

Conversely, assume that V is weakly stable. Then for each $(x, y) \in H^2$ and any $n \geq 1$, we have $\langle T^n x, y \rangle = \rho \langle V^n x, y \rangle \rightarrow 0$. Hence, T is weakly stable. \square

Corollary 2.26. *Let T_0 and T_1 be two operators in $C_\rho(H)$. Then, we have:*

- (i) *Assume that T_1 be Harnack dominated by T_0 in $C_\rho(H)$ and that T_0 is weakly stable (resp. stable). Then T_1 is also weakly stable (resp. stable).*
- (ii) *Let T_0 and T_1 be Harnack equivalent in $C_\rho(H)$. Then T_0 is weakly stable (resp. stable) if and only if T_1 is weakly stable (resp. stable).*

Proof. (i) Assume that T_0 is weakly stable. Using Lemma 2.25, we see that the minimal isometric ρ -dilation V_0 is weakly stable. Applying Proposition 2.24 (ii) and using the Blum-Hanson property as in the proof of Lemma 2.25, we deduce that V_1 is weakly stable. Using again Lemma 2.25, we obtain the weak stability of T_1 .

Now, suppose that T_0 is stable. We deduce from Lemma 3.5 of [10] that V_0 is stable. From Proposition 2.24 (ii) we derive that V_1 is stable. Then, by Lemma 3.5 of [10] we obtain the stability of T_1 .

The assertion (ii) is a direct consequence of (i). \square

Remark 2.27. 1) Concerning the stability of two Harnack equivalent ρ -contractions, the assertion (ii) is exactly Corollary 3.6 of [10].

2) Since any ρ -contraction T is similar to a contraction and power bounded, by [19, Proposition 8.5], the residual spectrum $\sigma_r(T)$ of T is included in \mathbb{D} . By [19, Proposition 8.4] it follows that if any ρ -contraction T is weakly stable then $\sigma_p(T) \subseteq \mathbb{D}$. In this case, according to Lemma 2.25, if V is the minimal isometric ρ -dilation of T , then $\Gamma(V) = \sigma_c(V)$. So, if there exist $\lambda \in \sigma_p(T)$ such that $|\lambda| = 1$ then T is not weakly stable and this ρ -contraction is in Harnack part of an operator with ρ -numerical radius one.

3. EXAMPLES OF HARNACK PARTS FOR SOME NILPOTENT MATRICES WITH NUMERICAL RADIUS ONE

In the following, we try to describe the Harnack parts a nilpotent matrices with numerical radius one. We begin by the nilpotent matrix of order one in the dimension two.

Theorem 3.1. *Let $T_0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in C_2(\mathbb{C}^2)$, then the Harnack parts of T_0 is reduced to $\{T_0\}$.*

Proof. Let $T \in C_2(\mathbb{C}^2)$ such that $T \stackrel{H}{\sim} T_0$, then by Theorem 1.1, there exists $\alpha, \beta > 0$ ($\alpha \leq 1, \beta \geq 1$) such that

$$\alpha K_z^2(T_0) \leq K_z^2(T) \leq \beta K_z^2(T_0), \quad \text{for all } z \in \mathbb{D}. \quad (3.1)$$

By Corollary 2.2, the operator T not admits eigenvalues in \mathbb{T} . Hence, $K_z^2(T_0)$ and $K_z^2(T)$ are uniformly bounded in \mathbb{D} and may be extended to a positive operators on $\overline{\mathbb{D}}$.

We have

$$K_z^2(T_0) = 2 \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix},$$

thus $\det(K_z^2(T_0)) = 4(1 - |z|^2)$ and $\text{Ker}(K_z^2(T_0))$ of rang one on \mathbb{T} . Let $v(z) = \begin{pmatrix} 1 \\ -z \end{pmatrix}$, then $K_z^2(T_0)v(z) = 0$ on \mathbb{T} . This implies by (3.1) that

$$0 = K_{1,\theta}^2(T)v(e^{i\theta}) = K_{1,\theta}^2(T)e_1 - e^{i\theta}K_{1,\theta}^2(T)e_2 = 0 \quad \text{for all } \theta \in \mathbb{R}. \quad (3.2)$$

Multiplying successively (3.2) by 1 and $e^{-i\theta}$, and integrating with respect the Haar measure m on the torus, we obtain: $Te_2 = 2e_1$ and $T^*e_1 = 2e_2$. Thus T take the form

$$T = \begin{pmatrix} 0 & 2 \\ b & 0 \end{pmatrix},$$

with $b \in \mathbb{C}$. Since $w(T) \leq 1$, we have

$$|2x_2\bar{x}_1 + bx_1\bar{x}_2| \leq |x_1|^2 + |x_2|^2.$$

If we take $x_1 = \frac{\sqrt{2}}{2}$ and $x_2 = \frac{\sqrt{2}}{2}e^{i\theta}$, we get

$$|1 + be^{-2i\theta}| \leq 1.$$

In particular, for $\theta = \frac{\arg b}{2}$

$$1 + |b| \leq 1$$

This implies that $b = 0$ and $T = T_0$. □

In the following result, we describe the Harnack parts of a nilpotent matrix of order two in $C_2(\mathbb{C}^n)$, $n \geq 3$, with numerical radius one.

Theorem 3.2. *Let $N \in C_2(\mathbb{C}^n)$, $n \geq 3$ such that*

$$N = \begin{pmatrix} 0 & 0 & \dots & a \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

with $|a| = 2$, then the Harnack parts of N is the set of all matrices of $C_2(\mathbb{C}^n)$ of the form

$$T = \begin{pmatrix} 0 & 0 & a \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3)$$

with $B \in C_2(\mathbb{C}^{n-2})$ such that $w(B) < 1$.

Proof. Let $T \in C_2(\mathbb{C}^n)$ such that $T \stackrel{H}{\sim} N$. By Corollary 2.2, the operator T not admits eigenvalues in \mathbb{T} . Hence, $K_z^2(N)$ and $K_z^2(T)$ are uniformly bounded in \mathbb{D} and may be extended to a positive operators on $\overline{\mathbb{D}}$. We have

$$K_z^2(N) = \begin{pmatrix} 2 & 0 & a\bar{z} \\ 0 & 2I_{n-2} & 0 \\ \bar{a}z & 0 & 2 \end{pmatrix},$$

with I_{n-2} denote the identity matrix on the linear space spanned by the vectors e_2, \dots, e_{n-1} of the canonical basis of \mathbb{C}^n . Then $\det(K_z^2(N)) = 2^{n-2}(4 - |a|^2|z|^2)$. Let $v(z) = -a\bar{z}e_1 + 2e_n$, then $K_z^2(N)v(z) = 0$ on \mathbb{T} . Thus by proposition 2.11, $K_z^2(T)v(z) = 0$ on \mathbb{T} . This implies that

$$-ae^{-i\theta}K_{1,\theta}^2(T)e_1 + 2K_{1,\theta}^2(T)e_n = 0 \quad \text{for all } \theta \in \mathbb{R}. \quad (3.4)$$

Multiplying successively (3.4) by 1, $e^{i\theta}$, $e^{-i\theta}$ and $e^{2i\theta}$, and integrating with respect m , we obtain:

$$T^*e_1 = \bar{a}e_n, Te_n = ae_1, T^*e_n = 0 \text{ and } Te_1 = 0. \quad (3.5)$$

By (3.5), the matrix T take the form (3.3). Hence

$$K_z^2(T) = \begin{pmatrix} 2 & 0 & a\bar{z} \\ 0 & K_z^2(B) & 0 \\ \bar{a}z & 0 & 2 \end{pmatrix} \quad (3.6)$$

By Theorem 2.17, we know that $\ker(K_z^2(T)) = \ker(K_z^2(N))$ for all $z \in \mathbb{T}$, it forces $\ker(K_z^2(B))$ to be equal to $\{0\}$ for every $z \in \mathbb{T}$. Using again Theorem 2.17, we deduce that B is Harnack equivalent to 0, thus $w(B) < 1$.

Conversely, Let $T \in C_2(\mathbb{C}^n)$ given by (3.3), then we can write $K_z^2(T)$ under the form given by (3.6). Since $B \in C_2(\mathbb{C}^{n-2})$ with $w(B) < 1$, B is Harnack equivalent to 0 in $C_2(\mathbb{C}^{n-2})$. Then by Theorem 1.1, there exists $\alpha, \beta > 0$ ($\alpha \leq 1$, $\beta \geq 1$) such that

$$2\alpha I_{n-2} \leq K_z^2(B) \leq 2\beta I_{n-2}, \quad \text{for all } z \in \mathbb{D}.$$

Thus

$$\alpha K_z^2(N) \leq K_z^2(T) \leq \beta K_z^2(N), \quad \text{for all } z \in \mathbb{D}.$$

This means that T is Harnack equivalent to N . □

Theorem 3.3. Let $N = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$ such that $|a| = \sqrt{2}$, then the Harnack parts

of N is the set of all matrices of $C_2(\mathbb{C}^3)$ of the form

$$T = a \begin{pmatrix} 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Proof. Let $T \in C_2(\mathbb{C}^3)$ such that $T \stackrel{H}{\sim} N$. By Corollary 2.2, the operator T not admits eigenvalues in \mathbb{T} . Hence, $K_z^2(N)$ and $K_z^2(T)$ are uniformly bounded in \mathbb{D}

and may be extended to a positive operators on $\overline{\mathbb{D}}$. Furthermore, by [7, Theorem 5.2] $T^2 \stackrel{H}{\sim} N^2$, then by Theorem 3.2, the operator T^2 takes the following form

$$T^2 = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $|b| < 1$. If $b \neq 0$ then $\text{Ker}T^2 = \mathbb{C}e_1$ is invariant by T , so $Te_1 = xe_1$ but $0 = T^2e_1 = x^2e_1$, this implies that $x = 0$ and $Te_1 = 0$. Similarly, $\mathbb{C}^3 \neq \text{Im}T \supseteq \text{Im}T^2 = \text{span}\{e_1, e_2\}$ is invariant by T , so $Te_2 = ue_1 + ve_2$ for some $u, v \in \mathbb{C}$. On the other hand, we have

$$K_z^2(N) = \begin{pmatrix} 2 & a\bar{z} & a^2\bar{z}^2 \\ \bar{a}z & 2 & a\bar{z} \\ \bar{a}^2z^2 & \bar{a}z & 2 \end{pmatrix},$$

thus $\det(K_z^2(N)) = 4(2 - |a|^2|z|^2)$, so $\text{Ker}(K_z^2(N))$ of rank one on \mathbb{T} . Let $v(z) = -a^2\bar{z}e_1 + 2ze_2$, then $K_z^2(N)v(z) = 0$ on \mathbb{T} . Thus by Proposition 2.11, $K_z^2(T)v(z) = 0$ on \mathbb{T} . This implies that

$$-a^2e^{-i\theta}K_{1,\theta}^2(T)e_1 + 2e^{i\theta}K_{1,\theta}^2(T)e_3 = 0 \quad \text{for all } \theta \in \mathbb{R}. \quad (3.7)$$

Using (3.7) in a similar way than before, we get

$$2Te_3 = a^2T^*e_1 \quad \text{and} \quad 2T^*e_3 = a^2Te_1. \quad (3.8)$$

By this we deduce that

$$\langle Te_3, e_1 \rangle = \frac{a^2}{2} \langle T^*e_1, e_1 \rangle = \frac{a^2}{2} \langle e_1, Te_1 \rangle = 0$$

and

$$\langle Te_3, e_3 \rangle = \frac{a^2}{2} \langle T^*e_1, e_3 \rangle = \frac{a^2}{2} \langle e_1, Te_3 \rangle = 0.$$

The matrix T take the form

$$T = \begin{pmatrix} 0 & u & 0 \\ 0 & v & w \\ 0 & 0 & 0 \end{pmatrix}.$$

By (3.8), $2w = a^2T^*e_1 = a^2\bar{u}e_2$, hence

$$\bar{a}w = a\bar{u}. \quad (3.9)$$

This implies that u and v must be not equal to 0. Now the fact that

$$T^2 = \begin{pmatrix} 0 & uv & uw \\ 0 & v^2 & vw \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

implies that $v = b = 0$ and

$$uw = a^2. \quad (3.10)$$

By (3.9) and (3.10) we can deduce that $u = ae^{-i\theta}$ and $w = ae^{i\theta}$, $\theta \in \mathbb{R}$.

Conversely, Let $T \in C_2(\mathbb{C}^3)$ given as above, then

$$K_z^2(T) = \begin{pmatrix} 2 & u\bar{z} & a^2\bar{z}^2 \\ \bar{u}z & 2 & w\bar{z} \\ \bar{a}^2z^2 & \bar{w}z & 2 \end{pmatrix}$$

By a simple calculus, we can see that

$$\det(\beta K_z^2(N) - K_z^2(T)) \geq \beta^3 + q(\beta)$$

with q is a polynomial with constant coefficient of degree two on \mathbb{T} . so for β sufficiently large, we assert that there exists a constant $r > 0$ such that

$$\det(\beta K_z^2(N) - K_z^2(T)) \geq \beta^3 + q(\beta) \geq r > 0.$$

This is exactly the principal minor of order 3 of $\beta K_z^2(N) - K_z^2(T)$. Similarly, we can prove that the principal minor of order 2 is also positive for β sufficiently large. By the Sylvester's criterion for the positive-semidefinite Hermitian matrices, we assert that

$$\beta K_z^2(N) - K_z^2(T) \geq 0 \quad \text{for all } z \in \overline{\mathbb{D}}.$$

We prove similarly the existence of $0 < \alpha \leq 1$ such that

$$K_z^2(T) - \alpha K_z^2(N) \geq 0 \quad \text{for all } z \in \overline{\mathbb{D}}.$$

□

Remark 3.4. We can see that the matrix T given in the above theorem takes the form

$$T = U_\theta^* N U_\theta \quad \text{with} \quad U_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Thus all the elements of the Harnack parts of N are unitary equivalent to N .

REFERENCES

1. T. Ando and K. Nishio, *Convexity properties of operator radii associated with unitary ρ -dilations*, Michigan Math. J. **20** (1973), 303–307.
2. T. Ando, I. Suciú and D. Timotin, *Characterization of some Harnack parts of contractions*, J. Operator Theory **2** (1979), 233–245.
3. C. Badea and G. Cassier, *Constrained von Neumann inequalities*, Adv. Math. **166** (2002), no. 2, 260–297.
4. C. Badea, D. Timotin and L. Suciú, *Classes of contractions and Harnack domination*, arXiv:1505.01972.
5. C. A. Berger, *A strange dilation theorem*, Notices Amer. Math. Soc. **12** (1965), 590.
6. G. Cassier *Ensembles K -spectraux et algèbres duales d'opérateurs* Preprint LAFP N2 (1991).
7. G. Cassier *Mapping formula for functional calculus, Julia's lemma for operator and applications* Acta Sci. Math. (Szeged) **74** (2008), no. 3-4, 783–805.
8. G. Cassier and T. Fack, *Un noyau pour divers calculs fonctionnels*, C. R. Acad. Sci. Paris Sér. I **317** (1993), 683–688.
9. G. Cassier and T. Fack, *Contractions in von Neumann algebras*, J. Funct. Anal. **55** (1996), no. 2, 297–338.
10. G. Cassier and N. Suciú, *Mapping theorems and Harnack ordering for ρ -contractions*, Indiana Univ. Math. J. **55** (2006), no. 2, 483–523.
11. G. Cassier and E. H. Zerouali, *operator matrices in class C_ρ* , Linear Algebra Appl. **420** (2007), 361–376.

12. C. Davis, *The shell of a Hilbert-space operator*, Acta Sci. Math. (Szeged) **29** (1968), 69–86.
13. M. A. Dritschel, S. McCullough and H. J. Woerdeman, *Model theory for ρ -contractions, $\rho \leq 2$* , J. Operator Theory **41** (1999), no. 2, 321–350.
14. E. Durszt, *On unitary ρ -dilations of operators*, Acta Sci. Math. (Szeged) **27** (1966), 247–250.
15. C. K. Fong and J. A. R. Holbrook, *Unitarily invariant operator norms*, Canad. J. Math. **35** (1983), no. 2, 274–299.
16. C. Foias, *On Harnack parts of contractions*, Rev. Roum. Math. Pures et Appl. **XIX** (1974), no. 3, 315–318.
17. J. A. R. Holbrook, *On the power-bounded operators of Sz.-Nagy and Foias*, Acta Sci. Math. (Szeged) **29** (1968), 299–310.
18. J. A. R. Holbrook, *Inequalities governing the operator radii associated with unitary ρ -dilations*, Michigan Math. J. **18** (1971), 149–159.
19. C. S. Kubrusly, *An introduction to Models and Decompositions in Operator Theory*, Birkhauser, Boston, 1997.
20. V. A. Khatskevich, Yu. L. Shmul’yan and V. S. Shul’man, *Preorder and equivalences in the operator sphere*, Sibirsk. Mat. Zh. **32** (1991), no. 3 (in Russian); English transl. : Siberian Math. J. **32** (1991), no. 3, 496–506.
21. T. Nakazi and K. Okubo, *ρ -contraction and 2×2 matrix*, Linear Algebra Appl. **283** (1998), no. 1-3, 165–169.
22. B. Sz.-Nagy, *Sur les contractions de l’espace de Hilbert*, Acta Sci. Math. (Szeged) **15** (1953), 87–92(French).
23. B. Sz.-Nagy and C. Foias, *On certain classes of power-bounded operators in Hilbert space*, Acta Sci. Math. (Szeged) **27** (1966), 17–25.
24. B. Sz.-Nagy and C. Foias, *Similitude des opérateurs de class C_ρ à des contractions*, C. R. Acad. Sci. Paris Sér. A-B **264** (1967), A1063–A1065 (French).
25. B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, *Harmonic analysis of operators on Hilbert space*. Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010.
26. J. P. Williams, *Schwarz norms for operators*, Pacific J. Math. **24** (1968), 181–188.
27. K. Yosida, S. Kakutani, *Operator-theoretical treatment of Markov’s process and mean ergodic theorem*, Ann. of Math., **42** (1941), 188–228.

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