On improper interval edge colourings

Peter Hudák, František Kardoš, Tomáš Madaras, Michaela Vrbjarová

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Peter Hudák, Košice, František Kardoš, Bordeaux, Tomáš Madaras, Košice, Michaela Vrbjarová, Košice,

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Abstract. In this paper, we study improper interval edge colourings, defined by the requirement that edge colours around each vertex form an integer interval. For the corresponding chromatic invariant (being the maximum number of colours in such a colouring), we present upper and lower bounds and discuss their qualities; also, we determine its values and estimates for graphs of various families, like wheels, prisms or complete graphs. The study of this parameter was inspired by interval colouring, introduced by A.S. Asratian, R.R. Kamalian: Interval colorings of edges of a multigraph. Applied Mathematics, Yerevan State University 5 (1987) 25–34. The difference is that we relax the requirement of the original colouring to be proper.

Keywords: edge colouring, interval colouring, improper colouring

MSC 2010: 05C15

1. Introduction

Throughout this paper, we consider simple connected graphs without loops or multiple edges; we use standard graph terminology from the book [4].

A proper edge colouring $c : V(G) \to \{1, \ldots, k\}$ of a graph G which uses each colour from $\{1, \ldots, k\}$ at least once is called an *interval colouring* if, for each vertex x of G, the set of colours of edges incident with x (the *palette* of x) forms an integer interval; we say that the graph G is an *interval k-colourable*. For an interval colourable graph

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G, let $t(G)$ denote the maximum number of colours used in any interval colouring of G. The notion of interval colouring was introduced by Asratian and Kamalian in [2] in the connection with specialized scheduling problems and, since then, it was further investigated in many papers, see, for example [3, 5, 6, 7, 8, 9, 10]. Not all graphs are proper interval colourable (this concerns, for example, graphs of Class 2); in fact, the problem of determining whether a graph has an interval colouring is NP-complete, even for bipartite graphs, see [1] and [11].

In our paper, we relax the requirement of the above defined colouring to be proper and introduce the parameter $\widehat{t}(G)$ being the maximum number of colours in an improper interval colouring of G. Note that $\widehat{t}(G)$ is defined for every graph (which is in sharp contrast with $t(G)$, see [3]) and is at least 3 for all graphs of order at least 3 different from K_3. Also, for a graph G which is interval colourable, $t(G) \leq \widehat{t}(G)$ holds. Compared to the range of results for interval colourings, it seems that improper interval colourings have not been studied yet. Our aim is to contribute to this topic by determining the exact values of $\widehat{t}(G)$ or their estimates for graphs of several classic families (this concerns also many graphs which do not possess proper interval colourings), and to establish upper and lower bounds on $\widehat{t}(G)$ in terms of graph diameter and maximum degree.

2. Properties and results

In the analysis of improper interval colourings, the following observations (which are easy to see) will be useful:

Proposition 2.1. *If a graph G is improperly interval k-colourable with $k \geq 3$, then it is also improperly interval k-colourable in such a way that the colours 1 and k are used exactly once.*

Proposition 2.2. *A graph G is improperly interval ℓ-colourable for each $1 \leq \ell \leq \widehat{t}(G)$.*

First, we present an upper bound on $\widehat{t}(G)$ in terms of maximum degree and diameter:

Lemma 2.3. *For any connected graph G with maximum degree $\Delta = \Delta(G)$, $\widehat{t}(G) \leq 1 + (\Delta - 1)(\mathrm{diam}(G) + 1)$.*

Proof. Let uv and xy be edges coloured by 1 and k, respectively, in an improper interval colouring of G using $\widehat{t}(G)$ colours. Observe that, in each such a colouring, the colours of each two adjacent edges differ by at most $\Delta - 1$. Now, take the shortest path P between the vertex sets $\{u, v\}$ and $\{x, y\}$. Then P has length at most $\mathrm{diam}(G)$; note, however, that the edges uv and xy need not belong to P.

It follows that the number of colour changes from uv to xy along P is at most $\mathrm{diam}(G)+1$ which implies the result. □

Note that this lemma generalizes the result of [2] where the right side of the above inequality estimated $t(G)$ from above. Using the same arguments, we can prove an analogous inequality with respect to the diameter of the line graph $L(G)$:

Lemma 2.4. *For any connected graph G with maximum degree $\Delta = \Delta(G)$, $\widehat{t}(G) \leq 1 + (\Delta - 1)(\mathrm{diam}(L(G))$; the bound is sharp.*

We also present strengthenings of two theorems from [2]:

Theorem 2.5. *For each triangle-free graph G on n vertices, $\widehat{t}(G) \leq n - 1$; the bound is sharp.*

Proof. We follow the same reasoning as in the proof of Theorem 1 from [2]; the difference is only in the estimate of the number of elements of the set $A(i)$ (see original proof, page 38): we obtain that, for improper interval t-colouring of G, the inequality $|A(i)| \geq f(e_i) - f(e_{i+1}) - 1, i = 1, \ldots, k-1$, holds (instead of equality). Then the last argument of the original proof rephrases as

$$n \geq k + 1 + \sum_{i=1}^{k-1} |A(i)| \geq k + 1 + \sum_{i=1}^{k-1} (f(e_i) - f(e_{i+1}) - 1) = \\ = k + 1 + t - 1 - (k-1) = 1 + t,$$

implying $t \leq n - 1$.

To show the sharpness of the bound, consider the graph of path on n vertices P_n, $n \geq 2$. It is easy to see that $\widehat{t}(P_n) = n - 1$. □

Since the original proof of the Proposition 4 of [2], page 39 does not involve a considered interval colourings to be proper, we obtain

Theorem 2.6. *For each graph G on n vertices, $\widehat{t}(G) \leq 2n - 1$.*

For the lower bound on $\widehat{t}(G)$, we have the following estimate:

Theorem 2.7. *For each graph G, $\widehat{t}(G) \geq 1 + \mathrm{diam}(L(G))$; the bound is sharp.*

Proof. Consider the line graph $L(G)$ of G and let x be a vertex of maximum eccentricity in $L(G)$. Then colour x with the colour 1 and each vertex $y \in V(L(G)), y \neq x$ with the colour equal to $1 + \mathrm{dist}_{L(G)}(x, y)$. This vertex colouring of $L(G)$ induces an edge colouring of G. The way the colouring of vertices of $L(G)$ was constructed gives that, in G, the palette of each vertex consists either of two consecutive colours or of a single colour; thus, it is an improper interval colouring of G having the highest colour equal to $\mathrm{diam}(L(G)) + 1$.

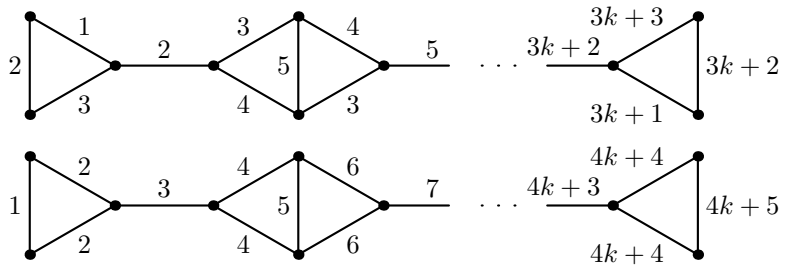

FIGURE 1. The graph SN_k and its proper and improper interval colourings.

To show the sharpness of the lower bound, consider, for an integer $k \geq 6$ the graph DB_k obtained from a k-vertex path $x_1x_2 \dots x_k$ by adding new edges x_1x_3 and $x_{k-2}x_k$. Then it is easy to check that $\widehat{t}(DB_k) = k-1$ and $\mathrm{diam}(L(DB_k)) = k-2$. □

The difference between $t(G)$ and $\widehat{t}(G)$ can be arbitrarily large. This can be seen on the graph SN_k formed from a chain of k copies of the graph K_4^- where both chain ends are closed with a different triangle, see Figure 1. It is easy to see that the graph SN_k has proper interval colouring; observe that, in each proper interval colouring of SN_k, the difference of colours of two consecutive bridges incident with the same copy of K_4^- is 0 or 3 while it is possible to construct an improper colouring of SN_k such that the colour difference on consecutive bridges is 4. Thus, we obtain that $\widehat{t}(SN_k) - t(SN_k) \geq k$. A similar construction can be used also for triangle-free graphs, where instead of copies of K_4^-, the 5-cycle with pendant edges incident with two nonadjacent vertices is used: the difference of colours on bridges in a proper interval colouring is at most 3 whereas it is possible to assign the colours to obtain the difference 4 in an improper interval colouring, see Figure 2.

The construction generalizes to classes of graphs of arbitrarily large girth. Moreover, one can consider several other suitable configurations to show that the colour difference on two selected edges can be greater in improper version of the colouring rather than in the proper one, and these configurations may be used to form other graphs (for example, 2-connected) with arbitrarily large difference of $t(G)$ and $\widehat{t}(G)$.

There is no known significant upper bound on difference $\widehat{t}(G) - t(G)$ in terms of number of vertices of G.

In the following, we establish the exact values and estimates of $\widehat{t}(G)$ of graphs from several standard graph families.

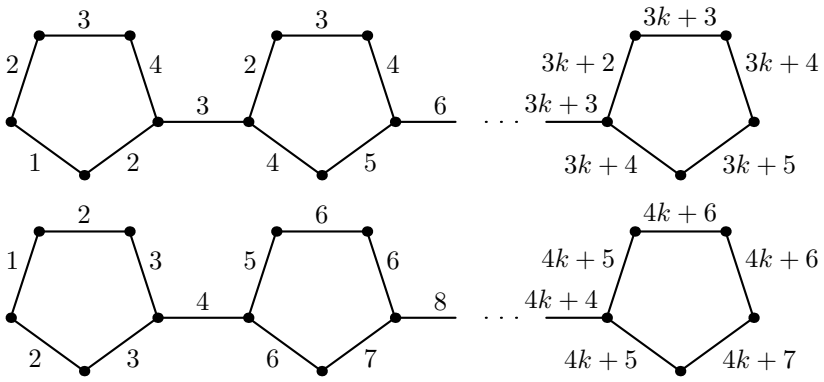

FIGURE 2. An analogous construction for triangle-free graphs.

Theorem 2.8. *For an n-wheel W_n,*

$$\widehat{t}(W_n) = \begin{cases} 4, & \text{if } n = 3; \\ n, & \text{if } n \geq 4. \end{cases}$$

Proof. Suppose first that $n = 3$. If $\widehat{t}(W_3) \geq 5$, then the unique edge of W_3 which is not adjacent to the edge of maximum colour would have a colour at least 2, so colour 1 is not used, a contradiction; on the other hand, an improper interval 4-colouring of W_3 is easy to find.

Now, let $n \geq 4$. An improper interval n-colouring of W_n can be constructed as follows: if x is the centre of W_n and $x_1, \ldots, x_n$ are its neighbours in counter clockwise order, assign to each edge $x_i x_{i+1}, 1 \leq i \leq \lceil \frac{n}{2} \rceil$ the colour $2i - 1$, to each edge $x_i x_{i+1}, \lceil \frac{n}{2} \rceil < i \leq n$ (indices taken modulo n) the colour $2n+4-2i$, and to each edge $xx_i, i \notin \{1, \lceil \frac{n}{2} \rceil + 1\}$ the colour equal to the arithmetic mean of colours of edges $x_{i-1}x_i, x_i x_{i+1}$ whereas the edge xx_1 receives colour 1 and the edge $xx_{\lceil \frac{n}{2} \rceil + 1}$ receives the colour n. It is easy to check that, in this colour assignment, the palettes of all vertices form integer intervals (note that the palette of x_1 is $[1, 2]$ and the palette of $x_{\lceil \frac{n}{2} \rceil + 1}$ is $[n - 1, n]$). See Figure 3 for illustration.

Assume now that, for some n, $\widehat{t}(W_n) \geq n+1$. Consider first the case when n is odd. Then at least one of colours 1 and $n + 1$ is used at a rim edge of W_n (otherwise the palette of the central vertex of W_n does not form an integer interval). Since replacing each colour c with the colour $n+2-c$ yields also an improper interval $(n+1)$-colouring of W_n, we can suppose, without loss of generality, that the colour 1 is used on a rim edge uv. Note that colours of two adjacent rim edges can differ by at most 2; this gives that two edges yw, wz which are – taking into account the bidirectional distance on the rim – most distant from uv have colour at most $1 + 2 \cdot \frac{n-1}{2} = n$. Thus, the

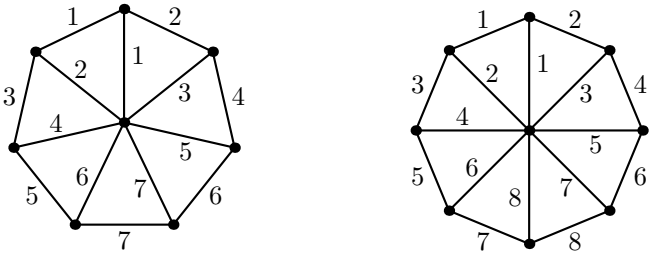

FIGURE 3. Improper interval n-colouring of a wheel W_n for $n = 7$ and $n = 8$.

colour $n + 1$ has to appear at a spoke edge of W_n and, repeating the above colour difference argument, we obtain that this spoke edge is incident with the vertex w. Due to the fact that, in the considered improper interval colouring, the colour 1 is unique, the edge uv is adjacent with at least two edges coloured by 2 and at least one of them – say, vq – is a rim edge (otherwise again, the palette of the central vertex is not an integer interval). Then the colour sequence of the rim path P from uv through vq ending in one of rim edges incident to w, is $1, 2, 4, \ldots, 2i, 2(i + 1), \ldots, n - 1$; this gives that one of yw and wz has colour $n - 1$ and the other one has the colour n. Now, if we take the rim path P' starting at uv, but with the opposite direction as P, we get that its colour sequence is $1, 3, 5, \ldots, 2i - 1, 2i + 1, \ldots, n$. But then the colour n does not appear at a spoke edge, hence, the palette of the central vertex is not an integer interval, a contradiction.

Consider now the case when n is even. Rephrasing the above arguments, we can suppose that the colour 1 is used on a rim edge uv. According to the possible position of the colour $n + 1$, we distinguish two cases:

Case 1: The colour $n + 1$ is on a rim edge. Then the colour sequences of both rim paths starting at uv are $1, 3, 5, \ldots, 2i - 1, 2i + 1, \ldots, n + 1$, which gives that colours of all spoke edges are even numbers, a contradiction.

Case 2: The colour $n + 1$ is on a spoke edge xw. Let yw and wz be rim edges adjacent to xw; without loss of generality, let yw be closer to uv than wz to uv. Then the colour sequence of a rim path starting at uv and ending at yw is $1, 3, \ldots, 2i - 1, 2i + 1, \ldots, n - 1$, which gives that wz has colour n. By the same argument as for n odd, the colour 2 has to be used on a rim edge incident with uv, hence, the colour sequence of the rim path between uv and wz is $2, 4, \ldots, 2i, 2(i+1), \ldots, n$. But then again, the colour n is missing at spoke edges, a contradiction.

□

Note that, by [3], only three wheels are proper interval colourable, namely W_3, W_6 and W_9.

Next, we present exact value for a graph $Y_n = C_n \Box K_2$, the graph of an n-sided prism.

Theorem 2.9. *For an n-prism graph Y_n with $n \geq 3$, $\widehat{t}(Y_n) = n + 2$.*

Proof. Let $Y_n = C_n \Box K_2$. In a plane drawing of Y_n, there are two n-gonal faces $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ interconnected by the edges $x_i y_i$, $i = 1, \ldots, n$ (the *side edges*); the edges of type $x_i x_{i+1}$ or $y_i y_{i+1}$ ($i = 1, \ldots, n$ indices are modulo n) will be called *base edges* in the sequel.

First we show that there exists an improper interval colouring of graph Y_n by $n+2$ colours. It can be constructed as follows: Assign the edge $x_1 y_1$ colour 1, the edges $x_1 x_2$, $y_1 y_2$ colour 2, the edges $x_1 x_n$, $y_1 y_n$ colour 3 and the edge $x_{\lceil \frac{n}{2} \rceil + 1} y_{\lceil \frac{n}{2} \rceil + 1}$ assign colour $n+2$. Now, assign edges $x_i y_i$ colour $2i-1$ for $i = 2, \ldots, \lceil \frac{n}{2} \rceil$. Remaining side edges $x_i y_i$ assign colour $2(n-i)+4$ for $i = n, \ldots, \lceil \frac{n}{2} \rceil + 2$. The colours of remaining base edges are now determined unambiguously, see Figure 4 for illustration.

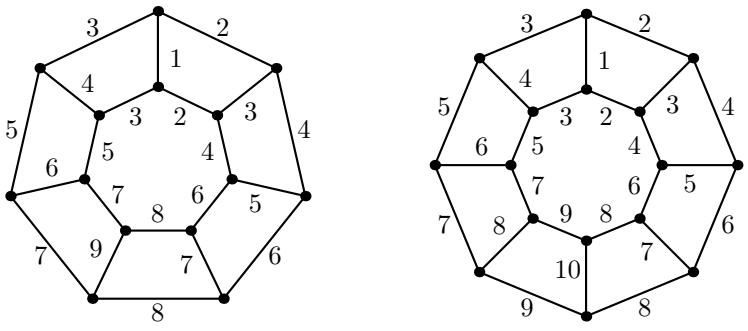

FIGURE 4. Improper interval $(n+2)$-colouring of a prism Y_n for $n = 7$ and $n = 8$.

Now we prove that the colouring is optimal. Observe that $\operatorname{diam}(L(Y_n)) = \lfloor \frac{n}{2} \rfloor + 1$, hence Lemma 2.4 gives an upper bound $\widehat{t}(Y_n) \leq 2 \cdot \lfloor \frac{n}{2} \rfloor + 3$. For n odd this meets the lower bound.

Suppose that n is even and there is an interval edge colouring of Y_n with $n+3$ colours. From the proof of Lemma 2.4 we know that for any pair of edges in G coloured 1 and $n+3$ the corresponding vertices in $L(G)$ are at distance $\operatorname{diam}(L(G))$. It is possible either for a pair of side edges $x_i y_i$ and $x_j y_j$ with $|i-j| = \frac{n}{2}$, or for a pair of base edges $x_i x_{i+1}$ and $y_j y_{j+1}$ with $|i-j| = \frac{n}{2}$. In both cases all edges incident with the edge assigned colour 1 lie on some path of optimal length between edges assigned 1 and $n+3$, so all these edges have to be assigned colour 3, which

is a contradiction, since there is no colour 2 in the palettes of vertices incident with the edge assigned colour 1. □

Finally, we discuss improper interval colourings of complete graphs which is of particular interest due to the ongoing intensive research in [9], [10].

Theorem 2.10. *For a complete graph K_n with $n \geq 5$, $\widehat{t}(K_n) \leq 2n-5$.*

Proof. By contradiction. Assume that there exists a positive integer $n \geq 5$ such that K_n is improperly interval $(2n-4)$-colourable. Let xy and uv be edges coloured by 1 and $2n-4$, respectively; note that xy and uv are not adjacent. Then the colour of xu (xv, yu or yv) is either $n-2$ or $n-1$. We claim that both palettes of x and y contain each of the colours $1,\ldots,n-1$ exactly once; similarly, both palettes of u and v contain each of the colours $n-2,\ldots,2n-4$ exactly once: If both xu and xv had colour $n-2$, then the palettes of u and v would contain each of the colours $n-2,\ldots,2n-4$ exactly once, hence, yu and yv would have colour $n-1$, which is impossible.

Let z be a vertex of K_n such that zx has colour 2. Then the colour of the edge zu is at least n (because of u); note, however, that the difference of the highest and the lowest colour (which is the colour 2) in the palette of z is at most $n-2$ which gives that the highest colour in the palette of z is at most n. Therefore, the colour of zu is equal to n. The same argument can be used for the edge zv, obtaining that its colour is also n. But then the palette of z is not an integer interval, a contradiction. □

Theorem 2.11. *For each n, $\widehat{t}(K_n) < \widehat{t}(K_{n+1})$.*

Proof. Let c be an improper interval $\widehat{t}$-colouring of K_n with $\widehat{t} = \widehat{t}(G)$, and let $w_1,\ldots,w_n$ be ordering of vertices of K_n such that w_1 is incident with an edge of colour $\widehat{t}$. Now, add to K_n a new vertex u and, for each $1 < i \leq n$, add new edge uw_i coloured with the colour $c(w_1w_i)+1$; the edge uw_1 will be then coloured with $\widehat{t}+1$. □

Theorem 2.12. *For each n, $\widehat{t}(K_{n+2}) - \widehat{t}(K_n) \geq 3$.*

Proof. Let c be an improper interval $\widehat{t}$-colouring of K_n with $\widehat{t} = \widehat{t}(G)$, and let $w_1,\ldots,w_n$ be ordering of vertices of K_n such that w_1 and w_2 are endvertices of an edge of colour $\widehat{t}$. Now, add to K_n two new vertices x, y and, for each $3 \leq i \leq n$, add new edges xw_i and yw_i coloured with the colour $c(w_iw_1)+1$ and new edges xw_1 and yw_2 coloured with $\widehat{t}+1$. In addition, add new edges yw_1 and xw_2 coloured with $\widehat{t}+2$ and the new edge xy coloured with $\widehat{t}+3$.

□

	$4j$	$4j+1$	$4j+2$	$4j+3$
$4i$	$3i+4j$	$3i+4j+1$	$3i+4j+2$	$3i+4j+3$
$4i+1$	$3i+4j+1$	$3i+4j$	$3i+4j+3$	$3i+4j+2$
$4i+2$	$3i+4j+2$	$3i+4j+3$	$3i+4j+1$	$3i+4j+4$
$4i+3$	$3i+4j+3$	$3i+4j+2$	$3i+4j+4$	$3i+4j+1$

TABLE 1. Colouring of a complete graph on $n = 4k$ vertices: the colour of an edge joining vertices $4i+r$ and $4j+s$ with $0 \leq i < j \leq k$ and $r, s \in \{0, 1, 2, 3\}$.

	$4i$	$4i+1$	$4i+2$	$4i+3$
$4i$		$7i+1$	$7i+2$	$7i+3$
$4i+1$			$7i+3$	$7i+2$
$4i+2$				$7i+4$
$4i+3$				

TABLE 2. Colouring of a complete graph on $n = 4k$ vertices: the colour of an edge joining vertices $4i+r$ and $4i+s$ with $0 \leq i \leq k$ and $0 \leq r < s \leq 3$.

Theorem 2.13. *For each n, $\widehat{t}(K_n) \geq \frac{7n-17}{4}$.*

Proof. We prove first that $\widehat{t}(K_{4k}) \geq 7k-3$ for every $k \geq 1$. The general bound is then implied by Theorems 2.11 and 2.12.

Let $n = 4k$ for $k \geq 1$, let $G = K_n$, and let $V(G) = \{0, 1, \ldots, n-1\}$. We define a colouring of the edges of G in the following manner: The colour of an edge joining vertices $4i+r$ and $4j+s$ with $0 \leq i < j \leq k$ and $r, s \in \{0, 1, 2, 3\}$ is given in Table 1, the colour of an edge joining vertices $4i+r$ and $4i+s$ with $0 \leq i \leq k$ and $0 \leq r < s \leq 3$ is given in Table 2. It is easy to observe that it is an (improper) interval colouring of G.

□

Note that, for proper interval colourings, it is not known whether the sequence $\{t(K_n)\}_{n=1}^{\infty}$ is monotone; also, any value of n for which $t(K_{n-1}) > t(K_n)$ yields that $\widehat{t}(K_n) \geq t(K_n) + 2$.

To conclude this part, we list exact values of $\widehat{t}(K_n)$ for some small values of n, see Table 3.

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n	2	3	4	5	6	7	8	9	10	11	12
lower bound from Theorem 2.13			4	5	7	8	11	12	14	15	18
$\widehat{t}(K_n)$	1	2	4	5	7	8	11	12	14	16	18
upper bound from Theorem 2.10				5	7	9	11	13	15	17	19

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Authors' addresses: *Peter Hudák*, Institute of Mathematics, Faculty of Sciences, University of P. J. Šafárik, Jesenná 5, 040 01 Košice, Slovak Republic e-mail: `hudakpeterr@gmail.com`. *František Kardoš*, LaBRI, University of Bordeaux, 351, cours de la Libération, 33405 Talence, France e-mail: `frantisek.kardos@labri.fr`. *Tomáš Madaras*, Institute of Mathematics, Faculty of Sciences, University of P. J. Šafárik, Jesenná 5, 040 01 Košice, Slovak Republic e-mail: `tomas.madaras@upjs.sk`. *Michaela Vrbjarová*, Institute of Mathematics, Faculty of Sciences, University of P. J. Šafárik, Jesenná 5, 040 01 Košice, Slovak Republic e-mail: `michaela.vrbjarova@student.upjs.sk`.