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► **To cite this version:**

Christine Chevallereau, Rabah Rabah. The Singular Value Decomposition and the Control of a Robot in Singular Configuration . CESA'96, IEEE-SMC-IMACS Multiconference, Symposium on Robotics and Cybernetics, Ecole Centrale de Lille, Jul 1996, Lille, France. pp.339-342. hal-01415991

HAL Id: hal-01415991

<https://hal.science/hal-01415991>

Submitted on 15 Dec 2016

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The Singular Value Decomposition and the Control of a Robot in Singular Configuration*

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Abstract

When a robot is at a singular configuration, i.e. when the Jacobian matrix of the system is not invertible, the classical control laws using the inverse or pseudo-inverse matrix of the Jacobian cannot be used. Nevertheless some trajectories can be followed with a finite input even along the degenerate direction. These trajectories, namely the feasible trajectories, are described and studied using the Singular Value Decomposition of the Jacobian matrix. This decomposition allows also to describe and to design the open loop control law which permits to leave the singular position. In a similar way a closed loop control law is given.

1 Introduction

The task of a robot is naturally described in the Cartesian space. The control is applied in the joint space. The relation between the Cartesian and the joint coordinates is given by the relation $x(t) = f(q(t))$, where $x = [x_1 \ x_2 \ \dots \ x_n]$, $q = [q_1 \ q_2 \ \dots \ q_n]$ are respectively the Cartesian coordinate of the end effector and the joint coordinate. The study is limited to the non-redundant robot. To define the joint velocity which is the control, we use the velocity model: $\dot{x} = J(q)\dot{q}$, where $J(q)$ is the Jacobian matrix of the function f . The functions \dot{x} and \dot{q} take values in \mathbb{R}^n .

The joint velocity $\dot{q}(t)$ may be obtained from the desired velocity $\dot{x}(t)$ using the relation:

$$\dot{q} = J^+(q)\dot{x}, \quad (1)$$

where $J^+(q)$ is the pseudo-inverse of the Jacobian matrix. At a regular configuration $J^+(q) = J^{-1}(q)$. A singular configuration, or more simply a *singularity*, is a configuration q_0 where the matrix $J(q_0)$ loses rank. In the corresponding position in the Cartesian space some directions, namely the directions which lies outside the image space of J , can not be followed using relation (1). The directions which are orthogonal to $\text{Im } J$ are called *degenerate directions*.

*CESA'96, IEEE-SMC-IMACS Multiconference, Symposium on Robotics and Cybernetics, July 1996, Lille, pp. 339-342. ISBN : 2-9510266-1-7.

Hence, relation (1) yields zero if the desired velocity is along a degenerate direction. As a result, the robot stops at the singularity. Nevertheless, as it has been shown by many authors [1, 4, 5, 7], some motions of the robot effector are possible from the singularity along the degenerate direction. Those motions are *feasible trajectories* (Cartesian path together with a timing law), i. e. trajectories which may be generated by finite joint velocities. Our purpose is to use the Singular Value Decomposition (SVD) of the matrix J to describe the degenerate direction, to characterize the feasible trajectories and to define the control law at the singularity. This control law is calculated in such a way that no discontinuity occurs.

In the first part of this paper we present an example where, in a particular case, it is shown how to leave the singularity along a feasible trajectory with a suitable control law. Then we give a presentation of the SVD. This decomposition permits an analysis of feasible trajectories. The last sections will be devoted to the calculation of the control law using the singular values. In Section 4 the main result is given: the open loop control law is expressed using the SVD in such a way that the control is continuous at and near the singular position. The Section 5 is concerned with the closed loop control law: an adequate gain is given. In the last section some simulation results are discussed.

We consider a singular position with one degenerate direction, the rank of $J(q)$ at the singularity is $n - 1$.

2 Preliminaries

First we consider an example. It permits to see how the method works in a particular case.

Example

Our example is taken from [2]. We consider a two rotative joint planar robot with link lengths $L_1 = 1$ and $L_2 = r$. The control inputs are the joint velocities w_1 and w_2 . The direct geometric relation between the Cartesian coordinates and the joint coordinates is given by:

$$\begin{cases} x_1 &= \cos q_1 + r \cos (q_1 + q_2) \\ x_2 &= \sin q_1 + r \sin (q_1 + q_2). \end{cases} \quad (2)$$

As $\dot{q}_i = w_i$, differentiating (2) we get the velocity model:

$$\begin{cases} \dot{x}_1 &= -w_1 \sin q_1 - r(w_1 + w_2) \sin(q_1 + q_2). \\ \dot{x}_2 &= w_1 \cos q_1 + r(w_1 + w_2) \cos(q_1 + q_2). \end{cases} \quad (3)$$

When $q_1 = q_2 = 0$ we have a singularity. In this singular configuration one has

$$\begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= (1+r)w_1 + rw_2. \end{cases} \quad (4)$$

In the direction x_1 , at the singularity, no velocity can be created. But the acceleration may be created using the control w_1 and w_2 :

$$\begin{cases} \ddot{x}_1 &= -w_1^2 - r(w_1 + w_2)^2 \\ \ddot{x}_2 &= (1+r)w_1 + rw_2. \end{cases} \quad (5)$$

If we assume a perfect tracking of a given trajectory $\xi = (\xi_1, \xi_2)$, then from (2) and (3) a simple calculation yields to:

$$\begin{aligned} w_1 &= \frac{r\dot{\xi}_1 \cos(q_1 + q_2) + (\xi_2 - \sin q_1)\dot{\xi}_2}{\xi_2 \cos q_1 - \xi_1 \sin q_1}, \\ w_2 &= \frac{-\xi_1 \dot{\xi}_1 - \xi_2 \dot{\xi}_2}{\xi_2 \cos q_1 - \xi_1 \sin q_1} \end{aligned} \quad (6)$$

out of the singularity.

At the singularity, the expression of w_1 and w_2 are indetermined because $\sin q_1 = 0$ and (2) gives $\xi_2 = 0$.

To leave the singularity we can choose a trajectory such that the desired first Cartesian coordinate ξ_1 satisfies the following conditions:

$$\xi_1(0) = 1 + r, \quad \dot{\xi}_1(0) = 0, \quad \ddot{\xi}_1(0) = \gamma_0 < 0.$$

Then one can take $\xi_1(t) = 1 + r + t^2\alpha(t)$, where $\alpha(t)$ is a differentiable function which may be chosen such that $\alpha(0) < 0$ and according to the desired final condition at a fixed time T . The initial time is $t = 0$.

From equation (2) we see that the second desired coordinate may be written as: $\xi_2(t) = t\beta(t)$ with a desired continuous function $\beta(t)$.

Taking into account the given desired trajectory (ξ_1, ξ_2) and after some simplifications we get finite values for w_1 and w_2 at the singularity. Moreover the expression are obtained by continuity. Indeed the functions $\sin q_1, \cos q_1$ and $\cos(q_1 + q_2)$ may be calculated from (2) via the desired trajectory. The function $\sin q_1$ may be expressed by $\sin q_1 = ts(t)$ with a smooth function $s(t)$ such that $s(0) \neq 0$. Then from (6) and after a simplification of the common factor t in the numerator and in the denominator, we get:

$$\begin{aligned} w_1(t) &= \frac{r(2\alpha(t) + t\dot{\alpha}(t)) \cos(q_1 + q_2) + (t\beta(t) - ts(t))\dot{\xi}_2}{\beta(t) \cos q_1 - (1+r+t^2\alpha(t))s(t)}, \\ w_2(t) &= \frac{-(1+r+t^2\alpha(t))(2\alpha(t) + t\dot{\alpha}(t)) - \beta(t)(\beta(t) + t\dot{\beta}(t))}{\beta(t) \cos q_1 - (1+r+t^2\alpha(t))s(t)}. \end{aligned}$$

We can see that no more indetermination occurs at $t = 0$. Hence, the control is well-defined at the singularity:

$$\begin{aligned} w_1 &= \frac{2r\alpha(0)}{\beta(0) + (1+r)s(0)}, \\ w_2 &= \frac{-2\alpha(0)(1+r) - \beta^2(0)}{\beta(0) + (1+r)s(0)}. \end{aligned}$$

In other words, for the given feasible trajectory, it is possible to calculate the corresponding control law.

The simplifications made here are very particular. What is the situation in a more general case, this will be the main subject of this paper. In order to see how the control law may be calculated in a more general case, we use the SVD.

The Singular Value Decomposition

The SVD is a powerful tool in several domains (cf. [6, 10]). Our purpose is to use this decomposition for the matrix $J(q)$ in order to describe degenerate direction and to compute the needed control law in a singular configuration along a degenerate direction. The matrix J may be decomposed as follows:

$$J = U\Sigma V^*, \quad \Sigma = \text{diag} [\sigma_1, \dots, \sigma_n],$$

the matrices $U = [u_1 \ u_2 \ \dots \ u_n]$ and $V = [v_1 \ v_2 \ \dots \ v_n]$ are unitary, σ_i are taken in decreasing order, σ_n being the smallest singular value. The vectors u_i and $v_i, i = 1, \dots, n$ are the eigenvectors of matrices JJ^* and J^*J respectively. The matrices U, V and Σ depend on $q(t)$. At the studied singularity, $t = 0$, we have $\sigma_n = 0, \sigma_i > 0$ for $i < n$ according to our assumption that rank deficiency of J is 1. At the singularity we denote the corresponding values of U and V by $U(0)$ and $V(0)$, instead of $U(q(0))$ and $V(q(0))$ since no confusion occurs. The corresponding column vectors are $u_i(0)$ and $v_i(0)$. The degenerate direction is given by the vector $u_n(0)$.

3 Feasible Trajectories

Here we give an analysis of the feasible trajectories using the SVD. Let us make a change of variables. Let us put $\dot{y} = U^*(0)\dot{x}$. Then $\dot{y} = U^*(0)Jw$, where $w = \dot{q}$ is the control. Hence, $\dot{y}_n = u_n^*(0)Jw$. At the singularity, $\dot{y}_n(0) = 0$. Let $L(q)$ be the row vector $u_n^*(0)J(q)$. That is $\dot{y}_n = Lw$. By differentiating, we get

$$\ddot{y}_n = \dot{L}w + L\dot{w} = w^* \partial L w + L\dot{w},$$

where

$$\partial L = \frac{\partial L}{\partial q} = \left[\frac{\partial l_j}{\partial q_i} \right]_{i,j=1}^n,$$

l_j being the components of L . At the singularity $L = 0$ and then $\ddot{y}_n(0) = w^*(0)\partial L(0)w(0)$.

From the velocity model and using the SVD, at the singularity, w may be expressed as

$$w(0) = J^+(0)\dot{x}(0) + zv_n(0),$$

where z is an arbitrary scalar and $J^+(0) = J^+(q(0))$. This yields to

$$\ddot{y}_n(0) = az^2 + \dot{x}^*(0)Bz + \dot{x}^*(0)C\dot{x}(0), \quad (7)$$

with well-defined constants a, B and C :

$$a = v_n^* \partial L v_n, \quad B = J^{+*} (\partial L + \partial L^*) v_n, \quad C = J^{+*} \partial L J^+,$$

calculated at $t = 0$. The feasible trajectories in the degenerate direction may be calculated from (7).

Proposition 3.1 *If $a \neq 0$, a trajectory is feasible along the degenerate direction. The desired acceleration may be chosen according to the sign of a .*

PROOF. For a desired trajectory along the degenerate direction, $\dot{x}(0)$ is orthogonal to $\text{Im } J$ and then $J^+ \dot{x}(0) = 0$. This implies that $\ddot{y}_n(0) = az^2$. Hence, there exists z which allows to choose $\ddot{y}_n(0)$ if this value has the same sign as a . So, there exists a control giving this acceleration. ■

An analogous result may be obtained for the projection of the acceleration on the degenerate direction. A simple calculation shows that the feasible trajectories are such that:

$$\begin{aligned} \ddot{y}_n(0) &\geq \frac{-(\dot{x}^*(0)B)^2 + 4a\dot{x}^*(0)C\dot{x}(0)}{4a} & \text{if } a > 0, \\ \ddot{y}_n(0) &\leq \frac{-(\dot{x}^*(0)B)^2 + 4a\dot{x}^*(0)C\dot{x}(0)}{4a} & \text{if } a < 0. \end{aligned}$$

Proposition 3.2 *If $a = 0$ a trajectory along the degenerate direction is not feasible with a non-zero initial acceleration.*

PROOF. If $a = 0$ one can choose $\ddot{y}_n(0)$ only if $\dot{x}^*(0)B \neq 0$. This implies $\dot{x}^* J^{+*}(0) \neq 0$ and then the tangent of the feasible trajectories are not along the degenerate direction.

If $a = 0$ and $\dot{x}^*(0)B = 0$ the acceleration cannot be chosen. For $C \neq 0$, it is imposed by the desired trajectory. If $C = 0$ the acceleration is zero along the degenerate direction for all possible trajectories. In this special case, we have to calculate the third derivative along the degenerate direction and to study the possibility of a choice of a jerk along the degenerate direction. The approach developed here may be extended in an analogous way. For more details see [1]. ■

4 Open Loop Control Law

Several methods were proposed to design a control law in a singular configuration. In [4] and [8], a motion in the kernel of J is proposed, however, if $a \neq 0$ those solutions are not very satisfying. Other authors define the control law near the singularity [3, 9] or only on the singularity [12].

Our method gives the control law on and near the singularity without discontinuity.

We consider the case $a \neq 0$. The other cases may be considered also in an analogous way.

Suppose that a desired sufficiently smooth feasible trajectory is given: $x(t) = \xi(t)$. Let us put $\dot{\eta}_n(t) = u_n^*(0)\dot{\xi}(t)$. As the initial velocity along the degenerate direction is zero, $\dot{\eta}_n(0) = 0$, one can take $\dot{\eta}_n(t) = t\alpha(t)$. The function $\alpha(t)$ is defined by

$$\begin{aligned} \alpha(t) &= \frac{\dot{\eta}_n(t)}{t}, \quad t > 0; \\ \alpha(0) &= \lim_{t \rightarrow +0} \alpha(t) = \ddot{\eta}_n(0). \end{aligned}$$

By construction $\alpha(t)$ is continuous. Under a smoothness condition, the same considerations may be used to write the smallest singular value σ_n and the corresponding vector u_n . We have $\sigma_n(t) = t\beta(t)$, where β is defined by:

$$\begin{aligned} \beta(t) &= \frac{\sigma_n(t)}{t}, \quad t > 0; \\ \beta(0) &= \lim_{t \rightarrow +0} \beta(t) = \dot{\sigma}_n(0), \end{aligned}$$

and $u_n(t) = u_n(0) + t\bar{u}_n(t)$, where $\bar{u}_n(t)$ is defined by:

$$\bar{u}_n(t) = \frac{u_n(q(t)) - u_n(0)}{t}, \quad \bar{u}_n(0) = \dot{u}_n(t).$$

Let us denote by ω the vector V^*w . Then we have the following result.

Theorem 4.1 *Assume that $\dot{\sigma}_n(0) \neq 0$. Then the following control law permits to leave the singularity along the degenerate direction with the desired trajectory $\xi(t)$: $w = V\omega$, with*

$$\omega_i(t) = \frac{u_i^*(q(t))\dot{\xi}(t)}{\sigma_i(t)}, \quad i < n, \quad t \geq 0, \quad (8)$$

$$\begin{aligned} \omega_n(t) &= \frac{u_n^*(q(t))\dot{\xi}(t)}{\sigma_n(t)}, \quad t > 0, \\ \omega_n(0) &= \frac{\alpha(0) + \bar{u}_n^*(0)\dot{\xi}(0)}{\beta(0)}. \end{aligned}$$

PROOF. For $t > 0$, one has $\dot{\xi} = U\Sigma V^*w = U\Sigma\omega$, and then $\omega = \Sigma^{-1}U^*\dot{\xi}$, which gives

$$\omega_i(t) = \frac{u_i^*(q(t))\dot{\xi}(t)}{\sigma_i(t)}, \quad i = 1, \dots, n, \quad t > 0.$$

At $t = 0$ the control is defined by continuity. For $i < n$, one has $\sigma_i(0) \neq 0$ and then we obtain (8). In order to define $\omega_n(0)$, we use the special form of σ_n and η_n introduced at the beginning of this section. This gives

$$\omega_n(t) = \frac{t\alpha(t) + t\bar{u}_n^*(t)\dot{\xi}(t)}{t\beta(t)} = \frac{\alpha(t) + \bar{u}_n^*(t)\dot{\xi}(t)}{\beta(t)}.$$

Hence, as $t \rightarrow 0$, we obtain

$$\omega_n(0) = \frac{\alpha(0) + \bar{u}_n^*(0)\dot{\xi}(0)}{\beta(0)}.$$

The control is then given by $w = V\omega$. ■

If the smoothness assumptions are satisfied, which is the case for a large class of systems, the given open loop control permits to leave the singularity and no discontinuity occurs. Note that this assumption may be checked via the calculation of $\frac{\partial \sigma}{\partial q}$. The condition $\dot{\sigma}_n(0) \neq 0$ is satisfied if the trajectory is feasible with a non zero initial acceleration. In fact we need that $\sigma_n(t)$ is not identically zero in a neighborhood of 0, this means that the desired feasible Cartesian trajectory is at a regular position for $t > 0$. If the desired trajectory is at a singular position for all $t \geq 0$, then one can use the pseudo-inverse to track this trajectory because this one belong to the image of J (see [11]).

5 Closed Loop Control Law

The open loop permits a perfect tracking of a trajectory but needs an exact knowledge of the model and of the current configuration and joint velocity. In fact some errors may occur. In this situation we can use a closed loop control law. This control law, at regular position, may be derived by: $w = J^{-1}(\xi - K(\xi - x))$, where ξ is the desired trajectory, x is the real trajectory and the gain K is defined as follows:

$$K(t) = U(0) \text{diag}[k_1, k_2, \dots, k_n] U^*(0),$$

with constant k_i , $i < n$ and $k_n(t)$ given by:

$$k_n(t) = \begin{cases} ct & 0 \leq t < T \\ cT & t \geq T. \end{cases}$$

This expression of the scalar gain k_n is chosen such that at a singular configuration $k_n(0) = 0$. The assumption that the desired velocity along the degenerate direction vanishes at $t = 0$ is necessary to define the control law. In the same way we have to assume that the projection of $K(\xi - x)$ along the degenerate direction also vanishes and the design of k_n is made in order to satisfy this assumption.

A simple calculation yields to an explicit expression of the control law near and at the singularity. Recall that $w = V\omega$ and ω is expressed, for all $0 \leq t \leq T$, as follows:

$$\omega_i(t) = \frac{u_i^*(q(t)) (\dot{\xi}(t) + K(t)(\xi(t) - x(t)))}{\sigma_i(t)}, \quad i < n,$$

$$\omega_n(t) = \frac{\alpha(t) + \bar{u}_n^*(t)\dot{\xi}(t)}{\beta(t)} + \frac{cu_n^*(0)(\xi(t) - x(t)) + \bar{u}_n^*(t)K(t)(\xi(t) - x(t))}{\beta(t)}.$$

For $t > T$, as the configuration is not singular, the classical closed loop control law can be used.

6 Simulation

The proposed control has been tested in simulation for the two link robot of our example above. The parameters are $L_1 = L_2 = 1$. Its initial (singular) position is in full extension: $q_1 = q_2 = 0$. The desired motion is along the degenerate direction. At $t = 0$ the velocity is zero and the acceleration is not zero. Setting in our example $\alpha(t) = -1$ and $\beta(t) = 0$ for all $t \geq 0$ we can calculate the needed control law for this desired trajectory: $\xi_1(t) = 2 - t^2$ and $\xi_2(t) = 0$. Without initial error the open loop control law gives a very good tracking of the desired trajectory. The Cartesian errors are near zero.

If some error occurs in the initial joint position, say $q_1 = 0$ and $q_2 = 10^{-3}$ radians, a Cartesian error exists if we use the open loop control law. Using the closed loop control law, we can reduce the Cartesian errors and we obtain a very good tracking of the desired trajectory.

7 Conclusion

Using the Singular Value Decomposition, we gave a general framework for the analysis of the feasible trajectories and the design of the open loop control law in a singular configuration. This method can also be developed in order to reach a singularity by a feasible trajectory which ends at a singular configuration. Moreover the given method is also used to define a closed loop control law with a time-varying gain. This control law is useful because some errors may occur in the initial joint position for example.

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