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Subsampling for nonstationary time series with non-zero mean function

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Abstract

In this paper a subsampling approach for nonstationary time series with a non-zero mean function is proposed. It is applied for periodically and almost periodically processes. Two statistical tests are constructed. An example with real data is presented.

Keywords: almost periodically correlated, autocovariance function, Fourier coefficients, periodicity, testing

MSC 2010: 62G09, 62G10, 62M10

1 Introduction and problem formulation

In this paper we discuss a subsampling approach for nonstationary time series, which have non-zero mean function.

As first we introduce some notation. Let $\{X_t : t \in \mathbb{Z}\}$ be a nonstationary time series and $\mu(t) = E(X_t)$ be its mean function. We assume that $\mu(t) \not\equiv 0$. Moreover, we consider the following decomposition of the mean function

$$\mu(t) = \mu_1(t) + \mu_2(t).$$

Functions $\mu_1(t)$ and $\mu_2(t)$ can be interpreted differently depending on the application. In economics they can represent e.g., seasonal or business fluctuations or a long-term trend pattern, while in telecommunications these can be two periodic functions e.g., the first one may be a pilot tone and the latter one the mean function of some periodic signal. Pilot tone is a signal transmitted for control, synchronization or reference purposes. In the following we assume that $\mu_1(t)$ is non-zero and we know how to estimate it. On the other hand, $\mu_2(t)$ may be a zero function and moreover we may not have complete knowledge of its form and properties. Such a model describes a real data situation in which only partial information about the mean function is available e.g., we know the nature of some structure contained in the data like seasonality, but other components require further investigation. The observed sample we denote by $\mathbf{X}_n = (X_1, X_2, \dots, X_n)$. Let $\hat{\mu}_{1,n}(t)$ be the estimator of $\mu_1(t)$ based on \mathbf{X}_n . We consider the modified sample $\tilde{\mathbf{X}}_n = (\tilde{X}_{1,n}, \tilde{X}_{2,n}, \dots, \tilde{X}_n)$, where

$$\tilde{X}_{t,n} = X_t - \hat{\mu}_{1,n}(t).$$

In the sequel, for the sake of simplicity we skip the subscript n and we denote elements of the triangular array $\tilde{\mathbf{X}}_n$ by $\tilde{X}_i, i = 1, \dots, n$. From now we will be using only the sample $\tilde{\mathbf{X}}_n$ i.e., we consider partially or entirely demeaned data. This corresponds to the standard procedures of data analysis in which the data is assumed to be demeaned before any further analysis (e.g. second-order) is performed.

In the next section we present how the subsampling idea for nonstationary time series (see chapter 4 in [9]) can be adapted to our setting. In Section 3 we construct two tests designed for time series with periodic and almost periodic structure. The first one is devoted to the first-order frequency detection, while the second one can be used for testing the second-order periodicity. Finally, Section 4 contains a real data application of our results.

2 Subsampling

In the sequel we use notation introduced by [9].

Let $\theta = \theta(\mathcal{P}) \in \mathbb{R}$ be the parameter of interest, where \mathcal{P} is the joint probability law governing $\{X_t, t \in \mathbb{Z}\}$. Parameter θ can simply relate to $\mu_2(t)$, but one may consider other characteristics of X_t . Moreover, let $\hat{\theta}_n = \hat{\theta}_n(\tilde{\mathbf{X}}_n)$ be the estimator of θ and $\hat{\theta}_{n,b,t} = \hat{\theta}_{n,b,t}(\tilde{X}_{t,n}, \tilde{X}_{t+1,n}, \dots, \tilde{X}_{t+b-1,n})$ be the estimator of θ based on subsample $\tilde{X}_t, \tilde{X}_{t+1}, \dots, \tilde{X}_{t+b-1}$. In addition, let $\hat{\theta}'_{n,b,t} = \hat{\theta}'_{n,b,t}(X'_t, X'_{t+1}, \dots, X'_{t+b-1})$, where $X'_i = X_i - \mu_1(i)$, $i \in \mathbb{Z}$.

By $J'_{b,t}(P)$ and $J_n(P)$ we denote the sampling distributions of $\tau_b(\hat{\theta}'_{n,b,t} - \theta)$ and $\tau_b(\hat{\theta}_n - \theta)$, where τ_b is an appropriate normalizing sequence. The corresponding cumulative distribution functions are defined as

$$J'_{b,t}(x, P) = P \left\{ \tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x \right\} \quad (1)$$

and

$$J_n(x, P) = P \left\{ \tau_b(\hat{\theta}_n - \theta) \leq x \right\}. \quad (2)$$

Finally, following [9] we assume that X_t is α -mixing.

Conditions below are the sufficient conditions for the subsampling consistency.

Assumption 2.1 *There exist a limiting law $J(P)$ such that*

- (i) $J_n(P)$ converges weakly to $J(P)$ as $n \rightarrow \infty$.
- (ii) For any continuity point x of $J(P)$ and for any sequences n, b with $n, b \rightarrow \infty$ and $b/n \rightarrow 0$, we have $1/q \sum_{t=1}^q J'_{b,t}(x, P) \rightarrow J(x, P)$, where $q = n - b + 1$.
OR
For any continuity point x of $J(\cdot, P)$ and for any index sequence $\{t_b\}$ we have that $J'_{b,t_b}(x, P) \rightarrow J(x, P)$ as $b \rightarrow \infty$.

- (iii) For any index sequence $\{t_b\}$ we have that $\tau_b \left| \hat{\theta}_{n,b,t_b} - \hat{\theta}'_{n,b,t_b} \right| \xrightarrow{P} 0$.

One may notice that comparing to [9] (see Assumption 4.2.1. in [9]) we need an additional assumption (iii). It is caused by the fact that in contrary to [9] we are using partially demeaned data. Finally, condition (i) is identical to the one proposed by Politis et al., while (ii) is adjusted to our problem.

The approximation of $J_n(x, P)$ is defined as

$$\tilde{L}_{n,b}(x) = \frac{1}{q} \sum_{t=1}^q \mathbf{1} \left\{ \tau_b \left(\hat{\theta}_{n,b,t} - \hat{\theta}_n \right) \leq x \right\},$$

where $\mathbf{1}\{A\}$ is an indicator function of the event A .

Below we state asymptotic validity of subsampling for general statistics.

Theorem 2.1 *Assume that X_t is α -mixing. Under Assumption 2.1 and taking $\tau_b/\tau_n \rightarrow 0$, $b = b(n) \rightarrow \infty$ such that $b/n \rightarrow 0$ as $n \rightarrow \infty$, we have*

- (i) if x is a continuity point of $J(\cdot, P)$, then $\tilde{L}_{n,b}(x) \xrightarrow{P} J(x, P)$,
- (ii) if $J(\cdot, P)$ is continuous, then $\sup_{x \in \mathbb{R}} |\tilde{L}_{n,b}(x) - J(x, P)| \xrightarrow{P} 0$,
- (iii) For $\alpha \in (0, 1)$, let $c(1-\alpha) = \inf\{x : J(x, P) \geq 1-\alpha\}$ and $\tilde{c}_{n,b}(1-\alpha) = \inf\{x : \tilde{L}_{n,b}(x) \geq 1-\alpha\}$. If $J(\cdot, P)$ is continuous at point $c(1-\alpha)$, then subsampling confidence intervals are asymptotically consistent i.e.,

$$P\left(\tau_n(\hat{\theta}_n - \theta) \leq \tilde{c}_{n,b}(1-\alpha)\right) \rightarrow 1-\alpha. \quad (3)$$

Remark: In our methodology we applied subsampling to partially de-measured data i.e., to $\tilde{\mathbf{X}}_n$. However, an alternative approach is possible. One may consider the usual subsampling and use subsamples $(X_t - \hat{\mu}_{1,n,b,t}(t), X_{t+1} - \hat{\mu}_{1,n,b,t}(t+1), \dots, X_{t+b-1} - \hat{\mu}_{1,n,b,t}(t+b-1))$, where $\hat{\mu}_{1,n,b,t}(\cdot)$ is the estimator of the function $\mu_1(\cdot)$ based on $(X_t, X_{t+1}, \dots, X_{t+b-1})$. In such cases the subsampling consistency can be obtained under the sufficient condition presented in [9] (see Assumption 4.2.1). However, one should be aware that the estimator $\hat{\mu}_{1,n,b,t}(\cdot)$ may have much higher variance than $\hat{\mu}_{1,n}(t)$, especially when b is small.

3 Applications

In this section we discuss possible applications of our subsampling approach. We focus on periodically correlated (PC) and almost periodically correlated (APC) time series, which are used for modeling signals in many different fields like economics, telecommunication, mechanics, vibroacoustics and hydrology (see e.g. [3], [7], [8]). Time series X_t is called PC with period T if it has periodic mean and autocovariance functions, i.e.

$$\mu(t) = \mu(t+T) \quad \text{and} \quad B(t, \tau) = \text{Cov}(X_t, X_{t+\tau}) = \text{Cov}(X_{t+T}, X_{t+\tau+T})$$

for each $t, \tau \in \mathbb{Z}$. For more details we refer the reader to [3]. Sometimes it may happen that period length is not known or a considered signal is a composition of two components with incommensurable periods. To model such data APC time series are used (see e.g. [7]). They are generalization of PC sequences. A time series that has finite second order moments is called APC if its mean and autocovariance functions are almost periodic in time. Finally, a function $f(t) : \mathbb{Z} \rightarrow \mathbb{R}$ of an integer variable is called almost periodic if for any $\epsilon > 0$ there exists an integer $L_\epsilon > 0$ such that among any L_ϵ consecutive integers there is an integer p_ϵ with the property

$\sup_{t \in \mathbb{Z}} |f(t + p_\epsilon) - f(t)| < \epsilon$ (see [2] for more details).

Analysis of PC and APC processes is often performed in the frequency domain. In such cases the following Fourier representations of the mean and the autocovariance functions are used

$$\mu(t) \sim \sum_{\psi \in \Psi} m(\psi) e^{i\psi t}, \quad B(t, \tau) \sim \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i\lambda t} \quad (4)$$

for any $\tau \in \mathbb{Z}$, where

$$m(\psi) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \mu(t) e^{-i\psi t}, \quad a(\lambda, \tau) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n B(j, \tau) e^{-i\lambda j}$$

and the sets $\Psi = \{\psi \in [0, 2\pi) : m(\psi) \neq 0\}$ and $\Lambda_\tau = \{\lambda \in [0, 2\pi) : a(\lambda, \tau) \neq 0\}$ are countable. Moreover, in the case of PC time series $\Psi, \Lambda_\tau \subseteq \{2k\pi/T, k = 0, 1, \dots, T-1\}$ for any $\tau \in \mathbb{Z}$. In the sequel we assume that the sets Ψ and Λ_τ are finite. Then in (4) we have equalities.

3.1 Testing significance of a frequency in mean function

Let us recall that we consider $\mu(t) = \mu_1(t) + \mu_2(t)$ and we know the estimator of $\mu_1(t)$. Moreover, functions $\mu_1(t)$ and $\mu_2(t)$ have the following Fourier expansions $\mu_1(t) = \sum_{\psi \in \Psi_1} m(\psi) e^{i\psi t}$ and $\mu_2(t) = \sum_{\psi \in \Psi_2} m(\psi) e^{i\psi t}$, where $\Psi = \Psi_1 \cup \Psi_2$, the set Ψ_1 is known and $\Psi_1 \cap \Psi_2 = \emptyset$. Then the natural estimator of $\mu_1(t)$ has the form (see e.g. [3])

$$\hat{\mu}_{1,n}(t) = \sum_{\psi \in \Psi_1} \left[\frac{1}{n} \sum_{j=1}^n X_j e^{-i\psi j} \right] e^{i\psi t}. \quad (5)$$

Our aim is to test if for some frequency $\psi_0 \notin \Psi_1$ we have $\psi_0 \in \Psi_2$. This problem can be equivalently expressed in terms of the corresponding Fourier coefficients i.e., we consider $H_0 : |m(\psi_0)| = 0$ vs $H_1 : |m(\psi_0)| > 0$. In this case $\theta = |m(\psi_0)|$ and the test statistics T_n based on $\tilde{\mathbf{X}}_n$ is of the form

$$T_n = \sqrt{n} |\hat{m}_n(\psi_0)| = \frac{1}{\sqrt{n}} \left| \sum_{j=1}^n (X_j - \hat{\mu}_{1,n}(j)) e^{-i\psi_0 j} \right|.$$

Large values of T_n suggests that H_1 is true. The quantiles of T_n can be approximated using the subsampling method. For that purpose below we state the subsampling validity for $|m(\psi_0)|$.

Theorem 3.1 *Assume that the assumptions of Theorem 2.1 from [4] hold. Then the subsampling based on $\tilde{\mathbf{X}}_n$ is consistent for $\theta = |m(\psi_0)|$ i.e., Theorem 2.1 holds.*

3.2 Testing if $B(t, \tau)$ is periodic

In this section we discuss the problem of testing periodicity of the autocovariance function in the class of PC time series with known period T . Note that in such a class of time series we have $\mu_2(t) \equiv 0$. Moreover, $\hat{\mu}_{1,n}(t)$ is the estimator of the periodic function $\mu(t)$ and is given by formula (5), where $\Psi_1 = \{2k\pi/T : k = 0, 1, 2, \dots, T-1\}$. To simplify the notation we will write $\hat{\mu}_n(t)$ instead of $\hat{\mu}_{1,n}(t)$. We take $\tau \in \mathbb{Z}$ and we consider the following test $H_0 : B(t, \tau)$ is constant in $t \in \mathbb{Z}$ vs $H_1 : B(t, \tau)$ is periodic in t . Note that H_0 can be now equivalently rewritten as $H_0 : |a(\lambda, \tau)| = 0$ for all $\lambda \in \{1, 2, \dots, T-1\}$. Assume that $\tau \geq 0$. The estimator of $a(\lambda, \tau)$ based on $\tilde{\mathbf{X}}_n$ has the form

$$\hat{a}_n(\lambda, \tau) = \frac{1}{n} \sum_{t=1}^{n-\tau} (X_t - \hat{\mu}_n(t))(X_{t+\tau} - \hat{\mu}_n(t+\tau))e^{-i\lambda t}. \quad (6)$$

Therefore we consider the following test statistics $U_n = \sqrt{n} \sum_{\lambda \in \mathcal{A}} |\hat{a}_n(\lambda, \tau)|$, where $\mathcal{A} = \{2k\pi/T, k = 1, \dots, T-1\}$. Large values of U_n suggests that H_1 is true. To approximate quantiles of U_n , we show subsampling consistency.

Theorem 3.2 *Assume that the assumptions of Theorem 2 of [5] hold. Then the subsampling based on $\tilde{\mathbf{X}}_n$ is consistent for $\theta = \sum_{\lambda \in \mathcal{A}} |a(\lambda, \tau)|$ i.e., Theorem 2.1 holds.*

4 Real data example

Below we present the real data application of the testing problem described in Section 3.1. For that purpose we use data published by Eurostat, concerning monthly industrial production (percentage change on previous period) for *Mining and quarrying; manufacturing; electricity, gas, steam and air conditioning supply* in three countries: France, Germany and Italy. In the first row of Figure 1 are visualized gross data i.e., data that were not modified in any way before publishing (neither seasonally nor calendar adjusted). The length of the analyzed sequence for each country is $n = 288$ (January 1992 - December 2015).

The considered data are known to contain two types of periodic structure. The first one represents the typical seasonal effect with period length equal to 12 months. The latter one is a consequence of the fact that in each month the number of working days can be different. As the result an additional period equal to 4800 months (400 years) is observed (see chapter 11.2 in [1]). The main theoretical frequency corresponding to the trading-day effect is $\psi_0 \cong 2.18733$ for the monthly data set. Our aim is to test its presence using tools introduced in Section 3.1. The second and the third row of Figure 1 contain the estimated values of $T_n = \sqrt{n}|\hat{m}_n(\psi)|$, $\psi \in [0, \pi]$ obtained using the raw data and partially demeaned data $\tilde{\mathbf{X}}_n$, respectively. One may note that in a case of the gross data ψ_0 is very close to the strong true frequency $\psi = 2\pi/3$. That fact could have a serious impact on the results of our test if we used simply the raw data. To check that we decided to apply our methodology to the data obtained after removing the seasonal effect and additionally to the raw data (based on [4]). Below we discuss results of both approaches. We set the significance level $\alpha = 5\%$ and subsample length $b \in \{15, 20, \dots, 40\}$. The obtained values of the test statistics for the raw data are: 0.397 (Germany), 0.371 (France) and 0.497 (Italy). In the case of partially demeaned data corresponding values are a bit lower: 0.382 (Germany), 0.303 (France) and 0.411 (Italy). For each considered country values of the subsampling quantiles are the highest for $b = 15$ and decrease when b increases. For example in the case of Germany taking $b = 15$ we got a subsampling quantile equal to 0.117 (raw data) and 0.06 (partially demeaned data). Corresponding values for $b = 40$ are: 0.094 and 0.042. Moreover, for the gross data quantiles have much higher values than those for the partially demeaned data. As a result in the case of France and Italy values of the subsampling quantiles for the raw data are close to the values of the test statistics. Moreover, when for Italy the raw data and $b = 15$ are considered, the null hypothesis is not rejected i.e., at a significance level 5%, frequency ψ_0 is not detected. Simultaneously for the partially demeaned data, independent of the chosen country and the subsample length b , the null hypothesis is strongly rejected, which means that at the considered significance level we always detect the frequency ψ_0 . Thus, using our subsampling approach allows for better frequency detection.

5 Appendix

Proof of Theorem 2.1. The proof is similar to the one proposed by [9] and hence we skip most of the details, indicating only some major differences.

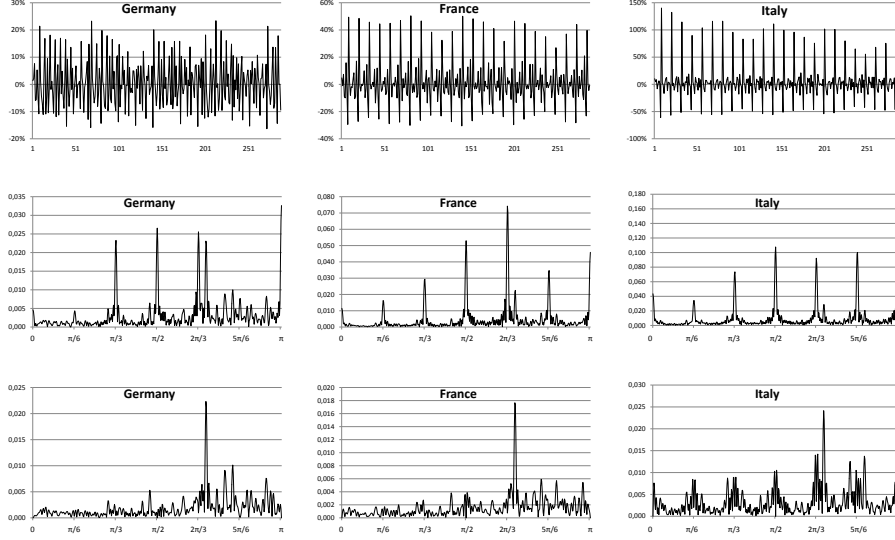


Figure 1: First row: Monthly industrial production (percentage change on previous period, gross data - neither seasonally nor calendar adjusted, source: Eurostat) from Jan. 1992 to Dec. 2015. Second and third row: values of test statistics $T_n = \sqrt{n}|\hat{m}_n(\psi)|$ for $\psi \in [0, \pi]$ obtained respectively for the gross data and for data got after subtracting estimates of seasonal means.

Let $q = n - b + 1$ and $U_n(x) = 1/q \sum_{t=1}^q \mathbf{1}\{\tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x\}$. Notice that $\tilde{L}_{n,b}(x) = 1/q \sum_{t=1}^q \mathbf{1}\{\tau_b[\hat{\theta}'_{n,b,t} - \theta] + \tau_b[\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}] - \tau_b[\hat{\theta}_n - \theta] \leq x\}$. We show that for any $x \in \mathbb{R}$ and $\epsilon > 0$

$$U_n(x - \epsilon)\mathbf{1}\{E_n\} \leq \tilde{L}_{n,b}(x)\mathbf{1}\{E_n\} \leq U_n(x + \epsilon), \quad (7)$$

where $E_n = \left\{ \max_{1 \leq t \leq q} |S_{n,b,t}| \leq \epsilon \right\}$ and $S_{n,b,t} = \tau_b \left([\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}] - [\hat{\theta}_n - \theta] \right)$ for $1 \leq t \leq q$. If $\mathbf{1}\{E_n\} = 0$, then (7) is obvious. If $\mathbf{1}\{E_n\} = 1$ then

$$\max_{1 \leq t \leq q} |S_{n,b,t}| = \tau_b \max_{1 \leq t \leq q} \left| [\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}] - [\hat{\theta}_n - \theta] \right| \leq \epsilon,$$

which means that uniformly in $1 \leq t \leq q$ we have $S_{n,b,t} \in [-\epsilon, \epsilon]$. Thus, for any $1 \leq t \leq q$ we get that

$$\begin{aligned} \mathbf{1}\{\tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x - \epsilon\} &\leq \mathbf{1}\{\tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x - S_{n,b,t}\} \\ &= \mathbf{1}\{\tau_b[\theta'_{n,b,t} - \theta] + \tau_b[\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}] - \tau_b[\hat{\theta}_n - \theta] \leq x\}, \end{aligned} \quad (8)$$

$$\begin{aligned} \mathbf{1}\{\tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x + \epsilon\} &\geq \mathbf{1}\{\tau_b(\hat{\theta}'_{n,b,t} - \theta) \leq x - S_{n,b,t}\} \\ &= \mathbf{1}\{\tau_b[\theta'_{n,b,t} - \theta] + \tau_b[\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}] - \tau_b[\hat{\theta}_n - \theta] \leq x\}. \end{aligned} \quad (9)$$

Summing inequalities (8), (9) over $t = 1, 2, \dots, q$ we get (7).

In the next step we show that $P(E_n) \rightarrow 1$. Note that

$$\max_{1 \leq t \leq q} |S_{n,b,t}| \leq \max_{1 \leq t \leq q} \tau_b |\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}| + \tau_b |\hat{\theta}_n - \theta|.$$

Using assumptions (iii) and (i) we obtain that $\max_{1 \leq t \leq q} \tau_b |\hat{\theta}_{n,b,t} - \hat{\theta}'_{n,b,t}| \xrightarrow{p} 0$ and $\tau_b |\hat{\theta}_n - \theta| \xrightarrow{p} 0$. Therefore $\max_{1 \leq t \leq q} |S_{n,b,t}| \xrightarrow{p} 0$ and we get that $P(E_n) \rightarrow 1$. \square

Proof of Theorem 3.1. Notice that under the appropriate regularity conditions (see Theorem 2.2 from [4]) (ii) holds. To show (i) note that

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n (X_t - \hat{\mu}_{1,n}(t)) e^{-i\psi_0 t} - m(\psi_0) &= \left(\frac{1}{n} \sum_{t=1}^n (X_t - \mu_1(t)) e^{-i\psi_0 t} - m(\psi_0) \right) \\ &+ \frac{1}{n} \sum_{t=1}^n (\mu_1(t) - E[\hat{\mu}_{1,n}(t)]) e^{-i\psi_0 t} + \frac{1}{n} \sum_{t=1}^n (E[\hat{\mu}_{1,n}(t)] - \hat{\mu}_{1,n}(t)) e^{-i\psi_0 t}. \end{aligned} \quad (10)$$

The second and the third summand on the right-hand side we denote by p_n and q_n , respectively. We have

$$\begin{aligned} \sqrt{n} p_n &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \sum_{\psi \in \Psi_1} \left(m(\psi) - \left[\frac{1}{n} \sum_{j=1}^n \sum_{\rho \in \Psi} m(\rho) e^{i\rho j} e^{-i\psi j} \right] \right) e^{i\psi t} e^{-i\psi_0 t} \\ &= \frac{1}{\sqrt{n}} \sum_{\psi \in \Psi_1} \left(\frac{1}{n} \sum_{\rho \in \Psi \setminus \{\psi\}} m(\rho) \frac{e^{i(\rho-\psi)} (e^{in(\rho-\psi)} - 1)}{e^{i(\rho-\psi)} - 1} \right) \frac{e^{i(\psi-\psi_0)} (e^{in(\psi-\psi_0)} - 1)}{e^{i(\psi-\psi_0)} - 1}. \end{aligned}$$

Therefore $|\sqrt{n}p_n| = O(n^{-3/2})$. Moreover, note that $|\sqrt{n}q_n|$ is equal to

$$\begin{aligned} & \left| \frac{1}{\sqrt{n}} \left(\sum_{\psi \in \Psi_1} \left[\frac{1}{n} \sum_{j=1}^n \mu(j) e^{-i\psi j} \right] - \sum_{\psi \in \Psi_1} \left[\frac{1}{n} \sum_{j=1}^n X_j e^{-i\psi j} \right] \right) \frac{e^{i(\psi-\psi_0)}(e^{in(\psi-\psi_0)} - 1)}{e^{i(\psi-\psi_0)} - 1} \right| \\ & \leq \frac{1}{\sqrt{n}} \sum_{\psi \in \Psi_1} \left[\frac{1}{n} \sum_{j=1}^n |\mu(j) - X_j| \right] \frac{2}{|e^{i(\psi-\psi_0)} - 1|}. \end{aligned}$$

Since $E|\mu(j) - X_j|$ is bounded by some constant independent of n uniformly at $j \in \mathbb{Z}$, we get that $E|\sqrt{n}q_n| = O(n^{-1/2})$ and finally that $\sqrt{n}q_n \xrightarrow{p} 0$.

To show (iii) we use the inequality $\|x\| - \|y\| \leq \|x - y\|$ (which holds for any $x, y \in \mathbb{C}$). Then $\tau_b |\hat{\theta}_{n,b,t_b} - \hat{\theta}'_{n,b,t_b}|$ can be bounded from above by

$$\begin{aligned} & \frac{1}{\sqrt{b}} \left| \sum_{j=t}^{t+b-1} (X_j - \hat{\mu}_{1,n}(j)) e^{-i\psi_0 j} - \sum_{j=t}^{t+b-1} (X_j - \mu_1(j)) e^{-i\psi_0 j} \right| \\ & = \frac{1}{\sqrt{b}} \left| \sum_{j=t}^{t+b-1} \left(\sum_{\psi \in \Psi_1} m(\psi) e^{i\psi j} - \sum_{\psi \in \Psi_1} \left[\frac{1}{n} \sum_{k=1}^n X_k e^{-i\psi k} \right] e^{i\psi j} \right) e^{-i\psi_0 j} \right| \\ & \leq \frac{1}{\sqrt{b}} \sum_{\psi \in \Psi_1} \left| m(\psi) - \frac{1}{n} \sum_{k=1}^n X_k e^{-i\psi k} \right| \frac{2}{|e^{i(\psi-\psi_0)} - 1|}. \end{aligned}$$

Since $|m(\psi) - 1/n \sum_{k=1}^n X_k e^{-i\psi k}| \rightarrow 0$ in probability as $n \rightarrow \infty$ for each $\psi \in \Psi_1$, we get that $\tau_b |\hat{\theta}_{n,b,t_b} - \hat{\theta}'_{n,b,t_b}| \rightarrow 0$ in probability as $n \rightarrow \infty$. \square

Proof of Theorem 3.2. Notice that under appropriate regularity conditions (see [5]) (ii) holds. To show (i) one needs to decompose $\sqrt{n}(\hat{\theta}_n - \theta)$ similarly to (10) and then to show the convergence to 0 in probability of the corresponding terms. For that purpose let us denote $p_n = 1/\sqrt{n} \sum_{t=1}^{n-\tau} (X_t - \mu(t))(\mu(t+\tau) - \hat{\mu}_n(t+\tau))e^{-i\lambda t}$, $q_n = 1/\sqrt{n} \sum_{t=1}^{n-\tau} (\mu(t) - \hat{\mu}_n(t))(X_{t+\tau} - \mu(t+\tau))e^{-i\lambda t}$ and $r_n = 1/\sqrt{n} \sum_{t=1}^{n-\tau} (\mu(t) - \hat{\mu}_n(t))(\mu(t+\tau) - \hat{\mu}_n(t+\tau))e^{-i\lambda t}$.

Note that using the same arguments as in the proof of the Theorem 2.1 in [4] (under the same assumptions), it can be shown that for any $\psi \in (0, \pi)$

and any $\delta_0 > 0$ we have $\|m(\psi) - \hat{m}_n(\psi)\|_2 = o(n^{-1/2+\delta_0})$ as $n \rightarrow \infty$. Thus,

$$E|p_n| \leq \sum_{\psi \in \Psi} \|m(\psi) - \hat{m}_n(\psi)\|_2 \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^{n-\tau} (X_t - \mu(t)) e^{-i\lambda t + i\psi(t+\tau)} \right\|_2 = o(1).$$

Similarly $E|q_n| \rightarrow 0$ as $n \rightarrow \infty$ and hence $p_n, q_n \rightarrow 0$ in probability as $n \rightarrow \infty$. Finally, $E|r_n|$ is bounded by

$$\begin{aligned} & \sum_{\psi_1 \in \Psi} \sum_{\psi_2 \in \Psi} \|m(\psi_1) - \hat{m}_n(\psi_1)\|_2 \|m(\psi_2) - \hat{m}_n(\psi_2)\|_2 \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n-\tau} e^{-i\lambda t + i\psi_1 t + i\psi_2(t+\tau)} \right| \\ &= \sum_{\psi_1 \in \Psi} \sum_{\psi_2 \in \Psi} \|m(\psi_1) - \hat{m}_n(\psi_1)\|_2 \|m(\psi_2) - \hat{m}_n(\psi_2)\|_2 O(\sqrt{n}) = o(1). \end{aligned}$$

To get (iii) note that using similar arguments as in the proof of Theorem 3.1, expression $\tau_b |\hat{\theta}_{n,b,t_b} - \hat{\theta}'_{n,b,t_b}|$ can be bounded from above by

$$\begin{aligned} & \frac{1}{\sqrt{b}} \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left| \sum_{j=t}^{t+b-1} X_j [\hat{\mu}_n(j+\tau) - \mu(j+\tau)] e^{-i\lambda j} \right| + \frac{1}{\sqrt{b}} \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left| \sum_{j=t}^{t+b-1} X_{j+\tau} [\hat{\mu}_n(j) \right. \\ & \quad \left. - \mu(j)] e^{-i\lambda j} \right| + \frac{1}{\sqrt{b}} \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left| \sum_{j=t}^{t+b-1} [\hat{\mu}_n(j) \hat{\mu}_n(j+\tau) - \mu(j) \mu(j+\tau)] e^{-i\lambda j} \right|. \end{aligned}$$

Let us denote summands above by a_n, b_n and c_n , respectively. Since $b/n \rightarrow 0$ as $n \rightarrow \infty$, there exists a positive constant δ' such that $\sqrt{b} = O(n^{1/2-\delta'})$. Thus,

$$\begin{aligned} E|a_n| &\leq \frac{1}{\sqrt{b}} \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \sum_{\psi \in \Psi} \|\hat{m}_n(\psi) - m(\psi)\|_2 \left\| \sum_{j=t}^{t+b-1} X_j e^{-i\lambda j + i\psi(j+\tau)} \right\|_2 \\ &= \sum_{\psi \in \Psi} \|\hat{m}_n(\psi) - m(\psi)\|_2 O(b^{1/2}) = O(n^{1/2-\delta'}) o(n^{-1/2+\delta_0}) = o(1) \end{aligned}$$

for any $\delta_0 < \delta'$. This means that $a_n \rightarrow 0$ in probability as $n \rightarrow \infty$. The corresponding convergence for b_n can be obtained analogically. Finally,

$$c_n \leq O(\sqrt{b}) \sum_{\psi_1 \in \Psi_T} \sum_{\psi_2 \in \Psi} |\hat{m}_n(\psi_1) \hat{m}_n(\psi_2) - m(\psi_1) m(\psi_2)|. \quad (11)$$

To finish the proof of the theorem it is enough to show that the each term of the sum in the right-hand side of (11) converges to 0 in probability as $n \rightarrow \infty$. For that purpose let us take any $\psi_1, \psi_2 \in \Psi$. If

$|m(\psi_1)m(\psi_2)| = 0$ (without loss of generality we assume that $|m(\psi_1)| = 0$), then $O(\sqrt{b})|\hat{m}_n(\psi_1)\hat{m}_n(\psi_2)| \rightarrow 0$ in probability as $n \rightarrow \infty$. The last convergence follows from the continuous mapping theorem, convergence $n^{1/2-\delta_0}|\hat{m}_n(\psi_1)| \rightarrow 0$ in probability as $n \rightarrow \infty$, convergence in probability of $\hat{m}_n(\psi_2)$ to $m(\psi_2)$ and Slutsky's lemma. If $|m(\psi_1)m(\psi_2)| \neq 0$, then from Theorem 2.1 in [6], the multivariate delta method, the continuous mapping theorem and Slutsky's lemma we have that $n^{1/2-\delta_0}|\hat{m}_n(\psi_1)\hat{m}_n(\psi_2) - m(\psi_1)m(\psi_2)| \xrightarrow{p} 0$ as $n \rightarrow \infty$ for any $\delta_0 > 0$. Taking $\delta_0 < \delta'$ we get the required convergence of the right-hand side of (11). \square

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