A uniqueness lemma with applications to regularization and incompressible fluid mechanics

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A UNIQUENESS LEMMA WITH APPLICATIONS TO REGULARIZATION AND INCOMPRESSIBLE FLUID MECHANICS.

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Abstract. In this paper, we extend our previous result from [16]. We prove that transport equations with rough coefficients do possess a uniqueness property. Our method relies strongly on duality and bears a strong resemblance with the well-known DiPerna-Lions theory first developed in [8]. As an application, we show a uniqueness result for the Euler and Navier-Stokes equations at the Leray regularity scale. In turn, this theorem stands as a barrier against the paradoxical weak solutions constructed in [17], [18], [19] and later reformulated in [6].

1. Introduction

In their seminal paper [8], R. J. DiPerna and P.-L. Lions proved the existence and uniqueness of solutions to transport equations on $\mathbb{R}^d$. We recall here a slightly simplified version of their statement.

Theorem 1 (DiPerna-Lions)**.** *Let $d \geq 1$ be an integer. Let $1 \leq p \leq \infty$ and p' its Hölder conjugate. Let a_0 be an initial condition in $L^p(\mathbb{R}^d)$. Let v be a fixed divergence free vector field in $L^1_{loc}(\mathbb{R}_+, \dot{W}^{1,p'}(\mathbb{R}^d))$. Then there exists a unique distributional solution a in $L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^d))$ of the Cauchy problem*

$$\left\{ \begin{array}{c} \partial_t a + \nabla \cdot (av) = 0 \\ a(0) = a_0, \end{array} \right. \tag{1}$$

with the initial condition understood in the sense of $\mathcal{C}^0(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^d))$. We recall that a is a distributional solution of the aforementioned Cauchy problem if and only if, for any φ belonging to $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ and any $T > 0$, there holds

$$\int_0^T \int_{\mathbb{R}^d} a(t,x) \left(\partial_t \varphi(t,x) + v(t,x) \cdot \nabla \varphi(t,x)\right) dxdt = \int_{\mathbb{R}^d} a(T,x)\varphi(T,x)dx - \int_{\mathbb{R}^d} a_0(x)\varphi(0,x)dx. \tag{2}$$

Beyond this theorem, many authors have since proved similar existence and (non-)uniqueness theorems, see for instance [1], [2], [7], [13], [15] and references therein. Our key result, which relies on the maximum principle for the *adjoint* equation, is both more general and more restrictive than the DiPerna-Lions theorem. The generality comes from the wider range of exponents allowed, along with the affordability of additional scaling-invariant and/or dissipative terms in the equation. On the other hand, we do not fully extend the original theorem, since we are unable to prove the existence of solutions in the uniqueness classes. Here is the precise statement.

Theorem 2. *Let $d \geq 1$ be an integer. Let $\nu \geq 0$ be a positive parameter. Let $1 \leq p, q \leq \infty$ be real numbers which we fix throughout this paper. Their Hölder conjugates are respectively denoted by p' and q'. Let $v = v(t,x)$ be a fixed, divergence free vector field in $L^{p'}(\mathbb{R}_+, \dot{W}^{1,q'}(\mathbb{R}^d))$. Given a time $T^* > 0$, let a be in $L^p([0,T^*], L^q(\mathbb{R}^d))$. Assume that a is a distributional solution of the Cauchy problem*

$$(C) \left\{ \begin{array}{c} \partial_t a + \nabla \cdot (av) - \nu \Delta a = 0 \\ a(0) = 0, \end{array} \right. \tag{3}$$

with the initial condition understood in the sense of $\mathcal{C}^0([0,T^*],\mathcal{D}'(\mathbb{R}^d))$. *That is, we assume that, for any function* φ *in* $\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^d)$ *and any* $T>0$, *there holds*

$$\int_{\mathbb{R}_+\times\mathbb{R}^d} a(t,x)\left(\partial_t\varphi(t,x)+v(t,x)\cdot\nabla\varphi(t,x)+\nu\Delta\varphi(t,x)\right)dxdt=\int_{\mathbb{R}^d}u(T,x)\varphi(T,x)dx. \tag{4}$$

Then a *is identically zero on* $[0,T^*]\times\mathbb{R}^d$.

Though one may fear that the lack of existence might render the theorem unapplicable in practice, it does not. For instance, when working with the Navier-Stokes equations, the vorticity of a Leray solution only belongs, a priori, to $L^\infty(\mathbb{R}_+,\dot{H}^{-1}(\mathbb{R}^d))\cap L^2(\mathbb{R}_+\times\mathbb{R}^d)$. In particular, the only Lebesgue-type space to which this vorticity belongs is $L^2(\mathbb{R}_+\times\mathbb{R}^d)$. Our theorem is well suited for solutions possessing *a priori* no integrable derivative whatsoever.

As such, our theorem appears a regularization tool. The philosophy is that, if an equation has smooth solutions, then any sufficiently integrable *weak* solution is automatically smooth. We illustrate our theorem with an application to the regularity result of J. Serrin [20] and subsequent authors [3], [4], [5], [9], [10], [11], [12], [21], [23]. Another consequence is the impossibility to construct paradoxical weak solutions to either the Euler or Navier-Stokes equations compactly supported in time belonging to the space $L^2(\mathbb{R}_+,\dot{H}^1(\mathbb{R}^d))$. In this sense, a rather weak regularity assumption is enough to forbid spontaneous creation of energy in incompressible fluid mechanics, even without assuming the relevant energy identity.

Before closing the introduction, we wish to discuss and shed some light on the results we proved. We first warn the reader that we did *not* prove that the Leray solutions are unique in their class and will not claim so. Indeed, the uniqueness stated in Theorem 4 is purely linear. In particular, it is completely oblivious to the strong link existing between Ω, a and v, in addition to the divergence free character of Ω, which was not needed.

However, on a more positive note, we were able to disprove the existence of paradoxical weak solutions to either the Euler or the Navier-Stokes equations. This statement holds at least when the initial condition is identically zero. Such a result is in sharp contrast with the existence result stated in [17], [18], [19] and [6]. In these papers, the authors build weak solutions to the Euler equations (in any dimension) having an arbitrary kinetic energy profile, provided it is continuous in time. Among the allowed profiles, the most striking ones are non identically zero and compactly supported in time. In view of these results, such profiles cannot exist at the Leray regularity scale, *even without assuming the relevant energy identity.* Bearing in mind that the paradoxical weak solutions do not satisfy the energy conservation anyway, it appears be that the energy identity in itself is not the primary information contained in the algebra of the equations. This remark has already been made, for instance, in [22]. The key point in our proof is the maximum principle of the *adjoint equation*, in turn partially depending on the vorticity equation having only differential operators rather than pseudodifferential ones.

Another standpoint on this theorem, which we owe to a private communication from N. Masmoudi, is that we now have two ways to recover the vorticity field from the velocity. We may either we use the defining identity $\Omega=\nabla\wedge u$, which only relies on the fact that u is divergence free, or recall that Ω is the unique solution of the vorticity equation. The second choice makes a strong use of the peculiar algebra of the Navier-Stokes equations, while the first one is absolutely general and requires no special assumption on u. Thus, we may hope to garner more information from the vorticity uniqueness, even though it may seem rather intricated. Embodied by Theorem 5 is our new ability to (re)prove more straightforwardly the smoothness of the Leray solutions under assumptions close in spirit to the theorem of J. Serrin and others. We hope, on a final note, to give a unified framework for such type of weak-strong uniqueness results.

Let us comment a bit on the strategy we shall use. First, because a lies in a low-regularity class of distributions, energy-type estimates seem out of reach. Thus, a duality argument is much more adapted to our situation. Given the assumptions on a, which for instance imply that Δa is in $L^p(\mathbb{R}_+,\dot{W}^{-2,q}(\mathbb{R}^d))$, a rather strong existence result is necessary, which we now state.

Theorem 3. *Let $\nu \geq 0$ be a positive real number. Let $v = v(t,x)$ be a fixed, divergence free vector field in $L^{p'}(\mathbb{R}_+, \dot{W}^{1,q'}(\mathbb{R}^d))$. Let φ_0 be a smooth, compactly supported function in $\mathbb{R}^d$. There exists a function φ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ solving*

$$(C') \left\{ \begin{array}{c} \partial_t \varphi - \nabla \cdot (\varphi v) - \nu \Delta \varphi = 0 \\ \varphi(0) = \varphi_0 \end{array} \right. \tag{5}$$

in the sense of distributions and satisfying the estimate

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}.$$

By considering $\varphi(T - \cdot)$ instead of φ, this amounts to build, for $T > 0$, a solution on $[0,T] \times \mathbb{R}^d$ of the Cauchy problem

$$(-C') \left\{ \begin{array}{c} -\partial_t \varphi - \nabla \cdot (\varphi v) - \nu \Delta \varphi = 0 \\ \varphi(T) = \varphi_0. \end{array} \right. \tag{6}$$

We emphasize that the theorem given is rather general, but does not cover all the possible extensions one may seek or need in practice. The most direct one is its analogue for diagonal systems : indeed, uniqueness in this case reduces to a finite number of applications of the scalar case to each component of the solution. Alternatively, one may add various linear, scaling invariant terms on the right hand side, or any dissipative term (such as a fractional laplacian) on the left hand side.

Among these numerous possibilities, we choose to present a particular one. Its purpose is to provide a unified uniqueness framework in incompressible fluid mechanics, beginning with the celebrated Navier-Stokes equations. We focus on the vorticity formulation, which is the cornerstone of the equations' peculiar algebra.

Theorem 4. *Let $d \geq 3$ be an integer. Let $\nu \geq 0$ be a positive real number. Let v, a be fixed vector fields in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^d))$ and assume that v is divergence free. Let Ω be in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. Assume that Ω is a distributional solution of the Cauchy problem*

$$(C) \left\{ \begin{array}{c} \partial_t \Omega + \nabla \cdot (\Omega \otimes v) - \nu \Delta \Omega = \nabla \cdot (a \otimes \Omega) \\ \Omega(0) = 0, \end{array} \right. \tag{7}$$

with the initial condition understood in the sense of $\mathcal{C}^0([0,T], \mathcal{D}'(\mathbb{R}^d))$. Then Ω is identically zero on $\mathbb{R}_+ \times \mathbb{R}^d$.

We wish to emphasize one fact. Since, when dealing with the incompressible Navier-Stokes and Euler equations, the solution Ω is divergence free, the equality

$$\nabla \cdot (a \otimes \Omega) = \Omega \cdot \nabla a$$

holds and both sides make sense in some distribution space. However, since we forget the divergence free character of Ω when we compute the adjoint equation, it is of utmost importance to stick with the divergence form of the left hand side. This divergence form is the only one with which we are able to get a maximum principle for the adjoint equation, an absolutely crucial feature of our proof. We briefly recall some notations here. If $a = (a_i)_i$ and $\Omega = (\Omega_j)_j$ are vectors, the tensor product $a \otimes \Omega$ is the matrix defined by

$$(a \otimes \Omega)_{i,j} = a_i \Omega_j.$$

The divergence of a matrix-valued function $A = (A_{i,j})_{i,j}$ is the vector-valued function defined by

$$(\nabla \cdot A)_i = \sum_j \partial_j A_{i,j}.$$

In particular, the following identity holds

$$(\nabla \cdot (a \otimes \Omega))_i = \sum_j \Omega_j \partial_j a_i + a_i \sum_j \partial_j \Omega_j = (\Omega \cdot \nabla a)_i + a_i \mathrm{div}\ \Omega.$$

We begin, as in the general theorem, by a dual existence result.

Theorem 5. *Let $d \geq 3$ be an integer. Let $\nu \geq 0$ be a positive real number. Let v, a be fixed vector fields in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^d))$ and assume that v is divergence free. There exists a solution φ to the following Cauchy problem*

$$(C') \begin{cases} \partial_t \varphi - \nabla \cdot (\varphi \otimes v) - \nu \Delta \varphi = -{}^t\nabla \varphi \cdot a \\ \varphi(0) = \varphi_0 \in \mathcal{D}(\mathbb{R}^d) \end{cases} \tag{8}$$

satisfying in addition, for almost every $t > 0$,

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}.$$

In the right hand side of the main equation, the quantity $-{}^t\nabla\varphi \cdot a$ is a shorthand for

$$-\nabla(\varphi \cdot a) +{}^t \nabla a \cdot \varphi$$

and this last expression makes sense in $L^2(\mathbb{R}_+, \dot{H}^{-1}_{loc}(\mathbb{R}^d)) + L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ provided that φ is bounded in space-time. Using coordinates, the different terms expand respectively as

$$({}^t\nabla\varphi \cdot a)_i = \sum_j \partial_i \varphi_j a_j;$$

$$(\nabla(\varphi \cdot a))_i = \sum_j \partial_i(\varphi_j a_j);$$

$$({}^t\nabla a \cdot \varphi)_i = \sum_j \partial_i a_j \varphi_j.$$

Although the left-hand sides of (C) and its adjoint equation (C') are almost identical, their right-hand sides are different. This discrepancy has striking consequences on their global behaviour, in that (C') does possess a maximum principle, while (C) does not. That fact is the core of our paper, without which no conclusion on the Navier-Stokes and Euler equations could have been drawn. Conversely, we are able to prove a uniqueness result for (C) while we do not expect any analogous result for (C'), at least at the present time.

As a consequence of Theorem 4, we give an alternative proof of the Serrin theorem in most cases. This new proof has the advantage of making a stronger use of the algebra of the Navier-Stokes equations than the previous one. To avoid technical details which would only obscure the proof, we choose to present it in the case of the three dimensional torus. An analogue exists when the regularity assumption is written on the whole space $\mathbb{R}^3$, or a subdmain thereof, with a similar proof and some minor adjustments.

Theorem 6. *Let $u = u(t,x)$ be a Leray solution of the Navier-Stokes equations*

$$(NS) \begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \in L^2(\mathbb{T}^3) \end{cases} \tag{9}$$

on $\mathbb{R}_+ \times \mathbb{T}^3$. Assume the existence of times $T_2 > T_1 > 0$ and exponents $2 \leq p < \infty, 3 < q \leq \infty$ such that u belongs to $L^p(]T_1, T_2[, L^q(\mathbb{T}^3))$. Then u belongs to $\mathcal{C}^\infty(]T_1, T_2[\times \mathbb{T}^3)$.

Besides reproving in a novel way the results of J. Serrin and his continuators, an immediate corollary of Theorem 4 is the following.

Theorem 7. *Let $d \geq 3$ be an integer. Let u be a divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^d))$. Let $\nu \geq 0$ be a positive real number. Assume that u is a weak solution of the generalized Navier-Stokes equations (degenerating to the Euler equations when $\nu = 0$)*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \nu \Delta u = -\nabla p \\ div\ u = 0 \\ u(0) = 0, \end{cases} \tag{10}$$

with the initial condition understood in the sense of $\mathcal{C}^0(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^d))$. Then, on $\mathbb{R}_+ \times \mathbb{R}^d$, u is identically 0.

2. Proofs

We state here a commutator lemma, similar to Lemma II.1 in [8], which we will use in the proof of Theorem 2.

Lemma 1. *Let $T > 0$. Let v be a fixed, divergence free vector field in $L^{p'}(\mathbb{R}_+, \dot{W}^{1,q'}(\mathbb{R}^d))$. Let a be a fixed function in $L^p(\mathbb{R}_+, L^q(\mathbb{R}^d))$. Let $\rho = \rho(x)$ be some smooth, positive and compactly supported function on $\mathbb{R}^d$. Normalize ρ to have unit norm in $L^1(\mathbb{R}^d)$ and define $\rho_\varepsilon := \varepsilon^{-d}\rho\left(\frac{\cdot}{\varepsilon}\right)$. Define the commutator C^ε by*

$$C^\varepsilon(t,x) := v(t,x)\cdot(\nabla\rho_\varepsilon * a(t))(x) - (\nabla\rho_\varepsilon * (v(t)a(t)))(x).$$

Then, as $\varepsilon \to 0$,

$$\|C^\varepsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \to 0.$$

Proof. For almost all (t,x) in $\mathbb{R}_+ \times \mathbb{R}^d$, we have

$$C^\varepsilon(t,x) = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} a(t,y) \frac{v(t,x)-v(t,y)}{\varepsilon}\cdot\nabla\rho\left(\frac{x-y}{\varepsilon}\right)dy.$$

Performing the change of variable $y = x + \varepsilon z$ yields

$$C^\varepsilon(t,x) = \int_{\mathbb{R}^d} a(t,x+\varepsilon z)\frac{v(t,x)-v(t,x+\varepsilon z)}{\varepsilon}\cdot\nabla\rho(z)dz.$$

Using the Taylor formula

$$v(\cdot,x+\varepsilon z) - v(\cdot,x) = \int_0^1 \nabla v(\cdot,x+r\varepsilon z)\cdot(\varepsilon z)dr,$$

which is true for smooth functions and extends to $\dot{W}^{1,q'}(\mathbb{R}^d)$ thanks to the continuity of both sides on this space and owing to Fubini's theorem to exchange integrals, we get the nicer formula

$$C^\varepsilon(t,x) = -\int_0^1\int_{\mathbb{R}^d} a(t,x+\varepsilon z)\nabla v(t,x+r\varepsilon z) : (\nabla\rho(z)\otimes z)dzdr,$$

where : denotes the contraction of rank two tensors. Because q and q' are dual Hölder exponents, at least one of them is finite. We assume for instance that $q < \infty$, the case $q' < \infty$ being completely similar.

Let

$$\widetilde{C}^\varepsilon(t,x) := -\int_0^1\int_{\mathbb{R}^d} a(t,x+r\varepsilon z)\nabla v(t,x+r\varepsilon z) : (\nabla\rho(z)\otimes z)dzdr.$$

We claim that, as $\varepsilon \to 0$,

$$\|C^\varepsilon - \widetilde{C}^\varepsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \to 0.$$

Integrating both in space and time and owing to Hölder's inequality, we have

$$\|C^\varepsilon - \widetilde{C}^\varepsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \leq \int_0^1\int_{\mathbb{R}^d}\int_0^\infty \|a(t,\cdot+\varepsilon z) - a(t,\cdot+r\varepsilon z)\|_{L^q(\mathbb{R}^d)}\|\nabla v(t)\|_{L^{q'}(\mathbb{R}^d)}|\nabla\rho(z)\otimes z|dtdzdr. \tag{11}$$

Since $a \in L^p(\mathbb{R}_+, L^q(\mathbb{R}^d))$ and $q < \infty$, for almost any $t \in \mathbb{R}_+$, for all $z \in \mathbb{R}^d$ and $r \in [0,1]$,

$$\|a(t,\cdot+\varepsilon z) - a(t,\cdot+r\varepsilon z)\|_{L^q(\mathbb{R}^d)} \to 0$$

as $\varepsilon \to 0$. Thanks to the uniform bound

$$\|a(t,\cdot+\varepsilon z) - a(t,\cdot+r\varepsilon z)\|_{L^q(\mathbb{R}^d)}\|\nabla v(t)\|_{L^{q'}(\mathbb{R}^d)}|\nabla\rho(z)\otimes z| \leq 2\|a(t)\|_{L^q(\mathbb{R}^d)}\|\nabla v(t)\|_{L^{q'}(\mathbb{R}^d)}|\nabla\rho(z)\otimes z|, \tag{12}$$

we may invoke the dominated convergence theorem to get the desired claim.

From this point on, we denote by $U(t,x)$ the quantity $a(t,x)\nabla v(t,x)$. We notice that U is a fixed function in $L^1(\mathbb{R}_+ \times \mathbb{R}^d)$ and that, by definition,

$$\widetilde{C}^\varepsilon(t,x) = -\int_0^1 \int_{\mathbb{R}^d} U(t,x+r\varepsilon z) : (\nabla\rho(z)\otimes z)dzdr.$$

The normalization on ρ yields the identity

$$-\int_{\mathbb{R}^d} \nabla\rho(z)\otimes z dz = \left(\int_{\mathbb{R}^d}\rho(z)dz\right) I_d = I_d,$$

where I_d is the $d-$dimensional identity matrix. This identity in turn entails that

$$\widetilde{C}^0(t,x) = a(t,x)\nabla v(t,x) : I_d = a(t,x) \text{ div } v(t,x) = 0.$$

A second application of the dominated convergence theorem to the function U gives

$$\|\widetilde{C}^\varepsilon - \widetilde{C}^0\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \to 0$$

as $\varepsilon \to 0$, from which the lemma follows. □

Proof of Theorem 3. Let us choose some mollifying kernel $\rho = \rho(x)$ and denote $v_\delta := \rho_\delta * v$, where $\rho_\delta(x) := \delta^{-d}\rho(\frac{x}{\delta})$. Let (C'_δ) be the Cauchy problem (C') where we replaced v by v_δ. The existence of a (smooth) solution φ^δ to (C'_δ) is then easily obtained thanks to, for instance, a Friedrichs method combined with heat kernel estimates. We now turn to the L^∞ bound uniform in δ.

Let $r \geq 2$ be a real number. Multiplying the equation on φ^δ by $\varphi^\delta|\varphi^\delta|^{r-2}$ and integrating in space and time, we get

$$\frac{1}{r}\|\varphi^\delta(t)\|^r_{L^r(\mathbb{R}^d)} + (r-1)\int_0^t \|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\|^2_{L^2(\mathbb{R}^d)}ds = \frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^d)}. \tag{13}$$

Discarding the gradient term, taking r-th root in both sides and letting r go to infinity gives

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}. \tag{14}$$

Thus, the family $(\varphi^\delta)_\delta$ is bounded in $L^\infty(\mathbb{R}_+\times\mathbb{R}^d)$. Up to an extraction, $(\varphi^\delta)_\delta$ converges weak$-*$ in $L^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ to some function φ.

As a consequence, because $v_\delta \to v$ strongly in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^d)$ as $\delta\to 0$, the following convergences hold :

$$\Delta\varphi^\delta \rightharpoonup^* \Delta\varphi \text{ in } L^\infty(\mathbb{R}_+, \dot{W}^{-2,\infty}(\mathbb{R}^d));$$
$$\varphi^\delta v^\delta \rightharpoonup \varphi v \text{ in } L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^d).$$

In particular, such a φ is a distributional solution of (C') with the desired regularity. □

We are now in position to prove the main theorem of this paper.

Proof of Theorem 2. Let $\rho = \rho(x)$ be a radial mollifying kernel and define $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\frac{x}{\varepsilon})$. Convolving the equation on a by ρ_ε gives, denoting $a_\varepsilon := \rho_\varepsilon * a$,

$$(C_\varepsilon)\quad \partial_t a_\varepsilon + \nabla\cdot(a_\varepsilon v) - \nu\Delta a_\varepsilon = C^\varepsilon, \tag{15}$$

where the commutator C^ε has been defined in Lemma 1. Notice that even without any smoothing in time, a_ε, $\partial_t a_\varepsilon$ lie respectively in $L^\infty(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^d))$ and $L^1(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^d))$, which is enough to make the upcoming computations rigorous. In what follows, we let φ^δ be a solution of the Cauchy problem $(-C'_\delta)$, where $(-C'_\delta)$ is $(-C')$ (defined in Theorem 5) with v replaced by v_δ. Let us now multiply, for $\delta,\varepsilon > 0$ the equation (C_ε) by φ^δ and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

$$\int_0^T\int_{\mathbb{R}^d} \partial_t a_\varepsilon(s,x)\varphi^\delta(s,x)dxds = \langle a_\varepsilon(T),\varphi_0\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} - \int_0^T\int_{\mathbb{R}^d} a_\varepsilon(s,x)\partial_t\varphi^\delta(s,x)dxds.$$

From this identity, it follows that

$$\langle a_\varepsilon(T),\varphi_0\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} = \int_0^T\int_{\mathbb{R}^d}\varphi^\delta(s,x)C^\varepsilon(s,x)dxds \\ -\int_0^T\int_{\mathbb{R}^d} a_\varepsilon(s,x)\left(-\partial_t\varphi^\delta(s,x)-\nabla\cdot(v(s,x)\varphi^\delta(s,x))-\nu\Delta\varphi^\delta(s,x)\right)dxds.$$

From Lemma 1, we know in particular that C^ε belongs to $L^1(\mathbb{R}_+\times\mathbb{R}^d)$ for each fixed $\varepsilon>0$. Thus, in the limit $\delta\to 0$, we have, for each $\varepsilon>0$,

$$\int_0^T\int_{\mathbb{R}^d}\varphi^\delta(s,x)C^\varepsilon(s,x)dxds\to\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)C^\varepsilon(s,x)dxds.$$

On the other hand, the definition of φ^δ gives

$$-\partial_t\varphi^\delta-\nabla\cdot(v\varphi^\delta)-\nu\Delta\varphi^\delta=\nabla\cdot((v_\delta-v)\varphi^\delta).$$

Thus, the last integral in the above equation may be rewritten, integrating by parts,

$$-\int_0^T\int_{\mathbb{R}^d}\varphi^\delta(v_\delta-v)\cdot\nabla a_\varepsilon(s,x)dxds.$$

For each fixed ε, the assumption on a entails that ∇a_ε belongs to $L^p(\mathbb{R}_+,L^q(\mathbb{R}^d))$. Furthermore, it is an easy exercise to show that

$$\|v_\delta-v\|_{L^{p'}(\mathbb{R}_+,L^{q'}(\mathbb{R}^d))}\le\delta\|\nabla v\|_{L^{p'}(\mathbb{R}_+,L^{q'}(\mathbb{R}^d))}\||\cdot|\rho\|_{L^1(\mathbb{R}^d)}.$$

Now, taking the limit $\delta\to 0$ while keeping $\varepsilon>0$ fixed, we have

$$\langle a_\varepsilon(T),\varphi_0\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)}=\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)C^\varepsilon(s,x)dxds. \tag{16}$$

Taking the limit $\varepsilon\to 0$ and using Lemma 1, we finally obtain

$$\langle a(T),\varphi_0\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)}=0. \tag{17}$$

This being true for any test function φ_0, $a(T)$ is the zero distribution and finally $a\equiv 0$. □

Proof of Theorem 5. We first assume that $\nu>0$. For simplicity, we reduce to the case $\nu=1$. Let $\rho=\rho(x)$ be a radial mollifying kernel and let us denote $\rho_\delta(x)=\delta^{-d}\rho(\frac{x}{\delta})$. Let $a_\delta=\rho_\delta*a$ and $v_\delta=\rho_\delta*v$. Let (C'_δ) be the Cauchy problem (C') with a,v replaced by a_δ,v_δ respectively. The existence of a smooth solution φ^δ to the Cauchy problem (C'_δ) is easy and thus omitted. We focus on the the relevant estimates. Let $r\ge 2$ be a real number. Multiplying the equation on φ^δ by $|\varphi^\delta|^{r-2}\varphi^\delta$ and integrating in space and time, we get

$$\frac{1}{r}\|\varphi^\delta(t)\|^r_{L^r(\mathbb{R}^d)}+(r-1)\int_0^t\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\|^2_{L^2(\mathbb{R}^d)}ds \\ =\frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^d)}-\int_0^t\int_{\mathbb{R}^d}|\varphi^\delta|^{r-2}\varphi^\delta\cdot({}^t\nabla\varphi^\delta\cdot a_\delta)dxds. \tag{18}$$

Denote by $I(t)$ the integral on the right hand side. From the Hölder inequality, we have

$$|I(t)|\le\int_0^t\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\|_{L^2(\mathbb{R}^d)}\|\varphi^\delta(s)\|^{\frac{r}{2}}_{L^r(\mathbb{R}^d)}\|a_\delta(s)\|_{L^\infty(\mathbb{R}^d)}ds. \tag{19}$$

Since a_δ is a convolution, we have

$$\|a_\delta(s)\|_{L^\infty(\mathbb{R}^d)}\le\|\rho_\delta\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\|a(s)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \tag{20}$$

$$\le\delta^{-\frac{d+2}{2}}\|\rho\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\|a(s)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}. \tag{21}$$

Thanks to the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, there exists a constant $C = C(d) > 0$ such that

$$|I(t)| \leq C(d)\delta^{-\frac{d+2}{2}}\|\rho\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}\int_0^t \|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\|_{L^2(\mathbb{R}^d)}\|\varphi^\delta(s)\|^{\frac{r}{2}}_{L^r(\mathbb{R}^d)}\|a(s)\|_{\dot{H}^1(\mathbb{R}^d)}ds. \tag{22}$$

for all $t \geq 0$. The Cauchy-Schwarz inequality then yields

$$|I(t)| \leq \frac{r}{2}\int_0^t \|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\|^2_{L^2(\mathbb{R}^d)}ds + \frac{C(d)^2}{2r\delta^{d+2}}\int_0^t \|\varphi^\delta(s)\|^r_{L^r(\mathbb{R}^d)}\|a(s)\|^2_{\dot{H}^1(\mathbb{R}^d)}ds. \tag{23}$$

Absorbing the gradient term in the left hand side of the L^r estimate and using Grönwall's inequality, we arrive at

$$\|\varphi^\delta(t)\|_{L^r(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^r(\mathbb{R}^d)}\exp\left(\frac{C(d)^2}{2r\delta^{d+2}}\int_0^t \|a(s)\|^2_{\dot{H}^1(\mathbb{R}^d)}ds\right). \tag{24}$$

Letting r go to infinity, we get

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}. \tag{25}$$

As in the proof of Theorem 3, the weak-$*$ limit points of the family $(\varphi^\delta)_\delta$ as $\delta \to 0$ solve the Cauchy problem (C'). The L^∞ bound is immediate from weak-$*$ convergence.

We now return to the case $\nu = 0$. From the previous case, we are able to build, for any $\nu > 0$, a solution $\varphi^{(\nu)}$ of the Cauchy problem (C'), satisfying, for almost every $t > 0$,

$$\|\varphi^{(\nu)}(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}.$$

Up to an extraction, $\varphi^{(\nu)}$ converges weak-$*$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ to some φ. For every $\nu > 0$, the function $\varphi^{(\nu)}$ solves

$$\partial_t\varphi^{(\nu)} - \nabla\cdot(\varphi^{(\nu)} \otimes v) - \nu\Delta\varphi^{(\nu)} = -{}^t\nabla\varphi^{(\nu)}\cdot a. \tag{26}$$

Since v and a are in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^d))$ and do not depend on ν, taking weak-$*$ limits as $\nu \to 0$ in the equation gives

$$\partial_t\varphi - \nabla\cdot(\varphi \otimes v) = -{}^t\nabla\varphi\cdot a. \tag{27}$$

From the weak-$*$ convergence, it is obvious that φ satisfies for almost every $t > 0$ the bound

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^d)}.$$

□

We now turn to the proof of the uniqueness theorem.

Proof of Theorem 4. Let $\rho = \rho(x)$ be a radial mollifying kernel and define $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\frac{x}{\varepsilon})$. Convolving the equation on Ω by ρ_ε gives, denoting $\Omega_\varepsilon := \rho_\varepsilon * \Omega$,

$$(C_\varepsilon)\quad \partial_t\Omega_\varepsilon + \nabla\cdot(\Omega_\varepsilon \otimes v) - \nu\Delta\Omega_\varepsilon = \nabla\cdot(a \otimes \Omega_\varepsilon) + C^\varepsilon + D^\varepsilon, \tag{28}$$

where the commutator C^ε has been defined in Lemma 1. The second commutator is defined by

$$D^\varepsilon := \rho_\varepsilon * \nabla\cdot(\Omega \otimes a) - \nabla\cdot(a \otimes \Omega_\varepsilon).$$

Similarly to what we proved for C^ε, we have

$$\|D^\varepsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \to 0 \text{ as } \varepsilon \to 0.$$

Notice that even without any smoothing in time, Ω_ε, $\partial_t\Omega_\varepsilon$ lie respectively in $L^\infty(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^d))$ and $L^1(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^d))$, which is enough to make the upcoming computations rigorous. In what follows, we let φ^δ be a solution of the Cauchy problem $(-C'_\delta)$, with $(-C'_\delta)$ being $(-C')$ where v and a are replaced by v_δ and a_δ. Let us now multiply, for $\delta, \varepsilon > 0$ the equation (C_ε) by φ^δ and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

$$\int_0^T\int_{\mathbb{R}^d} \partial_t\Omega_\varepsilon(s,x)\varphi^\delta(s,x)dxds = \langle\Omega_\varepsilon(T), \varphi_0\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} - \int_0^T\int_{\mathbb{R}^d} \Omega_\varepsilon(s,x)\partial_t\varphi^\delta(s,x)dxds.$$

From this identity, it follows that

$$\langle \Omega_\varepsilon(T), \varphi_0 \rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} = \int_0^T \int_{\mathbb{R}^d} \varphi^\delta(s,x)(C^\varepsilon + D^\varepsilon)(s,x)dxds$$
$$-\int_0^T \int_{\mathbb{R}^d} \Omega_\varepsilon(s,x)\left(-\partial_t\varphi^\delta(s,x) - \nabla\cdot(v(s,x)\varphi^\delta(s,x)) - \nu\Delta\varphi^\delta(s,x) +^t \nabla\varphi^\delta(s,x)\cdot a(s,x)\right)dxds.$$

From Lemma 1, we know in particular that C^ε belongs to $L^1(\mathbb{R}_+\times\mathbb{R}^d)$ for each fixed $\varepsilon>0$ and the same goes for D^ε. Thus, in the limit $\delta\to 0$, we have, for each $\varepsilon>0$,

$$\int_0^T \int_{\mathbb{R}^d} \varphi^\delta(s,x)(C^\varepsilon + D^\varepsilon)(s,x)dxds \to \int_0^T \int_{\mathbb{R}^d} \varphi(s,x)(C^\varepsilon + D^\varepsilon)(s,x)dxds.$$

On the other hand, the definition of φ^δ gives

$$-\partial_t\varphi^\delta - \nabla\cdot(v\varphi^\delta) - \nu\Delta\varphi^\delta +^t \nabla\varphi^\delta\cdot a = \nabla\cdot((v_\delta - v)\varphi^\delta) +^t \nabla\varphi^\delta\cdot(a-a_\delta).$$

Thus, the last integral in the above equation may be rewritten, integrating by parts,

$$-\int_0^T \int_{\mathbb{R}^d} \varphi^\delta(v_\delta - v)\cdot\nabla\Omega_\varepsilon(s,x)dxds + \int_0^T \int_{\mathbb{R}^d} \varphi^\delta(s,x)\nabla\cdot((a-a_\delta)(s,x)\otimes\Omega_\varepsilon(s,x))dxds.$$

For each fixed ε, the assumption on Ω entails that $\nabla\Omega_\varepsilon$ belongs to $L^2(\mathbb{R}_+\times\mathbb{R}^d)$. Furthermore, it is an easy exercise to show that

$$\|v_\delta - v\|_{L^2(\mathbb{R}_+\times\mathbb{R}^d)} \le \delta\|\nabla v\|_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}\||\cdot|\rho\|_{L^1(\mathbb{R}^d)}$$

and

$$\|\nabla\cdot((a-a_\delta)\otimes\Omega_\varepsilon)\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)} \to 0 \text{ as } \delta\to 0, \text{ for any fixed } \varepsilon.$$

Now, taking the limit $\delta\to 0$ while keeping $\varepsilon>0$ fixed, we have

$$\langle \Omega_\varepsilon(T), \varphi_0 \rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} = \int_0^T \int_{\mathbb{R}^d} \varphi(s,x)(C^\varepsilon + D^\varepsilon)(s,x)dxds. \tag{29}$$

Taking the limit $\varepsilon\to 0$ and using Lemma 1, we finally obtain

$$\langle \Omega(T), \varphi_0 \rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)} = 0. \tag{30}$$

This being true for any test function φ_0, $\Omega(T)$ is the zero distribution and finally $\Omega\equiv 0$. □

Proof of Theorem 6. Let $\Omega := \nabla\wedge u$ and $\Omega_0 := \nabla\wedge u_0$. The equation on Ω writes

$$(NSV)\begin{cases} \partial_t\Omega + \nabla\cdot(\Omega\otimes u) - \Delta\Omega = \nabla\cdot(u\otimes\Omega) \\ \Omega(0) = \Omega_0. \end{cases} \tag{31}$$

Let $\chi=\chi(t)$ be a smooth cutoff in time supported inside $]T_1,T_2[$. Let $\varphi=\varphi(t)$ be another smooth cutoff such that

$$\text{supp }\varphi \subset \{\chi\equiv 1\}.$$

Denoting $\Omega' = \chi\Omega$ and $u' = \varphi u$, we have

$$(NSV')\begin{cases} \partial_t\Omega' + \nabla\cdot(\Omega'\otimes u) - \Delta\Omega' = \nabla\cdot(u'\otimes\Omega') + \Omega\partial_t\chi \\ \Omega'(0) = 0. \end{cases} \tag{32}$$

Along the same lines as Theorem 5, we are able to build another solution to (NSV'), say Ω'', which belongs to $L^\infty(\mathbb{R}_+,\dot H^1(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,\dot H^2(\mathbb{T}^3))\cap L^\infty(\mathbb{R}_+\times\mathbb{T}^3)$. Thus, letting $\widetilde\Omega := \Omega' - \Omega''$, we see that $\widetilde\Omega$ solves

$$(NSV^0)\begin{cases} \partial_t\widetilde\Omega + \nabla\cdot(\widetilde\Omega\otimes u) - \Delta\widetilde\Omega = \nabla\cdot(u'\otimes\widetilde\Omega) \\ \widetilde\Omega(0) = 0. \end{cases} \tag{33}$$

We recall that u and u' belong to $L^2(\mathbb{R}_+,\dot H^1(\mathbb{T}^3))$. Moreover, the high regularity of Ω'' and the fact that u is a Leray solution of the Navier-Stokes equations together entail that $\widetilde\Omega$ belongs to $L^2(\mathbb{R}_+\times\mathbb{T}^3)$. These regularity assumptions allow us to invoke Theorem 4, from which we deduce that $\widetilde\Omega\equiv 0$. It follows that

$$\Omega\in L^\infty_{loc}(]T_1,T_2[\times\mathbb{T}^3)\cap L^\infty_{loc}(]T_1,T_2[,\dot H^1(\mathbb{T}^3))\cap L^2_{loc}(]T_1,T_2[,\dot H^2(\mathbb{T}^3)).$$

From there, it is a routine exercise to show that Ω (thus, also u) belongs to $\mathcal{C}^\infty(]T_1, T_2[\times\mathbb{T}^3)$. □

Proof of Theorem 7. Given the assumptions we made, we compute the vorticity equation by taking the curl on each side of the equation. Applying Theorem 4 to the unknown $\Omega := \nabla \wedge u$ with the forcing fields $v = a := u$, we deduce that Ω is identically 0 on $\mathbb{R}_+ \times \mathbb{R}^d$. Thus, on $\mathbb{R}_+ \times \mathbb{R}^d$, u is also identically 0. □

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