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Benoît Claudon, Andreas H\"oring. The fundamental group of compact Kähler threefolds. 2017. hal-01415323v2

**HAL Id: hal-01415323**

**<https://hal.science/hal-01415323v2>**

Preprint submitted on 24 Jul 2017 (v2), last revised 16 Dec 2020 (v4)

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# THE FUNDAMENTAL GROUP OF COMPACT KÄHLER THREEFOLDS

BENOÎT CLAUDON AND ANDREAS HÖRING

**ABSTRACT.** Let  $X$  be a compact Kähler manifold of dimension three. We prove that there exists a projective manifold  $Y$  such that  $\pi_1(X) \simeq \pi_1(Y)$ . We also prove the bimeromorphic existence of algebraic approximations for compact Kähler threefolds with algebraic dimension two. Together with the work of Graf and Lin this settles the bimeromorphic Kodaira problem for compact Kähler threefolds that are not uniruled.

## 1. INTRODUCTION

**1.A. Main results.** Compact Kähler manifolds arise naturally as generalisations of complex projective manifolds, and Kodaira's problem asked if every compact Kähler manifold is deformation equivalent to a projective manifold. A positive answer to this problem trivially implies that the larger class of Kähler manifolds realises the same topological invariants. The classification of analytic surfaces [Kod63] implies a positive answer to Kodaira's problem (cf. also [Buc08] for a different approach). However, Voisin's counterexamples [Voi04, Voi06] show that there exist compact Kähler manifold of dimension at least four that do not deform to projective ones. Nevertheless it is interesting to study Kodaira's problem at the level of some specific topological invariants, like the fundamental group:

**1.1. Conjecture.** *Let  $X$  be a compact Kähler manifold. Then the fundamental group  $\pi_1(X)$  is projective, i.e. there exists a projective manifold  $M$  such that  $\pi_1(X) \simeq \pi_1(M)$ .*

Note that unlike other problems on fundamental groups, this conjecture does not reduce to the case of surfaces: while, by the Lefschetz hyperplane theorem, the fundamental group of any projective manifold is realised by a projective surface, it is a priori not clear if the same holds in the Kähler category. Several partial results on Conjecture 1.1 have been obtained in the last years [CCE15], [CCE14, Théorème 0.2] and [Cla16, Corollary 1.3]. In this paper we give a complete answer in dimension three:

**1.2. Theorem.** *Let  $X$  be a smooth compact Kähler threefold. Then  $\pi_1(X)$  is a projective group.*

The proof of the result comes in several steps: if  $X$  is covered by rational curves, then its MRC-fibration/rational quotient  $X \dashrightarrow Z$  induces an isomorphism  $\pi_1(X) \simeq \pi_1(Z)$  [Kol93, Corollary 5.3] [BC15, Corollary 1], so we are done. If  $X$  is not covered by rational curves, we make a case distinction based on the algebraic dimension,

i.e. the transcendence degree of the field of meromorphic functions on  $X$ . The case  $a(X) = 0$  has been solved in [CC14] (see also [Gra16, Corollary 1.8]), and for the case<sup>1</sup>  $a(X) = 1$  we can describe in detail the structure of the fundamental group in terms of the natural fibration onto a curve  $X \rightarrow C$ . The most difficult case is when  $a(X) = 2$ , where the algebraic reduction defines an elliptic fibration  $X \rightarrow S$  over a surface. Here the structure of the fundamental group is not known, even for a projective threefold. The main contribution of this paper is to use the global theory of elliptic fibrations developed by Nakayama [Nak02a] to show that, modulo several reduction steps, and changing to a bimeromorphic model of  $X$ , the elliptic fibration is deformation equivalent over  $S$  to a projective fibration. These arguments work in any dimension:

**1.3. Theorem.** *Let  $X$  be a smooth compact Kähler manifold of dimension  $n$  and algebraic dimension  $a(X) = n - 1$ . Then  $\pi_1(X)$  is a projective group.*

**1.B. Algebraic approximation.** Voisin's examples [Voi06] show that there exist compact Kähler manifolds of dimension at least eight such that any *smooth* bimeromorphic model does not deform to a projective one. However, the philosophy of the minimal model program is that in higher dimension, the most natural bimeromorphic models have mild singularities. In this spirit Peternell and independently Campana proposed a more flexible, bimeromorphic version of Kodaira's problem:

**1.4. Conjecture.** *Let  $X$  be a compact Kähler manifold that is not uniruled. Then there exists a bimeromorphic map  $X \dashrightarrow X'$  to a normal compact Kähler space  $X'$  with terminal singularities that admits an algebraic approximation (cf. Definition 2.3).*

Algebraic approximation provides an explicit way to prove that a Kähler group is projective: since  $X'$  has terminal singularities, the fundamental group is invariant under the bimeromorphic map  $X \dashrightarrow X'$  [Tak03]. If we can always choose the algebraic approximation  $\mathcal{X} \rightarrow \Delta$  to be a locally trivial deformation in the sense of [FK87, p.627] [Ser06], then Conjecture 1.4 implies Conjecture 1.1.

Very recently Graf [Gra16] and Lin [Lin16, Lin17] have made tremendous progress on the Kodaira problem, by proving the existence of algebraic approximations for all smooth compact Kähler threefolds of Kodaira dimension zero or one. We prove Conjecture 1.4 when the algebraic dimension is two, hence in particular when the Kodaira dimension is two.

**1.5. Theorem.** *Let  $X$  be a compact Kähler threefold of algebraic dimension  $a(X) = 2$ . Then there exist a bimeromorphic map  $X' \dashrightarrow X$  such that  $X'$  is a smooth compact Kähler threefold that admits an algebraic approximation.*

A smooth compact Kähler threefold is not uniruled if and only if  $\kappa(X) \geq 0$  [HP16, Cor.1.4]. Thus Theorem 1.5 together with [Gra16, Lin16, Lin17] establishes Conjecture 1.4 for Kähler threefolds that are not uniruled. As far as we can see our techniques do not imply the existence of algebraic approximations for threefolds that are minimal models  $f : X \rightarrow S$  over a surface. In fact the birational map  $X' \dashrightarrow X$  appearing in our statement induces a base change  $S' \rightarrow S$  and blowing

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<sup>1</sup>After the submission of the first version of this paper to the arxiv, Hsueh-Yung Lin posted his preprint [Lin16] on algebraic approximation which implies this case. Our proof is completely different and should be useful for generalisations to higher dimension.

up might blow up some curves that are not contracted by  $X' \rightarrow S'$ . In this case it is difficult to relate the deformation theory of  $X$  and  $X'$ . Thus the biregular Kodaira problem is still open for non-uniruled threefolds. For Kähler threefolds that are uniruled, the situation seems to be quite complicated, see [Sch12] for some partial results. Even the weaker statement (analogous to Conjecture 1.4) asking for the existence of a (possibly singular) bimeromorphic model admitting algebraic approximations is not known in that case.

**Acknowledgements.** We would like to thank J. Cao, A. Dimca, P. Graf, J. Kollár, H.-Y. Lin and N. Nakayama for very helpful communications on the various technical problems related to this project. This work was partially supported by the Agence Nationale de la Recherche grant project Foliage (ANR-16-CE40-0008) and Hodgefun (ANR-16-CE40-0011-02).

## 2. NOTATION AND BASIC DEFINITIONS

All complex spaces are supposed to be of finite dimension, a complex manifold is a smooth irreducible complex space. A fibration is proper surjective morphism with connected fibres between complex spaces. A fibration  $\varphi : X \rightarrow Y$  is locally projective if there exists an open covering  $U_i \subset Y$  such that  $\varphi^{-1}(U_i) \rightarrow U_i$  is projective, i.e. admits a relatively ample line bundle.

We refer to [Gra62, Fuj79, Dem85] for basic definitions about  $(p, q)$ -forms and Kähler forms in the singular case.

**2.1. Definition.** Let  $\varphi : X \rightarrow Y$  be a fibration. A relative Kähler form is a smooth real closed  $(1, 1)$ -form  $\omega$  such that for every  $\varphi$ -fibre  $F$ , the restriction  $\omega|_F$  is a Kähler form. A class  $\alpha \in N^1(X)$  is a relative Kähler class if it can be represented by a relative Kähler form.

If  $\varphi$  admits a relative Kähler class we say that  $\varphi$  is a Kähler fibration.

**2.2. Remark.** If  $\varphi$  is a Kähler fibration over a Kähler base  $Y$ , then  $X_U := \varphi^{-1}(U)$  is Kähler for every relatively compact open set  $U \subset Y$ . In fact if  $\omega_X$  is a relative Kähler form on  $X$  and  $\omega_Y$  is a Kähler form on  $Y$ , then for all  $m \gg 0$  the form  $\omega_X + m\varphi^*\omega_Y$  is Kähler [Bin83, Proposition 4.6 (2)], [Fuj79].

**2.3. Definition.** Let  $X$  be a normal compact Kähler space. We say that  $X$  admits an algebraic approximation if there exists a flat fibration  $\pi : \mathcal{X} \rightarrow \Delta$  such that the scheme-theoretic fibre  $\pi^{-1}(0)$  is isomorphic to  $X$  and there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  converging to 0 such that  $\pi^{-1}(t_n)$  is a projective variety.

In general a flat deformation does not preserve the fundamental group, this holds however for deformations that are locally trivial in the sense of [FK87, p.627]. For the case of elliptic fibrations we will work with an even more restricted class of deformations:

**2.4. Definition.** Let  $\varphi : X \rightarrow S$  be a fibration between normal compact complex spaces. A locally trivial deformation of  $(X, \varphi)$  is a pair of fibrations

$$\pi : \mathcal{X} \rightarrow \Delta, \quad \Phi : \mathcal{X} \rightarrow S \times \Delta$$

such that  $\pi = p_S \circ \Phi$ , where  $p_S : S \times \Delta \rightarrow \Delta$  is the projection on the first factor and the following holds:

- The fibration  $\pi$  has integral reduced fibres and  $X \simeq \pi^{-1}(0)$ .

- We have  $\varphi = \Phi|_{\pi^{-1}(0)}$ .
- There exists an open cover  $(U_i)_{i \in I}$  of  $S$  such that (up to replacing  $\Delta$  by a smaller polydisc) we have

$$\Phi^{-1}(U_i \times \Delta) \simeq \varphi^{-1}(U_i) \times \Delta$$

for all  $i \in I$ .

Let us recall some basic definitions on geometric orbifolds introduced in [Cam04]. They are pairs  $(X, \Delta)$  where  $X$  is a complex manifold and  $\Delta$  a Weil  $\mathbb{Q}$ -divisor; they appear naturally as bases of fibrations to describe their multiple fibres: let  $\varphi : X \rightarrow Y$  be a fibration between compact Kähler manifolds and consider  $|\Delta| \subset Y$  the union of the codimension one components of the  $\varphi$ -singular locus. If  $D \subset |\Delta|$ , we can write

$$\varphi^*(D) = \sum_j m_j D_j + R,$$

where  $D_j$  is mapped onto  $D$  and  $\varphi(R)$  has codimension at least 2 in  $Y$ .

The integer  $m(\varphi, D) = \gcd_j(m_j)$  is called the classical multiplicity of  $\varphi$  above  $D$  and we can consider the  $\mathbb{Q}$ -divisor

$$(1) \quad \Delta = \sum_{D \subset |\Delta|} \left(1 - \frac{1}{m(\varphi, D)}\right) D.$$

The pair  $(Y, \Delta)$  is called the *orbifold base* of  $\varphi$ .

**2.5. Remark.** In Campana's work [Cam04] both the classical and the non-classical multiplicities  $\inf_j(m_j)$  play an important role. Let us note that for elliptic fibrations, these multiplicities coincide: the problem is local on the base, moreover we can reduce to the case of a relatively minimal elliptic fibration. Then it is sufficient to observe that in Kodaira's classification of singular fibres which are not multiple [Kod60], [BHPVdV04, Chapter V, Table 3], there is always at least one irreducible component of multiplicity one.

Let us recall what smoothness means for a geometric orbifold.

**2.6. Definition.** A geometric orbifold  $(X/\Delta)$  is said to be smooth if the underlying variety  $X$  is a smooth manifold and if the  $\mathbb{Q}$ -divisor  $\Delta$  has only normal crossings. If in a coordinate patch, the support of  $\Delta$  can be defined by an equation

$$\prod_{j=1}^r z_j = 0,$$

we will say that these coordinates are adapted to  $\Delta$ .

In the category of smooth orbifolds, there is a good notion of fundamental group. It is defined in the following way : if  $\Delta = \sum_{j \in J} (1 - \frac{1}{m_j}) \Delta_j$ , choose a small loop  $\gamma_j$  around each component  $\Delta_j$  of the support of  $\Delta$ . Consider now the fundamental group of  $X^* = X \setminus \text{Supp}(\Delta)$  and its normal subgroup generated by the loops  $\gamma_j^{m_j}$ :

$$\langle\langle \gamma_j^{m_j}, j \in J \rangle\rangle \leq \pi_1(X^*).$$

**2.7. Definition.** The fundamental group of  $(X/\Delta)$  is defined to be:

$$\pi_1(X/\Delta) := \pi_1(X^*) / \langle\langle \gamma_j^{m_j}, j \in J \rangle\rangle.$$

**2.8. Remark.** By definition the loops  $\gamma_j$  define torsion elements in  $\pi_1(X/\Delta)$ . Thus we see that if  $\pi_1(X/\Delta)$  is torsion-free, the natural surjection  $\pi_1(X/\Delta) \twoheadrightarrow \pi_1(X)$  is an isomorphism.

### 3. ELLIPTIC FIBRATIONS

The structure of elliptic fibrations and their deformation theory has been described in detail in the landmark paper of Kodaira [Kod60] and its generalisation to higher dimension by Nakayama [Nak02b, Nak02a]. For the convenience of the reader we review this theory and explain some additional properties that will be important in the proof of Theorem 1.3.

#### 3.A. Definitions and local structure.

**3.1. Definition.** A fibration  $f : X \rightarrow S$  is elliptic if the general fibre  $X_s$  is isomorphic to an elliptic curve. The elliptic fibration is relatively minimal if  $K_X$  is  $\mathbb{Q}$ -Cartier and there exists a line bundle  $L$  on  $S$  such that

$$mK_X \simeq f^*L$$

for some  $m \in \mathbb{N}^*$ .

We say that  $f$  has a meromorphic section in a point  $s \in S$  if there exists an analytic neighbourhood  $s \in U \subset S$  and a meromorphic map  $s : U \dashrightarrow X$  such that  $\varphi \circ s$  identifies to the identity on the open set  $U_0 \subset U$  where  $s$  is holomorphic.

The theory of elliptic fibrations is based on the use of Weierstraß fibrations:

**3.2. Definition.** Let  $S$  be a complex manifold.

- (a) A Weierstraß fibration over  $S$  consists in a line bundle  $\mathcal{L}$  on  $S$  and two sections  $\alpha \in H^0(S, \mathcal{L}^{(-4)})$  and  $\beta \in H^0(S, \mathcal{L}^{(-6)})$  such that  $4\alpha^3 + 27\beta^2$  is a non zero section of  $H^0(S, \mathcal{L}^{(-12)})$ . With these data, we can associate a projective family of elliptic curves:

$$\mathbb{W} := \mathbb{W}(\mathcal{L}, \alpha, \beta) = \{Y^2Z = X^3 + \alpha XZ^2 + \beta Z^3\} \subset \mathbb{P}$$

where

$$\mathbb{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$$

and  $X, Y$  and  $Z$  are canonical sections of  $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{(-2)}$ ,  $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{(-3)}$  and  $\mathcal{O}_{\mathbb{P}}(1)$  respectively. The restriction of the natural projection  $\mathbb{P} \rightarrow S$  to  $\mathbb{W}$  gives rise to a flat morphism  $p_{\mathbb{W}} : \mathbb{W} \rightarrow S$  whose fibres are irreducible cubic plane curves. This elliptic fibration is endowed with a distinguished section  $\{X = Z = 0\}$ .

- (b) A locally Weierstraß fibration is an elliptic fibration  $f : X \rightarrow S$  such that there exists an open covering  $(U_i)_{i \in I}$  of  $S$  such that the restriction of  $f$  to  $X_i := f^{-1}(U_i)$  is a Weierstraß fibration.

**3.3. Remark.** Since the total space of the Weierstraß fibration  $p_{\mathbb{W}} : \mathbb{W} \rightarrow S$  is by definition a hypersurface in a manifold, the complex space  $\mathbb{W}$  is Gorenstein, so the canonical sheaf is locally free. In case the divisor  $\text{div}(4\alpha^3 + 27\beta^2)$  is simple normal crossings, we know by [Nak88, Corollary 2.4] that  $\mathbb{W}$  has rational, hence canonical, singularities (and it holds of course for the total space of a locally Weierstraß fibration). Since  $p_{\mathbb{W}}$  is flat and the restriction of  $K_{\mathbb{W}}$  to every fibre is trivial, we have

$K_{\mathbb{W}} \simeq p_{\mathbb{W}}^* L$  for some line bundle  $L$  on  $S$ . Thus a locally Weierstraß fibration is relatively minimal.

It is well-known that smooth elliptic fibrations with a section are always Weierstraß:

**3.4. Theorem.** [Kod63] [Nak02b, Proposition 1.2.4] *Let  $S$  be a complex manifold, and let  $f : X \rightarrow S$  be a smooth elliptic fibration admitting a section  $s : S \rightarrow X$ . Then there exists a canonically defined isomorphism  $X \rightarrow \mathbb{W}$  over  $S$  to some Weierstraß fibration  $\mathbb{W} \rightarrow S$  sending  $s$  onto the distinguished section.*

For non-smooth fibrations we can only hope to work with bimeromorphic models:

**3.5. Definition.** *Let  $f : X \rightarrow S$  be an elliptic fibration admitting a meromorphic section  $s : S \dashrightarrow X$ . Denote by  $f^* : X^* \rightarrow S^*$  the restriction to a non-empty Zariski open subset  $S^* \subset S$  such that  $f$  is smooth. A Weierstraß model is a bimeromorphic morphism  $X \dashrightarrow \mathbb{W}$  to some Weierstraß fibration  $\mathbb{W} \rightarrow S$  such that the restriction to  $X^*$  identifies to the isomorphism from Theorem 3.4.*

*Let  $f : X \rightarrow S$  be an elliptic fibration having meromorphic sections in every point. A local Weierstraß model is a bimeromorphic morphism  $X \dashrightarrow \mathbb{W}$  over  $S$  such that for some open covering  $(U_i)_{i \in I}$  the restriction to  $X_i := f^{-1}(U_i)$  and some bimeromorphic section  $s_i : U_i \dashrightarrow X_i$  is a Weierstraß model for every  $i \in I$ .*

The existence of local meromorphic sections is an obvious necessary condition for the existence of a local Weierstraß model. Modulo some further geometric conditions it is also sufficient:

**3.6. Theorem.** [Nak88, Theorem 2.1, Theorem 2.5] [Nak02b, Theorem 3.2.3, Theorem 3.3.3] *Let  $S$  be a complex manifold, and let  $f : X \rightarrow S$  be an elliptic fibration. Suppose that  $f$  is smooth over the complement of an SNC divisor in  $S$ .*

- (a) *If  $f$  admits a (local) meromorphic section (in every point), then  $f$  has a (local) Weierstraß model.*
- (b) *If  $X$  is compact Kähler, then  $f$  is locally projective over  $S$  and we have*

$$R^j f_* \mathcal{O}_X = 0 \quad \forall j \geq 2.$$

The following example highlights the importance of the SNC condition for the theory of elliptic fibrations:

**3.7. Example.** Let  $S$  be a smooth non-algebraic compact Kähler surface that admits an elliptic fibration  $g : S \rightarrow \mathbb{P}^1$ . Let  $\mathbb{F}_1 \rightarrow \mathbb{P}^1$  be the first Hirzebruch surface, and set  $X := \mathbb{F}_1 \times_{\mathbb{P}^1} S$ . Then  $X$  is a smooth compact Kähler threefold, and we denote by  $f : X \rightarrow \mathbb{P}^2$  the composition of the elliptic fibration  $X \rightarrow \mathbb{F}_1$  with the blowdown  $\mathbb{F}_1 \rightarrow \mathbb{P}^2$ .

Then  $f$  is not locally projective since it has a two-dimensional fibre isomorphic to the non-projective surface  $S$ . Note however that  $g$  has at least 3 singular fibres ([Bea81, Proposition 1], cf. Proposition A.1 for a detailed proof in the Kähler case). Thus the discriminant locus of  $f$  consists of at least 3 lines meeting in one point. In particular it is not a SNC divisor and it is quite easy to see that  $R^2 f_* \mathcal{O}_X \neq 0$  in this case.

A general elliptic fibration does not admit local meromorphic sections, a fact that is the starting point of Nakayama's global theory of elliptic fibration using the  $\partial$ -étale

cohomology. For our needs we can use the strategy of Kodaira [Kod63] to reduce to this case via base change:

**3.8. Proposition.** *Let  $f : X \rightarrow S$  be a locally projective elliptic fibration onto a projective manifold  $S$ . Suppose that  $f$  is smooth over the complement of an SNC divisor in  $S$ . Then there exists a finite Galois cover  $\tilde{S} \rightarrow S$  by some projective manifold  $\tilde{S}$  such that*

$$X \times_S \tilde{S} \rightarrow \tilde{S}$$

*has a local meromorphic section in every point. The elliptic fibration  $X \times_S \tilde{S} \rightarrow \tilde{S}$  is smooth over the complement of an SNC divisor in  $\tilde{S}$ .*

This statement is a variant of [Nak02b, Corollary 4.3.3]: in our case  $S$  is projective, but we lose the control over the branch locus.

*Proof.* For every irreducible component  $D_i$  of  $D$  we denote by  $m_i \in \mathbb{N}$  the multiplicity of the generic fibre over  $D_i$ . By [Laz04, Proposition 4.1.12] we can choose a covering  $\tilde{S} \rightarrow S$  ramifying with multiplicity exactly  $m_i$  over  $D_i$  and the ramification divisor has simple normal crossings. By construction the elliptic fibration  $X \times_S \tilde{S} \rightarrow \tilde{S}$  has no multiple fibre in codimension one. Up to taking another finite cover and the Galois closure we can suppose that  $\tilde{S} \rightarrow S$  is Galois and the local monodromies are unipotent. Since the elliptic fibration is locally projective, we can now apply [Nak02b, Theorem 4.3.1, Theorem 4.3.2] to conclude that it has local meromorphic sections in every point.  $\square$

### 3.B. Global structure and deformation theory.

**3.B.1. Smooth fibrations.** Let  $S^*$  be a complex manifold, and let  $f^* : X^* \rightarrow S^*$  be a smooth elliptic fibration. We associate a variation of Hodge structures  $H$  (VHS for short in the sequel) of weight -1 over  $S^*$ , the underlying local system being given by the first homology group of the fibers  $H_1(X_s, \mathbb{Z}) \simeq \mathbb{Z}^2$ . A rank 2 and weight -1 VHS over  $S^*$  is equivalent to the following data: a holomorphic function to the upper half plane

$$\tau_H : \widetilde{S^*} \rightarrow \mathbb{H}$$

defined on the universal cover  $\widetilde{S^*} \rightarrow S^*$  which is equivariant under a representation

$$\rho_H : \pi_1(S^*) \longrightarrow \mathrm{SL}_2(\mathbb{Z}).$$

Now if  $\gamma \in \pi_1(S^*)$ , let us write

$$\rho_H(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

the image of  $\gamma$  under  $\rho_H$ . It is then straightforward to check that the following formula

$$((m, n), \gamma) \cdot (x, z) = \left( \gamma(x), \frac{z + m\tau_H(x) + n}{c_\gamma\tau_H(x) + d_\gamma} \right)$$

defines an action of the semi-direct product  $\mathbb{Z}^2 \rtimes \pi_1(S^*)$  on  $\tilde{S^*} \times \mathbb{C}$  which is fixed point free and properly discontinuous. We can then form the quotient to get a smooth elliptic fibration

$$p : \mathbf{J}(H) \longrightarrow S^*$$

which is endowed with a canonical section  $\sigma : S^* \rightarrow \mathbf{J}(H)$ . Following the terminology of [Kod63, Nak02a] we call  $p$  the *basic elliptic fibration* (associated with  $H$ ).



Note that  $f^*$  and  $p$  are locally isomorphic over  $S^*$ , but their global structure can be quite different.

Since  $p$  has a global section, its sheaf of holomorphic sections  $\mathcal{J}(H)$  is a well-defined sheaf of abelian groups and we have an exact sequence of sheaves

$$(2) \quad 0 \longrightarrow H \longrightarrow \mathcal{L}_H \longrightarrow \mathcal{J}(H) \longrightarrow 0$$

where

$$\mathcal{L}_H := R^1 p_* \mathcal{O}_{\mathbf{J}(H)} \simeq R^1 f_*^* \mathcal{O}_{X^*}.$$

Let us note that  $\mathcal{L}_H$  can also be interpreted as the  $(0, 1)$ -part of the Hodge filtration of  $H_{\mathbb{C}}$  and thus depends only on the VHS  $H$ . Since  $f^*$  is smooth, it has local sections in every point of  $S^*$  and the difference of two such sections on the intersection of their sets of definition can be seen as a section of  $\mathcal{J}(H)$ . In this way we just have associated with  $f^*$  a cohomology class

$$\eta(f^*) \in H^1(S^*, \mathcal{J}(H)).$$

The class  $\eta(f^*)$  can also be constructed in a more conceptual way. Pushing forward the exponential sequence on  $X^*$  by  $f^*$  yields a long exact sequence on  $S^*$ :

$$(3) \quad 0 \longrightarrow R^1 f_*^* \mathbb{Z}_X \longrightarrow R^1 f_*^* \mathcal{O}_X \longrightarrow R^1 f_*^* \mathcal{O}_X^* \longrightarrow R^2 f_*^* \mathbb{Z}_X \simeq \mathbb{Z}_{S^*} \longrightarrow 0$$

The class  $\eta(f^*)$  is the image of  $1 \in H^0(S^*, \mathbb{Z}_{S^*})$  under the connecting morphism

$$\delta : H^0(S^*, \mathbb{Z}_{S^*}) \longrightarrow H^1(S^*, \mathcal{J}(H)).$$

The set of isomorphism classes of smooth elliptic fibrations over  $S^*$  inducing  $H$  has the structure of a torsor over the basic elliptic fibration  $p$ :

**3.9. Theorem.** [Kod63, Theorem 10.1, Theorem 11.5] [Nak02b, Proposition 1.3.1, Proposition 1.3.3] *Let  $S^*$  be a complex manifold, and let  $H$  be a VHS of rank 2 and weight -1 over  $S^*$ .*

- (a) *There is a one-to-one correspondence between the isomorphism classes of smooth elliptic fibrations over  $S^*$  inducing  $H$  and elements of the cohomology group  $H^1(S^*, \mathcal{L}_H)$ .*
- (b) *A smooth elliptic fibration  $f^* : X^* \rightarrow S^*$  is a projective morphism if and only if  $\eta(f^*)$  is a torsion class.*

The short exact sequence (2) induces an exact sequence

$$(4) \quad H^1(S^*, \mathcal{L}_H) \xrightarrow{\text{exp}} H^1(S^*, \mathcal{J}(H)) \xrightarrow{c} H^2(S^*, H).$$

The vector space  $H^1(S^*, \mathcal{L}_H)$  appears as the deformation space of smooth elliptic fibrations over  $S^*$ :

**3.10. Theorem.** [Kod63]

- (a) *Let  $f_1^* : X_1^* \rightarrow S^*$  and  $f_2^* : X_2^* \rightarrow S^*$  be two smooth elliptic fibrations over  $S^*$  inducing  $H$ . Then  $f_1^*$  can be deformed into  $f_2^*$  over  $S^*$  if and only if  $c(\eta(f_1^*)) = c(\eta(f_2^*))$ .*
- (b) *Let  $f^* : X^* \rightarrow S^*$  be a smooth elliptic fibration such that  $X^*$  is Kähler. Then  $c(\eta(f^*))$  is torsion and  $f^*$  can be deformed to a projective fibration.*

*Proof of (b).* The fact that  $c(\eta(f^*))$  is torsion when the total space is Kähler is an easy consequence of the degeneracy of the Leray's spectral sequence (see for instance [Cla16, Proposition 2.5]).  $\square$

**3.C. Elliptic fibrations with local meromorphic sections.** In this subsection we always work under the following

**3.11. Assumption.** Let  $S$  be a complex manifold, and let  $f : X \rightarrow S$  be an elliptic fibration having local meromorphic sections in every point. We denote by  $j : S^* \subset S$  a Zariski open subset such that  $X^* := f^{-1}(S^*) \rightarrow S^*$  is smooth. Denote by  $H$  the VHS on  $S^*$  induced by the smooth elliptic fibration  $X^* \rightarrow S^*$ . We set

$$\mathcal{L} := \mathcal{L}_{H/S} := R^1 f_* \mathcal{O}_X.$$

We suppose that the complement  $D := S \setminus S^*$  is a SNC divisor.

Let  $p : \mathbf{J}(H) \rightarrow S^*$  be the basic elliptic fibration associated with  $H$ . By [Nak88, Theorem 2.5] (see Theorem 3.6) we can extend  $p$  to a unique Weierstraß model

$$p_{\mathbb{W}} : \mathbb{W} \rightarrow S.$$

We also denote by  $p : \mathbf{B}(H) \rightarrow S$  the composition of  $p_{\mathbb{W}}$  with a desingularisation  $\mathbf{B}(H) \rightarrow \mathbb{W}$ . Since  $p_{\mathbb{W}}$  has a canonical section, the elliptic fibration  $p$  has a global meromorphic section. We call  $p$  a basic elliptic fibration associated with  $H$  [Nak02a, p.549].

Let  $\mathbb{W}^{\#} \subset \mathbb{W}$  be the Zariski open set of points in which  $p_{\mathbb{W}}$  is smooth. The image of the canonical section  $\sigma$  is contained in  $\mathbb{W}^{\#}$  and defines a relative group structure on  $\mathbb{W}^{\#} \rightarrow S$  [Nak02a, Lemma 5.1.1]. The image of any local section of  $p_{\mathbb{W}}$  is contained in  $\mathbb{W}^{\#}$ , so its sheaf of sections  $\mathcal{J}(H)^{\mathbb{W}}$  has an abelian group structure. Denote by  $\mathcal{L}_{H/S} \rightarrow S$  the line bundle defining the Weierstraß model  $\mathbb{W}$ . By [Nak02a, p.550] we have an exact sequence

$$0 \rightarrow j_* H \rightarrow \mathcal{L}_{H/S} \rightarrow \mathcal{J}(H)^{\mathbb{W}} \rightarrow 0$$

which extends the exact sequence (2) defined on  $S^* \subset S$ . The following observation is crucial:

**3.12. Lemma.** [Nak02a, Lemma 5.1.4] *The sheaf  $\mathcal{J}(H)^{\mathbb{W}}$  is isomorphic to the sheaf of germs of automorphisms  $\varphi : \mathbb{W} \rightarrow \mathbb{W}$  over  $S$  such that  $\varphi^*|_{S^*}$  induces the identity on  $H$ .*

Thus we can associate with a cohomology class  $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  an elliptic fibration [Nak02a, p.550]:

**3.13. Construction.** *Fix an open cover  $(U_j)_{j \in \mathbb{N}}$  of  $S$  such that the class  $\eta$  is represented by a cocycle  $(\eta_{ij})_{i < j}$  where  $\eta_{ij} \in H^0(U_i \cap U_j, \mathcal{J}(H)^{\mathbb{W}})$ . By the lemma above we can identify the  $\eta_{ij}$  to automorphisms of  $\mathbb{W}|_{U_i \cap U_j}$  over  $S$ . The cocycle condition assures that we satisfy the condition of the gluing lemma [Har77, Chapter II, Exercise 2.12], so we can glue the elliptic fibrations  $\mathbb{W}|_{U_i} \rightarrow U_i$  to an elliptic fibration  $p^{\eta} : \mathbb{W}^{\eta} \rightarrow S$ . Since the gluing morphisms act by identity on the VHS, the VHS induced by  $p^{\eta}$  on  $S^*$  is  $H$ .*

The cohomology group  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  is thus a parameter set of elliptic fibrations over  $S$  with VHS  $H$ , but for classification purposes it is too small. We denote by  $\mathcal{J}(H)_{\text{mer}}$  the sheaf of meromorphic sections of a basic elliptic fibration  $p : \mathbf{B}(H) \rightarrow S$ . Since two basic elliptic fibrations are bimeromorphically equivalent the sheaf  $\mathcal{J}(H)_{\text{mer}}$  does not depend on the choice of the model. Moreover, since  $p$  has a

global meromorphic section, we see that  $\mathcal{J}(H)_{mer}$  has a group structure [Nak02b, p. 243-244]. There is a trivial inclusion of sheaves of abelian groups

$$(5) \quad \mathcal{J}(H)^{\mathbb{W}} \subset \mathcal{J}(H)_{mer}$$

which is an isomorphism on  $S^*$ : since  $\mathbb{W}$  is smooth over  $S^*$  we have  $\mathbf{J}(H) \simeq \mathbf{B}(H)|_{S^*} \simeq \mathbb{W}|_{S^*}$ , moreover any meromorphic section is holomorphic over  $S^*$  [Nak02b, Lemma 1.3.5]. In particular the quotient sheaf

$$\mathcal{Q}_H := \mathcal{J}(H)_{mer} / \mathcal{J}(H)^{\mathbb{W}}$$

is supported on  $D = S \setminus S^*$ . By [Nak02a, Theorem 5.4.9] we have a commutative diagram

$$(6) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & j_*H & \longrightarrow & \mathcal{L}_{H/S} & \longrightarrow & \mathcal{J}(H)^{\mathbb{W}} \longrightarrow 0 \\ & & & \downarrow & & & \\ & & 0 & \longrightarrow & \mathcal{J}(H)_{mer} & \xrightarrow{\Psi_f} & R^1 f_* \mathcal{O}_X^* / \mathcal{V}_X \longrightarrow \mathbb{Z}_S \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \downarrow \\ & & 0 & \longrightarrow & \mathcal{Q}_H & \longrightarrow & R^2 f_* \mathbb{Z}_X / \mathcal{V}_X \longrightarrow \mathbb{Z}_S \longrightarrow 0 \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

where

$$\mathcal{V}_X := \text{Ker} (R^1 f_* \mathcal{O}_X^* \longrightarrow j_*((R^1 f_* \mathcal{O}_X^*)|_{S^*}))$$

and  $\Psi_f$  is constructed from the local meromorphic sections of  $f$ .

**3.14. Definition.** *We define*

$$\eta(f) \in H^1(S, \mathcal{J}(H)_{mer})$$

*to be the image of  $1 \in H^0(S, \mathbb{Z}_S)$  under the connecting morphism of the long exact sequences associated with the second line of Diagram (6).*

By [Nak02a, Proposition 5.5.1] we have an injection

$$\mathcal{E}_0(S, D, H) \hookrightarrow H^1(S, \mathcal{J}(H)_{mer}),$$

where  $\mathcal{E}_0(S, D, H)$  is the set of bimeromorphic equivalence classes of elliptic fibrations  $X \rightarrow S$  having meromorphic sections in every point that are smooth over  $S^* := S \setminus D$  and induce the vhs  $H$  on  $S^*$ . By Construction 3.13 we have

$$H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \rightarrow \mathcal{E}_0(S, D, H) \hookrightarrow H^1(S, \mathcal{J}(H)_{mer})$$

but contrary to the smooth case it is not clear if the images coincide. If  $S$  is a curve the skyscraper sheaf  $\mathcal{Q}_H$  has no higher cohomology so the map

$$H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \longrightarrow H^1(S, \mathcal{J}(H)_{mer})$$

is surjective.

3.C.1. *The Kähler case.* From now on we will focus on the case where the total space of the elliptic fibration is compact Kähler. In that case, up to replacing multiplication in the fibers, the fibration has a local Weierstraß model (cf. [Nak02a, Proposition 7.4.2] for a more general statement).

**3.15. Lemma.** *In the situation of Assumption 3.11, suppose also that  $X$  is bimeromorphic to a compact Kähler manifold. Then the image of the class  $\eta(f) \in H^1(S, \mathcal{J}(H)_{\text{mer}})$  is torsion in  $H^1(S, \mathcal{Q}_H)$ . In particular there exists an integer  $m \geq 1$  such that*

$$m \cdot \eta(f) \in H^1(S, \mathcal{J}(H)^{\mathbb{W}}).$$

*Proof.* Since the class  $\eta(f)$  depends only on the bimeromorphic equivalence class of  $X \rightarrow S$  we can suppose that  $X$  is a compact Kähler manifold.

Let  $\omega \in H^2(X, \mathbb{R})$  be a Kähler class on  $X$ . By density, there exists a class  $\alpha \in H^2(X, \mathbb{Q})$  (in general not of type  $(1, 1)$ ) such that  $\alpha \cdot F \neq 0$  where  $F$  is a general  $f$ -fibre. The class  $\alpha$  defines a global section of  $R^2 f_* \mathbb{Q}_X$  and we can cancel the denominators in such a way that  $\alpha$  defines a section of  $R^2 f_* \mathbb{Z}_X$  and thus a non-zero element  $\bar{\alpha} \in H^0(S, R^2 f_* \mathbb{Z}_X / \mathcal{V}_X)$ . The third line of (6) induces an exact sequence

$$H^0(S, R^2 f_* \mathbb{Z}_X / \mathcal{V}_X) \xrightarrow{\tau} H^0(S, \mathbb{Z}_S) \xrightarrow{\delta} H^1(S, \mathcal{Q}_H)$$

and is straightforward to check that  $\tau(\bar{\alpha}) = F \cdot \alpha$ . It is then a positive multiple of the class  $1 \in H^0(S, \mathbb{Z}_S)$  and it proves  $\delta(1)$  is a torsion class in  $H^1(S, \mathcal{Q}_H)$ . A diagram chase in (6) shows that  $\delta(1)$  is the image of  $\eta(f)$  in  $H^1(S, \mathcal{Q}_H)$ .  $\square$

We can now generalise Theorem 3.10 (cf. also [Nak02a, Proposition 7.4.2]):

**3.16. Theorem.** *In the situation of Assumption 3.11, let us assume also that  $X$  is bimeromorphic to a compact Kähler manifold. Suppose also that  $\eta(f)$  is in the image of  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ . Denote by*

$$c : H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \rightarrow H^2(S, j_* H)$$

*the morphism defined by the first line of the exact sequence (6). Then the class  $c(\eta(f))$  is torsion in  $H^2(S, j_* H)$ .*

In order to simplify the notation we identified  $\eta(f) \in H^1(S, \mathcal{J}(H)_{\text{mer}})$  with a class of  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  mapping onto it.

*Proof.* By Theorem 3.10 we already know that  $c(\eta(f))|_{S^*}$  is torsion, say of order  $m \in \mathbb{N}$ . Fibrewise multiplication by  $m$  defines a natural morphism  $X^* \rightarrow (X^*)^{(m)}$  where  $(X^*)^{(m)} \rightarrow S^*$  is the smooth elliptic fibration corresponding to  $m\eta(f)|_{S^*}$ . By [Nak02a, Proposition 6.3.7] we can extend this map to a finite morphism  $X \rightarrow X^{(m)}$  where  $X^{(m)} \rightarrow S$  is a representative of the class  $m\eta(f)$ . By Varouchas's theorem [Var84] the complex space  $X^{(m)}$  is bimeromorphic to a compact Kähler manifold. Thus, up to replacing  $X$  by (some blowup of)  $X^{(m)}$ , we can assume  $X$  is Kähler and  $c(\eta(f))|_{S^*}$  vanishes.

Now we look at the piece of exact sequence given by the Leray spectral sequence of the inclusion  $j : S^* \hookrightarrow S$  (for the sheaf  $H$ ):

$$(7) \quad H^1(S^*, H) \rightarrow H^0(S, R^1 j_* H) \rightarrow H^2(S, j_* H) \rightarrow H^2(S^*, H).$$

Since we know that  $c(\eta(f))|_{S^*} = 0$ , there exists a class  $\beta \in H^0(S, R^1j_*H)$  mapped onto  $c(\eta(f))$ . Let us remind the reader how this class  $\beta$  is constructed. If  $U$  is a small ball in  $S$ , we note  $U^* := U \cap S^*$  and, due to the SNC assumption on  $S \setminus S^*$ , we can assume that  $U^*$  is a product of discs and punctured discs. In particular we have  $\pi_2(U^*) = 0$  and the homotopy exact sequence yields ( $E$  being the fiber of  $f$ ):

$$(8) \quad 1 \longrightarrow \pi_1(E) \longrightarrow \pi_1(X_{U^*} := f^{-1}(U^*)) \longrightarrow \pi_1(U^*) \longrightarrow 1.$$

It is classically known that abelian extensions are classified by the group

$$H^2(\pi_1(U^*), \pi_1(E)) = H^2(U^*, H)$$

and the corresponding class is nothing but  $c(\eta(f))|_{U^*}$ . By the vanishing of the latter, the sequence (8) is split and such an extension gives rise to an extension class  $\beta_U \in H^1(\pi_1(U^*), \pi_1(E)) = H^1(U^*, H)$ . The collection  $(\beta_U)_{U \subset S}$  provides us with the sought element in  $H^0(S, R^1j_*H)$ .

On the other hand, the Kähler assumption can be used once more: the fundamental group of the fiber  $E$  injects in the whole open set  $X^* := f^{-1}(S^*)$  (see [Cla10, Lemme 2.3]). We have thus a short exact sequence

$$1 \longrightarrow \pi_1(E) \longrightarrow \pi_1(X^*) \longrightarrow \pi_1(S^*) \longrightarrow 1$$

whose extension class vanishes. With this split extension we can associate a global class  $\alpha \in H^1(\pi_1(S^*), \pi_1(E)) = H^1(S^*, H)$ . By construction the class  $\alpha$  is mapped onto  $\beta$  in the sequence (7). The latter being an exact sequence, this shows that  $c(\eta(f))$  vanishes.  $\square$

**3.17. Remark.** The last result gives a direct proof of a phenomenon which was *observed* by Kodaira in the case  $\dim S = 1$ : he first proved that the cohomology group  $H^2(S, j_*H)$  is finite if the VHS is not trivial. He then computed the first Betti number of an elliptic surface when  $H$  is trivial and obtained in [Kod63, Theorem 11.9] that this quantity is even for the basic member (here a product  $S \times E$  where  $E$  is the fixed elliptic curve) and odd else. *A posteriori* we can conclude that an elliptic surface  $f : X \rightarrow S$  (without multiple fibres) is Kähler if and only if  $c(\eta(f))$  is torsion in  $H^2(S, j_*H)$ . We will now prove that this equivalence also holds in our setting:

**3.18. Proposition.** *In the situation of Assumption 3.11, suppose also that the base  $S$  is a compact Kähler manifold. Let  $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  be a class such that  $c(\eta)$  is torsion in  $H^2(S, j_*H)$ . Then the total space  $\mathbb{W}^\eta \rightarrow S$  is Kähler space.*

*Proof.* Recall that the Weierstraß model  $p_{\mathbb{W}} : \mathbb{W} \rightarrow S$  is a projective morphism. Since  $S$  is compact Kähler, the total space  $\mathbb{W}$  is Kähler by Remark 2.2. The Weierstraß fibration comes equipped with a full family of deformation (over  $S$ ) parametrized by the vector space  $H^1(S, \mathcal{L})$ . Let us denote it by  $\mathcal{W} \rightarrow S \times H^1(S, \mathcal{L}) \rightarrow S$  and recall that

$$\eta(\mathcal{W}_t \rightarrow S) = \exp(t) \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$$

for any  $t \in H^1(S, \mathcal{L})$ . By Remark 3.3 the complex spaces  $\mathbb{W}^\eta$  have at most canonical, hence rational, singularities. From [Nam01, Proposition 5] we know that any small flat deformation of compact Kähler space having rational singularities remains Kähler. Thus  $\mathbb{W}^{\exp(t)}$  is Kähler for  $t$  in a neighborhood  $U$  of  $0 \in H^1(S, \mathcal{L})$ .

Now if  $t$  is given in  $H^1(S, \mathcal{L})$  let us consider a positive integer  $m$  such  $t/m \in U$ . The  $m^{\text{th}}$  morphism (multiplication by  $m$  on the fibers) is a finite morphism

$$\mathbb{W}^{\exp(t/m)} \longrightarrow \mathbb{W}^{m \cdot \exp(t/m)} = \mathbb{W}^{\exp(t)}.$$

Since the total space is Kähler and the map finite, [Var89, Cor.3.2.2] implies that  $\mathbb{W}^{\exp(t)}$  is Kähler as well.

Now we treat the general case. If  $\eta \in H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  is such that  $c(\eta)$  is torsion then there exists an integer  $m$  and a element  $t \in H^1(S, \mathcal{L})$  such that  $m \cdot \eta = \exp(t)$ . Using again the  $m^{\text{th}}$  morphism, we get a finite map

$$\mathbb{W}^{\eta} \longrightarrow \mathbb{W}^{m \cdot \eta} = \mathbb{W}^{\exp(t)},$$

the target of this map being Kähler. We readily conclude that  $\mathbb{W}^{\eta}$  is a Kähler space (see for instance [Var89, Proposition 1.3.1]).  $\square$

For classes in  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  the deformation theory is analogous to the smooth case:

**3.19. Proposition.** *Under the Assumption 3.11, let  $\eta, \theta$  be cohomology classes in  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  such that  $c(\eta) = c(\theta)$ . Then the elliptic fibrations  $p^{\eta} : \mathbb{W}^{\eta} \rightarrow S$  and  $p^{\theta} : \mathbb{W}^{\theta} \rightarrow S$  can be deformed from one to each other by a deformation that is locally trivial over the base (cf. Definition 2.4).*

*Proof.* Since  $c(\eta) = c(\theta)$ , we can write

$$\eta = \theta + \exp(\xi)$$

for some  $\xi \in H^1(S, \mathcal{L})$ . This defines a normal complex space  $\mathcal{X}$  with canonical singularities and morphisms

$$\pi : \mathcal{X} \longrightarrow S \times \mathbb{C} \longrightarrow \mathbb{C}$$

such that the fibre  $\pi^{-1}(t)$  is isomorphic to the elliptic fibration

$$p^{\eta + \exp(t\xi)} : \mathbb{W}^{\eta + \exp(t\xi)} \rightarrow S.$$

In particular  $\mathbb{W}^{\eta} = \pi^{-1}(0)$  and  $\mathbb{W}^{\theta} = \pi^{-1}(1)$ , so this defines a flat deformation from  $\mathbb{W}^{\eta}$  to  $\mathbb{W}^{\theta}$ .

More precisely, we fix an open covering  $(U_i)_{i \in I}$  of  $S$  such that the cohomology classes are represented by Čech cocycles  $(\eta_{ij})$ ,  $(\theta_{ij})$  and  $(\xi_{ij})$  defined on the intersections  $U_i \cap U_j$ . The complex space  $\mathcal{X}$  is then obtained by gluing the families  $\mathbb{W}_i \times \mathbb{C}$  using the automorphisms corresponding to  $\eta_{ij} + \exp(t\xi_{ij})$ . Let  $U \subset S$  be an open subset such that the  $H^1(U, \mathcal{L}|_U) = 0$  (say  $U$  is a polydisc). Then the class of

$$(\eta + \exp(t\xi))|_U \in H^1(U, \mathcal{J}(H)^{\mathbb{W}})$$

does not depend on  $t$ . Thus the deformation is trivial over  $U$ .  $\square$

**3.C.2. Hodge theory of Weierstraß models.** The main purpose of this paragraph is to establish the following result.

**3.20. Theorem.** *In the situation of Assumption 3.11, suppose that  $S$  is a compact Kähler manifold. Then the canonical map*

$$(9) \quad H^1(S, j_* H_{\mathbb{R}}) \longrightarrow H^1(S, \mathcal{L})$$

*is surjective.*

When  $S^* = S$  this is a straightforward consequence of the existence of a pure Hodge structure of weight 2 on the lattice  $H^1(S, H)$  as constructed by Deligne (see [Zuc79, Theorem 2.9]).

**3.21. Remark.** Before giving the proof of Theorem 3.20, let us remark that this statement has a geometric counterpart. If  $\mathcal{W} \rightarrow S \times H^1(S, \mathcal{L}) \rightarrow H^1(S, \mathcal{L})$  is the tautological family, Theorem 3.20 is equivalent to the density of fibrations  $\mathcal{W}_t \rightarrow S$  that are projective over  $S$ . Indeed such an elliptic fibration is projective over  $S$  if and only if its cohomology class  $\eta(\mathcal{W}_t \rightarrow S)$  is torsion in  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$  (see [Nak02a, Theorem 6.3.8]). Using the first line of (6), the exact sequence

$$H^1(S, j_*H) \rightarrow H^1(S, \mathcal{L}) \rightarrow H^1(S, \mathcal{J}(H)^{\mathbb{W}})$$

shows that this happens exactly when  $t$  lies in the range of the map  $H^1(S, j_*H_{\mathbb{Q}}) \rightarrow H^1(S, \mathcal{L})$ . Now it is clear that density of projective elliptic fibrations is equivalent to surjectivity of the map (9).

To prove Theorem 3.20, we will establish the density of projective fibrations in the tautological family. The following criterion is reminiscent from Buchdahl's works [Buc06, Buc08]:

**3.22. Lemma.** *Let  $\pi : \mathcal{X} \rightarrow B$  be a smooth family of compact Kähler manifolds, and let  $\Phi : \mathcal{X} \rightarrow S \times B \rightarrow B$  be a fibration such that  $\pi = pr_B \circ \Phi$ . Consider the following VHS over  $B$ :*

$$\mathbb{V} := R^2\pi_*\mathbb{Q}/H^2(S, \mathbb{Q}).$$

*Let  $b \in B$  be a point, and  $[\omega]$  a Kähler class defined on  $X := \mathcal{X}_b$ . If the composition of the maps*

$$T_{B,b} \xrightarrow{\kappa_S} H^1(X, T_X) \xrightarrow{\bullet \wedge [\omega]} H^2(X, \mathcal{O}_X) \rightarrow \mathbb{V}_b^{0,2}$$

*is surjective, then the set of parameters  $u \in B$  such that the morphism  $\mathcal{X}_u \rightarrow S$  is projective is dense near  $b$ .*

In the statement above, the first arrow is the Kodaira-Spencer map associated with  $\pi$ , and the second one is induced by the contraction with the class  $\omega \in H^1(X, \Omega_X^1)$ .

*Proof.* This is nothing but [Voi02, Proposition 17.20, p.410] applied to the VHS  $\mathbb{V}$ .  $\square$

The deformation families provided by Nakayama's theory are not smooth, so in order to apply the relative Buchdahl criterion we have to pass to a smooth model. Kollár's theory of strong resolutions [Kol07, Chapter 3] gives a resolution in families:

**3.23. Lemma.** *Let  $p_0 : \mathbb{W}_0 \rightarrow S$  be a fibration from a normal compact complex space onto a compact complex manifold  $S$ , and  $p : \mathcal{W} \rightarrow S \times B$  be a locally trivial deformation of  $(\mathbb{W}_0, p_0)$  (cf. Definition 2.4). Then (up to replacing  $B$  by a smaller open set) there exists a resolution of singularities  $\mu : \mathcal{X} \rightarrow \mathcal{W}$  such that the family*

$$\pi := pr_B \circ p \circ \mu : \mathcal{X} \rightarrow B$$

*is a family of compact complex manifolds, and for every  $b \in B$  the map  $\mu_b : \mathcal{X}_b \rightarrow \mathbb{W}_b$  is a resolution of singularities that is functorial.*

*Proof.* By [Kol07, Theorem 3.35] (the analytic situation is dealt with in [Wlo09]) there exists a functorial resolution  $\mu : \mathcal{X} \rightarrow \mathcal{W}$ . This resolution commutes with smooth maps, so if  $(U_i)_{i \in I}$  is a finite open cover<sup>2</sup> of  $S$  such that  $p^{-1}(U_i \times B) \simeq p_0^{-1}(U_i) \times B$ , then the resolution is the product of the functorial resolution of  $p_0^{-1}(U_i)$  times the identity.  $\square$

We follow the convention of [Ser06, Appendix B, p.287]: given a morphism  $f : X \rightarrow S$  of normal varieties, we denote by  $T_{X/S}$  the dual of the sheaf of Kähler differentials  $\Omega_{X/S}$ . In particular  $T_{X/S}$  is always a reflexive sheaf.

**3.24. Lemma.** *Let  $p^\eta : \mathbb{W}^\eta \rightarrow S$  be a local Weierstraß fibration over a smooth base  $S$ . Then we have*

$$(10) \quad T_{\mathbb{W}^\eta/S} \simeq (p^\eta)^* L.$$

*Moreover, let  $\mu : X \rightarrow \mathbb{W}^\eta$  be a functorial resolution of singularities and set  $f := p^\eta \circ \mu$ . Then there exists a natural injection  $\mu^* T_{\mathbb{W}^\eta/S} \rightarrow T_{X/S}$  inducing a map*

$$(11) \quad H^1(\mathbb{W}^\eta, T_{\mathbb{W}^\eta/S}) \rightarrow H^1(X, T_{X/S}).$$

*Proof.* All the fibres of  $p^\eta$  are reduced plane cubics, so there exists a codimension two subset  $Z \subset \mathbb{W}^\eta$  such that  $p^\eta|_{\mathbb{W}^\eta \setminus Z}$  is a smooth fibration. On this smooth locus we have by construction  $T_{\mathbb{W}^\eta \setminus Z/S} \simeq (p^\eta|_{\mathbb{W}^\eta \setminus Z})^* L$ . Since  $T_{\mathbb{W}^\eta/S}$  and  $p^* L$  are both reflexive, the isomorphism extends to an isomorphism on  $\mathbb{W}^\eta$ . This shows (10), in particular  $T_{\mathbb{W}^\eta/S}$  is locally free.

Since the resolution  $\mu$  is functorial, the direct image sheaf  $\mu_*(T_X) \subset T_{\mathbb{W}^\eta}$  is reflexive [GKK10, Corollary 4.7]. Thus for any open subset  $U \subset \mathbb{W}^\eta$ , the restriction map

$$\Gamma(\mu^{-1}(U), T_X) \rightarrow \Gamma(\mu^{-1}(U) \setminus \text{Exc}(\mu), T_X)$$

is surjective. Using the exact sequence

$$0 \rightarrow T_{X/S} \rightarrow T_X \rightarrow f^* T_S$$

and the fact that  $f^* T_S$  is torsion-free, we obtain that

$$\Gamma(\mu^{-1}(U), T_{X/S}) \rightarrow \Gamma(\mu^{-1}(U) \setminus \text{Exc}(\mu), T_{X/S})$$

is surjective. Thus  $\mu_*(T_{X/S})$  is reflexive, and the natural map  $\mu_*(T_{X/S}) \rightarrow T_{\mathbb{W}^\eta/S}$  is an isomorphism. By applying the projection formula to the inverse  $T_{\mathbb{W}^\eta/S} \rightarrow \mu_*(T_{X/S})$  we obtain an injective morphism

$$\mu^* T_{\mathbb{W}^\eta/S} \rightarrow T_{X/S}.$$

Since  $T_{\mathbb{W}^\eta/S}$  is locally free and  $\mathbb{W}^\eta$  has rational singularities, we have an isomorphism

$$H^1(\mathbb{W}^\eta, T_{\mathbb{W}^\eta/S}) \simeq H^1(X, \mu^* T_{\mathbb{W}^\eta/S})$$

given by  $\mu^*$ . The statement follows by composing this isomorphism with the map  $H^1(X, \mu^* T_{\mathbb{W}^\eta/S}) \rightarrow H^1(X, T_{X/S})$ .  $\square$

Following [Ser06, Chapter 3.4.2] (cf. [FK87] for a presentation in the analytic setting) we consider the functor of locally trivial deformations of  $f : X \rightarrow S$  with fixed target  $S$ . By [Ser06, Lemma 3.4.7.b) and Theorem 3.4.8] we have an injection

$$H^1(X, T_{X/S}) \hookrightarrow D_{X/S},$$

---

<sup>2</sup>By Definition 2.4 these covers exist up to replacing  $B$  by a smaller open subset.



where  $D_{X/S}$  is the tangent space of the semiuniversal deformation. Given a locally trivial deformation

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & S \times B \\ & \searrow \pi & \downarrow pr_B \\ & & B \end{array}$$

parametrised by a smooth base  $B$ , we have for every  $b \in B$  a Kodaira-Spencer map

$$\kappa_{\zeta\Phi} : T_{B,b} \rightarrow D_{X/S}$$

associating a tangent vector with the corresponding first-order deformation.

*Proof of Theorem 3.20.* Let us first recall from Remark 3.21 that the surjectivity involved in the statement of Theorem 3.20 is equivalent to the density of projective fibrations in the tautological Weierstraß family

$$p : \mathcal{W} \rightarrow S \times H^1(S, \mathcal{L}).$$

Fix now  $b \in H^1(S, \mathcal{L})$ . By Lemma 3.23 there exists (up to replacing the base  $H^1(S, \mathcal{L})$  by a neighbourhood of the point  $b$ ) a simultaneous functorial resolution of the tautological family:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mu} & \mathcal{W} \\ \searrow \Phi & & \swarrow p \\ & S \times H^1(S, \mathcal{L}) & \\ \searrow \pi & \downarrow & \swarrow \\ & H^1(S, \mathcal{L}) & \end{array}$$

In order to simplify the notation, we replace  $p_b : \mathbb{W}^{\exp(b)} \rightarrow S$  by  $p_b : \mathbb{W}_b \rightarrow S$ . We have a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_b & \xrightarrow{\mu_b} & \mathbb{W}_b \\ \searrow f_b & & \swarrow p_b \\ & S & \end{array}$$

where  $\mu_b$  is a functorial resolution of singularities of  $\mathbb{W}_b$ .

*Step 1.* The Kodaira-Spencer map  $\kappa_{\zeta\Phi,b}$  is given by  $f_b^*$ . Using (10) one shows easily that the Kodaira-Spencer map  $\kappa_{\zeta p}$  for any point  $b \in H^1(S, \mathcal{L})$  identifies to the pull-back

$$p_b^* : H^1(S, \mathcal{L}) \rightarrow H^1(\mathbb{W}_b, p_b^* \mathcal{L}) \simeq H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}).$$

By functoriality of the Kodaira-Spencer map we have a factorisation

$$\begin{array}{ccc} H^1(S, \mathcal{L}) & \xrightarrow{\kappa_{\zeta\Phi,b}} & H^1(\mathcal{X}_b, T_{\mathcal{X}_b/S}) \\ & \searrow \kappa_{\zeta p,b} & \downarrow \\ & & H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}). \end{array}$$

By (11) the right column has an inverse  $H^1(\mathbb{W}_b, T_{\mathbb{W}_b/S}) \rightarrow H^1(\mathcal{X}_b, T_{\mathcal{X}_b/S})$  defined by  $\mu_b^*$ . Since  $\kappa_{\zeta_p, b}$  identifies to the pull-back  $p_b^*$  we obtain that  $\kappa_{\zeta_{\Phi}, b}$  identifies to

$$f_b^* : H^1(S, \mathcal{L}) \rightarrow H^1(\mathcal{X}_b, (f_b)^* \mathcal{L}).$$

*Step 2. Applying the relative Buchdahl criterion.* We want to apply Lemma 3.22 to the family  $\mathcal{X} \rightarrow S \times H^1(S, \mathcal{L}) \rightarrow H^1(S, \mathcal{L})$ . Fix a point  $b \in H^1(S, \mathcal{L})$ , and observe that

$$\mathbb{V}_b^{0,2} = H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) / H^2(S, \mathcal{O}_S).$$

We have to check that the composed map

$$(12) \quad H^1(S, \mathcal{L}) \xrightarrow{\kappa_{\zeta_{\Phi}, b}} H^1(\mathcal{X}_b, T_{\mathcal{X}_b}) \xrightarrow{\bullet \wedge^{[\omega]}} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \longrightarrow H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) / H^2(S, \mathcal{O}_S)$$

is surjective.

*Claim: the quotient  $H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) / H^2(S, \mathcal{O}_S)$  is isomorphic to  $H^1(S, \mathcal{L})$  and the projection  $H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \longrightarrow H^1(S, \mathcal{L})$  is induced by  $(f_b)_*$ .*

By Theorem 3.6 we have  $R^1(f_b)_* \mathcal{O}_{\mathcal{X}_b} \simeq \mathcal{L}$  and  $R^2(f_b)_* \mathcal{O}_{\mathcal{X}_b} = 0$ . Moreover, since  $\mathcal{X}_b$  is compact Kähler, the natural map  $H^2(S, \mathcal{O}_S) \rightarrow H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})$  is injective. Thus the Leray spectral sequence yields an exact sequence

$$0 \rightarrow H^2(S, \mathcal{O}_S) \rightarrow H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \rightarrow H^1(S, R^1(f_b)_* \mathcal{O}_{\mathcal{X}_b}) \simeq H^1(S, \mathcal{L}) \rightarrow 0.$$

This shows the claim.

Combining Step 1 and the claim the composed map (12) is given by:

$$H^1(S, \mathcal{L}) \xrightarrow{f_b^*} H^1(\mathcal{X}_b, T_{\mathcal{X}_b}) \xrightarrow{\bullet \wedge^{[\omega]}} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) \xrightarrow{(f_b)_*} H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) / H^2(S, \mathcal{O}_S).$$

Using Dolbeault representatives we see that

$$(f_b)_*(\omega \wedge f_b^*(\alpha)) = c_\omega \alpha$$

where  $c_\omega = \omega \cdot F$  with  $F$  a general  $f_b$ -fibre. Thus (12) is even an isomorphism.  $\square$

**3.D. Construction of the algebraic approximation.** We will now prove the existence of algebraic approximations modulo a technical condition. The proof of the main results then consists of rather simple reduction arguments.

Let  $X$  be a compact Kähler manifold and  $S$  a projective manifold. Let

$$f : X \rightarrow S$$

be an elliptic fibration that is smooth over  $S^*$  and assume that  $D := S \setminus S^*$  is a SNC divisor. By Theorem 3.6 the fibration  $f$  is locally projective. Thus we can apply Proposition 3.8 to see that there exists a Galois cover  $\tilde{S} \rightarrow S$  with Galois group  $G$  such that the induced elliptic fibration  $X \times_S \tilde{S} \rightarrow \tilde{S}$  has local meromorphic sections in every point. Let  $\tilde{X}$  be a  $G$ -equivariant resolution of singularities, then we obtain a  $G$ -equivariant elliptic fibration  $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$  with local meromorphic sections. The complex manifold  $\tilde{X}$  is Kähler, since it is a resolution of the Kähler space  $X \times_S \tilde{S} \subset X \times S$ . Moreover  $\tilde{f}$  is smooth over  $\tilde{j} : \tilde{S}^* \hookrightarrow \tilde{S}$ , the complement of a SNC divisor, since this holds already for  $X \times_S \tilde{S} \rightarrow \tilde{S}$  by Proposition 3.8. We denote by  $\tilde{H}$  the vhs induced by  $\tilde{f}$  on  $\tilde{S}^*$ , and by  $\tilde{\mathbb{W}}$  the Weierstraß model ( $\tilde{H}$  is nothing but the pull-back of  $H$  to  $\tilde{S}^*$ ).

The group  $G$  acts equivariantly on the fibration  $\tilde{f}$ , so it acts on the Weierstraß model and on the cohomology groups involved. By construction the class  $\eta(\tilde{f})$  is  $G$ -invariant, so  $G$  acts equivariantly on the normal complex space  $\mathbb{W}^{\eta(\tilde{f})}$  (cf. Construction 3.13 see also [Cla16, §2.3 and 2.4]), and we denote by

$$\mathbb{W}^{\eta(f)} := \mathbb{W}^{\eta(\tilde{f})} / G$$

the quotient. Note that we did not define the cohomology class  $\eta(f)$  for an elliptic fibration that does not have local meromorphic sections. However, using Nakayama's  $\partial$ -étale cohomology  $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$  [Nak02a, §6.3], one can show that  $\mathbb{W}^{\eta(f)}$  represents the  $\partial$ -étale cohomology class  $\eta(f)$ . We also denote by

$$p^{\eta(f)} : \mathbb{W}^{\eta(f)} \rightarrow S$$

the induced elliptic fibration. By Proposition 3.18 the complex space  $\mathbb{W}^{\eta(\tilde{f})}$  is Kähler, so  $\mathbb{W}^{\eta(f)}$  is also Kähler by [Var89, Corollary 3.2.2].

**3.25. Theorem.** *Let  $X$  be a compact Kähler manifold and  $S$  a projective manifold. Let  $f : X \rightarrow S$  be an elliptic fibration that is smooth over  $S^*$  and assume that  $D := S \setminus S^*$  is a SNC divisor. Using the notation above, suppose that we have*

$$(13) \quad \eta(\tilde{f}) \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{\mathbb{W}}}).$$

*Then the elliptic fibration  $p^{\eta(f)} : \mathbb{W}^{\eta(f)} \rightarrow S$  admits a locally trivial deformation*

$$p : \mathcal{W} \rightarrow S \times \Delta$$

*that is an algebraic approximation. Moreover the fundamental group of  $X$  is projective.*

*Proof of Theorem 3.25.* Let us first explain how the first statement applies that  $\pi_1(X)$  is projective: by Lemma 3.23 we can replace the family  $p : \mathcal{W} \rightarrow S \times \Delta$  by a simultaneous resolution  $\Phi : \mathcal{X} \rightarrow S \times \Delta$ . Since the central fibre  $\mathcal{X}_0$  is smooth and bimeromorphic to  $X$  we have  $\pi_1(\mathcal{X}_0) \simeq \pi_1(X)$ . Since the compact Kähler manifold  $\mathcal{X}_0$  has an algebraic approximation, the group  $\pi_1(\mathcal{X}_0)$  is projective.

*Proof of main statement.* By assumption we have  $\eta(\tilde{f}) \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{\mathbb{W}}})$ . So we can consider the class  $c(\eta(\tilde{f}))$  which is torsion of some order  $k \in \mathbb{N}$  by Theorem 3.16. Therefore we can write  $k\eta(f) = \exp(\xi)$  for some integer  $k \geq 1$  but the cohomology class  $\xi$  is not necessarily  $G$ -invariant. The exact sequence

$$H^1(\tilde{S}, \tilde{j}_* \tilde{H}) \longrightarrow H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}}) \xrightarrow{\exp} H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{\mathbb{W}}}) \xrightarrow{c} H^2(\tilde{S}, \tilde{j}_* \tilde{H})$$

induces a sequence of  $G$ -invariant spaces

$$(14) \quad H^1(\tilde{S}, \tilde{j}_* \tilde{H})^G \longrightarrow H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G \xrightarrow{\exp} H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{\mathbb{W}}})^G \xrightarrow{c} H^2(\tilde{S}, \tilde{j}_* \tilde{H})^G$$

and by construction  $\eta(\tilde{f}) \in H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{\mathbb{W}}})^G$ . Thus we can write  $k\eta(f) = \exp(\xi)$  for some  $G$ -invariant class  $\xi$ . Consider now the deformation family

$$\tilde{\Phi} : \tilde{\mathcal{X}} \longrightarrow \tilde{S} \times H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G$$

such that  $\tilde{\mathcal{X}}_t = \mathbb{W}^{\eta(\tilde{f}) - \exp(\frac{1}{k}t)}$ . Then  $\mathbb{W}^{\eta(\tilde{f})}$  is the central fibre of this family and we claim that this defines an algebraic approximation of  $\mathbb{W}^{\eta(\tilde{f})}$ . Assuming this for the time being, let us see how to conclude: the fibration  $\tilde{\Phi}$  is  $G$ -equivariant (where

the action on  $\tilde{S} \times H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G$  is the product of the natural  $G$ -action on  $\tilde{S}$  and the identity). Thus when we take the quotient by  $G$  we obtain a family

$$\Phi : \mathcal{X} \longrightarrow S \times H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G$$

such that the central fibre is  $\mathbb{W}^{\eta(f)}$ . Moreover this deformation is locally trivial over  $S$  in the sense<sup>3</sup> of Definition 2.4. Since the projective fibres are dense in the family  $\tilde{\Phi}$  and being projective is invariant under a finite group action, the projective fibres are dense in  $\Phi$ .

*Proof of the claim.* By Proposition 3.18 the local Weierstraß models are Kähler spaces. Since they have rational singularities (Remark 3.3), they are projective if and only if they are Moishezon [Nam02]. Since the base  $S$  is projective this holds if and only if the elliptic fibration is (birationally) a projective map. By [Nak02a, Theorem 6.3.8] the fibration  $\mathbb{W}^{\eta(\tilde{f}) - \exp(\frac{1}{k}t)} \rightarrow S$  is projective if and only if the class  $\eta(\tilde{f}) - \exp(\frac{1}{k}t)$  is torsion. This holds if and only if

$$k(\eta(\tilde{f}) - \exp(\frac{1}{k}t)) = \exp(\xi - t)$$

is torsion. By (14) this holds if and only if  $\xi - t$  is in the image of

$$H^1(\tilde{S}, \tilde{j}_* \tilde{H}_{\mathbb{Q}})^G \rightarrow H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G.$$

Yet by Theorem 3.20 the map  $H^1(\tilde{S}, \tilde{j}_* \tilde{H}_{\mathbb{R}}) \rightarrow H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})$  is surjective, so the same holds for the map of  $G$ -invariant vector spaces

$$H^1(\tilde{S}, \tilde{j}_* \tilde{H}_{\mathbb{R}})^G \rightarrow H^1(\tilde{S}, \mathcal{L}_{\tilde{H}/\tilde{S}})^G.$$

The rational points  $H^1(\tilde{S}, \tilde{j}_* \tilde{H}_{\mathbb{Q}})^G \subset H^1(\tilde{S}, \tilde{j}_* \tilde{H}_{\mathbb{R}})^G$  are dense, so this shows the claim.  $\square$

#### 4. PROOFS OF THE MAIN RESULTS

**4.A. Proof of Theorem 1.3.** Theorem 3.25 provides an algebraic approximation if we can check the technical condition (13). By Lemma 3.15 this condition holds up to replacing an elliptic fibration  $X$  by some power  $X^{(m)}$ . We will now relate the fundamental groups of these two elliptic fibrations.

**4.1. Lemma.** *Let  $X$  be a compact Kähler manifold, and let  $f : X \rightarrow S$  be a neat [Cam04, Definition 1.2] elliptic fibration onto a projective manifold. Denote by  $E$  a general fibre of  $f$ . Then the natural morphism  $\pi_1(E) \longrightarrow \pi_1(X)$*

- (i) *either trivial,*
- (ii) *or injective.*

*The second case occurs if and only if  $f$  is bimeromorphically almost smooth: there exists subset  $Z \subset S$  of codimension at least two such that for every  $s \in S \setminus Z$  the fibre  $f^{-1}(s)$  is either smooth or multiple and has an irreducible component whose reduction is a smooth elliptic curve.*

---

<sup>3</sup>We can mimic the proof of Proposition 3.19. Indeed, if  $p : \tilde{S} \rightarrow S$  is the finite base change, let us observe that the deformation  $\tilde{\mathcal{X}}$  is trivial over  $\tilde{U} := p^{-1}(U)$ : a finite cover of a Stein manifold is still Stein and  $H^1(\tilde{U}, p^* \mathcal{L}) = 0$ . This provides us with a  $G$ -equivariant trivialization of  $\tilde{\mathcal{X}}$  over  $\tilde{U}$ .

**4.2. Remark.** In the case where  $\pi_1(E) \longrightarrow \pi_1(X)$  is not trivial, the global structure of  $X \rightarrow S$  can be completely described: there is a finite étale cover  $\tilde{X} \rightarrow X$  such that the Stein factorisation of  $\tilde{X} \rightarrow S$  is bimeromorphic to a locally trivial elliptic fibration. Indeed it is easy to check that a bimeromorphically almost smooth elliptic fibration is isotrivial. Thus we can apply [CP00, Proposition 6.7], [Fuj83] to see that there exists a finite étale cover  $\tilde{X} \rightarrow X$  such that  $\tilde{X} \rightarrow \tilde{S}$  satisfies  $q(\tilde{X}) = q(\tilde{S}) + 1$ . In particular the relative Albanese map defines a bimeromorphic morphism to the elliptic bundle  $\text{Alb}(\tilde{X}/\tilde{S}) \rightarrow \tilde{S}$ .

*Proof. Step 1. Suppose that  $f$  is bimeromorphically almost smooth.* We set  $X^\circ := f^{-1}(S \setminus Z)$ . Since  $f$  is neat, we have  $\pi_1(X) \simeq \pi_1(X^\circ)$ . Up to replacing  $Z$  by larger set (still of codimension two), we can suppose that  $X^\circ$  has a relative minimal model  $X^\circ \rightarrow Y^\circ \rightarrow S$ . The fibres of  $Y^\circ \rightarrow S$  are classified in Kodaira's list and by our assumption the singular fibres are of type  ${}_mI_0$ . Thus we can apply [Hör13, Lemma 2.2] to see that for some finite étale cover of  $\tilde{Y}^\circ \rightarrow Y^\circ$  the fibration becomes smooth, in particular, by [Cla10, Lemme 2.3], we have an injection

$$\pi_1(E) \hookrightarrow \pi_1(\tilde{Y}^\circ) \hookrightarrow \pi_1(Y^\circ) \simeq \pi_1(X^\circ).$$

*Step 2. Suppose that the map  $\pi_1(E) \longrightarrow \pi_1(X)$  is not trivial.* Since  $S$  is projective, we can consider a general complete intersection curve  $C \subset S$  and its preimage  $X_C := f^{-1}(C)$ . We can choose  $E \subset X_C$  and by our assumption this implies that the left-hand side of the map

$$\pi_1(E) \longrightarrow \pi_1(X_C) \longrightarrow \pi_1(X)$$

is not trivial. By [CZ79, Lemma 1.39] this implies that  $X_C \rightarrow C$  has trivial local monodromy, so by [CZ79, p.14] the relative minimal model of  $X_C \rightarrow C$  has only singular fibres of type  ${}_mI_0$ , i.e. multiple smooth elliptic curves. Thus  $X_C \rightarrow C$  (and hence  $X \rightarrow S$ ) is bimeromorphically almost smooth. By Step 1 this shows that  $\pi_1(E) \longrightarrow \pi_1(X)$  is injective.  $\square$

Let  $X$  be a compact Kähler manifold admitting an elliptic fibration  $f : X \rightarrow S$  onto a projective manifold  $S$  that is smooth over  $S^*$ , the complement of a SNC divisor. For  $m \in \mathbb{N}$  denote  $X^{(m)}$  a compact complex manifold admitting an elliptic fibration  $f^{(m)} : X^{(m)} \rightarrow S$  corresponding to the class  $m\eta(f)$  (cf. [Nak02a, Prop.5.2.5] for the group structure on the set of elliptic fibrations). Up to blowing up  $X$  we have a generically finite map  $\mu_m : X \rightarrow X^{(m)}$  that extends the multiplication by  $m$  defined on the smooth locus  $X^* \rightarrow S^*$ .

**4.3. Corollary.** *In the situation above, the induced map*

$$\pi_1(X) \longrightarrow \pi_1(X^{(m)})$$

*is injective.*

*Proof.* Since  $X^{(m)}$  is dominated by the compact Kähler manifold  $X$ , it is in the Fujiki class. Thus, up to blowing up  $X$  and  $X^{(m)}$ , we can suppose that both are compact Kähler manifolds and the fibrations  $f$  and  $f^{(m)}$  are neat [Cam04, Definition 1.2]. Since  $f$  and  $f^{(m)}$  induce the same vhs  $H$ , the orbifold divisors coincide by Lemma 4.4.

By [Cam11, Corollary 11.9] we have exact sequences of fundamental groups:

$$\begin{array}{ccccc}
\pi_1(E) & \xrightarrow{i} & \pi_1(X) & & \\
\downarrow \cdot \times m & & \downarrow (\mu_m)_* & \searrow & \\
& & & & \pi_1(S, \Delta) \longrightarrow 1 \\
\pi_1(E) & \xrightarrow{j} & \pi_1(X^{(m)}) & \nearrow & \\
& & & & 
\end{array}$$

If  $\pi_1(E) \rightarrow \pi_1(X)$  is trivial this already shows the claim, so suppose that this map is not the zero one. By Lemma 4.1 this implies that  $i : \pi_1(E) \rightarrow \pi_1(X)$  is injective and  $f$  is bimeromorphically almost smooth. Since  $\mu_m : X \rightarrow X^{(m)}$  is generically finite, this implies that  $f^{(m)}$  is also bimeromorphically almost smooth. Thus we can apply Lemma 4.1 to see that  $j : \pi_1(E) \rightarrow \pi_1(X^{(m)})$  is also injective. In particular the composition of  $j$  with the multiplication  $\times m : \pi_1(E) \hookrightarrow \pi_1(E)$  is injective.  $\square$

**4.4. Lemma.** *Let  $f : X \rightarrow S$  be a locally projective elliptic fibration between complex manifolds and let  $H$  denote the VHS induced by  $f$  on the smooth locus  $S^* \subset S$ . Then the orbifold divisor  $\Delta$  defined by (1) depends only on  $H$ , and not on (the choice of) the elliptic fibration  $f$  inducing  $H$ .*

*Proof.* Since the orbifold divisor is determined in codimension one, we can assume without loss of generality that  $S \setminus S^*$  is a smooth divisor. Moreover, up to replacing  $S$  by  $S \setminus Z$  with  $\text{codim}_S Z \geq 2$ , we can suppose that there exists a smooth relative minimal model  $X \rightarrow Y \rightarrow S$ : the local construction of relative minimal models is given in [Nak02b, Section 5.3]; these local models glue to a global, not necessarily Kähler, model over  $S$ . Since the orbifold divisor does not depend on the choice of the bimeromorphic model, we can suppose without loss of generality that  $X$  is relatively minimal.

By [Nak02a, Theorem 6.3.6] (and its proof) we know that, for smooth  $S \setminus S^*$ , each bimeromorphic equivalence class is uniquely represented by a relative minimal model, and this model is locally isomorphic to  $(G_\alpha \backslash B_\alpha) \rightarrow S_\alpha$  where  $U_\alpha \rightarrow S_\alpha$  is a  $\partial$ -étale covering with Galois group  $G_\alpha$  and  $B_\alpha \rightarrow U_\alpha$  is the unique relatively minimal basic elliptic fibration [Nak02a, Lemma 5.3.1] determined by the pull-back of  $H|_{S_\alpha^*}$  to  $U_\alpha^*$ . In particular  $(G_\alpha \backslash B_\alpha) \rightarrow S_\alpha$  only depends on  $H$ , in particular its orbifold divisor only depends on  $H$ .  $\square$

*Proof of Theorem 1.3.* The algebraic reduction endows  $X$  with a structure of elliptic fibration  $X \dashrightarrow A(X)$ , so up to blowing up  $X$  and  $A(X)$  we can suppose without loss of generality that we have an elliptic fibration

$$f : X \rightarrow S$$

that is smooth over  $S^*$  and  $D := S \setminus S^*$  is a SNC divisor. Using the notation of Theorem 3.25 we must deal with the problem that the class  $\eta(\tilde{f})$  might not be in the image of  $H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{W}})$ .

By Lemma 3.15 there exists an integer  $m$  such that  $m\eta(\tilde{f})$  comes from the cohomology group  $H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\tilde{W}})$ . By construction the class  $\eta(\tilde{f})$  is  $G$ -invariant, so the

class  $m\eta(\tilde{f})$  is  $G$ -invariant. Hence  $G$  acts equivariantly on  $\mathbb{W}^{m\eta(\tilde{f})} \rightarrow \tilde{S}$ , and we denote by  $\mathbb{W}^{m\eta(f)} := \mathbb{W}^{m\eta(\tilde{f})}/G$  the quotient. We denote by

$$p^{m\eta(f)} : \mathbb{W}^{m\eta(f)} \rightarrow S$$

the induced elliptic fibration, and by  $f^{(m)} : X^{(m)} \rightarrow S$  the composition of  $p^{m\eta(f)}$  with a desingularisation. Over the open set  $S^* \subset S$  we have a natural map

$$X|_{S^*} \rightarrow X^{(m)}|_{S^*}$$

given by multiplication by  $m$  on the fibres. Up to replacing  $X$  by some blowup this maps extends to a surjective map  $\mu_m : X \rightarrow X^{(m)}$  which is étale over  $S^*$ .

By Corollary 4.3 the map

$$(\mu_m)_* : \pi_1(X) \longrightarrow \pi_1(X^{(m)})$$

is injective and the image has finite index. Now it is easy to conclude: by Theorem 3.25 the group  $\pi_1(X^{(m)})$  is projective, so let  $Z$  be a projective manifold such that  $\pi_1(Z) \simeq \pi_1(X^{(m)})$ . The finite index subgroup  $\pi_1(X) \hookrightarrow \pi_1(X^{(m)})$  defines an étale cover  $\tilde{Z} \rightarrow Z$  such that  $\pi_1(\tilde{Z}) \simeq \pi_1(X)$ . Thus  $\pi_1(X)$  is projective.  $\square$

#### 4.B. Proof of Theorem 1.5.

**4.5. Proposition.** *Let  $X$  be a smooth compact Kähler threefold, and let  $f : X \rightarrow S$  be an elliptic fibration onto a smooth surface  $S$  such that  $f$  is smooth over  $S^*$ , the complement of a SNC divisor  $D := S \setminus S^*$ . Suppose also that  $f$  has local meromorphic sections in every point. Then we have*

$$\eta(f) \in \text{Im} \left( H^1(S, \mathcal{J}(H)^{\mathbb{W}}) \rightarrow H^1(S, \mathcal{J}(H)_{\text{mer}}) \right).$$

*Proof.* By [HP16, Thm.1.1] we can run a relative MMP  $X \dashrightarrow X'$  over  $S$ , the outcome is a normal compact Kähler space  $f' : X' \rightarrow S$  with  $\mathbb{Q}$ -factorial terminal singularities such that  $K_{X'}$  is relatively nef. Note that  $f'$  is smooth over  $S^*$  since the restriction of the MMP to  $f'^{-1}(S^*) \dashrightarrow f'^{-1}(S^*)$  is an isomorphism. Since the class  $\eta(f)$  only depends on the bimeromorphic equivalence class of the fibration, we will suppose without loss of generality that  $X = X'$ .

Let  $Z$  be the union of  $D_{\text{Sing}}$  and  $f(X_{\text{Sing}})$ . Since  $D$  is a reduced curve and  $X$  is a threefold with terminal singularities, the set  $Z$  is finite. Choose now an open cover  $U_j$  of  $S$  by polydiscs such that  $X_j := f^{-1}(U_j) \rightarrow U_j$  has a meromorphic section  $s_j : U_j \dashrightarrow X_j$  and every point of  $Z$  is contained in exactly one of the open sets  $U_j$ . The elliptic fibration

$$X_j \setminus f^{-1}(Z) \rightarrow U_j \setminus Z$$

is isomorphic to the unique relative minimal model of  $f^{-1}(U_j \setminus Z) \rightarrow U_j \setminus Z$  [Nak02a, Lemma 5.3.1], in particular the restriction of the section  $s_j|_{U_j \setminus Z}$  is holomorphic [Nak02a, Lemma 5.3.2].

Now observe that by construction for all  $i \neq j$  we have  $(U_i \setminus Z) \cap (U_j \setminus Z) = U_i \cap U_j$ . Thus for all  $i \neq j$

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$$

defines a local *holomorphic* section of the basic elliptic fibration  $p : \mathbf{B}(H) \rightarrow S$ . Since  $\mathbf{B}(H)$  is a desingularisation of the Weierstraß model  $\mathbb{W}$ , this defines a local holomorphic section of  $\mathbb{W}$ . Thus we obtain that the class  $\eta(f)$  can be represented by the cocycle  $(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i > j}$ , which defines an element of  $H^1(S, \mathcal{J}(H)^{\mathbb{W}})$ .  $\square$

*Proof of Theorem 1.5.* The algebraic reduction  $X \dashrightarrow A(X)$  defines an elliptic fibration onto a projective surface  $A(X)$ , so up to blowing up  $X$  and  $A(X)$  we can suppose without loss of generality that we have an elliptic fibration  $f : X \rightarrow S$  over a smooth projective surface such that  $f$  is smooth over  $S^*$  and  $D := S \setminus S^*$  is a SNC divisor.

Using the notation of Theorem 3.25 we know by Proposition 4.5 that the class  $\eta(\tilde{f})$  is in the image of  $H^1(\tilde{S}, \mathcal{J}(\tilde{H})^{\overline{\mathbb{W}}})$ . Thus Theorem 3.25 yields a locally trivial deformation of the local Weierstraß model  $\mathbb{W}^{\eta(f)}$  that is an algebraic approximation. By Lemma 3.23 we can desingularise in locally trivial families, so we obtain an algebraic approximation of a (functorial) resolution of singularities  $X' \rightarrow \mathbb{W}^{\eta(f)}$ .  $\square$

**4.C. Proof of Theorem 1.2.** Let  $X$  be a compact Kähler manifold, and consider  $g : X \rightarrow Y$  a fibration onto a compact Kähler manifold  $Y$ . If  $F$  is a general fibre, denote by  $\pi_1(F)_X$  the image of the morphism  $\pi_1(F) \rightarrow \pi_1(X)$ . Up to blowing up  $X$  and  $Y$  we can suppose that the fibration  $g$  is neat in the sense of [Cam04, Definition 1.2]. By [Cam11, Corollary 11.9] we then have an exact sequence

$$(15) \quad 1 \rightarrow \pi_1(F)_X \rightarrow \pi_1(X) \rightarrow \pi_1(Y, \Delta) \rightarrow 1.$$

where  $\Delta$  is the orbifold divisor defined in (1).

Let us also recall that by [Cam94, Kol93] every compact Kähler manifold  $X$  admits a (unique up to bimeromorphic equivalence of fibrations) almost holomorphic fibration

$$g : X \rightarrow \Gamma(X)$$

with the following property: let  $Z$  be a subspace with normalisation  $Z' \rightarrow Z$  passing through a very general point  $x \in X$ . Then  $Z$  is contained in the fibre through  $x$  if and only if the natural map  $\pi_1(Z') \rightarrow \pi_1(X)$  has finite image. This fibration is called the  $\Gamma$ -reduction of  $X$  (Shafarevich map in the terminology of [Kol93]). Up to replacing  $\gamma$  by some neat holomorphic model we thus obtain a fibration such that  $\pi_1(F)_X$  is finite and such that the dimension of the base is minimal among all fibrations with this property. We call  $\gamma \dim(X) := \dim(\Gamma(X))$  the  $\gamma$ -dimension of  $X$ .

In geometric situations it is often necessary to replace  $X$  by some étale cover. It is easily seen that the situation can be made equivariant under the Galois group of the cover.

**4.6. Lemma.** *Let  $X$  be a compact Kähler manifold acted upon by a finite group  $G$ . Then there exists a proper modification  $\mu : \tilde{X} \rightarrow X$  and a holomorphic map  $g : \tilde{X} \rightarrow Y$  such that:*

- (i)  $\tilde{X}$  and  $Y$  are compact Kähler manifolds.
- (ii)  $G$  acts on  $\tilde{X}$  and  $Y$ .
- (iii)  $\mu$  and  $g$  are  $G$ -equivariant for these actions.
- (iv)  $g$  is a neat model [Cam04, Definition 1.2] of the  $\Gamma$ -reduction of  $X$ .

*Proof.* Let us consider  $S$  the (normalisation of the) irreducible component of the cycle space  $\mathcal{C}(X)$  which parametrizes the fibres of the  $\Gamma$ -reduction. By uniqueness of the latter, the group  $G$  acts on  $S$  and the natural meromorphic map  $X \dashrightarrow S$  is  $G$ -equivariant. Now it is enough to perform  $G$ -equivariant resolution of singularities for  $S$  and  $G$ -equivariant blow-ups on  $X$  in order to make the latter map holomorphic, neat and  $G$ -equivariant.  $\square$



The following remark allows to control the extensions appearing in these covers.

**4.7. Remark.** Let  $1 \rightarrow K \rightarrow H \rightarrow G \rightarrow 1$  be an exact sequence of groups. It is well known that this extension determines a morphism  $\varphi_H : G \rightarrow \text{Out}(K)$  to the group of outer automorphisms of  $K$  (induced by the conjugation in  $H$ ). To recover the extension, it is needed to prescribe an additional information: the class  $c_H \in H^2(G, Z(K))$  (see [Bro82, Chapter IV, §6] for details).

On the reverse direction, when  $G$  is a finite group acting by homeomorphisms on a topological space  $Z$ , we also have an induced morphism  $\varphi_Z : G \rightarrow \text{Out}(\pi_1(Z))$ . In this case, there is an extension

$$1 \longrightarrow \pi_1(Z) \longrightarrow H \longrightarrow G \longrightarrow 1$$

induced by the action of  $G$  on  $Z$ . It can be explicitly constructed in the following way. By [Ser58] there exists a projective simply connected manifold  $P$  on which  $G$  acts freely and we can look at the natural projection  $Z \times P \rightarrow (Z \times P)/G$ . This is a finite étale cover of Galois group  $G$  and the homotopy exact sequence is:

$$(16) \quad 1 \longrightarrow \pi_1(Z) \simeq \pi_1(Z \times P) \longrightarrow \pi_1((Z \times P)/G) \longrightarrow G \longrightarrow 1.$$

This is the sought exact sequence. It is stated in [Cla16, Lemma 3.9] that the extension (16) does not depend on  $P$  and that this extension is the usual one if  $G$  acts freely on  $Z$  (*i.e.* the one corresponding to  $Z \rightarrow Z/G$ ).

Using the construction above we obtain the following technical result<sup>4</sup>.

**4.8. Lemma.** *Let  $H := \pi_1(X)$  be a Kähler group and  $K \triangleleft H$  be finite index normal subgroup with quotient  $G := H/K$ . Let us denote by  $\tilde{X}$  the corresponding étale cover of  $X$ . Assume now that there exists a continuous map  $g : \tilde{X} \rightarrow Z$  to a projective manifold  $Z$  which is  $G$ -equivariant (so that  $G$  acts on  $Z$ ) and which induces an isomorphism at the level of fundamental groups. Then  $H$  is a projective group.*

*Proof.* The  $G$ -equivariant map

$$\tilde{X} \times P \xrightarrow{g} Z \times P$$

induces an isomorphism on the fundamental group and, using Remark 4.7, we infer that

$$\pi_1(X) \simeq \pi_1((Z \times P)/G)$$

is a projective group. □

Let us recall that a group  $G$  is virtually torsion-free if there exist a subgroup  $H \subset G$  of finite index that is torsion free.

**4.9. Lemma.** *Let  $X$  be a compact Kähler manifold admitting a fibration onto a curve  $f : X \rightarrow C$  such that the fundamental group  $\pi_1(F)$  of a general fibre  $F$  is abelian. Then the group  $\pi_1(X)$  is virtually torsion-free.*

*Proof.* Applying [Cam98, Appendix C] we can take a finite étale cover such that  $\pi_1(F)_X$  coincides with  $K := \ker(f_* : \pi_1(X) \rightarrow \pi_1(C))$  and the latter is thus finitely generated. If  $C \simeq \mathbb{P}^1$  we obtain that  $\pi_1(X) \simeq \pi_1(F)_X$  is abelian, so virtually

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<sup>4</sup>We thank H.-Y. Lin for pointing out a flaw in a previous version of this article.

torsion-free. We can thus suppose that  $g(C) \geq 1$ . By [Ara11, Theorem 5.1] the cohomology class  $e \in H^2(\pi_1(C), K)$  corresponding to the extension

$$1 \longrightarrow K \longrightarrow \pi_1(X) \longrightarrow \pi_1(C) \longrightarrow 1$$

is torsion. Then so is the class  $e' \in H^2(\pi_1(C), K/K_{tor})$  corresponding to the extension

$$1 \longrightarrow K/K_{tor} \longrightarrow \pi_1(X)/K_{tor} \longrightarrow \pi_1(C) \longrightarrow 1.$$

Arguing as in [CCE14, §2.1] we can assume that the latter cohomology class  $e'$  vanishes (up to replacing  $\pi_1(C)$  with a finite index subgroup). Using the following piece of long exact sequence of cohomology of  $\pi_1(C)$ -modules

$$\cdots \longrightarrow H^2(\pi_1(C), K_{tor}) \longrightarrow H^2(\pi_1(C), K) \longrightarrow H^2(\pi_1(C), K/K_{tor}) \longrightarrow \cdots$$

we see the cohomology class  $e$  comes from  $H^2(\pi_1(C), K_{tor})$ . It is then easily observed<sup>5</sup> that this class is annihilated when restricted to a finite index subgroup of  $\pi_1(C)$ . This means that the following exact sequence of groups

$$1 \longrightarrow K_{tor} \longrightarrow \pi_1(X) \longrightarrow \pi_1(X)/K_{tor} \longrightarrow 1$$

splits when restricted to a finite index subgroup and it proves that  $\pi_1(X)$  is virtually torsion-free since  $\pi_1(X)/K_{tor}$  is.  $\square$

Now we give some criteria to decide whether a Kähler group is projective.

**4.10. Lemma.** *Let  $X$  be a compact Kähler manifold having  $\gamma \dim(X) \leq 1$ . Its fundamental group is then projective.*

*Proof.* If  $\gamma \dim(X) = 0$  its fundamental group is finite so [Ser58] applies. If  $\gamma \dim(X) = 1$  we know from [Cla10, Théorème 1.2] (which is just a rephrase of [Siu87]) that there exists a finite étale Galois cover  $\pi : \tilde{X} \rightarrow X$  with group  $G$  such that the  $\Gamma$ -reduction of  $\tilde{X}$  is a fibration  $\tilde{g} : \tilde{X} \rightarrow C$  onto a curve inducing an isomorphism  $\pi_1(X) \simeq \pi_1(C)$ . It is also  $G$ -equivariant according to Lemma 4.6. We conclude by Lemma 4.8.  $\square$

**4.11. Lemma.** *Let  $X$  be a compact Kähler manifold such that  $\gamma \dim(X) = 2$ . If  $\pi_1(X)$  is virtually torsion-free it is a projective group.*

*Proof.* Let  $K \triangleleft \pi_1(X)$  be a finite index subgroup which is normal and torsion-free, and set  $G := \pi_1(X)/K$ . Applying Lemma 4.6 to the finite étale cover corresponding to  $G$ , we know that we can find  $\tilde{X} \rightarrow X$  which is a composition of a finite étale cover and a modification such that the  $\gamma$ -reduction  $g : \tilde{X} \rightarrow Y$  is neat,  $Y$  is smooth Kähler surface and  $g$  is equivariant for the natural actions of  $G$  on  $\tilde{X}$  and  $Y$ . Consider now the exact sequence (15): the group  $\pi_1(\tilde{X})$  is torsion-free and  $\pi_1(F)_X$  is finite, so we have  $\pi_1(F)_X = 1$ . Thus  $\pi_1(\tilde{X}) \simeq \pi_1(Y, \Delta^*(g))$  is torsion-free, by Remark 2.8 this implies that

$$\pi_1(\tilde{X}) \simeq \pi_1(Y, \Delta^*(g)) \simeq \pi_1(Y).$$

We can now argue according to the algebraic dimension of  $Y$ .

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<sup>5</sup>If  $A$  is any finite  $\pi_1(C)$ -module then there is a finite index subgroup  $\pi_1(C')$  of  $\pi_1(C)$  such that the whole of the cohomology group  $H^2(\pi_1(C), A)$  vanishes when restricted to  $\pi_1(C')$ . This is a consequence of the fact that a curve of positive genus admits finite étale covers of any given degree.

- 1.) If  $a(Y) = 0$  then by the classification of surfaces  $\pi_1(Y)$  is abelian. Thus  $\pi_1(X)$  is virtually abelian and [BR11, Theorem 1.4] applies.
- 2.) If  $a(Y) = 1$  then the algebraic reduction  $Y \rightarrow C$  is an elliptic fibration over a curve  $C$ . Since the algebraic reduction is unique, it is  $G$ -equivariant. By [Kod63, Theorems 14.1-3-5] we know that there exists a  $G$ -equivariant deformation of  $Y$  to an algebraic elliptic surface, so we can again conclude by Lemma 4.8.
- 3.) If  $a(Y) = 2$  the surface  $Y$  is projective and Lemma 4.8 applies.

□

**Remark.** Although Lemmas 4.10 and 4.11 are stated in a very similar manner, they are of different nature: the former is a group theoretic statement whereas the latter is not. Indeed as a consequence of [Siu87] it is known that the property  $\gamma d(X) = 1$  is equivalent to having a fundamental group commensurable with the fundamental group of a curve; this property does thus depend only on the fundamental group. In general it is however possible to realize a given Kähler group as the fundamental group of several manifolds having different  $\gamma$ -dimensions.

*Proof of Theorem 1.2.* We argue according to the algebraic dimension of  $X$ , the case  $a(X) = 3$  being trivial since a Kähler Moishezon manifold is projective.

- 1.) If  $a(X) = 0$  then  $X$  is special in the sense of Campana. Thus the fundamental group is virtually abelian [CC14, Theorem 1.1] and thus projective [BR11, Theorem 1.4].
- 2.) If  $a(X) = 1$ , we replace  $X$  by some blowup such that the algebraic reduction is a holomorphic fibration  $f : X \rightarrow C$  onto a curve. By [CP00] the general fibre of  $F$  is bimeromorphic to a K3 surface, torus or ruled surface over an elliptic curve, so its fundamental group is abelian. By Lemma 4.9 the group  $\pi_1(X)$  is virtually torsion free. If  $\gamma \dim(X) \leq 2$  we can thus apply Lemma 4.10 and Lemma 4.11. If  $\gamma \dim(X) = 3$  it is shown in [CZ05, Theorem 1] that up to bimeromorphic transformations and étale cover  $f$  is a smooth morphism. Thus we can apply [Cla16, Corollary 1.2].
- 3.) If  $a(X) = 2$  the algebraic reduction makes  $X$  into an elliptic fibre space over a projective surface and we can apply Theorem 1.3.

□

## APPENDIX A. ELLIPTIC SURFACES

**A.1. Proposition.** *Let  $S$  be a non-algebraic compact Kähler surface that admits an elliptic fibration  $f : S \rightarrow \mathbb{P}^1$ . Then  $f$  has at least three singular fibres.*

*Proof.* We can suppose without loss of generality that  $f$  is relatively minimal. Thus we know by the canonical bundle formula [Kod63] that

$$K_S \simeq f^*(K_{\mathbb{P}^1} + M + \sum_{c \in \mathbb{P}^1} m_c S_c)$$

where  $M$  is the modular part defined by the  $j$ -function and  $\sum_{c \in \mathbb{P}^1} m_c S_c$  the discriminant divisor. Recall that any non-algebraic Kähler surface has a pseudoeffective canonical bundle, so  $K_S$  is nef.

Suppose first that  $f$  is isotrivial, i.e. we have  $M \equiv 0$ . Then [BHPVdV04, Chapter V, Table 6] shows that the singular fibres are either multiples of smooth elliptic curves or of type  $I_0^*$ . For a multiple fibre we have  $m_c \leq \frac{1}{2}$  and for a fibre of type  $I_0^*$  we have  $m_c = \frac{1}{2}$ . Since  $K_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$  we see that there are at least 4 singular fibres.

Suppose now that  $f$  is not isotrivial. Then we can use the argument from [Bea81, Proposition 1]: let  $C^* \subset \mathbb{P}^1$  be the maximal open set over which  $f$  is smooth. The  $j$ -function defines a non-constant holomorphic map  $\tilde{C}^* \rightarrow \mathbb{H}$  from the universal cover  $\tilde{C}^* \rightarrow C^*$  to the upper half plane  $\mathbb{H}$ . In particular  $\tilde{C}^*$  is not  $\mathbb{C}$  or  $\mathbb{P}^1$ , hence  $\mathbb{P}^1 \setminus C^*$  has at least three points.  $\square$

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