

INFLUENCE OF DISORDER FOR THE POLYMER PINNING MODEL

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ABSTRACT. When studying disordered systems, the influence of disorder on the phase transition is a central question: one wants determine whether an arbitrary quantity of disorder modifies the critical properties of the system, with respect to the non-disordered case. We present here an overview of the mathematical results obtained to answer that question for the polymer pinning model. In the IID case, the picture of disorder relevance/irrelevance is by now established, and follows the so-called Harris criterion: disorder is irrelevant if $\nu^{\text{hom}} > 2$ and relevant if $\nu^{\text{hom}} < 2$, where ν^{hom} is the order of the homogeneous phase transition. In the correlated case, Weinrib and Halperin predicted that, if the two point correlation decays as a power law with exponent $a > 0$, then the Harris criterion would be modified if $a < 1$: disorder should be relevant whenever $\nu^{\text{hom}} < 2 \max(1, 1/a)$. It turns out that this prediction is not accurate: the key quantity is not the decay exponent a , but the occurrence of rare regions with atypical disorder. An *infinite disorder* regime may appear, in which the relevance / irrelevance picture is crucially modified. We also mention another recent approach to the question of the influence of disorder for the pinning model: the persistence of disorder when taking the scaling limit of the system.

INTRODUCTION AND PHYSICAL MOTIVATIONS

The study of random polymers interacting with their environment has been a very active field of research in the past fifty years, both in the physical literature, and more recently in the mathematical literature [18, 25, 27]. We focus here on the so-called polymer pinning model, which was introduced by Poland and Scheraga to describe the DNA denaturation phenomenon, but has also been used in other contexts: it has for example been used as an effective model for the wetting of an interface on a disordered substrate (see [25] and references therein for an overview of the physical motivations).

The model considers a directed polymer interacting with a defect line, and the main focus, as in general for disordered systems, is its phase transition. One wants to describe it, and in particular to determine the influence of inhomogeneities on its characteristics. Some predictions regarding this question exist in the physical literature. In his seminal paper, Harris [32] predicts that, for a d -dimensional system ($d = 1$ for the pinning model), an IID disorder is relevant if the order ν^{hom} of the corresponding non-disordered (or homogeneous) phase transition is smaller than $2/d$. In the case of a correlated disorder, with two-point correlations decaying as a power a of the distance, $|i - j|^{-a}$, Weinrib and Halperin [42] predict that the Harris criterion is not changed if $a > d$ (disorder is relevant if $\nu^{\text{hom}} < 2/d$), and modified if $a < d$ (disorder is relevant if $\nu^{\text{hom}} < 2/a$).

- In Section 1, we introduce the disordered pinning model and give its first properties.

- In Section 2, we give the results in the IID case, where the Harris criterion has been (almost completely) proven. We stress the influence of disorder fluctuations, and in particular the role of rare atypical regions in disorder relevance, see Section 2.2.

- In Section 3, we turn to a correlated environment, where Weinrib-Halperin's predictions are not valid in general: the important quantity to consider is not the two point correlation decay exponent, but the exponential or sub-exponential decay of the size of rare favorable regions.

- Finally, we conclude by mentioning another question in Section 4, which is a priori unrelated to disorder relevance as usually discussed, but brings a new approach: the study of the scaling limit of the system, and the persistence of disorder in the continuum limit.

1. THE DISORDERED PINNING MODEL

1.1. Definition of the model. The polymer. Let $\tau := \{\tau_0 = 0, \tau_1, \dots\}$ be a renewal process, whose law is denoted by \mathbf{P} : $\tau_0 := 0$ and $(\tau_i - \tau_{i-1})_{i \in \mathbb{N}}$ is a sequence of IID, \mathbb{N} -valued, random variables, with common inter-arrival distribution $K(n) := \mathbf{P}(\tau_1 = n)$. The set τ (with a slight abuse of notations) is interpreted as the set of contact points of the polymer with the defect line. We denote $\{n \in \tau\}$ the event that n is a renewal point, that is that there exists some $k \in \mathbb{N}$ such that $\tau_k = n$.

One is lead to make the following assumption on the inter-arrival distribution:

Assumption 1.1. *There exists a positive slowly-varying function $\varphi(\cdot)$ and some $\alpha \geq 0$ such that*

$$\mathbf{P}(\tau_1 = n) =: K(n) \stackrel{n \rightarrow \infty}{\sim} \varphi(n) n^{-(1+\alpha)}.$$

This assumption comes from the fact that, historically, the polymer was modeled by the graph of a directed random walk $(n, S_n)_{n \in \mathbb{N}}$, where S is a random walk on \mathbb{Z}^d , interacting with the defect line $\mathbb{N} \times \{0\}$: the renewal τ is the set of return times of S_n to 0. Then, if S is a symmetric simple random walk on \mathbb{Z}^d , the first hitting time of 0, τ_1 , is known to satisfy Assumption 1.1 (aside from periodicity considerations easily overcome by only considering even times of return): there are explicit constants c_d , such that, in dimension $d = 1$, $\alpha = 1/2$ and $\lim_{n \rightarrow \infty} \varphi(n) = c_1$; in dimension $d = 2$, $\alpha = 0$ and $\varphi(n) \stackrel{n \rightarrow \infty}{\sim} c_2(\log n)^{-2}$; and in dimension $d \geq 3$, $\alpha = \frac{d}{2} - 1$ and $\lim_{n \rightarrow \infty} \varphi(n) = c_d$. In dimension $d = 1$, Assumption 1.1 also covers the case of a random walks with IID increments in the domain of attraction of a stable law, and of Bessel-type random walks, which can give asymptotically any $\alpha \geq 0$ and $\varphi(\cdot)$, see [2].

The random composition of the defect line. Let $\omega = (\omega_n)_{n \in \mathbb{N}}$ be a sequence of random variables, whose law, denoted by \mathbb{P} , is assumed to be ergodic: to every site of the defect line, an ω_n is attached, modeling the inhomogeneities. We also assume for simplicity that the ω 's are centered, $\mathbb{E}[\omega_1] = 0$, and normalized, $\mathbb{E}[\omega_1^2] = 1$.

The polymer measure. Given a fixed realization of ω (quenched disorder), and parameters $\beta \geq 0$ (strength of the inhomogeneities) $h \in \mathbb{R}$ (homogeneous

interaction), we define the measures $\mathbf{P}_{N,h}^{\beta,\omega}$ on trajectories of τ up to length N , as a Gibbs transformation of \mathbf{P} :

$$(1) \quad \frac{d\mathbf{P}_{N,h}^{\beta,\omega}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}^{\beta,\omega}} \exp\left(\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{\{n \in \tau\}}\right) \mathbf{1}_{\{N \in \tau\}},$$

where $Z_{N,h}^{\beta,\omega}$ is the *partition function* of the system, used to normalize $\mathbf{P}_{N,h}^{\beta,\omega}$ to a probability,

$$(2) \quad Z_{N,h}^{\beta,\omega} = \mathbf{E} \left[\exp\left(\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{\{n \in \tau\}}\right) \mathbf{1}_{\{N \in \tau\}} \right].$$

The measure $\mathbf{P}_{N,h}^{\beta,\omega}$ corresponds to giving a reward (or a penalty, depending on its sign) $\beta\omega_n + h$ if there is a contact with the defect line at site n . The goal is then to study the properties of trajectories of τ under $\mathbf{P}_{N,h}^{\beta,\omega}$ as N goes to infinity, for typical realizations of ω , and how they depend on the parameters $\beta \geq 0$, $h \in \mathbb{R}$.

1.2. The free energy and the localization phase transition. A central physical quantity is the free energy, or energy per monomer, which encodes many properties of the system.

Proposition 1.2. *see [25], Ch. 4 The following limit exists \mathbb{P} -a.s. and in L^1 ,*

$$(3) \quad \mathbf{F}(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,h}^{\beta,\omega} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log Z_{N,h}^{\beta,\omega}.$$

The quantity $\mathbf{F}(\beta, h)$ is called the quenched free energy of the system, and the map $(\beta, h) \mapsto \mathbf{F}(\beta, h)$ is non-negative, convex, and non-decreasing in both variables.

With this proposition in hand, one observes a phase transition when making h vary: there exists a *quenched* critical point $h_c^{\text{que}}(\beta)$ such that $\mathbf{F}(\beta, h) > 0$ if and only if $h > h_c(\beta)$; $h = \inf\{h; \mathbf{F}(\beta, h) > 0\} = \sup\{h; \mathbf{F}(\beta, h) = 0\}$. How to actually interpret this phase transition may still be unclear, but one can derive some path properties from the free energy, and relate the positivity of the free energy to localization properties. Indeed, a straightforward computation yields that $\partial_h \log Z_{N,h}^{\beta,\omega} = \mathbf{E}_{N,h}^{\beta,\omega} \left[\sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} \right]$. Therefore, using the convexity of \mathbf{F} (and Proposition 1.2), we get that

$$(4) \quad \partial_h \mathbf{F}(\beta, h) = \lim_{N \rightarrow \infty} \mathbf{E}_{N,h}^{\beta,\omega} \left[\frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} \right], \quad \mathbb{P} - a.s.$$

when the derivative $\partial_h \mathbf{F}(\beta, h)$ exists (it is however known that $h \mapsto \mathbf{F}(\beta, h)$ is C^∞ except possibly at $h_c^{\text{que}}(\beta)$).

The critical point therefore marks the transition between a localized phase $\mathcal{L} = \{(\beta, h); \mathbf{F}(\beta, h) > 0\} = \{h > h_c^{\text{que}}(\beta)\}$, where there is (asymptotically) a positive density of contacts under $\mathbf{P}_{N,h}^{\beta,\omega}$, the polymer sticking to the defect line; and a delocalized phase $\mathcal{D} = \{(\beta, h); \mathbf{F}(\beta, h) = 0\} = \{h \leq h_c^{\text{que}}(\beta)\}$, where the density of contacts is (asymptotically) null, the polymer wandering away from the line. We give an overview of the phase diagram in Figure 1, with a sketch of typical trajectories in the localized and delocalized phase. Note also that the order of the phase transition ν^{que} , that one can formally define as $\nu^{\text{que}} = \lim_{h \downarrow h_c^{\text{que}}(\beta)} \log \mathbf{F}(\beta, h) / \log(h - h_c^{\text{que}}(\beta))$ (if the limit exists), reflects the growth of the contact fraction, see (4): if $\nu^{\text{que}} \leq 1$,

then there is a jump discontinuity in the contact fraction, and if $\nu^{\text{que}} > 1$, the contact fraction grows roughly like $(h - h_c(\beta))^{\nu^{\text{que}} - 1}$.

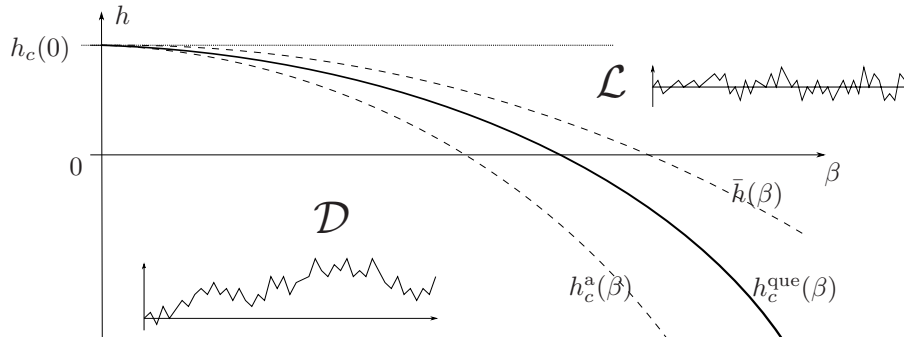


FIGURE 1. Since $(\beta, h) \mapsto F(\beta, h)$ is convex, the region \mathcal{D} is convex, so that $h_c(\beta)$ is a concave function of β . Moreover, $h_c^{\text{que}}(\beta) < h_c(0)$ for all $\beta > 0$ (if the disorder is IID, see [3]), showing that disorder helps for localization: indeed, a typical renewal trajectory may target positive ω 's. We also have some bounds on the critical curve: $h_c^{\text{que}}(\beta) \geq h_c^{\text{a}}(\beta)$, see Section 1.4 (note that, under some conditions (7), $h_c(0) - h_c^{\text{a}}(\beta) \sim cst.\beta^2$, see (15)). One can also show that there exists some $c > 0$ such that $h_c^{\text{que}}(\beta) \leq \bar{h}(\beta) = h_c(0) - c\beta^2$, see [25, § 5.2] (the method is presented in the IID case, but is very general). When $\beta \rightarrow \infty$, one is able to get the asymptotic behavior $h_c^{\text{que}}(\beta) \stackrel{\beta \uparrow \infty}{\sim} \frac{1}{1+\alpha} h_c^{\text{a}}(\beta)$ whenever the annealed model is defined for all $\beta > 0$, as in [40, Eq. (3.8)] or [7, Prop. 5.2.4].

Remark 1.3. Aside from the free energy, another central physical quantity of disordered systems is the correlation length $\ell(\beta, h) = 1/r(\beta, h)$, where $r(\beta, h)$ is the exponential decay rate of "correlations" in the system. In the context of pinning model, it is natural to define, for $h > h_c(\beta)$,

$$r(\beta, h) := \lim_{k \rightarrow \infty} -\frac{1}{k} \log \left(\mathbf{E}_{\infty, h}^{\beta, \omega} [\mathbf{1}_{\{i \in \tau\}} \mathbf{1}_{\{i+k \in \tau\}}] - \mathbf{E}_{\infty, h}^{\beta, \omega} [\mathbf{1}_{\{i \in \tau\}}] \mathbf{E}_{\infty, h}^{\beta, \omega} [\mathbf{1}_{\{i+k \in \tau\}}] \right) \quad \mathbb{P}\text{-a.s.},$$

where we denoted $\mathbf{E}_{\infty, h}^{\beta, \omega} [A] := \lim_{N \rightarrow \infty} \mathbf{E}_{N, h}^{\beta, \omega} [A]$, which is known to exist \mathbb{P} -a.s. for any bounded local observable A when $h > h_c^{\text{que}}(\beta)$, see [30].

In the homogeneous system, one has that $r(0, h) = F(0, h)$, see [26]. In the disordered system, Toninelli [38, Thm. 3.5] also showed that $r(\beta, h) = F(\beta, h)$, in the special case where the underlying renewal is the return times of a simple symmetric random walk S in dimension 1. The argument, however, works for any nearest-neighbor random walk in dimension 1, such as Bessel-type random walks [2]: one can therefore interpret the inverse of the free energy as the correlation length, for a wide class of renewals (with asymptotically any $\varphi(\cdot)$ and $\alpha \geq 0$ in Assumption 1.1).

The general goal is now to describe this localization/delocalization phase transition: can one compute explicitly or have sharp estimates on the critical point $h_c^{\text{que}}(\beta)$? what is the behavior of the free energy $F(\beta, h)$ close to criticality (when $h \downarrow h_c^{\text{que}}(\beta)$), and in particular can one find ν^{que} ? can one describe precisely path properties in the localized or delocalized phase? To that end, one compares the disordered system with its homogeneous and annealed counterparts, that we now define.

1.3. The homogeneous model. The homogeneous model is the non-disordered model, *i.e.* when $\beta = 0$, and its partition function reads $Z_{N,h}^{\beta=0,\omega} =: Z_{N,h}^{\text{hom}} = \mathbf{E}[e^{h \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} \mathbf{1}_{\{N \in \tau\}}]$. This model is exactly solvable: one has an explicit expression for the free energy, and can derive from it the value of the critical point and the critical behavior of the free energy.

Theorem 1.4 (cf. [25], Ch. 2). *The homogeneous free energy $F(0, h)$ is the solution in x of the equation*

$$(5) \quad \sum_{n \in \mathbb{N}} K(n) e^{-xn} = e^{-h},$$

if a solution exists, and $F(0, h) = 0$ otherwise.

From (5), one deduces that the critical point is $h_c(0) = -\log(\mathbf{P}(\tau_1 < +\infty)) \geq 0$. In particular, one has that $h_c(0) = 0$ if and only if τ is recurrent. One also has the critical behavior of the free energy: for every $\alpha > 0$ and $\varphi(\cdot)$, there exists an explicit slowly varying function $\widehat{\varphi}(\cdot)$ such that

$$(6) \quad F(0, h_c(0) + u) \stackrel{u \downarrow 0}{\sim} \widehat{\varphi}(1/u) u^{\nu^{\text{hom}}},$$

where $\nu^{\text{hom}} = \max(1, 1/\alpha)$ is the order of the homogeneous phase transition. When $\alpha = 0$, $F(0, h_c(0) + u)$ vanishes faster than any power of u , and the critical behavior is also explicit.

The homogeneous model is therefore very well understood, and the path properties can be described in details as well, cf. [25, Ch. 2] or [18, Ch. 7]: under $\mathbf{P}_{N,h}^{\beta=0}$, τ is a renewal with inter-arrival distribution $\widetilde{K}_h(n) = e^h K(n) e^{-F(0,h)n}$, conditioned to have $N \in \tau$.

Remark 1.5. In the following, we assume without loss of generality that the underlying renewal is recurrent, so that $h_c(0) = 0$. Indeed, considering the inter-arrival distribution $\bar{K}(n) := K(n)/\mathbf{P}(\tau_1 < +\infty)$ which is that of a recurrent renewal, and going from K to \bar{K} in (2) results only in a shift in the parameter h by $-\log \mathbf{P}(\tau_1 < +\infty)$.

1.4. The annealed model. The averaged partition function with respect to disorder is referred as the annealed model. In the pinning model, one has

$$\mathbb{E} Z_{N,h}^{\beta,\omega} = \mathbf{E}[e^{h \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} \mathbb{E}[e^{\beta \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}}}] \mathbf{1}_{\{N \in \tau\}}],$$

and the annealed measure $\mathbf{P}_{N,h}^{\beta,a}$ is defined accordingly. One also has the *annealed* free energy,

$$F^a(\beta, h) := \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} Z_{N,h}^{\beta,\omega},$$

and its *annealed* critical point, $h_c^a(\beta) := \inf\{h; F^a(\beta, h) > 0\}$. Then, Jensen's inequality gives that $F(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log Z_{N,h}^{\beta, \omega} \leq F^a(\beta, h)$, so that $h_c^{\text{que}}(\beta) \geq h_c^a(\beta)$.

Remark 1.6. For the comparison between the quenched and annealed system to be of interest, one needs the annealed model to be non-degenerated: one asks that there exists some β_0 such that,

$$(7) \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[\exp \left(\beta \sum_{n=1}^N \omega_n \right) \right] < +\infty \quad \text{for all } \beta \leq \beta_0.$$

Indeed, one has otherwise that $F^a(\beta, h) = +\infty$, and the inequality $F(\beta, h) \leq F^a(\beta, h)$ is trivial. In the IID case, (7) corresponds to asking that ω_1 has a finite exponential moment $M(\beta) := \mathbb{E}[e^{\beta \omega_1}] < +\infty$, for $\beta \leq \beta_0$.

Remark 1.7. We stress that if the ω_n 's are IID, then the annealed partition function is simply the partition function of an homogeneous model, with h replaced by $h + \log M(\beta)$, so that $h_c^a(\beta) = -\log M(\beta)$. The annealed system is then exactly solvable, and well understood, making it a cornerstone in the study of the influence of an IID disorder. When the ω_n 's are not IID anymore, the annealed system is not explicit, except in the case where $(\omega_n)_{n \in \mathbb{Z}}$ is a Gaussian stationary sequence with correlation function $(\rho_k)_{k \in \mathbb{Z}}$, $\rho_k := \mathbb{E}[\omega_0 \omega_k]$, $\rho_k = \rho_{-k}$, for which the annealed partition function is

$$\mathbb{E} Z_{N,h}^{\beta, \omega} = \mathbf{E} \left[\exp \left(h \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} + \frac{\beta^2}{2} \sum_{1 \leq i, j \leq N} \rho_{j-i} \mathbf{1}_{\{i \in \tau\}} \mathbf{1}_{\{j \in \tau\}} \right) \right].$$

One realizes that this is not the partition function of a standard homogeneous model, and its study is already challenging.

1.5. Disorder relevance/irrelevance? The central question to be answered is that of the influence of an arbitrarily small disorder on the phase transition. If, for all $\beta > 0$, the characteristics of the disordered and homogeneous (or annealed) phase transitions differ, then disorder is said to be *relevant*; otherwise, if there exists some $\beta < \beta_0$ such that the characteristics of the phase transitions are similar for all $\beta \leq \beta_0$, then disorder is said to be *irrelevant*.

- in terms of order of the phase transition: is the critical behavior of $F(\beta, h)$ as $h \downarrow h_c^{\text{que}}(\beta)$ different from that of the homogeneous system, see Theorem 1.4, for every $\beta > 0$? This is the original question of disorder relevance.
- in terms of critical points: if the annealed model is non-degenerated (see condition (7)), is the inequality $h_c^{\text{que}}(\beta) \geq h_c^a(\beta)$ strict for all $\beta > 0$?

2. ON THE INFLUENCE OF DISORDER IN THE IID CASE

In the context of disordered pinning model, we have that $d = 1$, and $\nu^{\text{hom}} = \max(1, 1/\alpha)$: Harris' criterion therefore predicts that disorder is irrelevant if $\alpha < 1/2$, and relevant if $\alpha > 1/2$. The case $\alpha = 1/2$, corresponding to the case of the simple symmetric random walk when $\varphi(\cdot)$ is a constant (see discussion after Assumption 1.1), is called *marginal*, and actually depends on the model considered.

2.1. Results in the IID case. In this section, we consider the case where $(\omega_n)_{n \in \mathbb{N}}$ is an IID sequence, with finite exponential moment $M(\beta) < +\infty$ for β smaller than some $\beta_0 > 0$, so that the annealed model is well defined, see (7). This case has been intensively studied in the past decade, and the Harris criterion is now (almost completely) settled.

Theorem 2.1 (Irrelevant disorder). *If $\alpha < 1/2$, or if $\alpha = 1/2$ and $\sum_{n \in \mathbb{N}} \frac{1}{n \varphi(n)^2} < +\infty$, then disorder is irrelevant: there exists some $\beta_1 > 0$ such that*

- for all $\beta \leq \beta_1$, $h_c^{\text{que}}(\beta) = h_c^{\text{a}}(\beta)$;
- for any $\beta \leq \beta_1$ and any $\eta > 0$, there exists some $c_{\eta, \beta}$ such that, for all $u \leq c_{\eta, \beta}$,

$$(8) \quad (1 + \eta) \geq \frac{\mathbf{F}(\beta, h_c^{\text{a}}(\beta) + u)}{\mathbf{F}(0, h_c(0) + u)} \geq (1 - \eta).$$

There exist several proofs of this result, with very different techniques: a second moment method in [1, 38], martingale techniques in [34], and a large deviation/variational formula approach in [17]. We stress that, in [5], the case $\alpha = 0$ is studied more in depth, and it is shown that then, one can take $\beta_1 = +\infty$: the annealed and quenched critical points are equal, whatever β is. On the other hand, it is shown in [39] that, when $\alpha > 0$, the annealed and quenched critical points differ when β is large enough (provided that the disorder is not bounded), so that $\beta_1 < +\infty$.

Theorem 2.2 (Relevant disorder). *If $\alpha > 1/2$, or if $\alpha = 1/2$ and $\sum_{n \in \mathbb{N}} \frac{1}{n \varphi(n)^2} = +\infty$, then disorder is relevant: for every $\beta > 0$, one has that*

- the critical points differ: $h_c^{\text{que}}(\beta) > h_c^{\text{a}}(\beta)$, and moreover

$$(9) \quad h_c^{\text{que}}(\beta) - h_c^{\text{a}}(\beta) \stackrel{\beta \downarrow 0}{\asymp} \begin{cases} \beta^2 & \text{if } \mathbf{E}[\tau_1] < +\infty \text{ (possible only if } \alpha \geq 1); \\ \psi(1/\beta) \beta^{\frac{2\alpha}{2\alpha-1}} & \text{if } 1/2 < \alpha \leq 1, \mathbf{E}[\tau_1] = +\infty \end{cases}$$

for some (explicit) slowly varying function $\psi(\cdot)$. In the case $\alpha = 1/2$, and if $\lim_{n \rightarrow \infty} \varphi(n) = c$, then one has that $-\log(h_c^{\text{que}}(\beta) - h_c^{\text{a}}(\beta)) \sim \frac{1}{2}c^2\beta^{-2}$ as $\beta \downarrow 0$.

- the phase transition is of order at least 2: for all $\beta > 0$ and $v > 0$, there exists a constant $C_{\beta, v}$, with $\lim_{\beta \downarrow 0, v \downarrow 0} C_{\beta, v} = 1$ such that, for all $u \leq v$,

$$(10) \quad \mathbf{F}(\beta, h_c^{\text{que}}(\beta) + u) \leq C_{\beta, v} \frac{1 + \alpha}{2\beta^2} u^2.$$

This result has been proven in a series of papers [4, 17, 19, 28, 29, 31] in several manners, the full necessary and sufficient condition for disorder relevance (together with the sharp critical point shift when $\alpha = 1/2$) being given only recently in [12]. In [12, 19, 28, 29], the authors estimate the fractional moment of the partition function up to the correlation length by a change of measure argument, and then use a coarse-graining procedure to glue these estimates together. A coarse-graining argument is also in order in [4], whereas in [17], a large deviation/variational formula approach is used. In [31, 15], the authors prove the smoothing inequality (10) via a rare-stretch strategy, that we outline in Section 2.2: it stresses the influence of rare regions in disorder relevance. For a more complete overview of the results and techniques employed, we refer to [27]. We now make a few comments about these results:

1. Theorem 2.2 shows that the critical points differ at all temperature, but also gives the right order of the gap between the critical points in the weak coupling

limit (*i.e.* when $\beta \downarrow 0$). We mention that in [10], the case $\mu := \mathbf{E}[\tau_1] < +\infty$ is considered, and the exact weak coupling behavior is found: $\lim_{\beta \downarrow 0} \frac{1}{\beta^2} (h_c^{\text{que}}(\beta) - h_c^{\text{a}}(\beta)) = \frac{\alpha}{2\mu(1+\alpha)}$. Note that it has a universal behavior, in the sense that the limit does not depend on the law \mathbb{P} of ω (provided that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$), or on the fine details of the inter-arrival distribution of τ . It is natural to conjecture that, also in the case $1/2 < \alpha < 1$, the limit $\lim_{\beta \downarrow 0} \frac{h_c^{\text{que}}(\beta) - h_c^{\text{a}}(\beta)}{\psi(1/\beta) \beta^{\frac{2\alpha}{2\alpha-1}}}$ exists and is universal.

A work in progress [16] shows that this limit exists, and depends only on α .

2. We mention that, remarkably, in [17], the authors are able to give a variational formula for the quenched critical point (and as a matter of fact, also for the annealed one). They deduce from it that the map $\beta \mapsto h_c^{\text{q}}(\beta) - h_c^{\text{a}}(\beta)$ is non-decreasing, so that disorder relevance is *monotone in β* (this was also found in [29, Section 6]): in particular, there exists a critical value $\beta_c \geq 0$ such that $h_c^{\text{que}}(\beta) = h_c^{\text{a}}(\beta)$ if $\beta \leq \beta_c$, and $h_c^{\text{que}}(\beta) > h_c^{\text{a}}(\beta)$ if $\beta > \beta_c$. One interprets β_c as the transition point between an irrelevant and relevant regime: in our words, disorder is relevant if one always lies in the relevant regime, that is if $\beta_c = 0$.

3. When disorder is relevant, one obtains that the order of the phase transition is larger than the homogeneous one (at least for $\alpha > 1/2$, $\nu^{\text{hom}} = \max(1, 1/\alpha) < 2$, the case $\alpha = 1/2$ being marginal), but more precise results are lacking, and the mystery concerning the order of the phase transition remains, even at the heuristic level. However, recently, Derrida and Retaux [21] studied a toy, hierarchical, pinning model, for which the quenched critical point is explicit, and they show that the disordered phase transition has an essential singularity of the type $e^{-c/\sqrt{z}}$, so that $\nu^{\text{que}} = \infty$. Even though their techniques rely strongly on the hierarchical structure, making the model more tractable (*i.e.* with a tractable renormalization group), it gives a first glimpse that relevant disorder may make the phase transition of infinite order.

4. The marginal case $\alpha = 1/2$, when the slowly varying function $\varphi(\cdot)$ is asymptotically equivalent to a constant (which is the case of a simple random walk in dimension $d = 1$), has long been subject to debates among physicists, with two different approaches and conjectures. In [24], the authors suggested that disorder was irrelevant, but then in [20], it was predicted that disorder was relevant, with a critical point shift was of order e^{-c/β^2} . The conjecture that, for $\alpha = 1/2$, disorder is relevant if and only if $\sum_{n \in \mathbb{N}} \frac{1}{n\varphi(n)^2} = +\infty$ has now been proven in a very recent paper [12], in which the authors also obtain the sharp critical point shift e^{-c/β^2} , the constant c being explicit: for example, $c = \pi/2$ in the case of the simple random walk in dimension $d = 1$.

Remark 2.3. The condition $\sum_{n \in \mathbb{N}} \frac{1}{n\varphi(n)^2} < +\infty$ appears naturally, since under Assumption (1.1), it is equivalent to the transience of the intersection $\tau \cap \tau'$ of two independent copies of the renewal. Indeed, $\mathbf{E}[|\tau \cap \tau' \cap [0, N]|] = \sum_{n=1}^N \mathbf{P}(n \in \tau \cap \tau') = \sum_{n=1}^N \mathbf{P}(n \in \tau)^2$, and [22, Thm. B] gives that (if $0 < \alpha < 1$) $\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} c_\alpha \varphi(n)^{-1} n^{-(1-\alpha)}$ for some constant c_α .

The intersection $\tau \cap \tau'$ appears for example when one computes the second moment of the partition function. If $h = h_c^a(\beta)$, one gets

$$(11) \quad \begin{aligned} \mathbb{E}[(Z_{N,h}^{\beta,\omega})^2] &= \mathbb{E} \left[\mathbf{E}^{\otimes 2} \left[\exp \left(\sum_{n=1}^N (\beta \omega_n - \log M(\beta)) (\mathbf{1}_{\{n \in \tau\}} + \mathbf{1}_{\{n \in \tau'\}}) \right) \mathbf{1}_{\{N \in \tau\}} \mathbf{1}_{\{N \in \tau'\}} \right] \right] \\ &= \mathbf{E}^{\otimes 2} \left[\exp \left(\sum_{n=1}^N (\log M(2\beta) - 2 \log M(\beta)) \mathbf{1}_{\{n \in \tau \cap \tau'\}} \right) \mathbf{1}_{\{N \in \tau \cap \tau'\}} \right], \end{aligned}$$

which is the partition function of an homogeneous system, with underlying renewal $\tau \cap \tau'$. Therefore, (11) diverges for all $\beta > 0$ if $\tau \cap \tau'$, and remains bounded when $N \rightarrow \infty$ if $\tau \cap \tau'$ is transient and provided that β is small, in which case there is a chance that the quenched and annealed system are close.

There is another interesting manifestation of the transience/recurrence of $\tau \cap \tau'$. Standing at the annealed critical point, $(Z_{N,h_c^a(\beta)}^{\beta,\omega})_{N \in \mathbb{N}}$ is a positive martingale: it converges \mathbb{P} -a.s. to some non-negative limit $Z_\infty^{\beta,\omega}$, and one has that $Z_\infty^{\beta,\omega} > 0$ \mathbb{P} -a.s. for small β if $\tau \cap \tau'$ is transient, and $Z_\infty^{\beta,\omega} = 0$ \mathbb{P} -a.s. for all $\beta > 0$ if $\tau \cap \tau'$ is recurrent, see [34, Prop. 1.6].

As far as path properties under $\mathbf{P}_{N,h}^{\beta,\omega}$ are concerned, we have already seen that one can extract the density of contacts from the free energy, and it is $\partial_h \mathbf{F}(\beta, h) > 0$ in the localized phase. In the delocalized phase, the answer is however not trivial. For the homogeneous case, one knows that the number of contact remains bounded if $h < h_c(0)$ [27, Ch. 7], whereas in the disordered case the situation is more subtle, since there might be some stretches of unusual disorder driving the polymer to have a growing number of contacts, even if $h < h_c^{\text{que}}(\beta)$. One a priori only knows that the number of contacts is $o(N)$ in the delocalized regime, but [35] gives that the number of contact under $\mathbf{P}_{N,h}^{\beta,\omega}$ is $O(\log N)$ \mathbb{P} -a.s. In [6], the authors show that the last contact occurs at distance $O(1)$ of the origin in probability (cf. Thm. 1.1), but they exhibit some parameters (β large enough, some $h_c^a(\beta) < h < h_c^{\text{que}}(\beta)$) for which the number of contacts is exactly of order $\log N$ \mathbb{P} -a.s.

Remark 2.4. If ω_1 does not admit any exponential moment, $\mathbb{E}[e^{\beta \omega_1}] = +\infty$ for all $\beta > 0$, we already mentioned that $\mathbf{F}^a(\beta, h) = +\infty$ for all $\beta > 0, h \in \mathbb{R}$. The behavior of the quenched system is however also degenerated: one is able to show that the quenched critical point is $h_c^{\text{que}}(\beta) = -\infty$, so that there is no phase transition anymore, and one obtains exactly the behavior of the free energy when $h \rightarrow -\infty$, $\mathbf{F}(\beta, h) \stackrel{h \rightarrow -\infty}{\sim} \beta \int_{x \geq |h|/\beta} \mathbb{P}(\omega_1 \geq x) dx$.

2.2. The rare-stretch strategy and the smoothing inequality. We now stress the importance of rare regions in disorder relevance, by sketching the rare stretch strategy. One divides the system in regions of size m , and denote $B_i := \{(i-1)m + 1, \dots, im\}$. Then, one defines the property $\mathcal{A}_m = \mathcal{A}(\omega_1, \dots, \omega_m)$ of being *favorable* (definition which has to be made specific at some point). Then, the rare-stretch strategy consists in imposing the renewal τ to visit only the blocks B_i for $i \in \mathcal{I}$, where $\mathcal{I} = \{i; (\omega_k)_{k \in B_i} \in \mathcal{A}\}$ are the indices of the favorable blocks, see Figure 2: this gives a non-trivial lower bound on the free energy.

Then, loosely speaking, the energetic gain (at the exponential level) on a favorable region is $\mathbb{E}[\log Z_{m,h}^{\beta,\omega} | \mathcal{A}_m]$. On the other hand, since the first favorable region

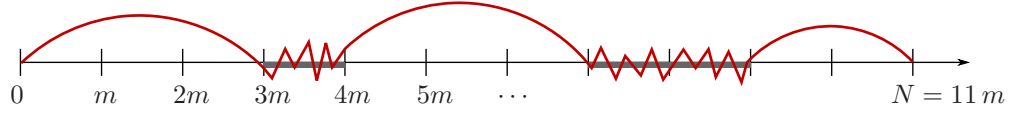


FIGURE 2. Overview of the rare-stretch strategy. Here, $N = 11m$, and the underlined regions are the ones identified as *favorables*: $\mathcal{I} = \{3, 7, 8\}$. Then, we impose the renewal trajectory to visit only these regions.

is at distance $\approx m\mathbb{P}(\mathcal{A}_m)^{-1}$, the entropic cost of aiming at a favorable block is $\log K(m\mathbb{P}(\mathcal{A}_m)^{-1}) \approx -(1 + \alpha)\log(\mathbb{P}(\mathcal{A}_m)^{-1})$. Then, in a system constituted of n blocks, one roughly has that $\log Z_{nm} \geq |\mathcal{I} \cap [0, n]| \times (\mathbb{E}[\log Z_{m,h}^{\beta,\omega} | \mathcal{A}_m] - (1 + \alpha)\log(\mathbb{P}(\mathcal{A}_m)^{-1}))$, and when taking $\lim_{n \rightarrow \infty} \frac{1}{mn} \log Z_{nm}$ one obtains

$$(12) \quad \mathbf{F}(\beta, h) \geq \mathbb{P}(\mathcal{A}_m) \left(\frac{1}{m} \mathbb{E}[\log Z_{m,h}^{\beta,\omega} | \mathcal{A}_m] - (1 + \alpha) \frac{1}{m} \log(\mathbb{P}(\mathcal{A}_m)^{-1}) \right).$$

Then, the key is to define properly \mathcal{A}_m , in order to optimize the lower bound (12).

For example, to obtain the smoothing inequality, the idea is to set, for $u \geq 0$, the property $\mathcal{A}_m = \{(\omega_1, \dots, \omega_m) \text{ has empirical mean } u/\beta\}$. Then, one naturally has that $\lim_{m \rightarrow \infty} \frac{1}{m} \mathbb{E}[\log Z_{m,h}^{\beta,\omega} | \mathcal{A}_m] = \mathbf{F}(\beta, h + u)$. Also, $\mathbb{P}(\mathcal{A}_m)$, the probability that the $(\omega_i)_{1 \leq i \leq m}$ are shifted by u/β , is approximately $e^{-C_{\beta,u} \frac{u^2}{2\beta^2} m}$, cf. [15]. In the end, if one takes $h = h_c^{\text{que}}(\beta)$, then $\mathbf{F}(\beta, h_c^{\text{que}}(\beta)) = 0$, and one obtains in (12), and letting m go to infinity,

$$(13) \quad 0 \geq \mathbf{F}(\beta, h_c^{\text{que}}(\beta) + u) - (1 + \alpha) C_{\beta,u} \frac{u^2}{2\beta^2}.$$

3. CASE OF A CORRELATED DISORDER

Once that one has studied the case of an IID environment, it is natural to turn to a correlated one, with correlations in ω decaying with the distance as a power law $|i - j|^{-a}$. In the context of the pinning model, Weinrib and Halperin prediction tells that disorder should be irrelevant if $\nu^{\text{hom}} > 2 \min(1, a)$, that is if $\alpha < \min(1, a)/2$.

3.1. Gaussian type environment with long-range correlations.

3.1.1. *The annealed model.* Since the annealed model is in general not explicit, the first step is to work with a stationary Gaussian environment, with correlation function $(\rho_k)_{k \in \mathbb{Z}}$, for which the annealed partition function is, as mentioned in Remark 1.7,

$$(14) \quad \mathbb{E} Z_{N,h}^{\beta,\omega} = \mathbf{E} \left[\exp \left(h \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} + \frac{\beta^2}{2} \sum_{1 \leq i, j \leq N} \rho_{j-i} \mathbf{1}_{\{i \in \tau\}} \mathbf{1}_{\{j \in \tau\}} \right) \right].$$

Here, the model is non degenerated if $\sum_{n \in \mathbb{Z}} |\rho_n| < +\infty$ (recall condition 7), and it is shown that the annealed free energy $\mathbf{F}^a(\beta, h)$ exists and is finite under that condition. Remark that, if $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$ for some $a > 0$, the condition $\sum_{n \in \mathbb{Z}} |\rho_n| < +\infty$ implies in particular that $a > 1$.

This annealed model is highly non-trivial, and it has been studied in [8, 36] via two different approaches. We collect here the main results.

Theorem 3.1. *Even though the critical point is not explicit, whenever $\sum_{n \in \mathbb{Z}} |\rho_n| < +\infty$ (or $a > 1$), one has*

$$(15) \quad h_c^a(\beta) \stackrel{\beta \downarrow 0}{\sim} -\frac{1}{2}\beta^2 \left(1 + 2 \sum_{n \in \mathbb{N}} \rho_n \mathbf{P}(n \in \tau) \right).$$

If moreover $\sum_{n \in \mathbb{N}} n|\rho_n| < +\infty$ (or $a > 2$), then the annealed critical behavior is similar to that of the homogeneous system: for any $\beta > 0$, there exists some constant $c_\beta \in (0, 1)$ such that, for all $u \in [0, 1]$, one has

$$(16) \quad c_\beta \mathbf{F}(0, u) \leq \mathbf{F}^a(\beta, h_c^a(\beta) + u) \leq \frac{1}{c_\beta} \mathbf{F}(0, u).$$

3.1.2. Disorder relevance/irrelevance. We focus on the case $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$ with $a > 1$: according to the Weinrib-Halperin predictions, one expects the disorder relevance/irrelevance picture to be similar to the IID case. Several partial results are now available in this direction.

One obtains a smoothing inequality analogous to (10), showing disorder relevance in terms of critical exponents when $\alpha > 1/2$, as in the IID case.

Proposition 3.2 (Theorem 5.6, [8]). *If $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$ with $a > 1$ (which is coherent with the fact that $\sum_{n \in \mathbb{N}} |\rho_n| < +\infty$), one has that for any $\beta > 0$ and $u \geq 0$,*

$$(17) \quad \mathbf{F}(\beta, h_c^{\text{que}}(\beta) + u) \leq \frac{1}{\sum_{n \in \mathbb{Z}} \rho_n} \times \frac{1 + \alpha}{2\beta^2} u^2.$$

The critical point shift when $\alpha > 1/2$ or equality when $\alpha < 1/2$ has not been proven in full generality in the correlated context. The only result is when $\mu := \mathbf{E}[\tau_1] < +\infty$, where, remarkably, one is able to get the asymptotic behavior of the quenched critical point [13, Thm. 9], as in the IID case (see Remark 1. after Theorem 2.2):

$$(18) \quad \lim_{\beta \downarrow 0} \frac{1}{\beta^2} (h_c^{\text{que}}(\beta) - h_c^a(\beta)) = \sum_{n \in \mathbb{Z}} \rho_n \times \frac{\alpha}{2\mu(1 + \alpha)}.$$

Remark 3.3. In [14], the authors study the hierarchical counterpart of this Gaussian correlated disordered pinning model. The hierarchical structure enables them to control the annealed measure close to criticality, without any knowledge on the annealed critical point: this trick works whenever $a > 1$, with the restriction that $a < 2 \min(1, \alpha)$. In particular, their results suggest that the annealed critical behavior should be different than the homogeneous one when $a < 2 \min(1, \alpha)$. In [14], the authors show disorder irrelevance when $\alpha < 1/2$, in terms of critical points and critical behavior, and are able to show the critical point shift if $a > 1$, $a < 2 \min(1, \alpha)$. We collect the results in the hierarchical and non-hierarchical context in Figure 3.

3.1.3. Non-summable correlations. The most interesting part comes when one considers non-summable correlations, or in Weinrib Halperin words, if $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$ with $a < 1$, when the Harris prediction should be modified, disorder relevant when $\alpha > a/2$. We already know that the annealed model is degenerated: $\mathbf{F}^a(\beta, h) = +\infty$

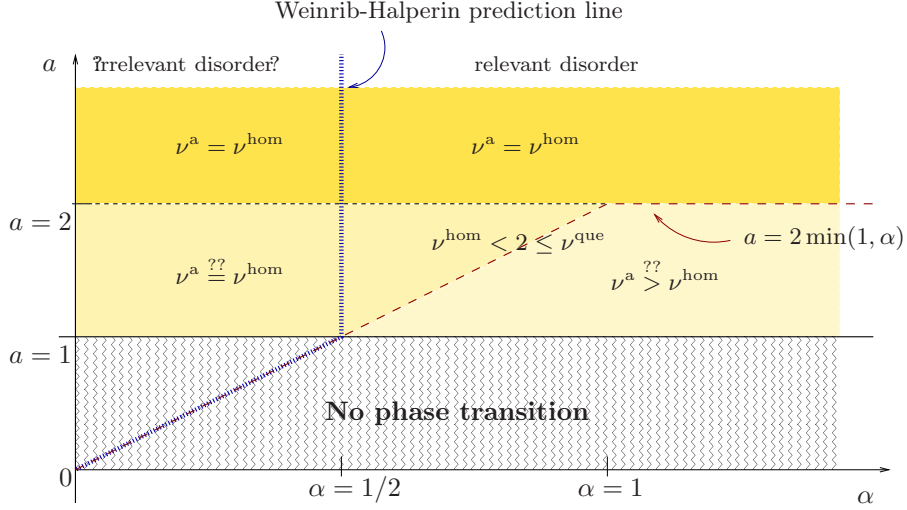


FIGURE 3. Overview of the results obtained in the standard pinning model with a Gaussian correlated environment with $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$, and of the results expected thanks to the comparison with its hierarchical counterpart [14]. When $a > 1$, (17) shows that disorder is relevant for $\alpha > 1/2$. The annealed and homogeneous critical behavior are shown to be the same when $a > 2$ (Theorem 3.1), and they are expected to be the same also when $a > 2 \min(1, \alpha)$, but to differ when $a > 1$, $a < 2 \min(1, \alpha)$. The proof that disorder is irrelevant if $\alpha < 1/2$, $a > 1$ is done only in the hierarchical setting. When $a < 1$ the annealed model is degenerated, and the disordered system is always in the localized phase, see Section 3.1.3.

for all $h \in \mathbb{R}$, $\beta > 0$. It is actually shown that the disordered model is also degenerated, in the sense that $h_c^{\text{que}}(\beta) = -\infty$: the system is always in the localized phase.

Theorem 3.4 (Thm. 6, [8]). *If $\rho_n \stackrel{n \rightarrow \infty}{\sim} n^{-a}$ with $a < 1$, then there exist constants $c, C > 0$ such that, for all $\beta > 0$ and $h \leq -1$,*

$$(19) \quad \mathbb{F}(\beta, h) \geq c \exp\left(-C \beta^{2/(1-a)} |h|^{-a/(1-a)}\right).$$

This is due to the fact that rare favorable regions appear much more frequently, because of stronger fluctuations of ω . One can turn this idea into a proof, thanks to the rare-stretch strategy exposed in Section 2.2, see (12).

Indeed, if one defines $\mathcal{A}_m := \{\omega_i \geq 2|h|/\beta; \text{ for } 1 \leq i \leq m\}$, so that $\beta\omega_n + h \geq |h|$ on favorable blocks, one obtains that $\mathbb{E}[\log Z_{m,h}^{\beta,\omega} | \mathcal{A}_m] \geq \log Z_{m,|h|}^{\text{hom}}$ which is approximately $m\mathbb{F}(0, |h|) \geq c''|h|m$ if $h \leq -1$. It is then just a matter of estimating $\mathbb{P}(\mathcal{A}_m)$, the cost of shifting all ω 's so that they are all larger than $2|h|/\beta$: Lemma 4.1 in [8] gives that $\mathbb{P}(\mathcal{A}_m) \geq e^{-c \frac{|h|^2}{\beta^2} m^a}$ if $a < 1$. This is the crucial point, and stresses that rare regions are much more frequent than in the IID case. Plugged in

(12), one gets that

$$(20) \quad \mathbf{F}(\beta, h) \geq e^{-c\frac{|h|^2}{\beta^2}m^a} \left(c''|h| - c'\frac{|h|^2}{\beta^2}m^{a-1} \right).$$

Now one just need to optimize the choice of m , taking $m^{a-1} = \text{cst.} \beta^2/|h|$, to get (17).

3.2. The infinite disorder regime. As seen in previous sections, the effect of rare regions on critical properties is crucial, and reveals the relevant character of disorder. Rare regions effects has been studied in depth in the physical literature, in particular via a renormalization group approach, see [41] for a review. If atypical regions occur very frequently (as a matter of fact, are only subexponentially rare, as it is the case in Theorem 3.4), their impact on the system blows up when taking larger and larger length scales: one lies in the *infinite disorder* (or strong randomness) regime. This regime has been investigated for various systems, for which the stong randomness renormalization group may in general be carried out explicitly, and many exact can be derived, see [33] for a brief review.

In the context of pinning model, we have seen a first occurrence of infinite disorder in Theorem 3.4. The above discussion suggests that

- a) the central quantity characterizing infinite disorder is not the two-point correlation, but the frequency of occurrence of rare atypical regions;
- b) whenever one lies in the infinite disorder regime, one can hope to obtain exact results on the critical behavior of the system.

Point a) is made rigourous in [9] in the case of a binary disorder ($\omega_i \in \{-1, 1\}$), but its results holds with more generality. Denote $\mathbf{M} := \text{ess sup } \omega_1$ ($\mathbf{M} = +\infty$ if ω is not bounded). Then, for any neighborhood V of \mathbf{M} , we define the event $\mathcal{A}_m^{(V)} := \{\omega_i \in V; \forall 1 \leq i \leq m\}$, of having a " V -favorable" region of size m : the frequency of their occurrence is $\mathbb{P}(\mathcal{A}_m^{(V)})$.

Definition 3.5. *One is said to have infinite disorder if*

$$(21) \quad \text{for any neighborhood } V \text{ of } \mathbf{M}, \text{ one has} \quad \liminf_{m \rightarrow \infty} -\frac{1}{m} \log \mathbb{P}(\mathcal{A}_m^{(V)}) = 0,$$

which corresponds to having arbitrarily favorable regions which are only subexponentially rare.

Theorem 3.6 ([9]). *If one has infinite disorder (i.e. (21) holds), then $h_c^{\text{que}}(\beta) = -\beta \mathbf{M}$ for all $\beta > 0$.*

Moreover, one has explicit lower and upper bounds on $\mathbf{F}(\beta, h)$, which do not match in general, but give that disorder is relevant, whatever the value of α is: if $\mathbf{M} < +\infty$, then for all $\beta > 0$

$$(22) \quad \mathbf{F}(\beta, h_c^{\text{que}}(\beta)) \stackrel{u \downarrow 0}{=} o(\mathbf{F}(0, u)).$$

Let us make some comments.

1. The definition 3.5 of infinite disorder has also some consequences on the annealed system. Indeed, one has that $\mathbb{E} Z_{N,h}^{\beta, \omega} \geq \mathbb{P}(\mathcal{A}_N^{(V)}) Z_{N, \beta \min V+h}^{\text{hom}}$, so that, when (21) holds, one gets that $\mathbf{F}^a(\beta, h) = \mathbf{F}(0, \beta \mathbf{M} + h)$: the annealed free energy

is degenerated. This has to be related to (7), since, under (21), one has

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[\exp \left(\beta \sum_{n=1}^N \omega_n \right) \right] = \beta \mathbf{M} \quad \text{for all } \beta > 0.$$

It is then natural to ask whether the annealed and disordered models are non-degenerated whenever (21) does not hold. This is proven in [9], under some additional assumption that correlations are non-increasing (in some specific sense, see Assumption 2.2 in [9]), and we can conjecture that this is true in full generality.

2. One realizes that the Weinrib-Halperin criterion is not valid in our context, for two reasons. First, the modification of the Harris criterion does not occur when the correlations decay exponent is $a < 1$, but with another weaker condition, (21). Secondly, when (21) holds, then, one does not have a relevance ($\alpha > a/2$) / irrelevance ($\alpha < a/2$) picture anymore: disorder is relevant for all $\alpha \geq 0$.

To illustrate this remark, we mention the work [11], in which the authors provide an example of a binary environment $\omega \in \{-1, 0\}$, with long-range correlations $\mathbb{E}[\omega_i \omega_{i+k}] \stackrel{k \rightarrow \infty}{\sim} k^{-a}$ with $a > 0$, but such that $\mathbb{P}(\omega_i = 0, \forall 1 \leq i \leq n) \stackrel{n \rightarrow \infty}{\sim} n^{-(2+a)}$. The environment is constituted of alternating blocks of 0's and -1's, the lengths of the blocks being independent random variables, with heavy-tail distribution of exponent $2 + a$, $a > 0$. Therefore, one has infinite disorder ((21) holds), whatever the two point correlation decay exponent $a > 0$ is. One has then $h_c(\beta) = 0$ for all $\beta > 0$ (because of Theorem 3.6), and the sharp critical behavior of the free energy is found: $\mathbf{F}(\beta, u) \asymp |\log u|^{-a} \mathbf{F}(0, u)^{1+a}$. The order of the disordered phase transition is therefore computed exactly, $(1+a)\nu^{\text{hom}} > \nu^{\text{pur}}$, and remarkably, the correct logarithmic corrections are also found. Disorder is then relevant whatever $a > 0$ and α are, disproving Weinrib and Halperin's prediction.

In the case of a Gaussian-based binary disorder (so that $h_c^{\text{que}}(\beta) > -\infty$), with correlation function $\rho_n \sim n^{-a}$, $a < 1$, we mention that [9] proves that $\nu^{\text{que}} = +\infty$, whatever α is. According to the physics literature, one may hope to obtain the exact behavior of the free energy whenever one lies in the infinite disorder regime, see Conjecture 7.3.3. in [7].

4. ANOTHER PERSPECTIVE ON DISORDER RELEVANCE: SCALING LIMITS

Another central question regarding disordered system is the one of the scaling limit: one might hope to be able to construct a continuum model which is the scaling limit of the system, in order to obtain for example universality results. We consider in this section the case $0 < \alpha < 1$, and give only the ideas of the results, for the sake of conciseness.

If we denote $\tau_{(N)}$ the set $\frac{1}{N}\tau \cap [0, N] \subset [0, 1]$, and $b_N \stackrel{N \rightarrow \infty}{\sim} c_\alpha N^\alpha / \varphi(N)$ with $c_\alpha = \alpha / \Gamma(1 - \alpha)$, one has that $\tau_{(N)} / b_N$ converges in distribution towards the *regenerative set* of index α . For the pinning model, the idea is to take, as the size N of the system goes to infinity, parameters β_N and h_N which scale properly with the system in order to obtain a non-trivial limit.

A first piece of work focuses on the scaling limit of the homogeneous pinning model, see [37]: there exists a *critical window* for the parameter h , for which the scaling limit is non-trivial. If one takes $h = \hat{h} / b_N$, one obtains that the set $\tau_{(N)}$, when distributed according to $\mathbf{P}_{N, h}$ converges in law to some specific limit $\mathcal{B}_{\alpha, \hat{h}}$ which depends only on \hat{h} and α , see [37, Thm. 3.1] for a precise statement.

The scaling limit of the disordered pinning model has been considered only recently, in [23], when the disorder is IID. The authors then show that, if $1/2 < \alpha < 1$, and setting $\beta = \widehat{\beta}N^{1/2}/b_N$, and $h - h_c^\alpha(\beta) = \widehat{h}/b_N$, then for all $t \geq 0$, the partition function $Z_{N,h}^{\beta,\omega}$ converges in distribution towards a *non-trivial* random variable $\mathbf{Z}_{t,\widehat{h}}^{\widehat{\beta},W}$ (which has an explicit dependence on the parameters), where $W(\cdot)$ denotes the white noise on \mathbb{R} : we refer to [23, Thm. 3.1] for a precise statement of the result. Their idea is to perform a so-called *polynomial chaos* expansion of the partition function $Z_{N,h}^{\beta,\omega}$, and prove the convergence of such an expansion.

The remarkable fact about this result is that one obtains a *disordered* continuum limit if $\alpha > 1/2$, and it is conjectured to be also the case when $\alpha = 1/2$ and $\sum \frac{1}{n\varphi(n)^2} < +\infty$: disorder *survives* when one takes the scaling limit, and it seems to survive exactly when disorder is relevant. This idea of having a disordered scaling limit is not directly related to disorder relevance as we discussed it in the previous sections, but it provides a new approach to the question of the influence of disorder on physical systems, and gives rise to new interesting continuum models.

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