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Surrogate Granular Materials for Modal Test of Fluid Filled Tanks

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ABSTRACT

Experimental vibration certification of launcher cryogenic tanks is an important issue in the aerospace industry. Liquid hydrogen is indeed too dangerous to be used in tank vibration tests. Unlike most fluids used in aerospace industry such as liquid oxygen, surrogate fluids cannot be used to approach the modal behavior of a tank filled with liquid hydrogen because of its particularly low mass density. However granular materials could safely replace liquid hydrogen for vibration testing of tanks.

This work aims at proposing an innovative methodology to determine the geometry and the material properties of substitution grains in order to keep the same mode shape and eigen frequencies of a vertical cylindrical tank studied considering 2D horizontal slices. Assuming that the tank slices are sufficiently far enough from the free-surface not to be affected by sloshing effects, the fluid-structure interaction is purely inertial. This methodology is based on the homogenization of a granular material composed of spheres with an elastic behavior. Thus the tank filled with grains can be modeled by a membrane surrounded by a circular beam. An experimental test bench developed to validate the methodology, as well as analytical and numerical simulation results of fluid-beam and membrane-beam interactions, are presented on the first modes.

Keywords : Cryogenic tank - Modal analysis - Rayleigh-Ritz - Granular materials

1 INTRODUCTION

Cryogenic stages are commonly used for launchers due to their efficiency. However, several technical issues make their development tough. One of these challenges is to determine the dynamic responses of tanks entirely or partially filled with cryogenic fluids. Indeed, it is dangerous and expensive to perform vibration tests on tanks filled with these fluids. Therefore, for some fluids such as liquid oxygen, surrogate fluids with close thermodynamic constants can be used. As explained by NASA ^[1], this approach is also used for liquid hydrogen, by using water even if the density ratio of these fluids is around 14. Looking for alternatives, this innovative work deals with the possibility to use pre-stressed granular materials to fill the tank in order to obtain experimentally the same modal response as a tank filled with liquid hydrogen for low frequencies.

A methodology based on a semi-analytical resolution to choose the granular material and the pre-development of an experimental test bench is presented: After the justification of 2D hypothesis, the aim will be to validate the modal equivalence between fluid filled tanks and grains filled tanks. In order to achieve this, another model based on a homogenization of the granular material is used as sketched in figure 1. After explaining the hypotheses associated with this homogeneous model, eigen-vectors and eigen-frequencies of the proposed mechanical model are determined thanks to the Rayleigh-Ritz method. To obtain the same modal parameters as fluid filled tank, grain material can be chosen. A test bench is under development to confirm this approach and identify some parameters such as the *coordination number* of the granular material.

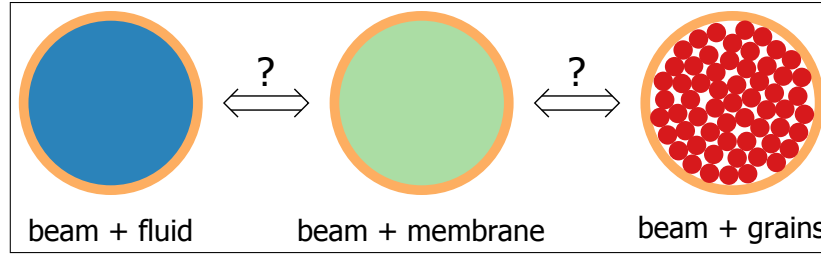


Figure 1: modal equivalence between fluid filled beam and grains filled beam using an homogeneous model for granular material

2 MODELING

This work focuses on cylindrical tanks such as the Ariane 5 cryogenic main stage tank. The first step is to define a model of tank filled with liquid hydrogen, in order to obtain an expression of the first eigen modes intended for the equivalence with the tank filled with grains.

2.1 Fluid-structure interaction

The tank is mainly composed of a long hollow cylinder closed by two spherical caps. Since the thickness is considerably smaller than the other dimensions, consequently the tank can be modeled by a shell. Moreover, the two spherical caps are non-developable surfaces, therefore the rigidity of these caps is much larger than the cylinder's one. We assume the caps do not participate in the first modes. Then, the model can be reduced to a cylindrical shell clamped. The shell geometry is defined by the radius R and the thickness h and its homogeneous, isotropic material by the Young modulus E_1 and the density ρ_1 .

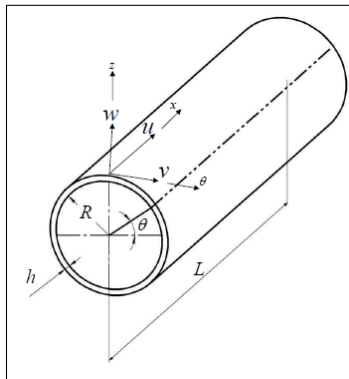


Figure 2: Cylindrical shell parametrization : $(\vec{u}, \vec{v}, \vec{w})$ is the local base of a current point defined by its cylindrical coordinates : (R, θ, x)

The first modes of this structure are flexural radial-orthoradial modes. Regardless of the shell theory chosen, with the parametrization of the figure 2, using the method of separation of variables and the invariance to rotation, the displacement can be written in

the form :

$$\begin{cases} u(\theta, x, t) = \sum_{i,j} u_{ij}(\theta, x, t) = \sum_{i,j} \mathcal{F}_{ij}^u(x) \cos(i\theta) \cos(\omega t) \\ v(\theta, x, t) = \sum_{i,j} v_{ij}(\theta, x, t) = \sum_{i,j} \mathcal{F}_{ij}^v(x) \sin(i\theta) \cos(\omega t) \\ w(\theta, x, t) = \sum_{i,j} w_{ij}(\theta, x, t) = \sum_{i,j} \mathcal{F}_{ij}^w(x) \cos(i\theta) \cos(\omega t) \end{cases} \quad (1)$$

The relation between i, j and ω depends on the shell theory used. In order to solve analytically the problem, an assumption of plane motion is used; accordingly $u(\theta, x, t) \ll v(\theta, x, t)$ and $w(\theta, x, t)$. Then, the tank can be studied slice by slice with an amplitude function of x . A tank's slice is modeled by a circular beam. As the radius is much larger than the thickness, the beam theory used is the Euler-Bernoulli's one^[2]; Love determined the flexural modes of a circular beam. These modes were obtained by adding to the Euler-Bernoulli equations, a closure relation imposing: $\epsilon_{\theta\theta} = 0 \Leftrightarrow \frac{\partial v}{\partial \theta} - w = 0$. This relation imposes that the work done by normal forces (membrane forces) is null. The i^{th} eigen-modes functions and eigen-frequencies equations are:

$$\begin{cases} v_i(\theta, t) = V_i(\theta) \cos(\omega t) = A_x \sin(i\theta) \cos(\omega t) \\ w_i(\theta, t) = W_i(\theta) \cos(\omega t) = i A_x \cos(i\theta) \cos(\omega t) \quad \forall i \in \mathbb{N} \\ \omega = \frac{i(i^2-1)}{R^2(i^2+1)^{1/2}} \left(\frac{E_1 h}{12 \rho_1} \right)^{1/2} \end{cases} \quad (2)$$

Thanks to these expressions, the generalized mass and stiffness for each mode by unit of length can be obtained as follows:

$$\begin{cases} K_g^B|_i = \frac{E_1 h^3}{12 R^3} \int_0^{2\pi} \left(\frac{d^2 W_i}{d\theta^2} + W_i \right)^2 d\theta = \frac{A_x^2 E_1 h^3 \pi}{12 R^3} (1 - i^2)^2 \\ M_g^B|_i = \rho_1 h R \int_0^{2\pi} (V_i^2 + W_i^2) d\theta = A_x^2 \rho_1 h R \pi (1 + \frac{1}{i^2}) \end{cases} \quad (3)$$

In this work, the tank is assumed totally filled, so there is no sloshing effect. To determine the importance of acoustic effects, orders of magnitude are used. The celerity of sound c is about 1000 m/s, the first frequency f of our system is about 10 Hz and the characteristic $2R$ is about 1 m. The Helmholtz number is defined for example in ^[3] as : $He = \left(\frac{2\pi f R}{c} \right)^2 \approx 10^{-2}$. As a consequence, the hypothesis **H1** is made :

- **H1** : Acoustic effects are negligible.

The Helmholtz equations simplify to Laplace equation in pressure^[3] :

$$\Delta(p) = 0 \quad (4)$$

To model the entire system, equations of elastic beams and equation 4 are used. The interaction is defined on the boundary of the liquid domain by :

$$\begin{cases} \overline{\sigma}(\vec{n}) = -p \vec{n} \\ grad(p) \cdot \vec{n} = \rho_F \omega^2 w \end{cases} \quad (5)$$

\vec{n} representing the normal vector from the fluid to the beam and ρ_F the density of the fluid. As shown for example by Axia in ^[4] for each mode, the mode shapes remain the same than for the beam without fluid, and the frequency decreases due to the inertia added by fluid. This phenomenon can be decreased by an added mass due to the fluid on each mode. In the case of a ring, this added mass by unit of length is defined by : $M_a^F|_{i,j} = \left(\mathcal{F}_{ij}^w(x) \right)^2 \frac{\rho_F \pi R^2}{i}$. Here $\mathcal{F}_{ij}^w(x)$ denotes the amplitude of the beam radial motion for the slice. In this 2D approach, it is a constant. Eigen-frequencies, can be obtained thanks to Rayleigh ratio with $M_a^F|_i = M_a^F|_{i,j}$ with $\mathcal{F}_{ij}^w(x) = 1$:

$$f_i = \frac{1}{2\pi} \sqrt{\frac{K_g^B|_i}{M_g^B|_i + M_a^F|_i}} \quad (6)$$

Thanks to this analysis, it is possible to sum up on modal objectives for the tank filled with grains as follows:

- mode shapes of the tank filled with grains must be the same as the empty tank's one;
- eigen-frequencies must follow relation 6, this paper focuses only on the first mode.

2.2 Homogenization of elastic granular materials

A granular material is composed of a lot of grains. The understanding of its different behaviours and the transitions between them, for such materials, constitutes an important research field. In this study the granular material is in its "static material" state: meaning the grains cannot flow. As explained for example by Nedderman^[5], this state can exist if the material is pre-compressed and the amplitude of pressure oscillations is low enough to prevent the formation of slip planes. According to the Mohr-Coulomb theory^[5], this failure mode happens when :

$$\left(\frac{1 - \sin(\delta)}{1 + \sin(\delta)}\right) \sigma_1 > \sigma_2 > \left(\frac{1 + \sin(\delta)}{1 - \sin(\delta)}\right) \sigma_1 \quad . \quad (7)$$

Here σ_1 and σ_2 are the principal stresses (the third principal stress being null in this 2D approach, the same kind of relation exists in 3D). δ is the effective angle of internal friction computed thanks to the geometry of grains and their assembly and to the friction coefficient between grains.

Given the foregoing hypothesis on principal stresses, the granular material can be described like a non-linear elastic solid. Indeed, even if the grain material is linearly elastic, according to Hertz theory^[6] the stiffness is non-linear due to the increase of contact area. If the grain's sizes is very small in front of the tank size, an homogeneous theory could be used. In this preliminary work, Boussinesq theory^[7] is used. The behaviour law is Hooke's law with material parameters depending on the pre-compression of grains : $\sigma_{ij} = K(\Delta)\Delta\delta_{ij} + 2G(\Delta)\tilde{\epsilon}_{ij}$ with Δ the volumic variation : $\Delta = -\frac{\delta V}{V} = -\epsilon_{kk}$. Material coefficients can be approximated combining both Hertz contact theory in 2D and Duffy-Mindlin model^[8] (pressure P is evenly distributed on contacts) as follows : $K \simeq A_1 Z E \Delta^{1/2} \simeq A_1 (Z E)^{2/3} P^{1/3}$ and $G \simeq A_2 Z E \Delta^{1/2} \simeq A_2 (Z E)^{2/3} P^{1/3}$. Where Z is the *coordination number* characterizing the average number of contacts by grain. The Z value depends on the assemblage configuration and the size disparity of grains. For a 2D model, $Z \in [3, 4]$ and for a 3D model, $Z \in [4, 6]$ as explained by Andreotti and al^[9]. A_1 and A_2 are constants depending on grain materials. To sum up, three hypotheses are made :

- **H2** : Pre-stressed pressure is high enough to prevent relative motions between grains.
- **H3** : Granular material can be described as an elastic material with material coefficient depending on pressure.
- **H4** : Pressure variation during vibration tests is low enough to linearize material coefficients.

The equivalent solid is described by its density ρ_2 , its Lamé parameters λ_2 and μ_2 . Referring to figure 1, the modal equivalence between a "beam+grains" model and a "beam+membrane" model in 2D is now possible. To obtain the modal equivalence between a "beam+membrane" model and the "beam+fluid" model in 2D, it's necessary to determine the mode shapes and eigen frequencies of the "beam+membrane" model. Thus the granular material properties and the pre-stress pressure could be determined to fit with this model's responses to the fluid structure developed in paragraph 2.1

2.3 2D Analytical modal determination of a circular membrane surrounded by a beam

The methodology developed to approximate the eigen modes of a circular membrane surrounded by a beam is based on the Rayleigh-Ritz method. A mode shape form is assumed, then by calculating the mass and stiffness operator, eigen-frequencies are determined. To validate the supposed mode shape form, these frequencies are compared to numerical results. With the currently used assumption of separation of spatial variables (r, θ) for mode shapes of disk and using the 2π -periodicity of the structure on θ , the following hypothesis **H5** can be made :

- **H5** : The radial displacements of the mode shape of a free circular beam, of a circular membrane on its free edge and of a beam filled by this circular membrane are the same.

To validate this assumption, a modal analysis has been made by the finite element method using COMSOL. The figure 3 shows the third mode of a beam, a membrane and a membrane circled by a beam obtained. The deformed shape of the free beam, of the membrane on free edge and of the beam on the coupled system seem to be quit close. Therefore, mode shapes of a free circular beam are the same as for a beam filled by a membrane. This result is coherent with the fluid filled beam.

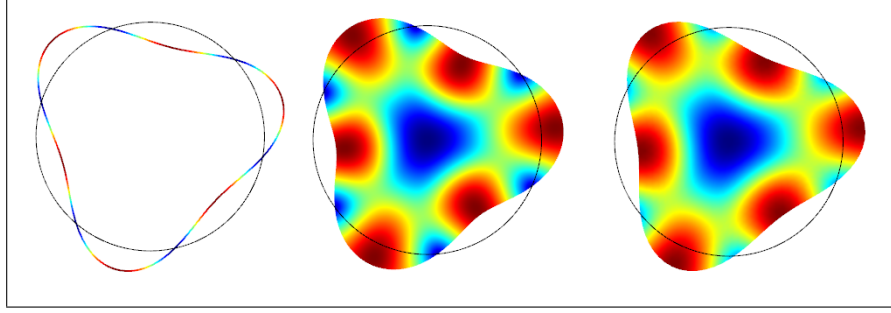


Figure 3: third mode of a circular beam, a circular membrane and circular membrane surrounded by a beam obtained by COMSOL to validate the hypothesis H5

As the mode shapes of the free beam and of the free membrane are barely compatible, for the coupled system these two mode shapes are used for the Rayleigh-Ritz approach with an holonomic condition of same amplitude in radial motion. Mode shapes of the beam have already been computed in section 2.1 as well as the mass and stiffness operators $K_g^B|_i$ $M_g^B|_i$. In-plane mode shapes of a free edge circular membrane have been less studied than out-plane vibration. In plane mode shapes of a circular membrane with elastic boundary conditions have been analytically obtained by Kim and al. ^[10]. By using the separation of

variables in polar coordinates, the i^{th} eigen-vector can be written as : $\vec{U} = \begin{pmatrix} U_r(r)|_i \cos i\theta \\ U_\theta(r)|_i \sin i\theta \\ 0 \end{pmatrix}$, with :

$$\begin{cases} U_r(r)|_i &= \frac{R}{2} \{A_1 [J_{i+1}(\alpha \frac{r}{R}) - J_{i-1}(\alpha \frac{r}{R})] + A_2 [J_{i+1}(c\alpha \frac{r}{R}) + J_{i-1}(c\alpha \frac{r}{R})]\} \\ U_\theta(r)|_i &= \frac{R}{2} \{A_1 [J_{i+1}(\alpha \frac{r}{R}) + J_{i-1}(\alpha \frac{r}{R})] + A_2 [J_{i+1}(c\alpha \frac{r}{R}) - J_{i-1}(c\alpha \frac{r}{R})]\} \\ c &= \sqrt{2/(1-\nu_1)} \\ \alpha &= \omega R \sqrt{\rho(1-\nu_2^2)/E_2} \end{cases} \quad (8)$$

where J_i is Bessel function of first kind of order i . The values of α are determined thanks to the dispersion relation :

$$2J_{n+2}(\alpha)J_{n-2}(c\alpha) + 2J_{n+2}(c\alpha)J_{n-2}(\alpha) - 2(c^2 - 1)J_n(\alpha) [J_{n+2}(c\alpha) + J_{n-2}(c\alpha)] = 0 \quad (9)$$

The amplitudes A_1 and A_2 are linked by the relation : $D_{22}D_{12}A_2 = -A_1D_{11}D_{21}$ with :

$$\begin{cases} D_{11} &= -\frac{\lambda}{2} [J_{i+2}(\lambda) + J_{i-2}(\lambda) - 2(c^2 - 1)J_i(\lambda)] \\ D_{22} &= -\frac{c\lambda}{2} [J_{i+2}(c\lambda) + J_{i-2}(c\lambda)] \\ D_{12} &= -\frac{c\lambda}{2} [J_{i+2}(c\lambda) - J_{i-2}(c\lambda)] \\ D_{21} &= -\frac{\lambda}{2} [J_{i+2}(\lambda) - J_{i-2}(\lambda)] \end{cases} \quad (10)$$

Using classical elasticity theory, the stiffness and mass operators per unit of length can be obtained as follows:

$$K_g^M|_i = \pi \int_0^R \frac{1}{r} [r^2 (U_r'|_i)^2 (\lambda_2 + 2\mu_2) + 2\lambda_2 r (U_r|_i + iU_\theta|_i) U_r'|_i + \mu_2 r^2 (U_\theta'|_i)^2 - 2\mu_2 r U_\theta'|_i (iU_r|_i + U_\theta|_i) + U_r|_i^2 ((2 + i^2)\mu_2 + \lambda_2) + 2i(\lambda_2 + 3\mu_2) U_r|_i U_\theta|_i + (i^2 + (\lambda_2 + 2\mu_2) + \mu_2) U_\theta|_i^2] dr \quad (11)$$

and

$$M_g^M|_i = \pi \rho_2 \int_0^R (U_r^2 + U_\theta^2) r dr \quad (12)$$

To evaluate $U_r'|_i$ and $U_\theta'|_i$ recurrence relation on Bessel functions is used : $J_{i+1}(x) - J_{i-1}(x) = -2\frac{dJ_i(x)}{dx}$ with $x = \alpha$ or $x = c\alpha$.

$$\begin{cases} U_r'(r)|_i &= \frac{\alpha}{4} \{A_1 [2J_i(\alpha \frac{r}{R}) - J_{i-2}(\alpha \frac{r}{R}) - J_{i+2}(\alpha \frac{r}{R})] + cA_2 [J_{i-2}(c\alpha \frac{r}{R}) - J_{i+2}(c\alpha \frac{r}{R})]\} \\ U_\theta'(r)|_i &= \frac{\alpha}{4} \{A_1 [J_{i-2}(\alpha \frac{r}{R}) - J_{i+2}(\alpha \frac{r}{R})] + cA_2 [2J_i(c\alpha \frac{r}{R}) - J_{i-2}(c\alpha \frac{r}{R}) - J_{i+2}(c\alpha \frac{r}{R})]\} \end{cases} \quad (13)$$

Integration of $M_g^M|_i$ and $K_g^M|_i$ are made numerically. Knowing $M_g^B|_i$, $K_g^B|_i$, $M_g^M|_i$ and $K_g^M|_i$, it is possible using Rayleigh-Ritz theory to determined eigen frequencies of the circular membrane surrounded by the beam:

$$f_i = \frac{1}{2\pi} \sqrt{\frac{K_g^B|_i + K_g^M|_i}{M_g^B|_i + M_g^M|_i}} \quad (14)$$

For the first mode of ovalization, this analytical frequency is compared to the frequency obtained by the finite element model for different radius and materials of membrane and beam. The maximal error obtained is around 5%. Therefore, this model can be used to find a modal equivalence between the circular membrane surrounded by the beam and the fluid filled beam.

3 METHODOLOGY OF CHOICE FOR GRANULAR MATERIAL AND FUTURE EXPERIMENTAL TEST BENCH

The objective is to choose a granular material in order to get the same eigen frequencies. Using expression (6) and (14), this objective can be written as :

$$\frac{K_g^B|_i + K_g^M|_i}{M_g^B|_i + M_g^M|_i} = \frac{K_g^B|_i}{M_g^B|_i + M_a^F|_i} \Leftrightarrow K_g^M|_i = \frac{K_g^B|_i}{M_g^B|_i + M_a^F|_i} M_g^M|_i + \frac{K_g^B|_i}{M_g^B|_i + M_a^F|_i} M_g^B|_i - K_g^B|_i \quad (15)$$

Fluid acts only as an added mass for the beam whereas the membrane acts as an added mass and an added stiffness. Therefore, the density of the membrane must be greater than the one of the liquid in a proportion depending on the membrane's stiffness as shown in relation (15). To choose the grain material, a density-Young modulus Ashby diagram^[11], on the figure 4, can be used. This graph represents the main material families, densities and young modulus. For a granular material, the average density ρ_2 is linked to the density of the material ρ thanks to the packing fraction c by the relation : $\rho_2 = c\rho$. The packing fraction value depends on the organisation of the grains. For a random compact arrangement, the value is closed to 0.7. Young modulus of a granular material E_2 is proportional to $E^{2/3}$ with E the material Young modulus. Moreover, $K_g^M|_i$ is proportional to E_2 and $M_g^M|_i$ is proportional to ρ_2 .

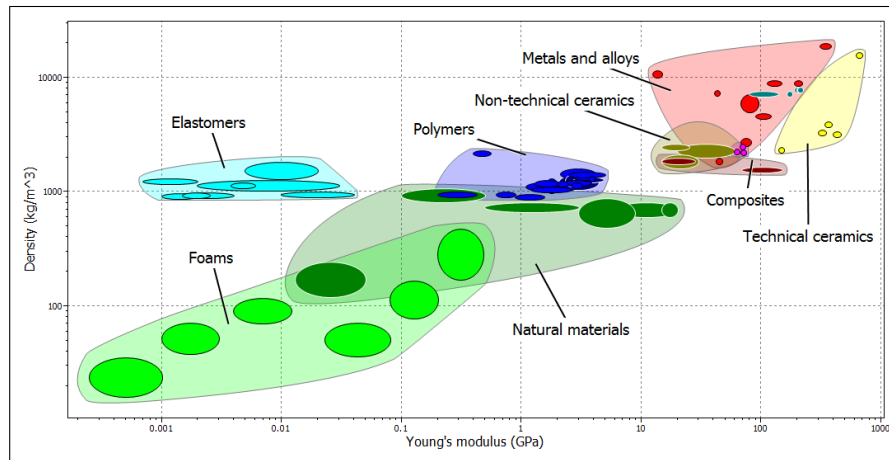


Figure 4: Density-Young modulus Ashby diagram obtained using CES-EduPack.

To verify equation (15) with existing materials, Young modulus needs to be quite low. According to Asbhy diagram, foams and elastomer could be interesting. By calculating the mass and stiffness operators of the membrane, for an aluminium beam with a radius of 1 meter, it is possible to determine the pre-stress pressure on grains needed to verify (15). To finalize the choice the pre-stress pressure obtained must be verified at several points during vibration tests:

- Each grain must remain in its elastic domain : $\sigma_{max} < Re \Rightarrow P_{Max}$
- Pressure must prevent slipping grain motions : equation 7 $\Rightarrow P_{Min}$

In order to validate the methodology and the semi-analytical approach, a test bench is under development. Even if the model is mainly $2D$, it is quite difficult to realize a $2D$ experimentation. The model presented on figure 1 of a granular material surrounded by a beam is unstable due to the possibility of out of slice-plane motions. To deal with this issue a solution is to use a long cylindrical tank to be in plane strain state for slices far away from boundaries conditions. The tank is fully filled with spherical grains pressurized by a membrane to get, thanks to a Jassen theory^[12], the pressure needed for the slice of interest. The tank is embedded on a horizontal plate directly connected to the shaker. Accelerometers are settled on the slice of interest. The figure 5 represents a schematic design of the test bench.

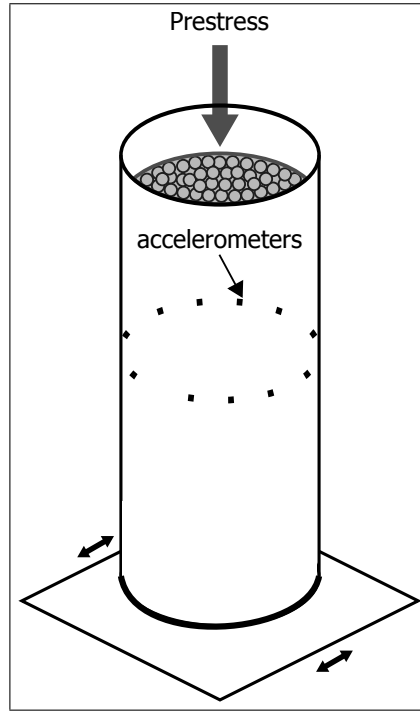


Figure 5: Schematic design of the test bench

4 CONCLUSION

This work has presented a new approach to perform vibration tests on liquid hydrogen filled tanks. Due to liquid hydrogen's low density, the use of surrogate fluid, as done up to now, presents some important issues. By using a granular material instead of water to fill the tank for vibration tests, it seems possible to reach mode shapes and eigen-frequencies which are unreachable with water.

After the justification of a $2D$ approach, the first mode of a cylindrical tank filled with liquid were analytically obtained under the hypothesis **H1** (incompressible fluid). Once these objective modes are known, an homogeneous approach of granular material based on several hypotheses (mainly **H2**, **H3** and **H4**) has been developed in order to semi-analytically find the modes of a tank filled with grains using assumption **H5**. At last, in order to fit the modes obtained for each tank, a methodology for choosing material and pre-stress pressure has been exposed. A test bench, currently under development, has been presented to validate this analytical approach. Theories used in this study are easy-to-apply like the Boussinesq's one. This choice was made to facilitate the analytical approach in order to get orders of magnitude to choose grain properties. The test bench could help us determine the validation range of these theories and to improve modeling if necessary. In future works, we will focus mainly on the impact and the modeling of damping in granular material using experimental results.

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