# On the connection between Tug-of-War Games and Nonlocal PDEs on graphs

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# On the connection between Tug-of-War Games and Nonlocal PDEs on graphs

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**Abstract**

In this paper, we are interested in the connection between some stochastic games, namely the Tug-of-War Games, and non-local PDEs on graphs. We consider a general formulation of Tug-of-War Games related to many continous PDEs. Using the framework of Partial difference Equation, we transcribe this formulation on graph, and show that it encompasses several PDEs on graphs such as $\infty$-Laplacian, Game $p$-Laplacian with and without gradient terms, and Eikonal equation. We then interpret these discrete games as non-local Tug-of-War Games. The proposed framework is illustrated with general interpolation problems on graphs.

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**Résumé**

**Sur la connexion entre certains jeux stochastiques et les EDPs sur graphes.**

Dans cet article, nous nous intéressons à la connexion entre certains jeux stochastiques et certaines Equations aux Dérivées Partielles (EDPs) sur graphes. Nous considérons une formulation générale des jeux de type Tug-of-War reliés à de nombreuses EDPs continues. En utilisant le cadre des Equations aux différences Partielles, nous transcrivons cette formulation, et montrons qu'elle inclue de nombreuses EDPs sur graphes telles que l'$\infty$-Laplacien, le Game $p$-Laplacien avec et sans termes de gradients, ainsi que l'équation Eikonale. Nous interprétons ensuite ces jeux discrets comme des jeux de type Tug-of-War non-locaux. La méthode proposée est illustrée à travers de nombreux problèmes d'interpolation sur graphe.

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*Keywords:* Tug-of-War games ; graph ; Partial Differential Equations ; local and non-local PDE on graphs ; Interpolation on graphs

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locales et non-locales sur graphes ; Interpolation sur graphes

### Version française abrégée

Les jeux stochastiques ou déterministes ont récemment émergé comme une nouvelle approche pour l'étude et l'approximation de nombreuses Équations aux Dérivées Partielles (EDPs) non linéaires. En particulier, les jeux de type *Tug-of-War* ont attiré beaucoup d'attention étant donné leurs liens avec l'infini-Laplacien ou avec le $p$-Laplacien. Ils ont été présenté pour la première fois par Peres, Schramm, Sheffiel, et Wilson dans [1, 2]. De nombreux travaux utilisent désormais ce type de jeu pour étudier l'existence ou la régularité des solutions de nombreuses EDPs (voir [3] et références inclues). La plupart de ces jeux sont généralement formulés en tant que fonctionnelles statistiques impliquant les opérateurs *moyenne*, *min*, ou *max*. Ils sont interprétés comme une approximation discrète de l'EDP sous-jacente, et résoudre cette dernière revient à considérer une solution limite appropriée du jeu discret.

Récemment, il y a un grand intérêt dans l'adaptation d'outils classiques en traitement du signal, comme les ondelettes ou les EDPs [4, 5, 6], aux graphes et réseaux. La nécessité de telles méthodes est motivée par des applications existantes et futures, comme l'apprentissage machine ou le traitement des images. En fait, tout type de données peut être représenté sous forme une forme de graphe abstrait pour lequel les sommets représentent les données, et les arcs les interactions entre les données.

Dans cet article, nous considérons une équation générale de programmation dynamique (équation 6) qui englobe de nombreuses versions du jeu de type *Tug-of-War* ainsi que leur EDPs adjointes, dont l'infini-Laplacien, le Game $p$-Laplacien, ou encore les équations de type Hamilton-Jacobi (Section 2). Nous montrons à la section 3 que, dans le cadre des Équations aux différences Partielles (EdPs) [7, 5], ces jeux discrets coincident avec des EDPs sur des graphes Euclidiens particuliers.

Les mêmes EDPs sur des graphes pondérés de topologie arbitraire conduisent à des fonctionnelles statistiques non-locales, incluant des opérateurs bien connus tels le moyennage non-local, la dilatation ou encore l'érosion non-locale [8]. Nous interprétons alors ces opérateurs comme des jeux de type *Tug-of-War* non-locaux, et montrons leurs liens avec des EDPs non-locales sur graphe Euclidien.

### 1. Introduction

Game theoretic stochastic or deterministic methods have recently emerged as a novel approach to study and to approximate various non-linear Partial Differential Equations (PDEs). In particular Tug-of-War Games (TOG) related to the $\infty$-Laplacian or to the $p$-Laplacian have attracted a lot of attention. They were first introduced by Peres, Schramm, Sheffiel, and Wilson in [1, 2]. It is now used in many works to study the existence or the regularity of solutions for many PDEs (see [3] and references therein). Many of these games generally are formulated as well-known statistical functionals such as *mean*, *min*, or *max* operators. They are interpreted as a discrete approximation of the underlying PDE and solving the latter leads to taking a suitable limit of the solution of the discrete game.

Recently, there is a high interest in adapting classical signal processing tools on graphs and networks such as wavelets or PDEs [4, 5, 6]. The demand for such methods is motivated by existing and potential future applications, such as in machine learning and mathematical image processing. Indeed, any kind of data can be represented by a graph in an abstract form in which the vertices are associated to the data and the edges correspond to relationships within data.

In this paper, we consider a general Dynamic Programming Equation that encompasses many versions Tug-of-War Games arising in the discrete game-theoretic interpretation for various non-linear PDEs including $\infty$-Laplacian, game $p$-Laplacian and Hamilton-Jacobi related equations. We show that under our framework of Partial difference Equations (PdEs) [7, 5], these discrete games coincide with PDEs on particular Euclidean graphs.

The same PDEs on weighted graph of arbitrary topology lead to non-local statistical functionals, including well-known non-local means, non-local dilation, and non-local erosion operators [8]. We interpret them as non-local Tug-of-War Games and we show their connections to non-local PDEs on Euclidean graphs.

This paper unfolds as follows: first, we briefly introduce several discrete games, their related PDEs, and the previously proposed Partial difference Equations (PdE) framework. Then, we naturally extend these games to non-local forms, and illustrate some interpolation problems on image and point cloud processing.

## 2. Tug-of-War Games and PDEs on Euclidean space

We first briefly review the principle of stochastic Tug-of-War Game (TOG) as described in [2], and one of its variants, namely *TOG with noise* [9]. A general formulation that encompasses these games is also described.

### *2.1. Tug-of-War Game (TOG) and continuous PDEs*

Let us briefly review the notion of Tug-of-War Game (TOG). Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, $h : \Omega \to \mathbb{R}$ the running payoff function, and $g : \partial\Omega \to \mathbb{R}$ the payoff function. Fix a number $\varepsilon > 0$. The dynamics of the game are as follows: a token is placed at an initial position $x_0 \in \Omega$. At the $k$th stage of the game, Player I and Player II select points $x_k^I$ and $x_k^{II}$, respectively, each belonging to a specified set $B_\varepsilon(x_{k-1}) \subseteq \Omega$ (where $B_\varepsilon(x_{k-1})$ is the $\varepsilon$-ball centered in $x_{k-1}$). The token is then moved to $x_k$, where $x_k$ can be either $x_k^I$ or $x_k^{II}$ with equal probability. In other words, a fair coin is tossed to determine where the token is placed (i.e., which player won this stage).

After the $k$th stage of the game, if $x_k \in \Omega$ then the game continue to stage $k+1$. Otherwise, if $x_k \in \partial\Omega$, the game ends and Player II pays Player I the amount $g(x_k) + \varepsilon^2 \sum_{j=0}^{k-1} h(x_j)$. Player I attempts to maximize the payoff while Player II attempts to minimize it. If both player are using optimal strategy, according to the Dynamic Programming Principle (DPP), the value functions for Player I and Player II for standard $\varepsilon$-turn Tug-of-War satisfy the relation:

$$\begin{cases} u^\varepsilon(x) &= \frac{1}{2}\left[\sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y\in B_\varepsilon(x)} u^\varepsilon(y)\right] + \varepsilon^2 h(x), \quad x\in\Omega, \\ u^\varepsilon(x) &= g(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x\in\partial\Omega. \end{cases} \tag{1}$$

The authors of [2] have shown that if the running payoff function $h$ is of constant sign, the value function $u^\varepsilon$ converges to the unique viscosity solution of the normalized $\infty$-Poisson equation:

$$\begin{cases} \Delta_\infty^N u(x) &= -h(x), \quad x\in\Omega, \\ u(x) &= g(x), \qquad x\in\partial\Omega, \end{cases} \tag{2}$$

where $\Delta_\infty^N u = \frac{1}{|\nabla u|^2}\Delta_\infty u$ is the normalized $\infty$-Laplacian and

$$\Delta_\infty u = |\nabla u|^{-2}\sum_{i,j} u_{x_i} u_{x_i x_j} u_{x_j}. \tag{3}$$

*2.2. TOG with noise*

In its version with noise, the game is modified as follows: at point $x_k$ in $\Omega$, player I and player II play $\epsilon$-step Tug-of-War game with probability $\beta\in[0,1]$, and a random point in ball of radius $\epsilon$ centered at $x_k$ is chosen with probability $1-\beta$. The value functions of the game satisfy the DPP:

$$u^\epsilon(x) = \frac{\beta}{2}\left[\sup_{y\in B_\varepsilon(x)} u^\epsilon(y) + \inf_{y\in B_\varepsilon(x)} u^\epsilon(y)\right] + \frac{1-\beta}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u^\epsilon(y)dy + \epsilon^2 h(x) \tag{4}$$

with the boundary condition $u^\epsilon(x) = g(x)$ for $x$ in $\delta\Omega$. A detailed proof for existence and uniqueness of these types of function was shown in [9].

Choosing the probability $\beta = \frac{p-2}{p+n}$, this DPP gives a connection to viscosity solutions of the following $p$-Laplace equation [10, 11, 3]:

$$\begin{cases} \Delta_p^N u(x) &= -h(x), \quad x\in\Omega, \\ u(x) &= g(x), \qquad x\in\partial\Omega, \end{cases} \tag{5}$$

for $p\geq 2$ with $\Delta_p^N u = \frac{1}{p}|\nabla u|^{2-p}\, div\left(|\nabla u|^{p-2}\nabla u\right)$.

*2.3. General formulation*

A general formulation that encompasses these previous games can be defined as follows: $x_{k+1}^I$ is chosen with a probability of $\alpha$, $x_{k+1}^{II}$ with a probability of $\beta$, and there is a probability equal to $\gamma$ that a random point in ball of radius $\epsilon$

centered at $x_k$ is chosen. In this setting, $\alpha+\beta+\gamma = 1$, and the value functions of the game satisfy:

$$u^\epsilon(x) = \alpha \sup_{y\in B_\varepsilon(x)} u^\epsilon(y) + \beta \inf_{y\in B_\varepsilon(x)} u^\epsilon(y) + \frac{\gamma}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u^\epsilon(y)dy + \epsilon^2 h(x) \quad (6)$$

This general formulation recovers the Tug-of-War Game ($\alpha = \beta$, $\gamma = 0$), its variant with noise ($\gamma > 0$, $\alpha = \beta$), and its biased version [12, 13] ($\alpha \neq \beta$, $\gamma = 0$).

## 3. From Discrete Tug-of-War Games to PDEs on graphs

### *3.1. Notations and definitions*

We recall here definitions and operators on graphs that constitute the basis of the Partial difference Equation (PdE) framework on graphs. These definitions are borrowed from [7, 5] and references therein.

A weighted graph $G = (V, E, w)$ is composed of a finite set $V = \{v_1, \ldots, v_N\}$ of $N$ vertices and a finite set $E \subset V \times V$ of weighted edges. $G$ is assumed to be simple (no self-loops nor multiple edges), connected, and undirected. Let $(u, v) \in E$ be the edge that connects two vertices $u$ and $v$ from $V$. Its weight, denoted by $w(u, v)$, represents the similarity between vertices $u$ and $v$. It is usually computed by using a positive symmetric function $w : V \times V \to \mathbb{R}^+$ satisfying $w(u, v) = 0$ if $(u, v) \notin E$. The notation $u \sim v$ is also used in the following to denote two adjacent vertices. In the following, we consider the real valued function $f$ defined on $V$.

**Définition** 3.1. Discrete upwind non-local weighted gradients *are defined as:*

$$(\nabla_w^\pm f)(u) = ((\partial_v^\pm f)(u))^T_{v\in V} \quad (7)$$

*where* $(\partial_v^\pm f)(u) = \left(\sqrt{w(u,v)}(f(v) - f(u))\right)^\pm$, *with* $(x)^+ = \max(x, 0)$ *and* $(x)^- = -min(x, 0)$.
*The* $\mathcal{L}_p$ *and* $\mathcal{L}_\infty$ *norms of these gradients are defined as:*

$$\begin{aligned} \|(\nabla_w^\pm f)(u)\|_p &= \left[\sum_{v\sim u} \sqrt{w(u,v)}^p (f(v) - f(u))^{p\pm}\right]^{\frac{1}{p}}, \\ \|(\nabla_w^\pm f)(u)\|_\infty &= \max_{v\sim u}\left(\sqrt{w(u,v)} \cdot (f(v) - f(u))^\pm\right) \end{aligned} \quad (8)$$

**Définition** 3.2. *The* 2*-Laplacian on graph is defined as:*

$$(\Delta_{w,2} f)(u) = \frac{\sum_{u\sim v} w(u,v) f(v)}{\sum_{u\sim v} w(u,v)} - f(u) \quad (9)$$

**Définition** 3.3. *The* $\infty$*-Laplacian on graph is defined as:*

$$(\Delta_{w,\infty} f)(u) = \frac{1}{2}\left[\|(\nabla_w^+ f)(u)\|_\infty - \|(\nabla_w^- f)(u)\|_\infty\right]. \quad (10)$$

**Définition** 3.4. *The game p-Laplacian on graph is defined as:*

$$(\Delta^G_{w,p} f)(u) = a(p)\Delta_{w,\infty} f(u) + b(p)\Delta_{w,2} f(u) \tag{11}$$

*with* $2 \leq p < \infty$, $a(p) = \frac{p-2}{p}$ *and* $b(p) = \frac{2}{p}$.

*3.2. Transcription of TOG on graphs*

Now, let us investigate the Euclidean graph $G = (V, E, w)$ with $V = \Omega \subset \mathbb{R}^n$, $E = \{(u, v) \in V \times V | w(u, v) > 0\}$, and

$$w(u, v) = \begin{cases} \frac{1}{\epsilon^4}, & \text{if } |v - u| < \epsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Using $w$ in the discrete upwind gradient $\mathcal{L}_\infty$-norm, we get

$$\begin{array}{rcl} \|(\nabla^+_w f)(u)\|_\infty & = & \max\limits_{v\sim u} \left(\sqrt{w(u,v)}(f(v) - f(u))\right) \\ & = & \frac{1}{\epsilon^2}\left(\max\limits_{v\sim u}(f(v)) - f(u)\right) \\ \|(\nabla^-_w f)(u)\|_\infty & = & \frac{1}{\epsilon^2}\left(f(u) - \min\limits_{v\sim u}(f(v))\right) \end{array} \tag{13}$$

We can define the min and max functions as:

$$\begin{array}{rcl} \max\limits_{v\sim u} f(v) & = & \epsilon^2 \|(\nabla^+_w f)(u)\|_\infty - f(u), \\ \min\limits_{y\sim u} f(v) & = & f(u) - \epsilon^2 \|(\nabla^-_w f)(u)\|_\infty. \end{array} \tag{14}$$

By replacing the min and max functions in equation 6 by their respective discrete gradient variants, we get the following PDE on graph:

$$\alpha\|(\nabla^+_w f)(u)\|_\infty - \beta\|(\nabla^-_w f)(u)\|_\infty + \gamma(\Delta_{w,2} f)(u) = -h(u) \tag{15}$$

This formulation recovers the Tug-of-War Game (equation 2) with $\alpha = \beta = 0.5, \gamma = 0$, the Biased TOG [12, 13] with $\alpha \neq \beta, \gamma = 0$, the Eikonal equation with $\alpha = \gamma = 0, \beta = 1$, and the TOG with noise (equation 5) with $\gamma \neq 0, \alpha = \beta$. Equation 15 can also be written as

$$2\min(\alpha,\beta)\Delta_{w,\infty} f(u) + \gamma\Delta_{w,2} f(u) + (\alpha-\beta)^+ |\nabla^+_w f(u)|_\infty - (\alpha-\beta)^- |\nabla^-_w f(u)|_\infty = -h(u) \tag{16}$$

that clearly exhibits the discrete approximations on graphs of $\infty$-Laplacian and game $p$-Laplacian with or without gradient terms.

## 4. Extension to non-local discrete TOG

In this section, we consider a general weighted connected graph $G = (V, E, w)$, and a null running payoff function $h(u) = 0, \forall u$.

*4.1. Interpolation and the associated Dirichlet problem*

Let $A \subset V$ be a subset of vertices, and $g : \partial A \to \mathbb{R}$ a function defined on the boundary of $A$. We consider the general PdE:

$$\begin{cases} \alpha\|(\nabla_w^+ f)(u)\|_\infty - \beta\|(\nabla_w^- f)(u)\|_\infty + \gamma(\Delta_{w,2} f)(u) = 0, & u \in A \\ f(u) = g(u), & u \in \partial A \end{cases} \tag{17}$$

Solving this general PdE is an interpolation problem aiming at determining the value of $f$ on the whole domain $A$ from known values defined on $\partial A$. The existence and uniqueness of the solution of equation 17 can be shown by using the Brouwer fixed point theorem and the comparison principle respectively [14, 15].

*4.2. Extension to non-local TOG*

Before interpreting the PDE (equation 15) as a non-local game, we first introduce 3 non-local operators defined on graphs, namely non-local dilation (NLD), non-local erosion (NLE), and non-local mean (NLM), by:

$$NLD(f) = \|\nabla_w^+\|_\infty + f \tag{18}$$

$$NLE(f) = f - \|\nabla_w^-\|_\infty \tag{19}$$

$$NLM(f) = \Delta_{w,2} f + f \tag{20}$$

Due to the generality of the weight function $w$ and the arbitrary graph topology, these operators do not necessarily act locally.

Incorporating these operators into equation 15 and setting $h(u) = 0, \forall u$ leads to the equation:

$$f(u) = \alpha NLD(f) + \beta NLE(f) + \gamma NLM(f) \tag{21}$$

This last equation can be interpreted as a non-local Tug-of-War Game as follows: the token at the $k$-th stage of the game is moved to a new destination $x_k^I \in \Omega$ with a probability

$$P_1 = \frac{\alpha\sqrt{w(u,y)}}{\alpha\sqrt{w(u,y)} + \beta\sqrt{w(u,z)} + \gamma} \tag{22}$$

$x_k = x_k^{II}$ with a probability $P_2 = \frac{\beta\sqrt{w(u,z)}}{\alpha\sqrt{w(u,y)}+\beta\sqrt{w(u,z)}+\gamma}$ and $x_k = x_k^{III}$ chosen randomly in $\{x_k \in V \mid x_k \sim x_{k-1}\}$ with a probability $1 - P_1 - P_2$.

*4.3. Connection to non-local PDEs on Euclidean graphs*

Let $G = (V, E, W)$ be an Euclidean graph with $V = \Omega \subset \mathbb{R}^n$, $E = \{(x, y) \in V \times V | w(x, y) > 0\}$.

• Setting $\gamma = 0$, $\alpha = \beta \neq 0$, and

$$w(x,y) = \begin{cases} \frac{1}{|x-y|^{2s}} & x \neq y, s \in [0,1] \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

equation 15 corresponds to the Hölder $\infty$-Laplacian equation proposed by Chambolle *et al.* in [16], which is given by

$$\Delta_{w,\infty} f(x) = \frac{1}{2}\left[\max_{y\in\Omega, y\neq x}\left(\frac{f(y)-f(x)}{|y-x|^s}\right) + \min_{y\in\Omega, y\neq x}\left(\frac{f(y)-f(x)}{|y-x|^s}\right)\right] \tag{24}$$

This operator is formally derived from the minimization of the following energy functional

$$\int_\Omega \int_\Omega \frac{|f(y)-f(x)|^p}{|y-x|^{p\times s}}\,\mathrm{d}x\mathrm{d}y \tag{25}$$

with $p \to \infty$.

• Setting $\alpha = \beta = 0$, and for a general weighting function $w$, it gets clear that equation 15 recovers the continuous non-local Laplacian equation:

$$\int_\Omega w(x,y)\cdot(f(y)-f(x))\,\mathrm{d}y = 0 \tag{26}$$

This operator has been recently used in many applications including continuum mechanics, population dynamics, and many different non-local diffusion problems [17]. A particular case is given by

$$w(x,y) = \begin{cases} \frac{1}{|y-x|^{n+2s}}, & x \neq y, s \in [0,1] \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Here, we recover the continuous fractional Laplacian which is commonly used to model anomalous diffusion $(-\Delta)^s f(x) = C_{n,s}\cdot\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+2s}}\mathrm{d}y$ where $C_{n,s}$ is a normalization constant.

• In a more general way, the proposed formulation of non-local graph $p$-Laplacian (eq. 15, with arbitrary values for $\alpha$, $\beta$, $\gamma$, and $w$) corresponds to PDEs that interpolates between the $\infty$-Laplacian, the Laplacian, and gradient terms.

## 5. Illustrations of interpolation problems on graphs

Our framework can handle many interpolation problems on graphs such as inpainting, colorization, and distance computation. For details about graph construction for images and point clouds, see [18] and references therein.

Figure 1 (rows 1 to 3) shows local and non-local inpaintings of a 2D image and a 3D point cloud for different values of $\alpha$, $\beta$, and $\gamma$. Distance computation on a mesh using $\beta = 1$ (Eikonal equation), and 3D point cloud colorization from scribbles using $\alpha = \beta = 0.5$ also illustrates the benefits of using PDEs on graphs (fourth row).

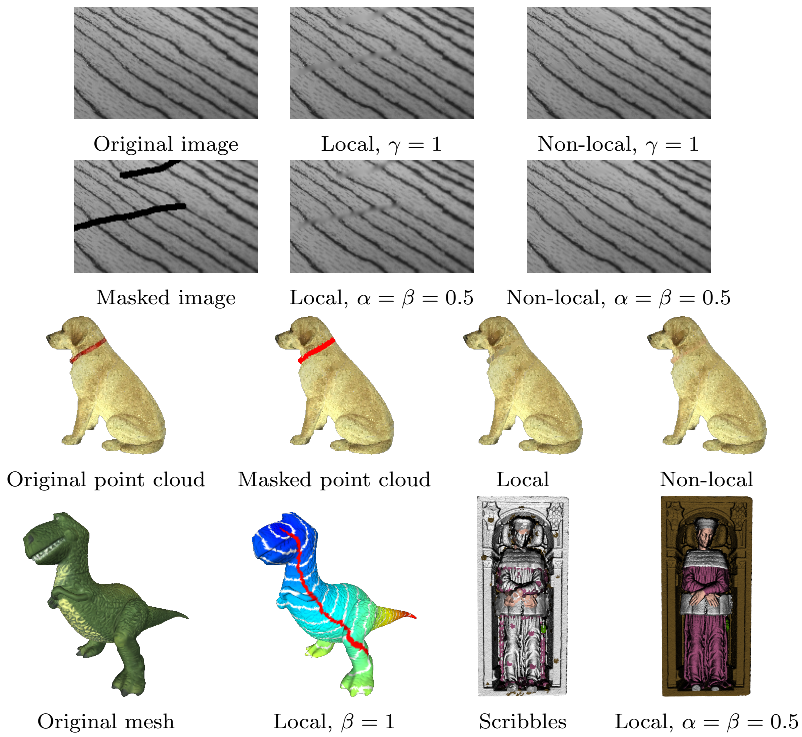


Figure 1: Illustrations of some interpolation problems on images and 3D point clouds: inpainting, distance computation, and colorization.

## 6. Conclusion and further work

We have proposed a transcription of several Dynamic Programming Equations arising in the discrete game-theoretic interpretation for various non-linear PDEs. Build upon our simple Partial difference Equations framework, we have shown that these discrete games coincide with PDEs on Euclidean graphs, and that their extension to non-local forms can be interpreted as non-local Tug-of-War games. This paper makes a bridge between stochastic games, continuous local and non-local PDEs and their graph-based counterparts. We plan to propose in the future new algorithms based on the min/max operators to solve efficiently these PDEs on graphs.

[1] Y. Peres, S. Sheffield, Tug-of-war with noise: A game-theoretic view of the $p$-laplacian, Duke Mathematical Journal 145 (1) (2008) 91–120.

[2] Y. Peres, O. Schramm, S. Sheffield, D. Wilson, Tug-of-war and the infinity laplacian, J. of the Am. Math. Soc. 22 (1) (2009) 167–210.

[3] E. Ruosteenoja, Local regularity results for value functions of tug-of-war with noise and running payoff, Adv. in Calculus of Variations 9 (1) (2016) 1–17.

[4] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, P. Vandergheynst, The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains, IEEE Signal Processing Magazine 30 (3) (2013) 83–98.

[5] A. Elmoataz, X. Desquesnes, Z. Lakhdari, O. Lézoray, Nonlocal infinity laplacian equation on graphs with applications in image processing and machine learning, Mathematics and Computers in Simulation 102 (2014) 153–163.

[6] A. L. Bertozzi, A. Flenner, Diffuse interface models on graphs for classification of high dimensional data, Multiscale Modeling & Simulation 10 (3) (2012) 1090–1118.

[7] A. Elmoataz, O. Lezoray, S. Bougleux, Nonlocal discrete regularization on weighted graphs: a framework for image and manifold processing, IEEE transactions on Image Processing 17 (7) (2008) 1047–1060.

[8] G. Gilboa, S. Osher, Nonlocal operators with applications to image processing, Multiscale Modeling & Simulation 7 (3) (2008) 1005–1028.

[9] J. J. Manfredi, M. Parviainen, J. D. Rossi, Dynamic programming principle for tug-of-war games with noise, ESAIM: Control, Optimisation and Calculus of Variations 18 (01) (2012) 81–90.

[10] J. J. Manfredi, M. Parviainen, J. D. Rossi, On the definition and properties of p-harmonious functions, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze-Serie V 11 (2) (2012) 215.

[11] M. Lewicka, J. J. Manfredi, Game theoretical methods in pdes, Bollettino dell'Unione Matematica Italiana 7 (3) (2014) 211–216.

[12] Q. Liu, A. Schikorra, A game-tree approach to discrete infinity laplacian with running costs, arXiv preprint arXiv:1305.7372.

[13] A. P. Sviridov, p-harmonious functions with drift on graphs via games, Electronic Journal of Differential Equations 2011 (114) (2011) 1–11.

[14] J. J. Manfredi, A. M. Oberman, A. P. Sviridov, Nonlinear elliptic partial differential equations and p-harmonic functions on graphs, Differential Integral Equations 28 (1–2) (2015) 79–102.

[15] A. Elmoataz, F. Lozes, M. Toutain, Nonlocal pdes on graphs: From tug-of-war games to unified interpolation on images and point clouds, Journal of Mathematical Imaging and Vision (2016) 1–21.

[16] A. Chambolle, E. Lindgren, R. Monneau, A hölder infinity laplacian, ESAIM: Control, Optimisation and Calculus of Variations 18 (2012) 799–835.

[17] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. J. Toledo-Melero, Nonlocal diffusion problems, Vol. 165, Am. Math. Soc., 2010.

[18] F. Lozes, A. Elmoataz, O. Lézoray, Partial difference operators on weighted graphs for image processing on surfaces and point clouds, IEEE Transactions on Image Processing 23 (9) (2014) 3896–3909.