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ON THE TANGENT GROUPOID OF A FILTERED MANIFOLD

ERIK VAN ERP AND ROBERT YUNCKEN

ABSTRACT. We give an intrinsic (coordinate-free) construction of the tangent groupoid of a filtered manifold.

1. INTRODUCTION

Connes' tangent groupoid,

$$\mathbb{T}M = TM \times \{0\} \sqcup (M \times M) \times \mathbb{R}^\times$$

provides a powerful conceptual framework for studying pseudo-differential operators. The power of the tangent groupoid lies in the smooth glueing of the fibres $M \times M$ (which carries the Schwartz kernel of a pseudodifferential operator) to the tangent space TM (which carries the principal cosymbol). Connes famously used this construction in a proof of the Atiyah-Singer Index Theorem [Con94]. In a recent paper we showed how the tangent groupoid can be used as the foundation for the *definition* of the classical pseudodifferential calculus [vEY16].

By now we have tangent groupoids associated to many different classes of pseudodifferential operators. One example is the Heisenberg calculus which is relevant to contact and CR geometries. Here the tangent bundle TM must be replaced by a bundle of Heisenberg groups $T_H M$ called the *osculating groupoid*. The Heisenberg calculus was developed by Folland-Stein [FS74] and Taylor [Tay] in the 1970s. An adaptation of the tangent groupoid to the Heisenberg calculus was developed in [vE05, Pon06], and this led to the index theorem for the Heisenberg calculus [vE10a, vE10b].

We have shown in [vEY16] that a pseudodifferential calculus for filtered manifolds, à la Melin [Mel82], can be easily produced from a tangent groupoid for filtered manifolds. The relevant tangent groupoid appears in Choi-Ponge [CP15], with a construction based

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on preferred local coordinate systems. For our purposes, we made an intrinsic (coordinate-free) construction of the same groupoid. The goal of this paper is to describe this construction.

We follow a classic strategy: just as it is easier to construct a Lie algebra than a Lie group, so it is easier to define a Lie algebroid than a Lie groupoid. We therefore begin by constructing the Lie algebroid $\mathfrak{t}_H M$. This is just a matter of glueing a filtered bundle to its associated graded bundle—a process which is most efficiently carried out by working with their modules of sections. We then appeal to Lie’s Third Theorem for Lie algebroids (on the existence of a Lie groupoid for a given Lie algebroid, under mild conditions) to produce the tangent groupoid $\mathbb{T}_H M$.

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Notation. We will be considering various fibrations over $M \times \mathbb{R}$, and often regard them as families of fibrations over M indexed by $t \in \mathbb{R}$. If $\pi : E \rightarrow M \times \mathbb{R}$ is a submersion, we shall denote by $E|_t = \pi^{-1}(M \times \{t\})$ its restriction to $t \in \mathbb{R}$. Similarly, if $\mathbb{X} : M \times \mathbb{R} \rightarrow E$ is a section of E , we will denote by \mathbb{X}_t its restriction to $M \times \{t\}$.

2. LIE ALGEBROIDS

We will construct the tangent groupoid of a filtered manifold by first constructing its Lie algebroid.

Recall that a *Lie algebroid* over a smooth manifold M is a vector bundle $\mathfrak{g} \rightarrow M$ equipped with two compatible structures:

- A Lie bracket on smooth sections $[\cdot, \cdot] : \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \rightarrow \Gamma(\mathfrak{g})$,
- A vector bundle map $\rho : \mathfrak{g} \rightarrow TM$, called the *anchor*,

such that

- (1) The induced map on sections $\rho : \Gamma(\mathfrak{g}) \rightarrow \Gamma(TM)$ is a Lie algebra homomorphism,
- (2) For any $X, Y \in \Gamma(\mathfrak{g})$, $f \in C^\infty(M)$,

$$[X, fY] = f[X, Y] + (\rho(X)f)Y.$$

To understand the tangent groupoid, two key examples are needed.

Example 1. The *tangent bundle* itself $TM \rightarrow M$ is a Lie algebroid over M , with the usual Lie bracket of vector fields and the identity map as anchor. This is the Lie algebroid of the pair groupoid $M \times M$.

Example 2. If the anchor of a Lie algebroid \mathfrak{g} is the zero map, then \mathfrak{g} is a *smooth bundle of Lie algebras*. This is because the Lie bracket

$[\cdot, \cdot]$ on sections $\Gamma(\mathfrak{g})$ is a $C^\infty(M)$ -linear map, and so restricts to a well-defined Lie bracket on each fibre \mathfrak{g}_x ($x \in M$).

In particular, if we equip $\Gamma(TM)$ with the zero Lie bracket, then the tangent bundle TM has an alternative Lie algebroid structure as a bundle of abelian Lie algebras.

3. CONNES' TANGENT GROUPOID

We start with a brief review of the classical tangent groupoid $\mathbb{T}M$ of Connes [Con94].

Let M be a smooth manifold without boundary. Algebraically, the tangent groupoid $\mathbb{T}M$ is the disjoint union

$$\mathbb{T}M = TM \times \{0\} \sqcup M \times M \times \mathbb{R}^\times.$$

The $t = 0$ fiber $TM \times \{0\}$ is glued to $M \times M \times \mathbb{R}^\times$ by blowing up the diagonal in $M \times M$. To make this precise, it is easier to first construct the Lie algebroid $\mathfrak{t}M$ of $\mathbb{T}M$.

As a vector bundle over $M \times \mathbb{R}$, the Lie algebroid $\mathfrak{t}M$ of the tangent groupoid $\mathbb{T}M$ is

$$\mathfrak{t}M = TM \times \mathbb{R} \rightarrow M \times \mathbb{R} \quad (1)$$

The bracket of two smooth sections $\mathbb{X}, \mathbb{Y} \in \Gamma(\mathfrak{t}M)$ is rescaled by t ,

$$[\mathbb{X}, \mathbb{Y}]_t = t [\mathbb{X}_t, \mathbb{Y}_t],$$

where the right hand bracket is the ordinary bracket of vector fields \mathbb{X}_t and \mathbb{Y}_t in $\Gamma(TM)$. The anchor on $\mathfrak{t}M$ is

$$\rho(\mathbb{X})_t = t \mathbb{X}_t$$

Note that multiplication by t is *injective* on *smooth* sections $\Gamma(\mathfrak{t}M)$, and that the image of this map is

$$\mathfrak{X} = \{\mathbb{X} \in \Gamma(TM \times \mathbb{R}) \mid \mathbb{X}|_{t=0} = 0\}$$

We may define $\mathfrak{t}M$ efficiently but indirectly as the vector bundle on $M \times \mathbb{R}$ underlying the $C^\infty(M \times \mathbb{R})$ module \mathfrak{X} . Note that the isomorphism

$$t : \Gamma(\mathfrak{t}M) \cong \mathfrak{X} \quad \mathbb{X} \mapsto t\mathbb{X}$$

intertwines the Lie algebroid bracket and anchor of $\mathfrak{t}M$ with the standard (constant in t) bracket and anchor of the product Lie algebroid $TM \times \mathbb{R}$. Thus, the $C^\infty(M \times \mathbb{R})$ -module \mathfrak{X} implicitly defines the Lie algebroid $\mathfrak{t}M$.

General results on integrability of Lie algebroids imply the existence of a minimal Lie groupoid with Lie algebroid $\mathfrak{t}M$ (see e.g. [CF03, Nis00, Deb01a, Deb01b]). This Lie groupoid is the tangent groupoid $\mathbb{T}M$.

Explicitly, the smooth structure of $\mathbb{T}M$ is as follows. Let ∇ be a connection on TM and $\exp_x^\nabla : T_x M \rightarrow M$ the associated geometric exponential maps. An exponential map from the Lie algebroid $\mathfrak{t}M$ to the Lie groupoid $\mathbb{T}M$ is

$$\begin{aligned} \mathbb{E}xp^\nabla : \mathfrak{t}M = TM \times \mathbb{R} &\rightarrow \mathbb{T}M; \\ (x, v, t) &\mapsto (\exp_x^\nabla(tv), x, t), & \text{if } t \neq 0 \\ (x, v, 0) &\mapsto (x, v, 0), & \text{if } t = 0. \end{aligned}$$

Note that $\mathbb{E}xp^\nabla$ is injective in a neighbourhood of $TM \times \{0\}$. The map $\mathbb{E}xp^\nabla$ is used to define the smooth structure of $\mathbb{T}M$ in such a neighborhood. Away from $t = 0$ the subset $M \times M \times \mathbb{R}^\times$ is an open subset of $\mathbb{T}M$ with the evident manifold structure.

4. FILTERED MANIFOLDS

A filtered manifold is a manifold for which the Lie algebra of vector fields is equipped with a filtration.¹ More precisely:

Definition 3. ([Mel82]) A *filtered manifold* is a smooth manifold M equipped with a filtration of the tangent bundle TM by vector bundles $M \times \{0\} = H^0 \subseteq H^1 \subseteq \dots \subseteq H^N = TM$ such that $\Gamma(H^\bullet)$ is a Lie algebra filtration, i.e.

$$[\Gamma(H^i), \Gamma(H^j)] \subseteq \Gamma(H^{i+j})$$

Here we are using the convention that $H^i = TM$ for $i > N$.

Sections of $\Gamma(H^i)$ may be referred to as *vector fields of order (less than or equal to) i* .

Of course, any manifold M can be equipped with the *trivial filtration* of depth one, i.e. $H^1 = TM$. In this trivial case the construction of the tangent groupoid that we describe below simplifies significantly, and reduces to the well-known construction outlined in section 3.

Any manifold equipped with a non-trivial subbundle $H \subset TM$ is a filtered manifold of depth 2, with $H^1 = H$ and $H^2 = TM$. Examples include foliations, contact manifolds, CR manifolds. More generally, such manifolds are called Heisenberg manifolds [FS74, BG88])

Parabolic manifolds are an example of filtered manifolds of depth greater than 2 (see [ČS09]).

¹Filtered manifolds are called *Carnot manifolds* by Choi and Ponge [CP15].

5. THE OSCULATING GROUPOID OF A FILTERED MANIFOLD

If M is a filtered manifold, we let

$$\mathfrak{t}_H M = \bigoplus_{i=1}^n H^i / H^{i-1}$$

be the associated graded vector bundle. We write $\sigma_i : H^i \rightarrow H^i / H^{i-1}$ for the quotient maps.

If $X \in \Gamma(H^i)$, $Y \in \Gamma(H^j)$ are vector fields of order i and j , respectively, then for any $f, g \in C^\infty(M)$,

$$\begin{aligned} [fX, gY] &= fg[X, Y] + f(\rho(X)g)Y - g(\rho(Y)f)X \\ &= fg[X, Y] \quad (\text{mod } \Gamma(H^{i+j-1})). \end{aligned}$$

Thus, although the Lie bracket on vector fields is not $C^\infty(M)$ -linear, it induces a $C^\infty(M)$ -linear bracket on sections of the associated graded bundle $\mathfrak{t}_H M$. Thus, $\mathfrak{t}_H M$ becomes a smooth bundle of graded nilpotent Lie algebras over M . Equipped with the zero anchor, $\mathfrak{t}_H M$ is thus a Lie algebroid.

Definition 4. The *osculating groupoid* $T_H M$ is the smooth bundle of connected, simply connected nilpotent Lie groups obtained by exponentiating the Lie algebroid $\mathfrak{t}_H M$.

Explicitly, $T_H M$ equals $\mathfrak{t}_H M$ as a smooth fibre bundle, and each fibre is equipped with the group law given by the Baker-Campbell-Hausdorff formula.

6. DILATIONS

Any graded vector bundle admits a canonical one-parameter family $(\delta_t)_{t \in \mathbb{R}}$ of bundle endomorphisms (automorphisms for $t \neq 0$) called the *dilations*. These generalize the homotheties of a trivially graded bundle.

For the osculating groupoid $\mathfrak{t}_H M$ the dilations are defined by

$$\delta_t : \zeta \mapsto t^i \zeta, \quad \text{for all } \zeta \in H^i / H^{i-1}.$$

For $t \neq 0$, the dilations δ_t are Lie algebroid homomorphisms.

7. THE H -TANGENT GROUPOID

Algebraically, the H -tangent groupoid is the disjoint union of the osculating groupoid $T_H M$ with a family of pair groupoids

$$\mathbb{T}_H M = (T_H M \times \{0\}) \sqcup (M \times M \times \mathbb{R}^\times) \quad (2)$$

This is to be understood as a family of Lie groupoids over \mathbb{R} , in that the allowable compositions are

$$\begin{aligned} (z, y, t)(y, x, t) &= (z, x, t), & x, y, z \in M, t \in \mathbb{R}^\times, \\ (x, \zeta, 0)(x, \eta, 0) &= (x, \zeta\eta, 0), & x \in M, \zeta, \eta \in T_H M_x. \end{aligned}$$

To glue the $t = 0$ fiber $T_H M \times \{0\}$ to $M \times M \times \mathbb{R}^\times$ we blow up the diagonal in $M \times M$ by means of the dilations δ_t^{-1} , as we shall now make precise.

Definition 5. A *degrading* of $\mathfrak{t}_H M$ is an isomorphism of vector bundles $\psi : \mathfrak{t}_H M \rightarrow TM$ such that for each i , the restriction of ψ to H^i/H^{i-1} is right inverse to the grading map $\sigma_i : H^i \rightarrow H^i/H^{i-1}$.

Fixing a choice of degrading allows us to transport the dilations δ_λ of $\mathfrak{t}_H M$ to endomorphisms of TM , $\delta_\lambda^\psi := \psi \circ \delta_\lambda \circ \psi^{-1}$. Let ∇ be a connection on TM that is compatible with the degrading, in the sense that ∇ commutes with δ_λ^ψ :

$$\nabla_X(\delta_\lambda^\psi Y) = \delta_\lambda^\psi \nabla_X(Y) \quad X, Y \in \Gamma(TM).$$

For such a choice of connection, use the exponential maps

$$\exp_x^\nabla : T_x M \rightarrow M$$

to define

$$\begin{aligned} \mathbb{E}xp^{\psi, \nabla} : \mathfrak{t}_H M \times \mathbb{R} &\longrightarrow T_H M \times \{0\} \sqcup M \times M \times \mathbb{R}^\times; & (3) \\ (x, v, t) &\longmapsto \begin{cases} (\exp_x^\nabla(\psi(\delta_t v)), x, t) & \text{if } t \neq 0, \\ (x, v, 0) & \text{if } t = 0, \end{cases} \end{aligned}$$

Via this map $\mathbb{E}xp^{\psi, \nabla}$, the smooth structure of $\mathfrak{t}_H M \times \mathbb{R}$ induces a smooth structure in an open neighborhood of the $t = 0$ fiber of $\mathbb{T}_H M$. Away from $t = 0$ the open subset $M \times M \times \mathbb{R}^\times$ of $\mathbb{T}_H M$ has the evident smooth manifold structure.

Theorem 6. *With the above charts $\mathbb{T}_H M$ is a smooth groupoid.*

We call $\mathbb{T}_H M$ the *H-tangent groupoid*. We shall prove Theorem 6 in the remaining sections.

8. THE H-TANGENT LIE ALGEBROID

As in Section 3, we shall construct the Lie algebroid $\mathfrak{t}_H M$ indirectly, starting with its module of sections.

Let \mathfrak{X}_H be the $C^\infty(M \times \mathbb{R})$ module of sections of the vector bundle $TM \times \mathbb{R} \rightarrow M \times \mathbb{R}$ whose degree of vanishing at $t = 0$ is controlled by the filtration as follows

$$\mathfrak{X}_H = \{\mathbb{X} \in \Gamma(TM \times \mathbb{R}) \mid \partial_t^k \mathbb{X}|_{t=0} \in \Gamma(H^k) \text{ for all } k \geq 0\} \quad (4)$$

where $\partial_t = \frac{\partial}{\partial t}$. Note that the $k = 0$ equation implies $\mathbb{X}|_{t=0} = 0$. The bundle $TM \times \mathbb{R} \rightarrow M \times \mathbb{R}$ is a Lie algebroid with constant bracket and anchor

$$\begin{aligned} [\mathbb{X}, \mathbb{Y}]_t &= [\mathbb{X}_t, \mathbb{Y}_t] & \mathbb{X}, \mathbb{Y} &\in \Gamma(TM \times \mathbb{R}) \\ \rho(\mathbb{X})_t &= \mathbb{X}_t \end{aligned}$$

The bracket and anchor of $\Gamma(TM \times \mathbb{R})$ restrict to \mathfrak{X}_H ,

$$[\cdot, \cdot] : \mathfrak{X}_H \times \mathfrak{X}_H \rightarrow \mathfrak{X}_H \quad (5)$$

$$\rho : \mathfrak{X}_H \rightarrow \Gamma(TM \times \mathbb{R}) \subset \Gamma(T(M \times \mathbb{R})). \quad (6)$$

We will show that the Lie algebra \mathfrak{X}_H is the module of sections of a Lie algebroid $\mathfrak{t}_H M \rightarrow M \times \mathbb{R}$. Thus, $\mathfrak{t}_H M$ will be constructed indirectly. We first need to show that \mathfrak{X}_H is indeed the module of sections of a vector bundle on $M \times \mathbb{R}$. This vector bundle is isomorphic to $\mathfrak{t}_H M \times \mathbb{R}$.

Lemma 7. *Fix a degrading ψ . The associated map*

$$\Delta^\psi : \Gamma(\mathfrak{t}_H M \times \mathbb{R}) \rightarrow \mathfrak{X}_H \quad (\Delta^\psi \mathbb{X})(x, t) = \psi(\delta_t(\mathbb{X}(x, t)))$$

is an isomorphism of $C^\infty(M \times \mathbb{R})$ -modules.

Proof. Let $H_i := \psi(H^i/H^{i+1})$. Then $TM = \bigoplus_{i=1}^N H_i$. For $\mathbb{X} \in \Gamma(\mathfrak{t}_H M \times \mathbb{R})$ write $\psi(\mathbb{X}) = \sum_{i=1}^N \mathbb{X}_i$ with $\mathbb{X}_i \in \Gamma(H_i \times \mathbb{R})$. We compute

$$\partial_t^k (\Delta^\psi \mathbb{X})|_{t=0} = \sum_{i=1}^N \partial_t^k (t^i \mathbb{X}_i)|_{t=0} = k! \mathbb{X}_k|_{t=0} \in \Gamma(H_k) \subseteq \Gamma(H^k),$$

so $\Delta^\psi \mathbb{X} \in \mathfrak{X}_H$.

Clearly, Δ^ψ is $C^\infty(M \times \mathbb{R})$ -linear. Since $\psi \circ \delta_t$ is invertible for $t \neq 0$, Δ^ψ is injective. We need to show it is surjective.

Let $\mathbb{Y} \in \mathfrak{X}_H$ and write $\mathbb{Y} = \sum_{i=1}^N \mathbb{Y}_i$ with $\mathbb{Y}_i \in \Gamma(H_i \times \mathbb{R})$. Since $\partial_t^k \mathbb{Y}|_{t=0} \in \Gamma(H^k)$, we have $\partial_t^k \mathbb{Y}_i|_{t=0} = 0$ for all $k < i$, and therefore $\mathbb{Y}_i(x, t) = t^i \mathbb{X}_i(x, t)$ for some $\mathbb{X}_i \in \Gamma(H_i \times \mathbb{R})$. Thus $\mathbb{Y} = \Delta^\psi \mathbb{X}$ for $\mathbb{X} = \sum_{i=1}^N \psi^{-1}(\mathbb{X}_i)$. \square

We claim that \mathfrak{X}_H is the module of sections of a Lie algebroid $\mathfrak{t}_H M$.

Let us introduce the restriction maps

$$\text{ev}_t : \mathfrak{X}_H \subset \Gamma(TM \times \mathbb{R}) \rightarrow \Gamma(TM); \quad \mathbb{X} \mapsto \mathbb{X}_t.$$

For $t \neq 0$, ev_t induces a *canonical* identification of Lie algebroids $\mathfrak{t}_H M|_t = TM$. At $t = 0$, ev_0 is zero on \mathfrak{X}_H . On the other hand, Lemma 7 gives a vector space isomorphism $\mathfrak{t}_H M|_0 \cong \mathfrak{t}_H M$ that turns out to be canonical as well, i.e. it does *not* depend on the choice of degrading ψ .

Lemma 8. *The map $\text{ev}_0^H : \mathfrak{X}_H \rightarrow \Gamma(\mathfrak{t}_H M)$ defined by*

$$\text{ev}_0^H = \text{ev}_0 \circ (\Delta^\psi)^{-1}$$

is independent of ψ . Moreover, ev_0^H is a Lie algebra homomorphism, where \mathfrak{X}_H has the bracket restricted from $\Gamma(TM \times \mathbb{R})$ as in equation (5).

Proof. We could define ev_0^H independently of ψ as the map

$$\text{ev}_0^H = \bigoplus_{i=1}^N \text{ev}_0^{(i)} : \mathfrak{X}_H \rightarrow \bigoplus_{i=1}^N \Gamma(H^i/H^{i-1}) = \Gamma(\mathfrak{t}_H M), \quad (7)$$

with components

$$\text{ev}_0^{(i)} : \mathfrak{X}_H \rightarrow \Gamma(H^i/H^{i-1}); \quad \mathbb{X} \mapsto \sigma_i \left(\frac{1}{i!} \partial_t^i \mathbb{X}|_{t=0} \right).$$

To see that $\text{ev}_0^H \circ \Delta^\psi = \text{ev}_0$ is a direct calculation, similar to the calculation in the proof of Lemma 7.

To verify that ev_0^H is a Lie algebra homomorphism, we compute:

$$\begin{aligned} \text{ev}_0^H[\mathbb{X}, \mathbb{Y}] &= \bigoplus_{i=1}^N \sigma_i \left(\frac{1}{i!} \partial_t^i [\mathbb{X}, \mathbb{Y}]|_{t=0} \right) = \bigoplus_{i=1}^N \sigma_i \left(\frac{1}{i!} \sum_{j=0}^i \binom{i}{j} [\partial_t^j \mathbb{X}, \partial_t^{i-j} \mathbb{Y}] \right) \\ &= \bigoplus_{i=1}^N \sum_{j=0}^i \left[\sigma_j \left(\frac{1}{j!} \partial_t^j \mathbb{X} \right), \sigma_{i-j} \left(\frac{1}{(i-j)!} \partial_t^{i-j} \mathbb{Y} \right) \right] = [\text{ev}_0^H \mathbb{X}, \text{ev}_0^H \mathbb{Y}]. \end{aligned}$$

□

The family of restrictions ev_0^H and ev_t ($t \neq 0$) therefore gives a *canonical* identification of Lie algebroids

$$\mathfrak{X}_H \cong \Gamma(\mathfrak{t}_H M); \quad \mathfrak{t}_H M = (\mathfrak{t}_H M \times \{0\}) \sqcup (TM \times \mathbb{R}^\times) \quad (8)$$

This defines the algebraic structure of $\mathfrak{t}_H M$.

The smooth structure of $\mathfrak{t}_H M$ is implicit in the identification $\mathfrak{X}_H \cong \Gamma(\mathfrak{t}_H M)$ and the isomorphism Δ^ψ . We can make this smooth structure explicit as follows. There is a *non-canonical* isomorphism of vector bundles that depends on ψ , and we use this to determine the smooth structure of the disjoint union in Equation 8. To be explicit, we declare that the following map be a diffeomorphism:

$$\begin{aligned} \Phi^\psi : \quad \mathfrak{t}_H M \times \mathbb{R} &\longrightarrow \mathfrak{t}_H M = \mathfrak{t}_H M \times \{0\} \sqcup TM \times \mathbb{R}^\times; \quad (9) \\ (x, v, t) &\longmapsto \begin{cases} (x, \psi(\delta_t v), t) & \text{if } t \neq 0, \\ (x, v, 0) & \text{if } t = 0, \end{cases} \end{aligned}$$

where $x \in M$, $v \in \mathfrak{t}_H M_x$, $t \in \mathbb{R}$. It is straightforward to verify that the smooth structure on $\mathfrak{t}_H M$ is independent of the choice of degrading ψ .

9. INTEGRATION OF THE H -TANGENT LIE ALGEBROID

In this section and the next we confirm that the groupoid $\mathbb{T}_H M$ of Equation 2 admits a unique smooth structure such that $\mathfrak{t}_H M$ is its Lie algebroid. Specifically, the map of Equation 3 is the composition of the map of equation 9 with an exponential map $\mathfrak{t}_H M \rightarrow \mathbb{T}_H M$ from the Lie algebroid to the Lie groupoid.

According to Lie's Third Theorem, every Lie algebra integrates to a unique simply connected Lie group. Similarly, under mild hypotheses, every Lie algebroid determines a unique s -simply connected Lie groupoid, i.e. a Lie groupoid whose source fibers are simply connected [CF03] [Nis00].

The two components of the Lie algebroid

$$\mathfrak{t}_H M = \mathfrak{t}_H M \times \{0\} \sqcup TM \times \mathbb{R}^\times$$

each determine an s -simply connected Lie groupoid. At $t = 0$, the Lie groupoid $T_H M$ is s -simply connected, since each of the nilpotent groups $T_H M_x$ is, by construction, simply connected. The s -simply connected groupoid that integrates TM is not the pair groupoid $M \times M$, but the *fundamental groupoid* Π_M . Elements in Π_M are homotopy classes of paths in M with fixed end points. Source and range of an arrow $\gamma = [c]$ represented by a path $c: [0, 1] \rightarrow M$ are the end points of the path: $s(\gamma) = c(0)$, $r(\gamma) = c(1)$. Composition of paths gives the groupoid multiplication. The covering map

$$\Pi_M \rightarrow M \times M \quad \gamma \mapsto (r(\gamma), s(\gamma))$$

is a groupoid homomorphism, and a local diffeomorphism. In a neighborhood of the unit space the groupoids Π_M and $M \times M$ are naturally isomorphic and diffeomorphic.

By Theorem 2 and 3 in [Nis00], the disjoint union

$$\tilde{\mathcal{G}} = T_H M \times \{0\} \sqcup \Pi_M \times \mathbb{R}^\times$$

has a unique smooth manifold structure that makes it a Lie groupoid with Lie algebroid $\mathfrak{t}_H M$. Following [Nis00], the smooth structure of $\tilde{\mathcal{G}}$ is induced by an exponential map from the Lie algebroid $\mathfrak{t}_H M$ to $\tilde{\mathcal{G}}$ as follows.

Let ∇ be a connection on TM that is compatible with a degrading $\psi: \mathfrak{t}_H M \rightarrow TM$ in the sense that ∇ commutes with the dilations δ_t^ψ . The connection $\nabla + \frac{d}{dt}$ on $TM \times \mathbb{R}$ induces a connection on $\mathfrak{t}_H M$ via the degrading and the non-canonical isomorphism Δ^ψ .

A connection on a Lie algebroid induces an exponential map to an associated Lie groupoid. See [Nis00] for details of the general

construction. We describe here the exponential map

$$\exp^{\tilde{\mathcal{G}}} : \mathfrak{t}_H M \rightarrow \tilde{\mathcal{G}}.$$

When restricted to $t = 0$ the exponential map $\mathfrak{t}_H M \rightarrow T_H M$ does not depend on ∇ , and is simply the identification $\mathfrak{t}_H M = T_H M$.

Because ∇ commutes with the dilation automorphisms δ_t^ψ , the restriction of the connection on $\mathfrak{t}_H M$ to $TM \times \{t\}$ is again ∇ , independent of $t \neq 0$. Therefore for $t \in \mathbb{R}^\times$ the exponential map

$$\exp_t^{\tilde{\mathcal{G}}} : TM \rightarrow \Pi_M$$

is fixed, and is the lift of the exponential map

$$\text{Exp}^\nabla : TM \rightarrow M \times M \quad (x, v) \mapsto (\exp_x^\nabla(v), x) \quad (10)$$

We obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{t}_H M & \xrightarrow{\exp^{\tilde{\mathcal{G}}}} & \tilde{\mathcal{G}} \\ \uparrow & & \downarrow \\ \mathfrak{t}_H M \times \mathbb{R} & \xrightarrow{\mathbb{E}\text{xp}^{\psi, \nabla}} & T_H M \end{array}$$

with the left vertical arrow as in Equation 9, and the bottom arrow as in Equation 3.

The condition for integrability² of the Lie algebroid $\mathfrak{t}_H M$ is that the exponential map $\exp^{\tilde{\mathcal{G}}} : \mathfrak{t}_H M \rightarrow \tilde{\mathcal{G}}$ be *injective* in a neighborhood of the zero section of $\mathfrak{t}_H M$.

Let $U \subseteq TM$ be an open neighbourhood of the zero section such that the exponential map $\text{Exp}^\nabla|_U$ is injective. Then $\mathbb{E}\text{xp}^{\psi, \nabla}$ is injective on the open set

$$\{(x, v, t) \in \mathfrak{t}_H M \times \mathbb{R} \mid (x, \psi(\delta_t v)) \in U\}, \quad (11)$$

with image

$$\mathbb{U} := (T_H M \times \{0\}) \sqcup (\text{Exp}^\nabla(U) \times \mathbb{R}^\times) \subset T_H M. \quad (12)$$

Such a set \mathbb{U} is an open neighbourhood of both the unit space $T_H M^{(0)} \cong M \times \mathbb{R}$ and the osculating groupoid $T_H M|_{t=0} = T_H M \times \{0\}$.

By the results in [Nis00], $\tilde{\mathcal{G}}$ is a smooth groupoid with the smooth structure in a neighborhood of $t = 0$ induced by $\exp^{\tilde{\mathcal{G}}}$. Theorem 6 is an immediate corollary: just observe that $\mathbb{U} \subset T_H M$ contains the entire $t = 0$ fiber $T_H M$, and that in this neighborhood the groupoid operations of $T_H M$ agree with those of $\tilde{\mathcal{G}}$.

²This necessary hypothesis is not explicitly mentioned in [Nis00], but the correction was published in [BN03].

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