# Extinction in a finite time for solutions of large classes of Parabolic Equations involving p-Laplacian 

Y Belaud, A Shishkov

## To cite this version:

Y Belaud, A Shishkov. Extinction in a finite time for solutions of large classes of Parabolic Equations involving p-Laplacian. 2016. hal-01413930

HAL Id: hal-01413930

## https://hal.science/hal-01413930

Preprint submitted on 11 Dec 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Extinction in a finite time for solutions of large classes of Parabolic Equations involving $p$-Laplacian 

Y. Belaud, A. Shishkov

December 11, 2016


#### Abstract

We study the property of extinction in a finite time for nonnegative solutions of $\frac{1}{q} \frac{\partial}{\partial t}\left(u^{q}\right)-\nabla\left(|\nabla u|^{p-2} \nabla u\right)+a(x) u^{\lambda}=0$ for the Dirichlet Boundary Conditions when $q>\lambda>0, p \geq 1+q, p \geq 2, a(x) \geq 0$ and $\Omega \subset \mathbb{R}^{N}$

Abstract a bounded domain of $\mathbb{R}^{N}(N \geq 1)$. Necessary and sufficient conditions are provided with the help of a family of infimum. When $p>1+q$, the threshold is for power functions but for $p=1+q$, it happens extinction in a finite time for very flat functions.


## 1 Introduction

Let $q>\lambda>0, p \geq 1+q, p \geq 2$ and $\Omega \subset \mathbb{R}^{N}$ a bounded domain of $\mathbb{R}^{N}(N \geq 1)$. We consider $u$ a nonnegative solution of

$$
\left\{\begin{align*}
\frac{1}{q} \frac{\partial}{\partial t}\left(u^{q}\right)-\nabla\left(|\nabla u|^{p-2} \nabla u\right)+a(x) u^{\lambda} & =0  \tag{1}\\
u(x, 0) & =u_{0}(x)
\end{align*}\right.
$$

for the Dirichlet Boundary Condition. In this article, we study the fact that the solutions vanish.
Definition 1.1 Let say that problem (1) has the extinction in finite time property if for arbitrary solution $u$, it exists some positive $T$ such as $u(x, t)=0$ a.e. in $\Omega, \forall t \geq T$.

So, we introduce the following notation :
(2) $T\left(u_{0}\right)=\sup \left\{t>0, \forall \tau \in[0, t],\|u(\tau)\|_{L^{2}(\Omega)}>0\right\}$.

This problem has been intensively studied by many authors by means of differents methods. Among these methods, one can mention energy methods. In [21], the authors introduce a new energy method called semi-classical method. They transform a parabolic problem into a sequence of elliptic problems, namely, the behaviour in large time for solutions of some parabolic equations is linked with the asymptotic behaviour of a family of first eigenvalues in the semi-classical limit. That is to say, for $q=1, p=2$, for the Neumann Boundary Conditions, they introduce

$$
\mu_{n}=\inf \left\{\int_{\Omega}\left(|\nabla u|^{2}+2^{n} a(x) u^{\lambda+1}\right) d x: u \in W^{1,2}(\Omega), \int_{\Omega} u^{2} d x=1\right\} .
$$

By iterations and using the maximum principle, they find a sufficient condition. This later was improved in [4] thank to Lieb-Thirring formula [22]. Moreover, in the same article, the authors have given a necessary condition using a Kaplan's like-method [20], [25] and the idea of homothetic test-functions. Unfortunatly, these two conditions were differents. In [2], for the first time, in only two cases for $p>2, q=1$ and $p=2, q<1$, for the sufficient condition, the critical power $\delta_{0}=\frac{p(q-\lambda)}{p-(1+q)}$ appeared under the condition $N>p$. No necessary conditions were provided because the Kaplan's like-method used the fact that $-\Delta$ is a self-adjoint operator. Some properties of the first eigenvalue and the first eigenfunction are studied in [3].
A Diny-like condition has appeared for the first time in this kind of problem in [5] and high-order operators are
studied in [6]. The main drawback of Kondratiev-Véron method is based on the comparaison principle. So we introduce a new semi-classical method : we give a sufficient condition for $p>1+q$ and $p=1+q$ and a necessary condition for $p>1+q$ and $p=2=1+q$, i.e., $q=1$.
There are two steps for the sufficient condition :

1. We transform the parabolic problem into a problem of behaviour for some kind of nonlinear eigenvalues. There is extinction in a finite time for solutions of (1) if an integral depending of some Raleight quotients is finite. We define

$$
\mu(h)=\inf _{v \in W_{0}^{1, p}(\Omega),\|v\|_{L^{1+q}(\Omega)}^{1+q}=h} \int_{\Omega}|\nabla v|^{p} d x+\int_{\Omega} a(x)|v|^{1+\lambda} d x .
$$

We proof under assumptions that there is extinction in a finite time if

$$
\begin{equation*}
\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty \tag{3}
\end{equation*}
$$

This condition is valid for a large class of maximum monotone operators of Hilbert Spaces.
But it seems that the genuine notion is a kind of uniform upper bound for the extinction time, i.e., we define for all $h_{0}>0$,
(4) $T^{+}\left(h_{0}\right)=\sup \left\{T\left(u_{0}\right): u_{0} \in W_{0}^{1, p}(\Omega), 0<\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q} \leq h_{0}\right\}$.

With this new definition, condition (3) implies that

$$
T^{+}\left(h_{0}\right)<+\infty \quad \forall h_{0}>0, \quad \text { and } \quad \lim _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)=0
$$

2. Now, the point is to find estimates of $\mu(h)$. The main tool is Sobolev injections, that's why we have the restriction $N>p$. The quantity $\mu(h)$ is estimated in two differents ways depending on the value of $p$ with respect to $1+q$, i.e., $p>1+q$ or $p=1+q$.

For the necessary condition, it can be expected that if the norm of the initial condition $u_{0}$ is small then its extinction time is also small. But we prove that

$$
\begin{equation*}
T^{+}\left(h_{0}\right) \geq \sup _{0<h \leq h_{0}} \frac{(1+\lambda)}{p(q-\lambda)} \frac{h}{\mu(h)} . \tag{5}
\end{equation*}
$$

This condition is also valid for a large class of maximum monotone operators of Hilbert Spaces. There are two important cases :
1.
(6) $\lim _{h \rightarrow 0} \frac{h}{\mu(h)}=+\infty \quad \Longrightarrow \quad T^{+}\left(h_{0}\right)=+\infty$,
means that for any $h_{0}>0$ and $T>0$, there is an initial data $u_{0}$ such as $T\left(u_{0}\right)>T$. But without any further assumption, it seems difficult to get a $u_{0}$ such as $T\left(u_{0}\right)=+\infty$ in the general case.
When $p=1+q$, (6) never happens.
2.

$$
\begin{equation*}
\liminf _{h \rightarrow 0} \frac{h}{\mu(h)} \geq C>0 \quad \Longrightarrow \quad \liminf _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)>0 \tag{7}
\end{equation*}
$$

implies that for any $h_{0}>0$, one can find an initial data $u_{0}$ such as $\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}=h_{0}$ and $T\left(u_{0}\right) \geq C$ for some fixed $C>0$. An initial data with a positive norm as small as we want exists but with an extinction time greater than a fixed positive constant.
When $p=1+q$, (7) happens only when $a(x)=0$ a.e. in $\Omega$ which is a trivial case.

For $p>1+q$, by means of "good" homothetic test-function, we estimate $\mu(h)$ by above.
We have seen that for $p=1+q$, condition (5) isn't relevant. For this reason, when $p=2$ (and so $q=1$ ), we use some kind of asymptotic Kaplan's method, i.e., we prove that under condition of $a(x)$, for all time $T>0$, a "good" first eigenfunction $\varphi$ such as $\int_{\Omega} u(x, T) \varphi(x) d x>0$ exists.
Here, we have to split $p>1+q$ and $p=1+q$ (called critical case). Calculations and estimates are differents due to the nature of (1). Indeed, for $p>1+q$, there are three different degrees of homogeneity while when $p=1+q$, there are only two. It provides two radically different thresholds (power functions for $p>1+q$ and very flat functions for the critical case).
We define for $p>1+q$,

$$
\delta_{0}=\frac{p(q-\lambda)}{p-(1+q)} .
$$

We assume
(8) $\quad a(x)=\frac{|x|^{\delta_{0}}}{\omega(|x|)}$ for $x$ small enough and otherwise $a(x) \geq C>0$.

For the sufficient condition, we assume that
(9) $\quad \rho \mapsto \frac{\rho^{\delta_{0}}}{\omega(\rho)}$ is a continuous nondecreasing function when $\rho>0$ is small enough,
(10) $\rho \mapsto \omega(\rho)$ is a nonincreasing function when $\rho>0$ is small enough,
(11) $\forall \gamma_{1} \in(0,1), \exists C>0, \exists \gamma_{2}>0, \forall h \in(0,1], \omega\left(h^{\gamma_{1}}\right) \geq C(\omega(h))^{\gamma_{2}}$,
(12) $\exists \eta>0, \forall h \in\left(0, h_{0}\right], \omega(h) \leq h^{-\eta}$ for some $h_{0}>0$.

For the necessary condition, we assume that
(13) $\omega(\rho)=\frac{1}{(-\ln \rho)^{\alpha}}$ for $\alpha>\frac{p-(1+\lambda)}{p-(1+q)}+\frac{\delta_{0}}{N}$.

## Theorem 1.1

Assume that $p>1+q, O \in \Omega$, (8).

1. Assume (9).
(a) If

$$
\liminf _{\rho \rightarrow 0} \omega(\rho)>0,
$$

then

$$
\liminf _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)>0 .
$$

In particular, it happens when

$$
\omega(\rho)=\frac{1}{(-\ln \rho)^{\alpha}},
$$

for

$$
\alpha \leq 0 .
$$

(b) If we assume (10), (11), (12) and

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=+\infty,
$$

then

$$
\forall h_{0}>0, T^{+}\left(h_{0}\right)=+\infty
$$

In particular, it happens when

$$
\omega(\rho)=\frac{1}{(-\ln \rho)^{\alpha}}
$$

for

$$
\alpha<0
$$

2. Assume that $N>p$. The function $\omega$ satisfies (13). Then all the solutions vanish in a finite time. More precisely,

$$
\forall h_{0}>0, T^{+}\left(h_{0}\right)<+\infty \text { and } \lim _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)=0
$$

For $p=1+q$, we define
(14) $a(x)=\exp \left(-\frac{\omega(|x|)}{|x|^{p}}\right)$.

We assume for the function $\omega$,
$\left(A_{1}\right) \omega(r)$ is a continuous and nondecreasing function $\forall r \geq 0$,
$\left(A_{2}\right) \omega(0)=0, \omega(r)>0, \forall r>0$,
$\left(A_{3}\right) \omega(r) \leq \omega_{0}<\infty, \forall r>0$,
$\left(A_{4}\right) \omega(r) \geq r^{p-\delta}, \forall s \in\left(0, s_{0}\right), s_{0}>0, p>\delta>0$,
$\left(A_{5}\right)$ the function $\frac{\omega(r)}{r^{p}}$ is nonincreasing on $\left(0, s_{0}\right)$.
Theorem 1.2 Assume that $p=1+q$ and $O \in \Omega$. We assume $(14),\left(A_{1}\right)-\left(A_{5}\right)$ and $N>p$. If

$$
\int_{0}^{c} \frac{\omega(r)}{r} d r<+\infty
$$

then all the solutions vanish in a finite time. More precisely,

$$
\forall h_{0}>0, T^{+}\left(h_{0}\right)<+\infty \text { and } \lim _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)=0
$$

Moreover, assume only (14) and $p=2$ (i.e. $q=1$ ).

1. If

$$
\liminf _{\rho \rightarrow 0} \omega(\rho)>0
$$

then

$$
\liminf _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)>0
$$

2. If

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=+\infty
$$

then

$$
\forall h_{0}>0, T^{+}\left(h_{0}\right)=+\infty
$$

The proofs of Theorems 1.1 and 1.2 are based on an abstract theorem, a proposition and four lemmas.

## 2 Proof

The next theorem is the key-stone for necessary condition and sufficient condition for $p>1+q$ and sufficient condition for $p=1+q$.
Let $H$ be an Hilbert Space for $q=1$ and $L^{2}(\Omega)$ for $q \neq 1$ (it is just a matter of algebra structure).
When $q=1,\| \|$ is the usual norm on $H$ and when $q \neq 1,\| \|=\| \|_{L^{1+q}(\Omega)}$ is the $L^{1+q}(\Omega)$-norm on $L^{2}(\Omega)$.
We always denote by $J$ be a proper convex lower semi-continuous function on $H$ for $q=1$ and on $L^{2}(\Omega)$ for $q \neq 1$.
We define $A=\partial J$ as the subdifferential of $J$. Let's denote the domain of $A$ by $D(A)$.
Let's remember that $D(J)=\{u \in H, J(u)<+\infty\}$.
For simplicity's sake, we assume that $A$ is a single-valued operator.
In what follows, we consider that $H=L^{2}(\Omega)$ but it can be easily extend to all Hilbert Space when $q=1$.
Let $u$ be a solution of
(15) $\frac{1}{q} \frac{\partial}{\partial t}\left(|u|^{q-1} u\right)+A u=0, \quad u(0)=u_{0}$.

We define for all $u_{0} \in D(A) \backslash\{0\}$,
(16) $T\left(u_{0}\right)=\sup \{t>0, \forall \tau \in[0, t], u(\tau) \neq 0\}$,
and for all $h_{0}>0$,
(17) $T^{+}\left(h_{0}\right)=\sup \left\{T\left(u_{0}\right): u_{0} \in D(A) \bigcap L^{1+q}(\Omega), 0<\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q} \leq h_{0}\right\}$.

The following assumptions can be made :

$$
\begin{aligned}
& H_{0}: \inf _{v \in D(J)} J(v)=J(0)=0 \\
& H_{1}: \forall h>0, D(A) \bigcap\left\{v \in L^{1+q}(\Omega),\|v\|_{L^{1+q}(\Omega)}^{1+q}=h\right\} \neq \emptyset .
\end{aligned}
$$

Let's define for all $h>0$,

$$
\begin{align*}
& \mu(h)=\inf _{v \in D(A) \cap\left\{v \in L^{1+q}(\Omega),\|v\|_{L^{1+q}(\Omega)}^{1+q}=h\right\}}<A v, v>.  \tag{18}\\
& H_{2}: \forall h>0, \mu(h)>0 . \\
& H_{3}: \exists p>1, \forall v \in D(A),<A v, v>\leq p J(v) . \\
& H_{4}: \exists \lambda \in[0, p-1), \forall v \in D(A),<A v, v>\geq(1+\lambda) J(v) .
\end{align*}
$$

## Remark 2.1

1. Assumption $H_{0}$ implies $A 0=0$.
2. Assumptions $H_{2}$ and $H_{3}$ imply $J(v)>0$ for all $v \in D(J) \backslash\{0\}$.
3. Assumption $H_{4}$ is always true for $\lambda=0$.
4. To some extend, $H_{3}$ means that $J$ is some kind of polynome in $u$ and of all its derivative.

Theorem 2.1 Let $u_{0} \in D(A) \backslash\{0\}$. Under $H_{0}-H_{4}$, for solutions of (15), we have the following results :

$$
\begin{equation*}
\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty \quad \Longrightarrow \quad\left(\forall h_{0}>0, T^{+}\left(h_{0}\right)<+\infty \text { and } \lim _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)=0\right) \tag{19}
\end{equation*}
$$

and
(20) $\left(\forall h_{0}>0, T^{+}\left(h_{0}\right)<+\infty\right.$ and $\left.\lim _{h_{0} \rightarrow 0} T^{+}\left(h_{0}\right)=0\right) \Longrightarrow \quad \lim _{h_{0} \rightarrow 0} \frac{h_{0}}{\mu\left(h_{0}\right)}=0$.

If it's true, we have the following estimates :
(21) $T\left(u_{0}\right) \leq T^{+}\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right) \leq \frac{1}{1+q} \int_{0}^{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}} \frac{d h}{\mu(h)}$,
and
(22) $T^{+}\left(h_{0}\right) \geq \sup _{0<h<h_{0}} \frac{(1+\lambda)}{p(q-\lambda)} \frac{h}{\mu(h)}$.

Proof: For the sufficient condition, we assume that $\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty$.
Taking the scalar product with $u$ leads to
(23) $\frac{1}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)+<A u, u>=0$.

But by definition of $\mu(h)$,

$$
<A u, u>\geq \mu\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)
$$

So,

$$
\frac{1}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)+\mu\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right) \leq 0
$$

Since $\mu(h)>0$ for all $h>0$,

$$
\frac{\frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)}{\mu\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)} \leq-(1+q)
$$

We integrate between 0 and $t<T\left(u_{0}\right)$.

$$
\int_{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}}^{\|u\|_{L^{1+q}}^{1+q}} \frac{d h}{\mu(h)} \leq-(1+q) t
$$

Hence,

$$
\int_{\|u\|_{L^{1+q}(\Omega)}^{1+q}}^{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}} \frac{d h}{\mu(h)} \geq(1+q) t
$$

We deduce that for all $t<T\left(u_{0}\right)$,

$$
\int_{0}^{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}} \frac{d h}{\mu(h)} \geq(1+q) t
$$

Going to the limit gives

$$
\int_{0}^{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}} \frac{d h}{\mu(h)} \geq(1+q) T\left(u_{0}\right)
$$

and therefore $T\left(u_{0}\right)<+\infty$. Moreover,

$$
T^{+}(h) \leq \frac{1}{1+q} \int_{0}^{h} \frac{d \theta}{\mu(\theta)}
$$

and

$$
T^{+}(h) \rightarrow 0
$$

as $h \rightarrow 0$.
For the necessary condition, since $u$ is regular, we have

$$
\frac{d}{d t} J(u(t))=<u_{t}, A u>=-<u_{t}, u_{t}|u|^{q-1}>=-\left\|u_{t}|u|^{\frac{q-1}{2}}\right\|_{L^{2}(\Omega)}^{2}
$$

According to Cauchy-Schwarz-Buniakowsky inequality,

$$
\left.\left|<u_{t}, u\right| u\right|^{q-1}>\left.\right|^{2}=\left.\left|<u_{t}\right| u\right|^{\frac{q-1}{2}}, u|u|^{\frac{q-1}{2}}>\left.\right|^{2} \leq\left\|u_{t}|u|^{\frac{q-1}{2}}\right\|_{L^{2}(\Omega)}^{2}\|u\|_{L^{1+q}(\Omega)}^{1+q}
$$

Hence,

$$
(<A u, u>)^{2} \leq\left(-\frac{d}{d t} J(u(t))\right)\|u\|_{L^{1+q}(\Omega)}^{1+q}
$$

So,
(24) $\frac{d}{d t} J(u(t)) \leq-\frac{(<A u, u>)^{2}}{\|u\|_{L^{1+q}(\Omega)}^{1+q}}=\frac{1}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right) \frac{<A u, u>}{\|u\|_{L^{1+q}(\Omega)}^{1+q}}$.

Then, using $H_{4}$,

$$
\frac{d}{d t} J(u(t)) \leq \frac{1+\lambda}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right) \frac{J(u(t))}{\|u\|_{L^{1+q}(\Omega)}^{1+q}}
$$

By integration, we obtain
(25) $J(u(t)) \leq \frac{J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}$.

$$
\frac{1}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)+p J(u(t)) \geq 0
$$

$J(u(t)) \leq \frac{J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}$
$\frac{1}{1+q} \frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)+\frac{p J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}} \geq 0$.
$\frac{\frac{d}{d t}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)}{\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}}+\frac{(1+q) p J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} \geq 0$.
$\frac{1+q}{q-\lambda}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}}-\frac{1+q}{q-\lambda}\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}}+\frac{(1+q) p J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} t \geq 0$.
$\frac{1}{q-\lambda}\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}} \geq \frac{1}{q-\lambda}\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}}-\frac{p J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} t$.
$\frac{p J\left(u_{0}\right)}{\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} T\left(u_{0}\right) \geq \frac{1}{q-\lambda}\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}}$.
$T\left(u_{0}\right) \geq \frac{1}{p(q-\lambda)} \frac{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}}{J\left(u_{0}\right)}$

$$
\begin{aligned}
& T\left(u_{0}\right) \geq \frac{(1+\lambda)}{p(q-\lambda)} \frac{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}}{<A u_{0}, u_{0}>} \\
& T^{+}\left(h_{0}\right)=\sup _{0<\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q} \leq h_{0}} T\left(u_{0}\right)
\end{aligned}
$$

Let $h_{0}>0$.
(26) $T^{+}\left(h_{0}\right) \geq \sup _{0<\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q} \leq h_{0}} \frac{(1+\lambda)}{p(q-\lambda)} \frac{\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}}{<A u_{0}, u_{0}>}$.

For all $u_{0} \in D(A) \backslash\{0\}$, there is a sequence $\left(u_{n}\right)$ such as $\left\|u_{n}\right\|_{L^{1+q}(\Omega)}^{1+q}=\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}$ and $<A u_{n}, u_{n}>\rightarrow \mu\left(\left\|u_{0}\right\|_{L^{1+q}(\Omega)}^{1+q}\right)$. When we go to the limit in (26),

$$
T^{+}\left(h_{0}\right) \geq \sup _{0<h \leq h_{0}} \frac{(1+\lambda)}{p(q-\lambda)} \frac{h}{\mu(h)}
$$

In $L^{2}(\Omega)$ when $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with $O \in \Omega$, let's say

$$
J(u)=\frac{1}{p} \int_{\Omega}|\nabla u|^{p} d x+\frac{1}{1+\lambda} \int_{\Omega} a(x)|u|^{1+\lambda} d x
$$

for $0<\lambda<q \leq p-1$ with $D(J)=W_{0}^{1, p}(\Omega)$ for $p \geq 2$.
Proposition 1 The functional $J$ satisfies assumptions $H_{0}-H_{4}$.
Proof: Assumptions $H_{0}$ and $H_{1}$ are satisfied when $D(J)=W_{0}^{1, p}(\Omega)$. Using Poincaré's inequality and Hölder's inequality,

$$
<A v, v>\geq C \int_{\Omega}|v|^{p} d x \geq C^{\prime}\left(\int_{\Omega}|v|^{1+q} d x\right)^{\frac{p}{1+q}}
$$

which means $\mu(h) \geq C^{\prime} h^{\frac{p}{1+q}}>0$ for all $h>0$. So $H_{2}$ is valid. Moreover, it is clear that $H_{3}$ and $H_{4}$ are also valid.

Lemma 1 We assume that $p>1+q, O \in \Omega$, (8) and (9).

1. If

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=0
$$

then
(27) $\mu(h) \leq C \frac{h}{\omega\left(h^{\varepsilon_{0}}\right)}$.
for some $C>0$ when

$$
\varepsilon_{0}=\frac{p-(1+\lambda)}{N(p-(1+\lambda))+\left(\delta_{0}+p\right)(1+q)}
$$

2. If

$$
\liminf _{\rho \rightarrow 0} \omega(\rho)>0
$$

then
(28) $\mu(h) \leq C h$,
for some $C>0$.
3. If we assume (10), (11), (12) and

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=+\infty
$$

then
(29) $\lim _{h \rightarrow 0} \frac{h}{\mu(h)}=+\infty$.

Proof: Since $O \in \Omega, \rho_{0}>0$ exists such as $B_{\rho_{0}}$ (the ball of center $O$ and radius $\rho_{0}$ ) is included in $\Omega$.
Let $v \in C_{0}^{\infty}\left(B_{1}\right)$ be with $v \geq 0$ and $\int_{B_{1}} v^{1+q}=1$.
Let $\rho$ be in $\left(0, \rho_{0}\right)$. We define $v_{\rho}(x)=v\left(\frac{x}{\rho}\right)$. Then,

$$
\int_{\Omega} v_{\rho}^{1+q}(x) d x=\int_{B_{\rho}} v_{\rho}^{1+q}(x) d x=\int_{B_{\rho}} v^{1+q}\left(\frac{x}{\rho}\right) d x=\rho^{N} \int_{B_{1}} v^{1+q}(y) d y=\rho^{N}
$$

As a consequence, $\left\|\frac{v_{\rho}}{\rho^{\frac{N}{1+q}}}\right\|_{L^{1+q}(\Omega)}=1$. On the other hand,

$$
\nabla_{x} v_{\rho}(x)=\rho^{-1} \nabla_{y} v\left(\frac{x}{\rho}\right) .
$$

Hence,

$$
\int_{\Omega}|\nabla v|^{p} d x=\int_{B_{\rho}}\left|\nabla v_{\rho}\right|^{p} d x=\rho^{N-p} \int_{B_{1}}|\nabla v|^{p} d y
$$

By using $h^{\frac{1}{1+q}} \frac{v_{\rho}}{\rho^{\frac{N}{1+q}}}$ in the definition of $\mu(h)$,

$$
\mu(h) \leq h^{\frac{p}{1+q}} \rho^{-\frac{p N}{1+q}} \int_{\Omega}\left|\nabla v_{\rho}\right|^{p} d x+h^{\frac{1+\lambda}{1+q}} \rho^{-N \frac{1+\lambda}{1+q}} \int_{\Omega} a(x) v_{\rho}^{1+\lambda} d x
$$

So,
(30) $\mu(h) \leq h^{\frac{p}{1+q}} \rho^{N-p-\frac{p N}{1+q}} \int_{B_{1}}|\nabla v|^{p} d y+h^{\frac{1+\lambda}{1+q}} \rho^{N-N \frac{1+\lambda}{1+q}} \int_{B_{1}} a(\rho y) v^{1+\lambda}(y) d y$.

Since $\rho \mapsto \frac{\rho^{\delta_{0}}}{\omega(\rho)}$ is a nondecreasing function for $h$ small enough,

$$
a(\rho y)=\frac{(\rho\|y\|)^{\delta_{0}}}{\omega(\rho\|y\|)} \leq \frac{\rho^{\delta_{0}}}{\omega(\rho)}
$$

for $y \in B_{1}$. Therefore,

$$
\mu(h) \leq C(p, q, \lambda, v)\left(h^{\frac{p}{1+q}} \rho^{N-p-\frac{p N}{1+q}}+h^{\frac{1+\lambda}{1+q}} \rho^{N-N \frac{1+\lambda}{1+q}+\delta_{0}} \frac{1}{\omega(\rho)}\right)
$$

A deducing can be made :
(31) $\mu(h) \leq C(p, q, \lambda, v)\left(h^{\frac{p}{1+q}} \rho^{-\frac{N(p-(1+q))}{1+q}-p}+h^{\frac{1+\lambda}{1+q}} \rho^{N \frac{q-\lambda}{1+q}+\delta_{0}} \frac{1}{\omega(\rho)}\right)$.

We take $\rho=h^{\varepsilon_{0}}$ for $\varepsilon_{0}>0$. So,

$$
\mu(h) \leq C(p, q, \lambda, v)\left(h^{\frac{p}{1+q}-\varepsilon_{0} \frac{N(p-(1+q))}{1+q}-\varepsilon_{0} p}+h^{\frac{1+\lambda}{1+q}+\varepsilon_{0} N \frac{q-\lambda}{1+q}+\varepsilon_{0} \delta_{0}} \frac{1}{\omega\left(h^{\varepsilon_{0}}\right)}\right)
$$

We choose $\varepsilon_{0}$ such as $\frac{p}{1+q}-\varepsilon_{0} \frac{N(p-(1+q))}{1+q}-\varepsilon_{0} p=\frac{\lambda+1}{1+q}+\varepsilon_{0} N \frac{q-\lambda}{1+q}+\varepsilon_{0} \delta_{0}$, i.e.,

$$
\varepsilon_{0}=\frac{p-(1+\lambda)}{N(p-(1+\lambda))+\left(\delta_{0}+p\right)(1+q)} .
$$

Hence,

$$
\begin{aligned}
& \frac{1+\lambda}{1+q}+\varepsilon_{0} N \frac{q-\lambda}{1+q}+\varepsilon_{0} \delta_{0} \\
& =\frac{(1+\lambda) N(p-(1+\lambda))+(1+\lambda)\left(\delta_{0}+p\right)(1+q)+N(p-(1+\lambda))(q-\lambda)+\delta_{0}(1+q)(p-(1+\lambda))}{(1+q)\left(N(p-(1+\lambda))+\left(\delta_{0}+p\right)(1+q)\right)} \\
& =\frac{(1+q) N(p-(1+\lambda))+p(1+\lambda)(1+q)+p(1+q) \delta_{0}}{(1+q)\left(N(p-(1+\lambda))+\left(\delta_{0}+p\right)(1+q)\right)} .
\end{aligned}
$$

But According to the definition of $\delta_{0}$,

$$
p(1+\lambda)+p \delta_{0}=\left(\delta_{0}+p\right)(1+q)
$$

which leads to

$$
\frac{p}{1+q}-\varepsilon_{0} \frac{N(p-(1+q))}{1+q}-\varepsilon_{0} p=\frac{\lambda+1}{1+q}+\varepsilon_{0} N \frac{q-\lambda}{1+q}+\varepsilon_{0} \delta_{0}=1 .
$$

With this $\varepsilon_{0}$, we obtain

$$
\mu(h) \leq C(p, q, \lambda, v) h\left(1+\frac{1}{\omega\left(h^{\varepsilon_{0}}\right)}\right) .
$$

If

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=0
$$

then for $h$ small enough,

$$
\mu(h) \leq C \frac{h}{\omega\left(h^{\varepsilon_{0}}\right)},
$$

for some $C>0$.
If

$$
\liminf _{\rho \rightarrow 0} \omega(\rho)>0,
$$

then

$$
\mu(h) \leq C h,
$$

for some $C>0$.
Let's go back to (31). We take $\rho=h^{\varepsilon_{0}} \omega(h)^{\alpha}$ for $\alpha>0$. This value of $\varepsilon_{0}$ implies that

$$
\frac{\mu(h)}{h} \leq C(p, q, \lambda, v)\left((\omega(h))^{-\alpha\left(\frac{N(p-(1+q))}{1+q}+p\right)}+(\omega(h))^{\alpha\left(N^{\frac{q-\lambda}{1+q}+\delta_{0}}\right)} \frac{1}{\omega\left(h^{\varepsilon_{0}} \omega(h)^{\alpha}\right)}\right)
$$

Using (12), it exists $h_{0}>0$ such as for all $0<h \leq h_{0}$,

$$
h^{\varepsilon_{0}} \omega(h)^{\alpha} \leq h^{\varepsilon_{0}-\alpha \eta} .
$$

Moreover, $\omega$ is a nonincreasing function hence,

$$
\omega\left(h^{\varepsilon_{0}} \omega(h)^{\alpha}\right) \geq \omega\left(h^{\varepsilon_{0}-\alpha \eta}\right)
$$

It is always possible to assume that $\varepsilon_{0}-\alpha \eta \geq \frac{\varepsilon_{0}}{2}$. So, if $h$ belongs to $\left(0, \min \left(h_{0}, 1\right)\right)$,

$$
h^{\varepsilon_{0}-\alpha \eta} \leq h^{\frac{\varepsilon_{0}}{2}} .
$$

Always for the same reason,

$$
\omega\left(h^{\varepsilon_{0}-\alpha \eta}\right) \geq \omega\left(h^{\frac{\varepsilon_{0}}{2}}\right)
$$

which leads to

$$
\omega\left(h^{\varepsilon_{0}} \omega(h)^{\alpha}\right) \geq \omega\left(h^{\frac{\varepsilon_{0}}{2}}\right) .
$$

The previous inequality is true for all $0<h \leq \min \left(h_{0}, 1\right)$ and for all $\alpha \in\left(0, \frac{\varepsilon_{0}}{2 \eta}\right)$.
So, we use assumption (11) for $\gamma_{1}=\frac{\varepsilon_{0}}{2}$. There exist $C>0$ and $\gamma_{2}>0$ such as

$$
\omega\left(h^{\frac{\varepsilon_{0}}{2}}\right) \geq C(\omega(h))^{\gamma_{2}}
$$

Therefore,

$$
\begin{aligned}
& \omega\left(h^{\varepsilon_{0}} \omega(h)^{\alpha}\right) \geq C(\omega(h))^{\gamma_{2}} . \\
& \frac{(\omega(h))^{\alpha}\left(N^{\frac{q-\lambda}{1+q}}+\delta_{0}\right)}{\omega\left(h^{\varepsilon_{0}} \omega(h)^{\alpha}\right)} \leq \frac{1}{C} \frac{1}{\left.(\omega(h))^{\gamma_{2}-\alpha(N(N-\lambda} \frac{q-\lambda}{1+q}+\delta_{0}\right)}
\end{aligned}
$$

We can take $\alpha>0$ small enough such as

$$
\gamma_{2}-\alpha\left(N \frac{q-\lambda}{1+q}+\delta_{0}\right)>0
$$

Finally,

$$
\lim _{h \rightarrow 0} \frac{\mu(h)}{h}=0
$$

since

$$
\lim _{\rho \rightarrow 0} \omega(\rho)=0.1
$$

Lemma 2 We assume that $p>1+q, N>p, O \in \Omega$, (8) and (13). Then

$$
\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty .
$$

Proof: For all $h>0$, there is $v_{h} \in W_{0}^{1, p}(\Omega)$ such as $\left\|v_{h}\right\|_{L^{1+q}(\Omega)}^{1+q}=h$ and

$$
2 \mu(h) \geq \int_{\Omega}\left|\nabla v_{h}\right|^{p} d x+\int_{\Omega} a(x) v_{h}^{1+\lambda} d x .
$$

To simplify, we drop $h$ for $v_{h}:=v$. So,

$$
\int_{\Omega}|\nabla v|^{p} d x \leq \frac{2 \mu(h)}{h} \int_{\Omega} v^{1+q} d x-\int_{\Omega} a(x) v^{1+\lambda} d x .
$$

Since $p>1+q$,

$$
\int_{\Omega}|\nabla v|^{p} d x \leq \int_{\Omega} v^{p}\left(\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}}-\frac{a(x)}{v^{p-(1+\lambda)}}\right) d x .
$$

Sobolev injection and Hölder inequality when $N>p$ give

$$
C_{S}\left(\int_{\Omega} v^{p^{*}} d x\right)^{\frac{p}{p^{*}}} \leq\left(\int_{\Omega} v^{p^{*}} d x\right)^{\frac{p}{p^{*}}}\left(\int_{\Omega}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}}-\frac{a(x)}{v^{p-(1+\lambda)}}\right]^{+}\right)^{\frac{N}{p}} d x\right)^{\frac{p}{N}}
$$

when $\frac{1}{p^{*}}=\frac{1}{p}-\frac{1}{N}$. Hence,

$$
\begin{aligned}
& C_{S}^{\frac{N}{p}} \leq \int_{\Omega}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right]^{+}\right)^{\frac{N}{p}} d x \\
& =\int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} v^{q-\lambda}(x)\right\}}\left(\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right)^{\frac{N}{p}} d x
\end{aligned}
$$

We define for $v>0$,

$$
f(v)=\frac{\beta}{v^{p-(1+q)}}-\frac{\alpha}{v^{p-(1+\lambda)}}
$$

when $\alpha, \beta>0$. By a study of the variations of $f$, we obtain

$$
\max _{v>0} f(v)=f\left(v_{M}\right)
$$

when

$$
v_{M}=\left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)}\right)^{\frac{1}{q-\lambda}}
$$

and so,

$$
\begin{aligned}
& f\left(v_{M}\right)=\frac{\beta}{\left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)}\right)^{\frac{p-(1+q)}{q-\lambda}}}-\frac{\alpha}{\left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)}\right)^{\frac{p-(1+\lambda)}{q-\lambda}}} \\
& =\left[\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+q)}{q-\lambda}}-\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+\lambda)}{q-\lambda}}\right] \frac{\beta^{\frac{p-(1+\lambda)}{q-\lambda}}}{\alpha^{\frac{p-(1+q)}{q-\lambda}}}
\end{aligned}
$$

Since meas $\{x: a(x)=0\}=0$, we deduce that for almost all $x \in\left\{x: a(x) \leq \frac{2 \mu(h)}{h} v^{q-\lambda}(x)\right\}$,

$$
\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)} \leq\left[\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+q)}{q-\lambda}}-\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+\lambda)}{q-\lambda}}\right]\left(\frac{2 \mu(h)}{h}\right)^{\frac{p-(1+\lambda)}{q-\lambda}} \frac{1}{a(x)^{\frac{p-(1+q)}{q-\lambda}}}
$$

Moreover, let $\varepsilon>0$. We have

$$
h=\|v\|_{L^{1+q}(\Omega)}^{1+q}=\int_{\Omega} v^{1+q} d x \geq \int_{v^{1+q} \geq \varepsilon} v^{1+q} d x \geq \varepsilon \operatorname{meas}\left\{x: v^{1+q} \geq \varepsilon\right\}
$$

By taking $\varepsilon=\varepsilon(h)$, we obtain

$$
\frac{h}{\varepsilon(h)} \geq \operatorname{meas}\left\{x: v^{1+q} \geq \varepsilon(h)\right\}=\operatorname{meas}\left\{x: v \geq \varepsilon(h)^{\frac{1}{1+q}}\right\}
$$

So,

$$
C_{S}^{\frac{N}{p}} \leq \int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} v^{q-\lambda}(x)\right\} \cap\left\{x: v \geq \varepsilon(h)^{\frac{1}{1+q}}\right\}}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right]^{+}\right)^{\frac{N}{p}} d x
$$

$$
+\int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} v^{q-\lambda}(x)\right\} \cap\left\{x: v<\varepsilon(h)^{\frac{1}{1+q}}\right\}}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right]^{+}\right)^{\frac{N}{p}} d x
$$

It follows that

$$
\begin{aligned}
& C_{S}^{\frac{N}{p}} \leq \int_{\left\{x: v \geq \varepsilon(h)^{\frac{1}{1+q}}\right\}}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right]^{+}\right)^{\frac{N}{p}} d x \\
& +\int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right\}}\left(\left[\frac{2 \mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)}-\frac{a(x)}{v^{p-(1+\lambda)}(x)}\right]^{+}\right)^{\frac{N}{p}} d x
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& C_{S}^{\frac{N}{p}} \leq \int_{\left\{x: v \geq \varepsilon(h)^{\frac{1}{1+q}}\right\}}\left(\frac{2 \mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} d x \\
+ & {\left[\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+q)}{q-\lambda}}-\left(\frac{p-(1+q)}{p-(1+\lambda)}\right)^{\frac{p-(1+\lambda)}{q-\lambda}}\right]^{\frac{N}{p}} \int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right\}}\left(\left(\frac{2 \mu(h)}{h}\right)^{\frac{p-(1+\lambda)}{q-\lambda}} \frac{1}{a(x)^{\frac{p-(1+q)}{q-\lambda}}}\right)^{\frac{N}{p}} d x . }
\end{aligned}
$$

We obtain

$$
\begin{aligned}
& 0<C=C(\Omega, N, p, q, \lambda) \leq\left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} \\
& +\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right\}} \frac{d x}{a(x)^{\frac{N}{\delta_{0}}}}
\end{aligned}
$$

It leads to

$$
0<C \leq\left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)}+\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{\left\{x: a(x) \leq \frac{2 \mu(h)}{h} \varepsilon(h)^{\left.\frac{q-\lambda}{1+q}\right\}}\right.} \frac{\omega\left(\left.|x|\right|^{\frac{N}{\delta_{0}}}\right.}{|x|^{N}} d x
$$

We define $\rho>0$ small enough,

$$
\phi: \rho \mapsto \frac{\rho}{\omega(\rho)^{\frac{N}{\delta_{0}}}} \quad \text { and } \quad \psi: \rho \mapsto \frac{\rho^{N}}{\omega(\rho)^{\frac{N}{\delta_{0}}}}
$$

Both functions are continuous, increasing and $\psi(\rho)=\rho^{N-1} \phi(\rho)$. Since the function $a(x)=\psi(|x|)^{\frac{\delta_{0}}{N}}$ is radial, we use a change of variables in radial coordinates :

$$
0<C \leq\left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)}+\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{0}^{\psi^{-1}\left(\left(\frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right)^{\frac{N}{\delta_{0}}}\right)} \frac{1}{\phi(\rho)} d \rho
$$

Using (27), we have

$$
\mu(h) \leq C \frac{h}{\omega\left(h^{\varepsilon_{0}}\right)}
$$

which leads to,

$$
0<C \leq\left(\frac{1}{\omega\left(h^{\varepsilon_{0}}\right)} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)}+\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{0}^{\psi^{-1}\left(\left(\frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right)^{\frac{N}{\delta_{0}}}\right)} \frac{1}{\phi(\rho)} d \rho
$$

We define

$$
\gamma_{0}=\frac{p(1+q)}{N(p-(1+q))+p(1+q)}
$$

We take $\gamma<\gamma_{0}$ and $\varepsilon(h)=h^{\gamma}$. So, since $\omega(\rho) \gg \rho^{\varepsilon}$ for all $\varepsilon>0$,

$$
\left(\frac{1}{\omega\left(h^{\varepsilon_{0}}\right)} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}}\right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} \rightarrow 0
$$

so,

$$
0<C \leq\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{0}^{\psi^{-1}\left(\left(\frac{2 \mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right)^{\frac{N}{\delta_{0}}}\right)} \frac{1}{\phi(\rho)} d \rho
$$

By definition, for $\rho>0$ small enough,

$$
\psi(\rho)^{\frac{1}{N}}=\frac{\rho}{\omega(\rho)^{\frac{1}{\delta_{0}}}}
$$

We deduce that for $r>0$ small enough,

$$
\psi^{-1}(r)=r^{\frac{1}{N}} \omega\left(\psi^{-1}(r)\right)^{\frac{1}{\delta_{0}}} \leq r^{\frac{1}{N}} \omega_{0}^{\frac{1}{\delta_{0}}}
$$

Hence,

$$
0<C \leq\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{0}^{K h^{\frac{\gamma(q-\lambda)}{(1+q) \delta_{0}}}} \frac{1}{\phi(\rho)} d \rho
$$

for another $\gamma \in\left(0, \gamma_{0}\right), K>0$ and $h>0$ small enough. For

$$
\omega(\rho)=\frac{1}{(-\ln \rho)^{\alpha}}
$$

we must have $\alpha>\frac{\delta_{0}}{N}$ to insure that

$$
\int_{0}^{1} \frac{\omega(\rho)^{\frac{N}{\delta_{0}}}}{\rho} d \rho<+\infty
$$

So, for $C>0$ and $h>0$ small enough,

$$
0<C \leq\left(\frac{\mu(h)}{h}\right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}}(-\ln h)^{\frac{\delta_{0}-\alpha N}{\delta_{0}}}
$$

Hence,

$$
\frac{C}{\mu(h)} \leq \frac{1}{h(-\ln h)^{\frac{\alpha N-\delta_{0}}{\delta_{0}} \frac{p(q-\lambda)}{N(p-(1+\lambda))}}}=\frac{1}{h(-\ln h)^{\frac{\left(\alpha N-\delta_{0}\right)(p-(1+q))}{N(p-(1+\lambda)}}} .
$$

For

$$
\alpha>\frac{p-(1+\lambda)}{p-(1+q)}+\frac{\delta_{0}}{N}
$$

we obtain

$$
\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty
$$

Lemma 3 We assume that $p=1+q=2, O \in \Omega$ and (14). Then there are $u_{0} \in W_{0}^{1,2}(\Omega)$ and $C>0$ such as for all $r>0$ small enough,

$$
T\left(u_{0}\right) \geq C \omega(r)
$$

Proof: The following proof is a simpler version of [4]. Let's take $\alpha>1, r>0$ and $u_{0} \in W_{0}^{1,2}(\Omega)$ such as $\left\|u_{0}\right\|_{L^{\infty}(\Omega)}=1$ and $u_{0}=1$ on a neighbourhood of $O$.
We define $\varphi_{r}$ as the first function for $-\Delta$ on the ball of centre 0 and radius $r$ for the Dirichlet Boundary Condition. We normalise it to 1 , i.e., $\left\|\varphi_{r}\right\|_{L^{2}(\Omega)}=1$. The first eigenvalue is denoted by $\lambda_{1, r}$.
The function $\varphi_{r}^{\alpha}$ and its normal derivative is equal to zero on the sphere of radius $r$. So,

$$
\frac{d}{d t} \int_{\Omega} u(x, t) \varphi_{r}^{\alpha}(x) d x-\int_{\Omega} u(x, t) \Delta\left(\varphi_{r}^{\alpha}\right)(x) d x+\int_{\Omega} a(x) u^{\lambda}(x, t) \varphi_{r}^{\alpha}(x) d x=0 .
$$

But

$$
\Delta\left(\varphi_{r}^{\alpha}\right)=\alpha(\alpha-1) \varphi_{r}^{\alpha-2}\left|\nabla \varphi_{r}\right|^{2}+\alpha \varphi_{r}^{\alpha-1} \Delta \varphi_{r}=\alpha(\alpha-1) \varphi_{r}^{\alpha-2}\left|\nabla \varphi_{r}\right|^{2}-\alpha \lambda_{1, r} \varphi_{r}^{\alpha} .
$$

Therefore,

$$
\begin{aligned}
& \frac{d}{d t} \int_{\Omega} u(x, t) \varphi_{r}^{\alpha}(x) d x+\alpha \lambda_{1, r} \int_{\Omega} u(x, t) \varphi_{r}^{\alpha}(x) d x+\int_{\Omega} a(x) u^{\lambda}(x, t) \varphi_{r}^{\alpha}(x) d x \\
& =\alpha(\alpha-1) \int_{\Omega} u(x, t) \varphi_{r}^{\alpha-2}(x)\left|\nabla \varphi_{r}\right|^{2}(x) d x \geq 0
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\int_{\Omega} a(x) u^{\lambda}(x, t) \varphi_{r}^{\alpha}(x) d x \leq\left(\int_{\Omega} u(x, t) \varphi_{r}^{\alpha}(x) d x\right)^{\lambda}\left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_{r}^{\alpha}(x) d x\right)^{1-\lambda}
$$

We define

$$
y(t)=\int_{\Omega} u(x, t) \varphi_{r}^{\alpha}(x) d x,
$$

and $\lambda_{1,1}=\lambda_{1}$ the first eigenvalue of the unit ball. Hence,

$$
y^{\prime}(t)+\frac{\alpha \lambda_{1}}{r^{2}} y(t)+\left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_{r}^{\alpha}(x) d x\right)^{1-\lambda} y(t)^{\lambda} \geq 0
$$

Since $r \mapsto a(r)$ is a non decreasing function,

$$
\begin{aligned}
& \left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_{r}^{\alpha}(x) d x\right)^{1-\lambda}=\left(\int_{B_{r}} a(x)^{\frac{1}{1-\lambda}} \varphi_{r}^{\alpha}(x) d x\right)^{1-\lambda} \leq a(r)\left(\int_{B_{r}} \varphi_{r}^{\alpha}(x) d x\right)^{1-\lambda} \\
& =a(r) r^{N(1-\lambda)}\left(\int_{B_{1}} \varphi_{1}^{\alpha}(x) d x\right)^{1-\lambda} .
\end{aligned}
$$

We define

$$
C_{1}=\left(\int_{B_{1}} \varphi_{1}^{\alpha}(x) d x\right)^{1-\lambda}
$$

Finally,

$$
y^{\prime}(t)+\frac{\alpha \lambda_{1}}{r^{2}} y(t)+C a(r) r^{N(1-\lambda)} y(t)^{\lambda} \geq 0
$$

We define

$$
z(t)=y(t)^{1-\lambda} .
$$

So,

$$
\frac{z^{\prime}(t)}{1-\lambda}+\frac{\alpha \lambda_{1}}{r^{2}} z(t)+C_{1} a(r) r^{N(1-\lambda)} \geq 0 .
$$

Hence,

$$
\begin{aligned}
& z^{\prime}(t)+\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} z(t)+C_{1}(1-\lambda) a(r) r^{N(1-\lambda)} \geq 0 . \\
& \frac{d}{d t}\left(z(t) \exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right)\right)+C_{1}(1-\lambda) a(r) r^{N(1-\lambda)} \exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right) \geq 0 .
\end{aligned}
$$

We define

$$
\zeta(t)=z(t) \exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right) .
$$

So, with this new notation,

$$
\zeta^{\prime}(t)+C_{1}(1-\lambda) a(r) r^{N(1-\lambda)} \exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right) \geq 0 .
$$

We integrate between 0 and $t>0$.

$$
\zeta(t)-\zeta(0)+C_{1} \frac{a(r) r^{N(1-\lambda)+2}}{\alpha \lambda_{1}}\left(\exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right)-1\right) \geq 0 .
$$

So, $\zeta(0)=C_{1}$ implies that

$$
\zeta(t) \geq C_{1}\left(1-\frac{a(r) r^{N(1-\lambda)+2}}{\alpha \lambda_{1}}\left(\exp \left(\frac{\alpha(1-\lambda) \lambda_{1}}{r^{2}} t\right)-1\right)\right) .
$$

We have $\zeta(t)>0$ for all $t \in\left[0, T_{0}(r)\right)$ when

$$
T_{0}(r)=\frac{r^{2}}{\alpha(1-\lambda) \lambda_{1}} \ln \left(1+\frac{\alpha \lambda_{1}}{a(r) r^{N(1-\lambda)+2}}\right) .
$$

Hence, for $r>0$ small enough,

$$
T_{0}(r) \geq C \omega(r)
$$

Now, it is clear that if

$$
\lim _{r \rightarrow 0} \omega(r)=+\infty,
$$

then

$$
T\left(u_{0}\right) \geq \lim _{r \rightarrow 0} T_{0}(r)=+\infty,
$$

which implies that

$$
T^{+}(h)=+\infty,
$$

for all $h>0$.
Lemma 4 We assume that $p=1+q=2, O \in \Omega$, (14) and $\left(A_{1}\right)-\left(A_{5}\right)$. If

$$
\int_{0}^{c} \frac{\omega(r)}{r} d r<+\infty,
$$

then

$$
\int_{0}^{1} \frac{d h}{\mu(h)}<+\infty .
$$

Proof: We start with (30) for $p=1+q$ so, for all $\rho>0$,

$$
\mu(h) \leq h \rho^{-p} \int_{B_{1}}|\nabla v|^{p} d y+h^{\frac{1+\lambda}{p}} \rho^{N-N \frac{1+\lambda}{p}} \int_{B_{1}} a(\rho y) v^{1+\lambda}(y) d y .
$$

We estimate $a(\rho y)$ by $\exp \left(\frac{-\omega(\rho)}{\rho^{p}}\right)$, so for $h$ small enough,
(32) $\mu(h) \leq C(p, \lambda, v)\left(h \rho^{-p}+h^{\frac{1+\lambda}{p}} \rho^{N-N \frac{1+\lambda}{p}} \exp \left(\frac{-\omega(\rho)}{\rho^{p}}\right)\right)$.

Using assumption $\left(A_{5}\right)$, for $h>0$ small enough, there is a function $\rho(h)$ such as

$$
h=\exp \left(\frac{-\omega(\rho(h))}{\rho^{p}(h)}\right) .
$$

Hence,

$$
\ln h=\frac{-\omega(\rho(h))}{\rho^{p}(h)} \geq \frac{-\omega_{0}}{\rho^{p}(h)},
$$

which gives

$$
\rho(h) \leq\left(\frac{\omega_{0}}{-\ln h}\right)^{\frac{1}{p}} .
$$

Moreover,

$$
\ln h=\frac{-\omega(\rho(h))}{\rho^{p}(h)} \leq-\frac{1}{\rho^{\delta}(h)} .
$$

So,

$$
\rho(h) \geq\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}
$$

We have

$$
\rho^{-p}(h)=\frac{-\ln h}{\omega(\rho(h))} \leq \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)} .
$$

Now,

$$
\mu(h) \leq C(p, \lambda, v)\left(\frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)}+h^{\frac{1+\lambda}{p}}\left(\frac{\omega_{0}}{-\ln h}\right)^{\frac{N}{p}-N \frac{1+\lambda}{p^{2}}} h\right) .
$$

Clearly, for $h>0$ small enough,
(33) $\mu(h) \leq C h \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)}$.

Now after an upper bound, the lower bound. For all $h>0$, there is $v_{h} \in W_{0}^{1, p}(\Omega)$ such as $\|v\|_{L^{1+q}(\Omega)}^{1+q}=h$ and

$$
2 \mu(h) \geq \int_{\Omega}|\nabla v|^{p} d x+\int_{\Omega} a(x) v^{1+\lambda} d x .
$$

To simplify, we drop $h$ for $v_{h}:=v$. So,

$$
\int_{\Omega}|\nabla v|^{p} d x \leq \frac{2 \mu(h)}{h} \int_{\Omega} v^{1+q} d x-\int_{\Omega} a(x) v^{1+\lambda} d x
$$

Since $p=1+q$,

$$
\int_{\Omega}|\nabla v|^{p} d x \leq \int_{\Omega} v^{p}\left(\frac{2 \mu(h)}{h}-\frac{a(x)}{v^{p-(1+\lambda)}}\right) d x .
$$

Sobolev injection and Hölder inequality when $N>p$ give

$$
C_{S}\left(\int_{\Omega} v^{p^{*}} d x\right)^{\frac{p}{p^{*}}} \leq\left(\int_{\Omega} v^{p^{*}} d x\right)^{\frac{p}{p^{*}}}\left(\int_{\Omega}\left(\left[\frac{2 \mu(h)}{h}-\frac{a(x)}{v^{p-(1+\lambda)}}\right]^{+}\right)^{\frac{N}{p}} d x\right)^{\frac{p}{N}}
$$

when $\frac{1}{p^{*}}=\frac{1}{p}-\frac{1}{N}$. Hence,

$$
C_{S}^{\frac{N}{p}} \leq \int_{\Omega}\left(\left[\frac{2 \mu(h)}{h}-\frac{a(x)}{v^{p-(1+\lambda)}}\right]^{+}\right)^{\frac{N}{p}} d x .
$$

Therefore,

$$
C_{S}^{\frac{N}{p}} \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \text { meas }\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} .
$$

Let $\varepsilon>0$. We have

$$
h=\|v\|_{L^{1+q}(\Omega)}^{1+q}=\int_{\Omega} v^{1+q} d x \geq \int_{v^{1+q} \geq \varepsilon} v^{1+q} d x \geq \varepsilon \operatorname{meas}\left\{x: v^{1+q} \geq \varepsilon\right\} .
$$

By taking $\varepsilon=h^{\gamma}$ with $0<\gamma<1$, we obtain

$$
h^{1-\gamma} \geq \operatorname{meas}\left\{x: v^{1+q} \geq h^{\gamma}\right\}=\operatorname{meas}\left\{x: v^{p-(1+\lambda)} \geq h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right\} .
$$

So,

$$
\begin{aligned}
& \text { meas }\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\}=\operatorname{meas}\left(\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} \bigcap\left\{x: v^{p-(1+\lambda)} \geq h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right\}\right) \\
& + \text { meas }\left(\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} \bigcap\left\{x: v^{p-(1+\lambda)}<h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right\}\right) .
\end{aligned}
$$

It follows that

$$
\text { meas }\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} \leq h^{1-\gamma}+\operatorname{meas}\left(\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} \bigcap\left\{x: v^{p-(1+\lambda)}<h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right\}\right) .
$$

We obtain

$$
\operatorname{meas}\left\{x: \frac{2 \mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}}\right\} \leq h^{1-\gamma}+\text { meas }\left\{x: \frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \geq a(x)\right\} .
$$

Now, from (33), for $h>0$ small enough,

$$
\mu(h) \leq C h \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)} .
$$

By $\left(A_{4}\right), \omega(r) \geq r^{p-\delta}$ for $r>0$ small enough. It leads to
(34) $\mu(h) \leq C h \frac{-\ln h}{\left(\left(\frac{1}{-\ln h}\right)^{\frac{p-\delta}{\delta}}\right)}=C h(-\ln h)^{\frac{p}{\delta}}$.

There are two consequences :

$$
\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} h^{1-\gamma} \rightarrow 0
$$

and

$$
\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \rightarrow 0
$$

when $h \rightarrow 0$. We deduce that there is $C>0$ such as for $h>0$ small enough,

$$
C \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \operatorname{meas}\left\{x: \frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \geq \exp \left(-\frac{\omega(|x|)}{|x|^{p}}\right)\right\}
$$

So,

$$
C \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \text { meas }\left\{x: \ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq-\frac{\omega(|x|)}{|x|^{p}}\right\}
$$

We take $x$ such as $\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq-\frac{\omega(|x|)}{|x|^{p}}$. Since $\omega$ is bounded, it satisfies

$$
\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq-\frac{\omega_{0}}{|x|^{p}}
$$

i.e.,

$$
|x| \leq\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}
$$

By monotonicity of $\omega$ (assumption $\left(A_{1}\right)$ ),

$$
\omega(|x|) \leq \omega\left(\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)
$$

Clearly,

$$
C \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \operatorname{meas}\left\{x: \ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq-\frac{\omega\left(\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)}{|x|^{p}}\right\}
$$

Consequently,

$$
C \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \operatorname{meas}\left\{x:|x|^{p} \leq \frac{\omega\left(\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right\}
$$

Hence,

$$
C \leq\left(\frac{2 \mu(h)}{h}\right)^{\frac{N}{p}} \operatorname{meas}\left\{x:|x|^{N} \leq\left(\frac{\omega\left(\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{N}{p}}\right\}
$$

It leads to

$$
C \leq\left(\frac{2 \mu(h)}{h}\right) \frac{\left.\omega\left(\frac{\omega_{0}}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)}{-\ln \left(\frac{2 \mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}
$$

for another $C>0$. Now, by (34), for $h>0$ small enough,

$$
\frac{\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \leq h^{\varepsilon}
$$

for some $\varepsilon>0$. We obtain

$$
C \leq\left(\frac{\mu(h)}{h}\right) \frac{\omega\left(\left(\frac{K}{-\ln h}\right)^{\frac{1}{p}}\right)}{-\ln h}
$$

for $C, K>0$. We end by using a change of variables.

## 3 Proof of Theorem 1.1 and Theorem 1.2

We use Lemma 1 (resp. Lemma 3) to prove Proposition 1. By Theorem 2.1 and Lemmas 1 and 2 (resp. Lemmas 3 and 4), we obtain the conclusion.

## References

[1] C. Bandle, I. Stakgold, The formation of the dead core in parabolic reaction-diffusion problems, Trans. Amer. Math. Soc. 286 (1984), 275-293.
[2] Y. Belaud, Time-vanishing properties of solutions of some degenerate parabolic equations with strong absorption, Advanced Nonlinear Studies 1, 2 : 117-152, 2001.
[3] Y. Belaud, Asymptotic estimates for a variational problem involving a quasilinear operator in the semiclassical limit, Annals of global analysis and geometry 26:271-313, 2004.
[4] Y. Belaud, B. Helffer, L. Véron, Long-time vanishing properties of solutions of sublinear parabolic equations and semi-classical limit of Schrödinger operator, Ann. Inst. Henri Poincaré Anal. nonlinear 18, 1 (2001), 43-68.
[5] Y. Belaud, A. E. Shishkov, Long-time extinction of solutions of some semilinear parabolic equations, J. Diff. Equations 238, (2007), 64-86.
[6] Y. Belaud, A. E. Shishkov, Extinction of solutions of some semilinear higher ordre parabolic equations with degenerate absorption potential, J. Evol. Equ. 10, (2010), 857-882.
[7] L. Bers, F. John and M. Schechter, Partial differential equations, American Mathematical Society, Interscience publishers, John Wiley and Sons, 1964.
[8] Chen Xu-Yan, H. Matano, M. Mimura, Finite point extinction and continity of interfaces in nonlinear diffusion equations with strong absorption, J. Reine Angew. Math. 459:1 (1995), 1-36.
[9] J.I. Díaz and L. Véron, Local vanishing properties of solutions of elliptic and parabolic quasilinear equations, Trans. Amer. Math. Soc. 290 (1985), 787-814.
[10] A. Friedman, M. Herrero, Extinction properties of semilinear heat equation with strong absorption, J. Math. Anal. Appl. 124 (1987), 530-546.
[11] B. Helffer, Semi-classical analysis for the Schrödinger operator and applications, Lecture Notes in Math. 1336, Springer-Verlag, 1989.
[12] Junjie Li, Qualitative properties for solutions of semilinear heat equations with strong absorption, J. Math. Anal. Appl. 281 (2003), 382-394.
[13] Junjie Li, Qualitative properties for solutions to semilinear heat equations with singular initial data, Electronic J. of Differ Equat. (2004), N 53, 1-12.
[14] A. S. Kalashnikov, The propagation of disturbances of nonlinear heat equation with absorption, USSR Comput. Math. and Math Phys. 14 (1974), 70-85.
[15] A. S. Kalashnikov, Instantaneous shinking of the support for solutions to certain parabolic equations and systems, Rend, Mt. Acc. Lincei. Serie 9, 8 (1997), 263-272.
[16] R. Kersner, Nonlinear heat conduction with absorption: space localization and extinction in finite time, SIAM J. Appl. Math. Vol. 436 (1983), 1274-1285.
[17] R. Kersner, F. Nicolosi, The nonlinear heat equation with absorption: effect of variable coefficients, J. Math. Anal. Appl. 170 (1992), 551-566.
[18] R. Kersner, A. Shishkov, Instantaneous shrinking of the support of energy solutions, J. Math. Anal. Appl. 198 (1996), 729-750.
[19] B. F. Knerr, The behaviour of the support of solution of the equation of nonlinear heatconduction with absorption in one dimension. Trans. Amer. Math. Soc. 249 (1979), N 2, 409-424.
[20] S. Kaplan, On the growth of solutions of quasi-linear parabolic equations, Comm. Pure Appl. Math. 16 (1963), 305-330.
[21] V.A. Kondratiev and L. Véron, Asymptotic behaviour of solutions of some nonlinear parabolic or elliptic equations, Asymptotic Analysis 14 (1997), 117-156.
[22] E.H. Lieb, W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relations to Sobolev Inequalities, In Studies in Math. Phys., essay in honour of V. Bargmann, Princeton Univ. Press, 1976.
[23] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod Gauthier-Villars, 1969.
[24] L. E. Payne, Improperly posed problems in partial differential equations, SIAM Regional Conference Seeries in Applied Math, 22 (1975).
[25] P. Quittner, P. Souplet, Superlinear Parabolic Problems, Birkhäuser Advanced Texts, 2007.
[26] A. E. Shishkov, Dead cores and instantaneous compactification of the supports of energy solutions of quasilinear parabolic equations of arbitrary order, Sbornik: Mathematics, 190:12, (1999), 1843-1869.
[27] I. V. Skrypnik, Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, Translations of Mathematical Monographs, volume 139, AMS, 1994.
[28] B. Straughan, Instability, nonexistence and weighted energy methods in fluid dynamics and related theories. Research Notes in Mathematics 74 (London: Pitman, 1982).

