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Extinction in a finite time for solutions of large classes of Parabolic Equations involving p -Laplacian

Y. Belaud, A. Shishkov

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Abstract We study the property of extinction in a finite time for nonnegative solutions of $\frac{1}{q} \frac{\partial}{\partial t}(u^q) - \nabla(|\nabla u|^{p-2} \nabla u) + a(x)u^\lambda = 0$ for the Dirichlet Boundary Conditions when $q > \lambda > 0$, $p \geq 1 + q$, $p \geq 2$, $a(x) \geq 0$ and $\Omega \subset \mathbb{R}^N$ a bounded domain of \mathbb{R}^N ($N \geq 1$). Necessary and sufficient conditions are provided with the help of a family of infimum. When $p > 1 + q$, the threshold is for power functions but for $p = 1 + q$, it happens extinction in a finite time for very flat functions.

1 Introduction

Let $q > \lambda > 0$, $p \geq 1 + q$, $p \geq 2$ and $\Omega \subset \mathbb{R}^N$ a bounded domain of \mathbb{R}^N ($N \geq 1$). We consider u a nonnegative solution of

$$(1) \quad \begin{cases} \frac{1}{q} \frac{\partial}{\partial t}(u^q) - \nabla(|\nabla u|^{p-2} \nabla u) + a(x)u^\lambda & = 0 \\ u(x, 0) & = u_0(x) \end{cases}$$

for the Dirichlet Boundary Condition. In this article, we study the fact that the solutions vanish.

Definition 1.1 *Let say that problem (1) has the extinction in finite time property if for arbitrary solution u , it exists some positive T such as $u(x, t) = 0$ a.e. in Ω , $\forall t \geq T$.*

So, we introduce the following notation :

$$(2) \quad T(u_0) = \sup \{ t > 0, \forall \tau \in [0, t], \|u(\tau)\|_{L^2(\Omega)} > 0 \}.$$

This problem has been intensively studied by many authors by means of different methods. Among these methods, one can mention energy methods. In [21], the authors introduce a new energy method called semi-classical method. They transform a parabolic problem into a sequence of elliptic problems, namely, the behaviour in large time for solutions of some parabolic equations is linked with the asymptotic behaviour of a family of first eigenvalues in the semi-classical limit. That is to say, for $q = 1$, $p = 2$, for the Neumann Boundary Conditions, they introduce

$$\mu_n = \inf \left\{ \int_{\Omega} (|\nabla u|^2 + 2^n a(x)u^{\lambda+1}) dx : u \in W^{1,2}(\Omega), \int_{\Omega} u^2 dx = 1 \right\}.$$

By iterations and using the maximum principle, they find a sufficient condition. This later was improved in [4] thank to Lieb-Thirring formula [22]. Moreover, in the same article, the authors have given a necessary condition using a Kaplan's like-method [20], [25] and the idea of homothetic test-functions. Unfortunately, these two conditions were different. In [2], for the first time, in only two cases for $p > 2$, $q = 1$ and $p = 2$, $q < 1$, for the sufficient condition, the critical power $\delta_0 = \frac{p(q - \lambda)}{p - (1 + q)}$ appeared under the condition $N > p$. No necessary conditions were provided because the Kaplan's like-method used the fact that $-\Delta$ is a self-adjoint operator. Some properties of the first eigenvalue and the first eigenfunction are studied in [3].

A Diny-like condition has appeared for the first time in this kind of problem in [5] and high-order operators are

studied in [6]. The main drawback of Kondratiev-Véron method is based on the comparison principle. So we introduce a new semi-classical method : we give a sufficient condition for $p > 1 + q$ and $p = 1 + q$ and a necessary condition for $p > 1 + q$ and $p = 2 = 1 + q$, i.e., $q = 1$.

There are two steps for the sufficient condition :

1. We transform the parabolic problem into a problem of behaviour for some kind of nonlinear eigenvalues. There is extinction in a finite time for solutions of (1) if an integral depending of some Raleigh quotients is finite. We define

$$\mu(h) = \inf_{v \in W_0^{1,p}(\Omega), \|v\|_{L^{1+q}(\Omega)}^{1+q} = h} \int_{\Omega} |\nabla v|^p dx + \int_{\Omega} a(x) |v|^{1+\lambda} dx.$$

We proof under assumptions that there is extinction in a finite time if

$$(3) \quad \int_0^1 \frac{dh}{\mu(h)} < +\infty,$$

This condition is valid for a large class of maximum monotone operators of Hilbert Spaces.

But it seems that the genuine notion is a kind of uniform upper bound for the extinction time, i.e., we define for all $h_0 > 0$,

$$(4) \quad T^+(h_0) = \sup \left\{ T(u_0) : u_0 \in W_0^{1,p}(\Omega), 0 < \|u_0\|_{L^{1+q}(\Omega)}^{1+q} \leq h_0 \right\}.$$

With this new definition, condition (3) implies that

$$T^+(h_0) < +\infty \quad \forall h_0 > 0, \quad \text{and} \quad \lim_{h_0 \rightarrow 0} T^+(h_0) = 0.$$

2. Now, the point is to find estimates of $\mu(h)$. The main tool is Sobolev injections, that's why we have the restriction $N > p$. The quantity $\mu(h)$ is estimated in two differents ways depending on the value of p with respect to $1 + q$, i.e., $p > 1 + q$ or $p = 1 + q$.

For the necessary condition, it can be expected that if the norm of the initial condition u_0 is small then its extinction time is also small. But we prove that

$$(5) \quad T^+(h_0) \geq \sup_{0 < h \leq h_0} \frac{(1 + \lambda) h}{p(q - \lambda) \mu(h)}.$$

This condition is also valid for a large class of maximum monotone operators of Hilbert Spaces.

There are two important cases :

- 1.

$$(6) \quad \lim_{h \rightarrow 0} \frac{h}{\mu(h)} = +\infty \quad \implies \quad T^+(h_0) = +\infty,$$

means that for any $h_0 > 0$ and $T > 0$, there is an initial data u_0 such as $T(u_0) > T$. But without any further assumption, it seems difficult to get a u_0 such as $T(u_0) = +\infty$ in the general case.

When $p = 1 + q$, (6) never happens.

- 2.

$$(7) \quad \liminf_{h \rightarrow 0} \frac{h}{\mu(h)} \geq C > 0 \quad \implies \quad \liminf_{h_0 \rightarrow 0} T^+(h_0) > 0,$$

implies that for any $h_0 > 0$, one can find an initial data u_0 such as $\|u_0\|_{L^{1+q}(\Omega)}^{1+q} = h_0$ and $T(u_0) \geq C$ for some fixed $C > 0$. An initial data with a positive norm as small as we want exists but with an extinction time greater than a fixed positive constant.

When $p = 1 + q$, (7) happens only when $a(x) = 0$ a.e. in Ω which is a trivial case.

For $p > 1 + q$, by means of “good” homothetic test-function, we estimate $\mu(h)$ by above.

We have seen that for $p = 1 + q$, condition (5) isn’t relevant. For this reason, when $p = 2$ (and so $q = 1$), we use some kind of asymptotic Kaplan’s method, i.e., we prove that under condition of $a(x)$, for all time $T > 0$, a “good” first eigenfunction φ such as $\int_{\Omega} u(x, T) \varphi(x) dx > 0$ exists.

Here, we have to split $p > 1 + q$ and $p = 1 + q$ (called critical case). Calculations and estimates are different due to the nature of (1). Indeed, for $p > 1 + q$, there are three different degrees of homogeneity while when $p = 1 + q$, there are only two. It provides two radically different thresholds (power functions for $p > 1 + q$ and very flat functions for the critical case).

We define for $p > 1 + q$,

$$\delta_0 = \frac{p(q - \lambda)}{p - (1 + q)}.$$

We assume

$$(8) \quad a(x) = \frac{|x|^{\delta_0}}{\omega(|x|)} \text{ for } x \text{ small enough and otherwise } a(x) \geq C > 0.$$

For the sufficient condition, we assume that

$$(9) \quad \rho \mapsto \frac{\rho^{\delta_0}}{\omega(\rho)} \text{ is a continuous nondecreasing function when } \rho > 0 \text{ is small enough,}$$

$$(10) \quad \rho \mapsto \omega(\rho) \text{ is a nonincreasing function when } \rho > 0 \text{ is small enough,}$$

$$(11) \quad \forall \gamma_1 \in (0, 1), \exists C > 0, \exists \gamma_2 > 0, \forall h \in (0, 1], \omega(h^{\gamma_1}) \geq C (\omega(h))^{\gamma_2},$$

$$(12) \quad \exists \eta > 0, \forall h \in (0, h_0], \omega(h) \leq h^{-\eta} \text{ for some } h_0 > 0.$$

For the necessary condition, we assume that

$$(13) \quad \omega(\rho) = \frac{1}{(-\ln \rho)^\alpha} \text{ for } \alpha > \frac{p - (1 + \lambda)}{p - (1 + q)} + \frac{\delta_0}{N}.$$

Theorem 1.1

Assume that $p > 1 + q$, $O \in \Omega$, (8).

1. Assume (9).

(a) If

$$\liminf_{\rho \rightarrow 0} \omega(\rho) > 0,$$

then

$$\liminf_{h_0 \rightarrow 0} T^+(h_0) > 0.$$

In particular, it happens when

$$\omega(\rho) = \frac{1}{(-\ln \rho)^\alpha},$$

for

$$\alpha \leq 0.$$

(b) If we assume (10), (11), (12) and

$$\lim_{\rho \rightarrow 0} \omega(\rho) = +\infty,$$

then

$$\forall h_0 > 0, T^+(h_0) = +\infty.$$

In particular, it happens when

$$\omega(\rho) = \frac{1}{(-\ln \rho)^\alpha},$$

for

$$\alpha < 0.$$

2. Assume that $N > p$. The function ω satisfies (13). Then all the solutions vanish in a finite time. More precisely,

$$\forall h_0 > 0, T^+(h_0) < +\infty \text{ and } \lim_{h_0 \rightarrow 0} T^+(h_0) = 0.$$

For $p = 1 + q$, we define

$$(14) \quad a(x) = \exp\left(-\frac{\omega(|x|)}{|x|^p}\right).$$

We assume for the function ω ,

(A₁) $\omega(r)$ is a continuous and nondecreasing function $\forall r \geq 0$,

(A₂) $\omega(0) = 0, \omega(r) > 0, \forall r > 0$,

(A₃) $\omega(r) \leq \omega_0 < \infty, \forall r > 0$,

(A₄) $\omega(r) \geq r^{p-\delta}, \forall s \in (0, s_0), s_0 > 0, p > \delta > 0$,

(A₅) the function $\frac{\omega(r)}{r^p}$ is nonincreasing on $(0, s_0)$.

Theorem 1.2 Assume that $p = 1 + q$ and $O \in \Omega$. We assume (14), (A₁) – (A₅) and $N > p$. If

$$\int_0^c \frac{\omega(r)}{r} dr < +\infty,$$

then all the solutions vanish in a finite time. More precisely,

$$\forall h_0 > 0, T^+(h_0) < +\infty \text{ and } \lim_{h_0 \rightarrow 0} T^+(h_0) = 0$$

Moreover, assume only (14) and $p = 2$ (i.e. $q = 1$).

1. If

$$\liminf_{\rho \rightarrow 0} \omega(\rho) > 0,$$

then

$$\liminf_{h_0 \rightarrow 0} T^+(h_0) > 0.$$

2. If

$$\lim_{\rho \rightarrow 0} \omega(\rho) = +\infty,$$

then

$$\forall h_0 > 0, T^+(h_0) = +\infty.$$

The proofs of Theorems 1.1 and 1.2 are based on an abstract theorem, a proposition and four lemmas.

2 Proof

The next theorem is the key-stone for necessary condition and sufficient condition for $p > 1 + q$ and sufficient condition for $p = 1 + q$.

Let H be an Hilbert Space for $q = 1$ and $L^2(\Omega)$ for $q \neq 1$ (it is just a matter of algebra structure).

When $q = 1$, $\| \cdot \|$ is the usual norm on H and when $q \neq 1$, $\| \cdot \| = \| \cdot \|_{L^{1+q}(\Omega)}$ is the $L^{1+q}(\Omega)$ -norm on $L^2(\Omega)$.

We always denote by J be a proper convex lower semi-continuous function on H for $q = 1$ and on $L^2(\Omega)$ for $q \neq 1$.

We define $A = \partial J$ as the subdifferential of J . Let's denote the domain of A by $D(A)$.

Let's remember that $D(J) = \{u \in H, J(u) < +\infty\}$.

For simplicity's sake, we assume that A is a single-valued operator.

In what follows, we consider that $H = L^2(\Omega)$ but it can be easily extend to all Hilbert Space when $q = 1$.

Let u be a solution of

$$(15) \quad \frac{1}{q} \frac{\partial}{\partial t} (|u|^{q-1}u) + Au = 0, \quad u(0) = u_0.$$

We define for all $u_0 \in D(A) \setminus \{0\}$,

$$(16) \quad T(u_0) = \sup \{t > 0, \forall \tau \in [0, t], u(\tau) \neq 0\},$$

and for all $h_0 > 0$,

$$(17) \quad T^+(h_0) = \sup \left\{ T(u_0) : u_0 \in D(A) \cap L^{1+q}(\Omega), 0 < \|u_0\|_{L^{1+q}(\Omega)}^{1+q} \leq h_0 \right\}.$$

The following assumptions can be made :

$$H_0 : \inf_{v \in D(J)} J(v) = J(0) = 0,$$

$$H_1 : \forall h > 0, D(A) \cap \left\{ v \in L^{1+q}(\Omega), \|v\|_{L^{1+q}(\Omega)}^{1+q} = h \right\} \neq \emptyset.$$

Let's define for all $h > 0$,

$$(18) \quad \mu(h) = \inf_{v \in D(A) \cap \left\{ v \in L^{1+q}(\Omega), \|v\|_{L^{1+q}(\Omega)}^{1+q} = h \right\}} \langle Av, v \rangle.$$

$$H_2 : \forall h > 0, \mu(h) > 0.$$

$$H_3 : \exists p > 1, \forall v \in D(A), \langle Av, v \rangle \leq pJ(v).$$

$$H_4 : \exists \lambda \in [0, p-1), \forall v \in D(A), \langle Av, v \rangle \geq (1 + \lambda)J(v).$$

Remark 2.1

1. Assumption H_0 implies $A0 = 0$.
2. Assumptions H_2 and H_3 imply $J(v) > 0$ for all $v \in D(J) \setminus \{0\}$.
3. Assumption H_4 is always true for $\lambda = 0$.
4. To some extend, H_3 means that J is some kind of polynome in u and of all its derivative.

Theorem 2.1 *Let $u_0 \in D(A) \setminus \{0\}$. Under $H_0 - H_4$, for solutions of (15), we have the following results :*

$$(19) \quad \int_0^1 \frac{dh}{\mu(h)} < +\infty \implies \left(\forall h_0 > 0, T^+(h_0) < +\infty \text{ and } \lim_{h_0 \rightarrow 0} T^+(h_0) = 0 \right),$$

and

$$(20) \quad \left(\forall h_0 > 0, T^+(h_0) < +\infty \text{ and } \lim_{h_0 \rightarrow 0} T^+(h_0) = 0 \right) \implies \lim_{h_0 \rightarrow 0} \frac{h_0}{\mu(h_0)} = 0.$$

If it's true, we have the following estimates :

$$(21) \quad T(u_0) \leq T^+ \left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q} \right) \leq \frac{1}{1+q} \int_0^{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}} \frac{dh}{\mu(h)},$$

and

$$(22) \quad T^+(h_0) \geq \sup_{0 < h < h_0} \frac{(1+\lambda)h}{p(q-\lambda)\mu(h)}.$$

Proof: For the sufficient condition, we assume that $\int_0^1 \frac{dh}{\mu(h)} < +\infty$.

Taking the scalar product with u leads to

$$(23) \quad \frac{1}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right) + \langle Au, u \rangle = 0.$$

But by definition of $\mu(h)$,

$$\langle Au, u \rangle \geq \mu \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right).$$

So,

$$\frac{1}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right) + \mu \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right) \leq 0.$$

Since $\mu(h) > 0$ for all $h > 0$,

$$\frac{\frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right)}{\mu \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right)} \leq -(1+q).$$

We integrate between 0 and $t < T(u_0)$.

$$\int_{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}}^{\|u\|_{L^{1+q}(\Omega)}^{1+q}} \frac{dh}{\mu(h)} \leq -(1+q)t.$$

Hence,

$$\int_{\|u\|_{L^{1+q}(\Omega)}^{1+q}}^{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}} \frac{dh}{\mu(h)} \geq (1+q)t.$$

We deduce that for all $t < T(u_0)$,

$$\int_0^{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}} \frac{dh}{\mu(h)} \geq (1+q)t.$$

Going to the limit gives

$$\int_0^{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}} \frac{dh}{\mu(h)} \geq (1+q)T(u_0),$$

and therefore $T(u_0) < +\infty$. Moreover,

$$T^+(h) \leq \frac{1}{1+q} \int_0^h \frac{d\theta}{\mu(\theta)},$$

and

$$T^+(h) \rightarrow 0,$$

as $h \rightarrow 0$.

For the necessary condition, since u is regular, we have

$$\frac{d}{dt} J(u(t)) = \langle u_t, Au \rangle = - \langle u_t, u_t |u|^{q-1} \rangle = - \| |u_t| |u|^{\frac{q-1}{2}} \|_{L^2(\Omega)}^2.$$

According to Cauchy-Schwarz-Buniakowsky inequality,

$$| \langle u_t, u |u|^{q-1} \rangle |^2 = | \langle u_t |u|^{\frac{q-1}{2}}, u |u|^{\frac{q-1}{2}} \rangle |^2 \leq \| |u_t| |u|^{\frac{q-1}{2}} \|_{L^2(\Omega)}^2 \|u\|_{L^{1+q}(\Omega)}^{1+q}$$

Hence,

$$(\langle Au, u \rangle)^2 \leq \left(-\frac{d}{dt} J(u(t))\right) \|u\|_{L^{1+q}(\Omega)}^{1+q}.$$

So,

$$(24) \quad \frac{d}{dt} J(u(t)) \leq -\frac{(\langle Au, u \rangle)^2}{\|u\|_{L^{1+q}(\Omega)}^{1+q}} = \frac{1}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right) \frac{\langle Au, u \rangle}{\|u\|_{L^{1+q}(\Omega)}^{1+q}}.$$

Then, using H_4 ,

$$\frac{d}{dt} J(u(t)) \leq \frac{1+\lambda}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q} \right) \frac{J(u(t))}{\|u\|_{L^{1+q}(\Omega)}^{1+q}}.$$

By integration, we obtain

$$(25) \quad J(u(t)) \leq \frac{J(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}.$$

$$\frac{1}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right) + pJ(u(t)) \geq 0.$$

$$J(u(t)) \leq \frac{J(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}$$

$$\frac{1}{1+q} \frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right) + \frac{pJ(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}} \geq 0.$$

$$\frac{\frac{d}{dt} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)}{\left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} + \frac{(1+q)pJ(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} \geq 0.$$

$$\frac{1+q}{q-\lambda} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}} - \frac{1+q}{q-\lambda} \left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}} + \frac{(1+q)pJ(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} t \geq 0.$$

$$\frac{1}{q-\lambda} \left(\|u\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}} \geq \frac{1}{q-\lambda} \left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}} - \frac{pJ(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} t.$$

$$\frac{pJ(u_0)}{\left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{1+\lambda}{1+q}}} T(u_0) \geq \frac{1}{q-\lambda} \left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q}\right)^{\frac{q-\lambda}{1+q}}.$$

$$T(u_0) \geq \frac{1}{p(q-\lambda)} \frac{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}}{J(u_0)}$$

$$T(u_0) \geq \frac{(1+\lambda)}{p(q-\lambda)} \frac{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}}{\langle Au_0, u_0 \rangle}.$$

$$T^+(h_0) = \sup_{0 < \|u_0\|_{L^{1+q}(\Omega)}^{1+q} \leq h_0} T(u_0).$$

Let $h_0 > 0$.

$$(26) \quad T^+(h_0) \geq \sup_{0 < \|u_0\|_{L^{1+q}(\Omega)}^{1+q} \leq h_0} \frac{(1+\lambda)}{p(q-\lambda)} \frac{\|u_0\|_{L^{1+q}(\Omega)}^{1+q}}{\langle Au_0, u_0 \rangle}.$$

For all $u_0 \in D(A) \setminus \{0\}$, there is a sequence (u_n) such as $\|u_n\|_{L^{1+q}(\Omega)}^{1+q} = \|u_0\|_{L^{1+q}(\Omega)}^{1+q}$ and $\langle Au_n, u_n \rangle \rightarrow \mu \left(\|u_0\|_{L^{1+q}(\Omega)}^{1+q} \right)$. When we go to the limit in (26),

$$T^+(h_0) \geq \sup_{0 < h \leq h_0} \frac{(1+\lambda)}{p(q-\lambda)} \frac{h}{\mu(h)}. \blacksquare$$

In $L^2(\Omega)$ when $\Omega \subset \mathbb{R}^N$ is a bounded domain with $O \in \Omega$, let's say

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{1+\lambda} \int_{\Omega} a(x) |u|^{1+\lambda} dx,$$

for $0 < \lambda < q \leq p-1$ with $D(J) = W_0^{1,p}(\Omega)$ for $p \geq 2$.

Proposition 1 *The functional J satisfies assumptions $H_0 - H_4$.*

Proof: Assumptions H_0 and H_1 are satisfied when $D(J) = W_0^{1,p}(\Omega)$. Using Poincaré's inequality and Hölder's inequality,

$$\langle Av, v \rangle \geq C \int_{\Omega} |v|^p dx \geq C' \left(\int_{\Omega} |v|^{1+q} dx \right)^{\frac{p}{1+q}},$$

which means $\mu(h) \geq C' h^{\frac{p}{1+q}} > 0$ for all $h > 0$. So H_2 is valid. Moreover, it is clear that H_3 and H_4 are also valid. \blacksquare

Lemma 1 *We assume that $p > 1+q$, $O \in \Omega$, (8) and (9).*

1. *If*

$$\lim_{\rho \rightarrow 0} \omega(\rho) = 0,$$

then

$$(27) \quad \mu(h) \leq C \frac{h}{\omega(h^{\varepsilon_0})}.$$

for some $C > 0$ when

$$\varepsilon_0 = \frac{p - (1+\lambda)}{N(p - (1+\lambda)) + (\delta_0 + p)(1+q)}.$$

2. *If*

$$\liminf_{\rho \rightarrow 0} \omega(\rho) > 0,$$

then

$$(28) \quad \mu(h) \leq C h,$$

for some $C > 0$.

3. If we assume (10), (11), (12) and

$$\lim_{\rho \rightarrow 0} \omega(\rho) = +\infty,$$

then

$$(29) \quad \lim_{h \rightarrow 0} \frac{h}{\mu(h)} = +\infty.$$

Proof: Since $O \in \Omega$, $\rho_0 > 0$ exists such as B_{ρ_0} (the ball of center O and radius ρ_0) is included in Ω .

Let $v \in C_0^\infty(B_1)$ be with $v \geq 0$ and $\int_{B_1} v^{1+q} = 1$.

Let ρ be in $(0, \rho_0)$. We define $v_\rho(x) = v(\frac{x}{\rho})$. Then,

$$\int_{\Omega} v_\rho^{1+q}(x) dx = \int_{B_\rho} v_\rho^{1+q}(x) dx = \int_{B_\rho} v^{1+q}\left(\frac{x}{\rho}\right) dx = \rho^N \int_{B_1} v^{1+q}(y) dy = \rho^N.$$

As a consequence, $\left\| \frac{v_\rho}{\rho^{\frac{N}{1+q}}} \right\|_{L^{1+q}(\Omega)} = 1$. On the other hand,

$$\nabla_x v_\rho(x) = \rho^{-1} \nabla_y v\left(\frac{x}{\rho}\right).$$

Hence,

$$\int_{\Omega} |\nabla v|^p dx = \int_{B_\rho} |\nabla v_\rho|^p dx = \rho^{N-p} \int_{B_1} |\nabla v|^p dy.$$

By using $h^{\frac{1}{1+q}} \frac{v_\rho}{\rho^{\frac{N}{1+q}}}$ in the definition of $\mu(h)$,

$$\mu(h) \leq h^{\frac{p}{1+q}} \rho^{-\frac{pN}{1+q}} \int_{\Omega} |\nabla v_\rho|^p dx + h^{\frac{1+\lambda}{1+q}} \rho^{-N \frac{1+\lambda}{1+q}} \int_{\Omega} a(x) v_\rho^{1+\lambda} dx.$$

So,

$$(30) \quad \mu(h) \leq h^{\frac{p}{1+q}} \rho^{N-p-\frac{pN}{1+q}} \int_{B_1} |\nabla v|^p dy + h^{\frac{1+\lambda}{1+q}} \rho^{N-N \frac{1+\lambda}{1+q}} \int_{B_1} a(\rho y) v^{1+\lambda}(y) dy.$$

Since $\rho \mapsto \frac{\rho^{\delta_0}}{\omega(\rho)}$ is a nondecreasing function for h small enough,

$$a(\rho y) = \frac{(\rho \|y\|)^{\delta_0}}{\omega(\rho \|y\|)} \leq \frac{\rho^{\delta_0}}{\omega(\rho)},$$

for $y \in B_1$. Therefore,

$$\mu(h) \leq C(p, q, \lambda, v) \left(h^{\frac{p}{1+q}} \rho^{N-p-\frac{pN}{1+q}} + h^{\frac{1+\lambda}{1+q}} \rho^{N-N \frac{1+\lambda}{1+q} + \delta_0} \frac{1}{\omega(\rho)} \right).$$

A deducing can be made :

$$(31) \quad \mu(h) \leq C(p, q, \lambda, v) \left(h^{\frac{p}{1+q}} \rho^{-\frac{N(p-(1+q))}{1+q}-p} + h^{\frac{1+\lambda}{1+q}} \rho^{N \frac{q-\lambda}{1+q} + \delta_0} \frac{1}{\omega(\rho)} \right).$$

We take $\rho = h^{\varepsilon_0}$ for $\varepsilon_0 > 0$. So,

$$\mu(h) \leq C(p, q, \lambda, v) \left(h^{\frac{p}{1+q} - \varepsilon_0 \frac{N(p-(1+q))}{1+q} - \varepsilon_0 p} + h^{\frac{1+\lambda}{1+q} + \varepsilon_0 N \frac{q-\lambda}{1+q} + \varepsilon_0 \delta_0} \frac{1}{\omega(h^{\varepsilon_0})} \right).$$

We choose ε_0 such as $\frac{p}{1+q} - \varepsilon_0 \frac{N(p-(1+q))}{1+q} - \varepsilon_0 p = \frac{\lambda+1}{1+q} + \varepsilon_0 N \frac{q-\lambda}{1+q} + \varepsilon_0 \delta_0$, i.e.,

$$\varepsilon_0 = \frac{p-(1+\lambda)}{N(p-(1+\lambda)) + (\delta_0+p)(1+q)}.$$

Hence,

$$\begin{aligned} & \frac{1+\lambda}{1+q} + \varepsilon_0 N \frac{q-\lambda}{1+q} + \varepsilon_0 \delta_0 \\ &= \frac{(1+\lambda)N(p-(1+\lambda)) + (1+\lambda)(\delta_0+p)(1+q) + N(p-(1+\lambda))(q-\lambda) + \delta_0(1+q)(p-(1+\lambda))}{(1+q)(N(p-(1+\lambda)) + (\delta_0+p)(1+q))} \\ &= \frac{(1+q)N(p-(1+\lambda)) + p(1+\lambda)(1+q) + p(1+q)\delta_0}{(1+q)(N(p-(1+\lambda)) + (\delta_0+p)(1+q))}. \end{aligned}$$

But According to the definition of δ_0 ,

$$p(1+\lambda) + p\delta_0 = (\delta_0+p)(1+q),$$

which leads to

$$\frac{p}{1+q} - \varepsilon_0 \frac{N(p-(1+q))}{1+q} - \varepsilon_0 p = \frac{\lambda+1}{1+q} + \varepsilon_0 N \frac{q-\lambda}{1+q} + \varepsilon_0 \delta_0 = 1.$$

With this ε_0 , we obtain

$$\mu(h) \leq C(p, q, \lambda, v) h \left(1 + \frac{1}{\omega(h^{\varepsilon_0})}\right).$$

If

$$\lim_{\rho \rightarrow 0} \omega(\rho) = 0,$$

then for h small enough,

$$\mu(h) \leq C \frac{h}{\omega(h^{\varepsilon_0})},$$

for some $C > 0$.

If

$$\liminf_{\rho \rightarrow 0} \omega(\rho) > 0,$$

then

$$\mu(h) \leq C h,$$

for some $C > 0$.

Let's go back to (31). We take $\rho = h^{\varepsilon_0} \omega(h)^\alpha$ for $\alpha > 0$. This value of ε_0 implies that

$$\frac{\mu(h)}{h} \leq C(p, q, \lambda, v) \left((\omega(h))^{-\alpha \left(\frac{N(p-(1+q))}{1+q} + p \right)} + (\omega(h))^\alpha \left(N \frac{q-\lambda}{1+q} + \delta_0 \right) \frac{1}{\omega(h^{\varepsilon_0} \omega(h)^\alpha)} \right).$$

Using (12), it exists $h_0 > 0$ such as for all $0 < h \leq h_0$,

$$h^{\varepsilon_0} \omega(h)^\alpha \leq h^{\varepsilon_0 - \alpha \eta}.$$

Moreover, ω is a nonincreasing function hence,

$$\omega(h^{\varepsilon_0} \omega(h)^\alpha) \geq \omega(h^{\varepsilon_0 - \alpha \eta}).$$

It is always possible to assume that $\varepsilon_0 - \alpha\eta \geq \frac{\varepsilon_0}{2}$. So, if h belongs to $(0, \min(h_0, 1))$,

$$h^{\varepsilon_0 - \alpha\eta} \leq h^{\frac{\varepsilon_0}{2}}.$$

Always for the same reason,

$$\omega(h^{\varepsilon_0 - \alpha\eta}) \geq \omega(h^{\frac{\varepsilon_0}{2}}),$$

which leads to

$$\omega(h^{\varepsilon_0} \omega(h)^\alpha) \geq \omega(h^{\frac{\varepsilon_0}{2}}).$$

The previous inequality is true for all $0 < h \leq \min(h_0, 1)$ and for all $\alpha \in \left(0, \frac{\varepsilon_0}{2\eta}\right)$.

So, we use assumption (11) for $\gamma_1 = \frac{\varepsilon_0}{2}$. There exist $C > 0$ and $\gamma_2 > 0$ such as

$$\omega(h^{\frac{\varepsilon_0}{2}}) \geq C (\omega(h))^{\gamma_2}.$$

Therefore,

$$\omega(h^{\varepsilon_0} \omega(h)^\alpha) \geq C (\omega(h))^{\gamma_2}.$$

$$\frac{(\omega(h))^\alpha \left(N \frac{q-\lambda}{1+q} + \delta_0\right)}{\omega(h^{\varepsilon_0} \omega(h)^\alpha)} \leq \frac{1}{C} \frac{1}{(\omega(h))^{\gamma_2 - \alpha \left(N \frac{q-\lambda}{1+q} + \delta_0\right)}}.$$

We can take $\alpha > 0$ small enough such as

$$\gamma_2 - \alpha \left(N \frac{q-\lambda}{1+q} + \delta_0\right) > 0.$$

Finally,

$$\lim_{h \rightarrow 0} \frac{\mu(h)}{h} = 0,$$

since

$$\lim_{\rho \rightarrow 0} \omega(\rho) = 0. \blacksquare$$

Lemma 2 *We assume that $p > 1 + q$, $N > p$, $O \in \Omega$, (8) and (13). Then*

$$\int_0^1 \frac{dh}{\mu(h)} < +\infty.$$

Proof: For all $h > 0$, there is $v_h \in W_0^{1,p}(\Omega)$ such as $\|v_h\|_{L^{1+q}(\Omega)}^{1+q} = h$ and

$$2\mu(h) \geq \int_{\Omega} |\nabla v_h|^p dx + \int_{\Omega} a(x) v_h^{1+\lambda} dx.$$

To simplify, we drop h for $v_h := v$. So,

$$\int_{\Omega} |\nabla v|^p dx \leq \frac{2\mu(h)}{h} \int_{\Omega} v^{1+q} dx - \int_{\Omega} a(x) v^{1+\lambda} dx.$$

Since $p > 1 + q$,

$$\int_{\Omega} |\nabla v|^p dx \leq \int_{\Omega} v^p \left(\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}} - \frac{a(x)}{v^{p-(1+\lambda)}} \right) dx.$$

Sobolev injection and Hölder inequality when $N > p$ give

$$C_S \left(\int_{\Omega} v^{p^*} dx \right)^{\frac{p}{p^*}} \leq \left(\int_{\Omega} v^{p^*} dx \right)^{\frac{p}{p^*}} \left(\int_{\Omega} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}} - \frac{a(x)}{v^{p-(1+\lambda)}} \right]^+ \right)^{\frac{N}{p}} dx \right)^{\frac{p}{N}}.$$

when $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. Hence,

$$\begin{aligned} C_S^{\frac{N}{p}} &\leq \int_{\Omega} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right]^+ \right)^{\frac{N}{p}} dx \\ &= \int_{\{x: a(x) \leq \frac{2\mu(h)}{h} v^{q-\lambda}(x)\}} \left(\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right)^{\frac{N}{p}} dx. \end{aligned}$$

We define for $v > 0$,

$$f(v) = \frac{\beta}{v^{p-(1+q)}} - \frac{\alpha}{v^{p-(1+\lambda)}},$$

when $\alpha, \beta > 0$. By a study of the variations of f , we obtain

$$\max_{v>0} f(v) = f(v_M),$$

when

$$v_M = \left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)} \right)^{\frac{1}{q-\lambda}},$$

and so,

$$\begin{aligned} f(v_M) &= \frac{\beta}{\left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)} \right)^{\frac{p-(1+q)}{q-\lambda}}} - \frac{\alpha}{\left(\frac{\alpha}{\beta} \frac{p-(1+\lambda)}{p-(1+q)} \right)^{\frac{p-(1+\lambda)}{q-\lambda}}}. \\ &= \left[\left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+q)}{q-\lambda}} - \left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+\lambda)}{q-\lambda}} \right] \frac{\beta^{\frac{p-(1+\lambda)}{q-\lambda}}}{\alpha^{\frac{p-(1+q)}{q-\lambda}}}. \end{aligned}$$

Since $\text{meas}\{x : a(x) = 0\} = 0$, we deduce that for almost all $x \in \left\{ x : a(x) \leq \frac{2\mu(h)}{h} v^{q-\lambda}(x) \right\}$,

$$\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \leq \left[\left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+q)}{q-\lambda}} - \left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+\lambda)}{q-\lambda}} \right] \left(\frac{2\mu(h)}{h} \right)^{\frac{p-(1+\lambda)}{q-\lambda}} \frac{1}{a(x)^{\frac{p-(1+q)}{q-\lambda}}}.$$

Moreover, let $\varepsilon > 0$. We have

$$h = \|v\|_{L^{1+q}(\Omega)}^{1+q} = \int_{\Omega} v^{1+q} dx \geq \int_{v^{1+q} \geq \varepsilon} v^{1+q} dx \geq \varepsilon \text{meas}\{x : v^{1+q} \geq \varepsilon\}.$$

By taking $\varepsilon = \varepsilon(h)$, we obtain

$$\frac{h}{\varepsilon(h)} \geq \text{meas}\{x : v^{1+q} \geq \varepsilon(h)\} = \text{meas}\{x : v \geq \varepsilon(h)^{\frac{1}{1+q}}\}.$$

So,

$$C_S^{\frac{N}{p}} \leq \int_{\{x: a(x) \leq \frac{2\mu(h)}{h} v^{q-\lambda}(x)\} \cap \{x: v \geq \varepsilon(h)^{\frac{1}{1+q}}\}} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right]^+ \right)^{\frac{N}{p}} dx$$

$$+ \int_{\{x:a(x) \leq \frac{2\mu(h)}{h} v^{q-\lambda}(x)\} \cap \{x:v < \varepsilon(h)^{\frac{1}{1+q}}\}} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right]^+ \right)^{\frac{N}{p}} dx.$$

It follows that

$$C_S^{\frac{N}{p}} \leq \int_{\{x:v \geq \varepsilon(h)^{\frac{1}{1+q}}\}} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right]^+ \right)^{\frac{N}{p}} dx$$

$$+ \int_{\{x:a(x) \leq \frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\}} \left(\left[\frac{2\mu(h)}{h} \frac{1}{v^{p-(1+q)}(x)} - \frac{a(x)}{v^{p-(1+\lambda)}(x)} \right]^+ \right)^{\frac{N}{p}} dx.$$

Hence,

$$C_S^{\frac{N}{p}} \leq \int_{\{x:v \geq \varepsilon(h)^{\frac{1}{1+q}}\}} \left(\frac{2\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} dx$$

$$+ \left[\left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+q)}{q-\lambda}} - \left(\frac{p-(1+q)}{p-(1+\lambda)} \right)^{\frac{p-(1+\lambda)}{q-\lambda}} \right]^{\frac{N}{p}} \int_{\{x:a(x) \leq \frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\}} \left(\left(\frac{2\mu(h)}{h} \right)^{\frac{p-(1+\lambda)}{q-\lambda}} \frac{1}{a(x)^{\frac{p-(1+q)}{q-\lambda}}} \right)^{\frac{N}{p}} dx.$$

We obtain

$$0 < C = C(\Omega, N, p, q, \lambda) \leq \left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)}$$

$$+ \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{\{x:a(x) \leq \frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\}} \frac{dx}{a(x)^{\frac{N}{\delta_0}}}$$

It leads to

$$0 < C \leq \left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} + \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_{\{x:a(x) \leq \frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\}} \frac{\omega(|x|)^{\frac{N}{\delta_0}}}{|x|^N} dx.$$

We define $\rho > 0$ small enough,

$$\phi : \rho \mapsto \frac{\rho}{\omega(\rho)^{\frac{N}{\delta_0}}} \quad \text{and} \quad \psi : \rho \mapsto \frac{\rho^N}{\omega(\rho)^{\frac{N}{\delta_0}}}.$$

Both functions are continuous, increasing and $\psi(\rho) = \rho^{N-1}\phi(\rho)$. Since the function $a(x) = \psi(|x|)^{\frac{\delta_0}{N}}$ is radial, we use a change of variables in radial coordinates :

$$0 < C \leq \left(\frac{\mu(h)}{h} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} + \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_0^{\psi^{-1}\left(\left(\frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right)^{\frac{N}{\delta_0}}\right)} \frac{1}{\phi(\rho)} d\rho.$$

Using (27), we have

$$\mu(h) \leq C \frac{h}{\omega(h^{\varepsilon_0})},$$

which leads to,

$$0 < C \leq \left(\frac{1}{\omega(h^{\varepsilon_0})} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} + \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_0^{\psi^{-1}\left(\left(\frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}}\right)^{\frac{N}{\delta_0}}\right)} \frac{1}{\phi(\rho)} d\rho.$$

We define

$$\gamma_0 = \frac{p(1+q)}{N(p-(1+q)) + p(1+q)}.$$

We take $\gamma < \gamma_0$ and $\varepsilon(h) = h^\gamma$. So, since $\omega(\rho) \gg \rho^\varepsilon$ for all $\varepsilon > 0$,

$$\left(\frac{1}{\omega(h^{\varepsilon_0})} \frac{1}{\varepsilon(h)^{\frac{p-(1+q)}{1+q}}} \right)^{\frac{N}{p}} \frac{h}{\varepsilon(h)} \rightarrow 0,$$

so,

$$0 < C \leq \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_0^{\psi^{-1} \left(\left(\frac{2\mu(h)}{h} \varepsilon(h)^{\frac{q-\lambda}{1+q}} \right)^{\frac{N}{\delta_0}} \right)} \frac{1}{\phi(\rho)} d\rho.$$

By definition, for $\rho > 0$ small enough,

$$\psi(\rho)^{\frac{1}{N}} = \frac{\rho}{\omega(\rho)^{\frac{1}{\delta_0}}}.$$

We deduce that for $r > 0$ small enough,

$$\psi^{-1}(r) = r^{\frac{1}{N}} \omega(\psi^{-1}(r))^{\frac{1}{\delta_0}} \leq r^{\frac{1}{N}} \omega_0^{\frac{1}{\delta_0}}.$$

Hence,

$$0 < C \leq \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} \int_0^{K h^{\frac{\gamma(q-\lambda)}{(1+q)\delta_0}}} \frac{1}{\phi(\rho)} d\rho,$$

for another $\gamma \in (0, \gamma_0)$, $K > 0$ and $h > 0$ small enough. For

$$\omega(\rho) = \frac{1}{(-\ln \rho)^\alpha},$$

we must have $\alpha > \frac{\delta_0}{N}$ to insure that

$$\int_0^1 \frac{\omega(\rho)^{\frac{N}{\delta_0}}}{\rho} d\rho < +\infty.$$

So, for $C > 0$ and $h > 0$ small enough,

$$0 < C \leq \left(\frac{\mu(h)}{h} \right)^{\frac{N(p-(1+\lambda))}{p(q-\lambda)}} (-\ln h)^{\frac{\delta_0 - \alpha N}{\delta_0}}.$$

Hence,

$$\frac{C}{\mu(h)} \leq \frac{1}{h (-\ln h)^{\frac{\alpha N - \delta_0}{\delta_0} \frac{p(q-\lambda)}{N(p-(1+\lambda))}}} = \frac{1}{h (-\ln h)^{\frac{(\alpha N - \delta_0)(p-(1+q))}{N(p-(1+\lambda))}}}.$$

For

$$\alpha > \frac{p-(1+\lambda)}{p-(1+q)} + \frac{\delta_0}{N},$$

we obtain

$$\int_0^1 \frac{dh}{\mu(h)} < +\infty. \blacksquare$$

Lemma 3 We assume that $p = 1 + q = 2$, $O \in \Omega$ and (14). Then there are $u_0 \in W_0^{1,2}(\Omega)$ and $C > 0$ such as for all $r > 0$ small enough,

$$T(u_0) \geq C \omega(r).$$

Proof: The following proof is a simpler version of [4]. Let's take $\alpha > 1$, $r > 0$ and $u_0 \in W_0^{1,2}(\Omega)$ such as $\|u_0\|_{L^\infty(\Omega)} = 1$ and $u_0 = 1$ on a neighbourhood of O .

We define φ_r as the first function for $-\Delta$ on the ball of centre 0 and radius r for the Dirichlet Boundary Condition.

We normalise it to 1, i.e., $\|\varphi_r\|_{L^2(\Omega)} = 1$. The first eigenvalue is denoted by $\lambda_{1,r}$.

The function φ_r^α and its normal derivative is equal to zero on the sphere of radius r . So,

$$\frac{d}{dt} \int_{\Omega} u(x, t) \varphi_r^\alpha(x) dx - \int_{\Omega} u(x, t) \Delta(\varphi_r^\alpha)(x) dx + \int_{\Omega} a(x) u^\lambda(x, t) \varphi_r^\alpha(x) dx = 0.$$

But

$$\Delta(\varphi_r^\alpha) = \alpha(\alpha - 1)\varphi_r^{\alpha-2}|\nabla\varphi_r|^2 + \alpha\varphi_r^{\alpha-1}\Delta\varphi_r = \alpha(\alpha - 1)\varphi_r^{\alpha-2}|\nabla\varphi_r|^2 - \alpha\lambda_{1,r}\varphi_r^\alpha.$$

Therefore,

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u(x, t) \varphi_r^\alpha(x) dx + \alpha\lambda_{1,r} \int_{\Omega} u(x, t) \varphi_r^\alpha(x) dx + \int_{\Omega} a(x) u^\lambda(x, t) \varphi_r^\alpha(x) dx \\ &= \alpha(\alpha - 1) \int_{\Omega} u(x, t) \varphi_r^{\alpha-2}(x) |\nabla\varphi_r|^2(x) dx \geq 0. \end{aligned}$$

Using Hölder's inequality, we have

$$\int_{\Omega} a(x) u^\lambda(x, t) \varphi_r^\alpha(x) dx \leq \left(\int_{\Omega} u(x, t) \varphi_r^\alpha(x) dx \right)^\lambda \left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_r^\alpha(x) dx \right)^{1-\lambda}.$$

We define

$$y(t) = \int_{\Omega} u(x, t) \varphi_r^\alpha(x) dx,$$

and $\lambda_{1,1} = \lambda_1$ the first eigenvalue of the unit ball. Hence,

$$y'(t) + \frac{\alpha\lambda_1}{r^2}y(t) + \left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_r^\alpha(x) dx \right)^{1-\lambda} y(t)^\lambda \geq 0.$$

Since $r \mapsto a(r)$ is a non decreasing function,

$$\begin{aligned} & \left(\int_{\Omega} a(x)^{\frac{1}{1-\lambda}} \varphi_r^\alpha(x) dx \right)^{1-\lambda} = \left(\int_{B_r} a(x)^{\frac{1}{1-\lambda}} \varphi_r^\alpha(x) dx \right)^{1-\lambda} \leq a(r) \left(\int_{B_r} \varphi_r^\alpha(x) dx \right)^{1-\lambda} \\ &= a(r) r^{N(1-\lambda)} \left(\int_{B_1} \varphi_1^\alpha(x) dx \right)^{1-\lambda}. \end{aligned}$$

We define

$$C_1 = \left(\int_{B_1} \varphi_1^\alpha(x) dx \right)^{1-\lambda}.$$

Finally,

$$y'(t) + \frac{\alpha\lambda_1}{r^2}y(t) + C a(r) r^{N(1-\lambda)} y(t)^\lambda \geq 0.$$

We define

$$z(t) = y(t)^{1-\lambda}.$$

So,

$$\frac{z'(t)}{1-\lambda} + \frac{\alpha\lambda_1}{r^2}z(t) + C_1 a(r) r^{N(1-\lambda)} \geq 0.$$

Hence,

$$z'(t) + \frac{\alpha(1-\lambda)\lambda_1}{r^2}z(t) + C_1(1-\lambda) a(r) r^{N(1-\lambda)} \geq 0.$$

$$\frac{d}{dt} \left(z(t) \exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right) \right) + C_1(1-\lambda) a(r) r^{N(1-\lambda)} \exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right) \geq 0.$$

We define

$$\zeta(t) = z(t) \exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right).$$

So, with this new notation,

$$\zeta'(t) + C_1(1-\lambda) a(r) r^{N(1-\lambda)} \exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right) \geq 0.$$

We integrate between 0 and $t > 0$.

$$\zeta(t) - \zeta(0) + C_1 \frac{a(r) r^{N(1-\lambda)+2}}{\alpha\lambda_1} \left(\exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right) - 1 \right) \geq 0.$$

So, $\zeta(0) = C_1$ implies that

$$\zeta(t) \geq C_1 \left(1 - \frac{a(r) r^{N(1-\lambda)+2}}{\alpha\lambda_1} \left(\exp \left(\frac{\alpha(1-\lambda)\lambda_1}{r^2} t \right) - 1 \right) \right).$$

We have $\zeta(t) > 0$ for all $t \in [0, T_0(r)]$ when

$$T_0(r) = \frac{r^2}{\alpha(1-\lambda)\lambda_1} \ln \left(1 + \frac{\alpha\lambda_1}{a(r) r^{N(1-\lambda)+2}} \right).$$

Hence, for $r > 0$ small enough,

$$T_0(r) \geq C \omega(r)$$

Now, it is clear that if

$$\lim_{r \rightarrow 0} \omega(r) = +\infty,$$

then

$$T(u_0) \geq \lim_{r \rightarrow 0} T_0(r) = +\infty,$$

which implies that

$$T^+(h) = +\infty,$$

for all $h > 0$. ■

Lemma 4 *We assume that $p = 1 + q = 2$, $O \in \Omega$, (14) and $(A_1) - (A_5)$. If*

$$\int_0^c \frac{\omega(r)}{r} dr < +\infty,$$

then

$$\int_0^1 \frac{dh}{\mu(h)} < +\infty.$$

Proof: We start with (30) for $p = 1 + q$ so, for all $\rho > 0$,

$$\mu(h) \leq h\rho^{-p} \int_{B_1} |\nabla v|^p dy + h^{\frac{1+\lambda}{p}} \rho^{N-N\frac{1+\lambda}{p}} \int_{B_1} a(\rho y) v^{1+\lambda}(y) dy.$$

We estimate $a(\rho y)$ by $\exp\left(\frac{-\omega(\rho)}{\rho^p}\right)$, so for h small enough,

$$(32) \quad \mu(h) \leq C(p, \lambda, v) \left(h\rho^{-p} + h^{\frac{1+\lambda}{p}} \rho^{N-N\frac{1+\lambda}{p}} \exp\left(\frac{-\omega(\rho)}{\rho^p}\right) \right).$$

Using assumption (A_5) , for $h > 0$ small enough, there is a function $\rho(h)$ such as

$$h = \exp\left(\frac{-\omega(\rho(h))}{\rho^p(h)}\right).$$

Hence,

$$\ln h = \frac{-\omega(\rho(h))}{\rho^p(h)} \geq \frac{-\omega_0}{\rho^p(h)},$$

which gives

$$\rho(h) \leq \left(\frac{\omega_0}{-\ln h}\right)^{\frac{1}{p}}.$$

Moreover,

$$\ln h = \frac{-\omega(\rho(h))}{\rho^p(h)} \leq -\frac{1}{\rho^\delta(h)}.$$

So,

$$\rho(h) \geq \left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}.$$

We have

$$\rho^{-p}(h) = \frac{-\ln h}{\omega(\rho(h))} \leq \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)}.$$

Now,

$$\mu(h) \leq C(p, \lambda, v) \left(h \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)} + h^{\frac{1+\lambda}{p}} \left(\frac{\omega_0}{-\ln h}\right)^{\frac{N}{p}-N\frac{1+\lambda}{p^2}} h \right).$$

Clearly, for $h > 0$ small enough,

$$(33) \quad \mu(h) \leq C h \frac{-\ln h}{\omega\left(\left(\frac{1}{-\ln h}\right)^{\frac{1}{\delta}}\right)}.$$

Now after an upper bound, the lower bound. For all $h > 0$, there is $v_h \in W_0^{1,p}(\Omega)$ such as $\|v\|_{L^{1+q}(\Omega)}^{1+q} = h$ and

$$2\mu(h) \geq \int_{\Omega} |\nabla v|^p dx + \int_{\Omega} a(x) v^{1+\lambda} dx.$$

To simplify, we drop h for $v_h := v$. So,

$$\int_{\Omega} |\nabla v|^p dx \leq \frac{2\mu(h)}{h} \int_{\Omega} v^{1+q} dx - \int_{\Omega} a(x)v^{1+\lambda} dx.$$

Since $p = 1 + q$,

$$\int_{\Omega} |\nabla v|^p dx \leq \int_{\Omega} v^p \left(\frac{2\mu(h)}{h} - \frac{a(x)}{v^{p-(1+\lambda)}} \right) dx.$$

Sobolev injection and Hölder inequality when $N > p$ give

$$C_S \left(\int_{\Omega} v^{p^*} dx \right)^{\frac{p}{p^*}} \leq \left(\int_{\Omega} v^{p^*} dx \right)^{\frac{p}{p^*}} \left(\int_{\Omega} \left(\left[\frac{2\mu(h)}{h} - \frac{a(x)}{v^{p-(1+\lambda)}} \right]^+ \right)^{\frac{N}{p}} dx \right)^{\frac{p}{N}}.$$

when $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. Hence,

$$C_S^{\frac{N}{p}} \leq \int_{\Omega} \left(\left[\frac{2\mu(h)}{h} - \frac{a(x)}{v^{p-(1+\lambda)}} \right]^+ \right)^{\frac{N}{p}} dx.$$

Therefore,

$$C_S^{\frac{N}{p}} \leq \left(\frac{2\mu(h)}{h} \right)^{\frac{N}{p}} \text{meas} \left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\}.$$

Let $\varepsilon > 0$. We have

$$h = \|v\|_{L^{1+q}(\Omega)}^{1+q} = \int_{\Omega} v^{1+q} dx \geq \int_{v^{1+q} \geq \varepsilon} v^{1+q} dx \geq \varepsilon \text{meas}\{x : v^{1+q} \geq \varepsilon\}.$$

By taking $\varepsilon = h^\gamma$ with $0 < \gamma < 1$, we obtain

$$h^{1-\gamma} \geq \text{meas}\{x : v^{1+q} \geq h^\gamma\} = \text{meas}\{x : v^{p-(1+\lambda)} \geq h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\}.$$

So,

$$\begin{aligned} \text{meas} \left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} &= \text{meas} \left(\left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} \cap \left\{ x : v^{p-(1+\lambda)} \geq h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right\} \right) \\ &+ \text{meas} \left(\left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} \cap \left\{ x : v^{p-(1+\lambda)} < h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right\} \right). \end{aligned}$$

It follows that

$$\text{meas} \left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} \leq h^{1-\gamma} + \text{meas} \left(\left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} \cap \left\{ x : v^{p-(1+\lambda)} < h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right\} \right).$$

We obtain

$$\text{meas} \left\{ x : \frac{2\mu(h)}{h} \geq \frac{a(x)}{v^{p-(1+\lambda)}} \right\} \leq h^{1-\gamma} + \text{meas} \left\{ x : \frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \geq a(x) \right\}.$$

Now, from (33), for $h > 0$ small enough,

$$\mu(h) \leq C h \frac{-\ln h}{\omega \left(\left(\frac{1}{-\ln h} \right)^{\frac{1}{\delta}} \right)}.$$

By (A₄), $\omega(r) \geq r^{p-\delta}$ for $r > 0$ small enough. It leads to

$$(34) \quad \mu(h) \leq C h \frac{-\ln h}{\left(\left(\frac{1}{-\ln h}\right)^{\frac{p-\delta}{\delta}}\right)} = C h (-\ln h)^{\frac{p}{\delta}}.$$

There are two consequences :

$$\left(\frac{2\mu(h)}{h}\right)^{\frac{N}{p}} h^{1-\gamma} \rightarrow 0,$$

and

$$\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \rightarrow 0,$$

when $h \rightarrow 0$. We deduce that there is $C > 0$ such as for $h > 0$ small enough,

$$C \leq \left(\frac{2\mu(h)}{h}\right)^{\frac{N}{p}} \text{meas} \left\{ x : \frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \geq \exp\left(-\frac{\omega(|x|)}{|x|^p}\right) \right\}.$$

So,

$$C \leq \left(\frac{2\mu(h)}{h}\right)^{\frac{N}{p}} \text{meas} \left\{ x : \ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq -\frac{\omega(|x|)}{|x|^p} \right\}.$$

We take x such as $\ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq -\frac{\omega(|x|)}{|x|^p}$. Since ω is bounded, it satisfies

$$\ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq -\frac{\omega_0}{|x|^p},$$

i.e.,

$$|x| \leq \left(\frac{\omega_0}{-\ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)} \right)^{\frac{1}{p}}.$$

By monotonicity of ω (assumption (A₁)),

$$\omega(|x|) \leq \omega\left(\left(\frac{\omega_0}{-\ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right).$$

Clearly,

$$C \leq \left(\frac{2\mu(h)}{h}\right)^{\frac{N}{p}} \text{meas} \left\{ x : \ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right) \geq -\frac{\omega\left(\left(\frac{\omega_0}{-\ln\left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}}\right)}\right)^{\frac{1}{p}}\right)}{|x|^p} \right\}.$$

Consequently,

$$C \leq \left(\frac{2\mu(h)}{h} \right)^{\frac{N}{p}} \text{meas} \left\{ x : |x|^p \leq \frac{\omega \left(\left(\frac{\omega_0}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)} \right)^{\frac{1}{p}} \right)}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)} \right\}.$$

Hence,

$$C \leq \left(\frac{2\mu(h)}{h} \right)^{\frac{N}{p}} \text{meas} \left\{ x : |x|^N \leq \left(\frac{\omega \left(\left(\frac{\omega_0}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)} \right)^{\frac{1}{p}} \right)}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)} \right)^{\frac{N}{p}} \right\}.$$

It leads to

$$C \leq \left(\frac{2\mu(h)}{h} \right) \frac{\omega \left(\left(\frac{\omega_0}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)} \right)^{\frac{1}{p}} \right)}{-\ln \left(\frac{2\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \right)},$$

for another $C > 0$. Now, by (34), for $h > 0$ small enough,

$$\frac{\mu(h)}{h} h^{\frac{\gamma(p-(1+\lambda))}{1+q}} \leq h^\varepsilon,$$

for some $\varepsilon > 0$. We obtain

$$C \leq \left(\frac{\mu(h)}{h} \right) \frac{\omega \left(\left(\frac{K}{-\ln h} \right)^{\frac{1}{p}} \right)}{-\ln h},$$

for $C, K > 0$. We end by using a change of variables. ■

3 Proof of Theorem 1.1 and Theorem 1.2

We use Lemma 1 (resp. Lemma 3) to prove Proposition 1. By Theorem 2.1 and Lemmas 1 and 2 (resp. Lemmas 3 and 4), we obtain the conclusion. ■

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