



**HAL**  
open science

# Generalized Kähler-Einstein metric along Q-Fano fibration

Hassan Jolany

► **To cite this version:**

| Hassan Jolany. Generalized Kähler-Einstein metric along Q-Fano fibration. 2017. hal-01413759v3

**HAL Id: hal-01413759**

**<https://hal.science/hal-01413759v3>**

Preprint submitted on 16 Sep 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Generalized Kähler-Einstein metric along $\mathbb{Q}$ -Fano fibration

Hassan Jolany

University of Lille1, France

Correspondence to be sent to: hassan.jolany@math.univ-lille1.fr

In this paper, we show that along  $\mathbb{Q}$ -Fano fibration, when general fibres, base and central fiber (with at worst Kawamata log terminal singularities) are K-poly stable then there exists a relative Kähler-Einstein metric. We introduce the fiberwise Kähler-Einstein foliation and we mention that the main difficulty to obtain higher estimates is to solve relative CMA equation along such foliation. We propose a program such that for finding a pair of canonical metric  $(\omega_X, \omega_B)$ , which satisfies in  $Ric(\omega_X) = \pi^*\omega_B + \pi^*(\omega_{WP}) + [N]$  on K-poly stable degeneration  $\pi : X \rightarrow B$ , where  $Ric(\omega_B) = \omega_B$ , we need to have Canonical bundle formula.

## 1 Introduction

Let  $(X, L)$  be a polarized projective variety. Given an ample line bundle  $L \rightarrow X$ , then a test configuration for the pair  $(X, L)$  consists of:

- a scheme  $\mathfrak{X}$  with a  $\mathbb{C}^*$ -action
- a flat  $\mathbb{C}^*$ -equivariant map  $\pi : \mathfrak{X} \rightarrow \mathbb{C}$  with fibres  $X_t$ ;
- an equivariant line bundle  $\mathfrak{L} \rightarrow \mathfrak{X}$ , ample on all fibres;
- for some  $r > 0$ , an isomorphism of the pair  $(\mathfrak{X}_1, \mathfrak{L}_1)$  with the original pair  $(X, L^r)$ .

Let  $U_k = H^0(\mathfrak{X}_0, \mathfrak{L}_0^k|_{\mathfrak{X}_1})$  be vector spaces with  $\mathbb{C}^*$ -action, and let  $A_k : U_k \rightarrow U_k$  be the endomorphisms generating those actions. Then

$$\dim U_k = a_0 k^n + a_1 k^{n-1} + \dots$$

$$\text{Tr}(A_k) = b_0 k^{n+1} + b_1 k^n + \dots$$

Then the Donaldson-Futaki invariant of a test configuration  $(\mathfrak{X}, \mathfrak{L})$  is

$$Fut(\mathfrak{X}, \mathfrak{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0}.$$

A Fano variety  $X$  is K-stable (respectively K-poly stable) if  $Fut(\mathfrak{X}, \mathfrak{L}) \geq 0$  for all normal test configurations  $\mathfrak{X}$  such that  $X_0 \neq X$  and equality holds if  $(\mathfrak{X}, \mathfrak{L})$  is trivial (respectively,  $\mathfrak{X} = X \times \mathbb{A}^1$ ). See [5, 6, 7, 8, 9] and references therein.

Let  $X$  be a projective variety with canonical line bundle  $K_X \rightarrow X$  of Kodaira dimension

$$\kappa(X) = \limsup \frac{\log \dim H^0(X, K_X^{\otimes \ell})}{\log \ell}$$

This can be shown to coincide with the maximal complex dimension of the image of  $X$  under pluri-canonical maps to complex projective space, so that  $\kappa(X) \in \{-\infty, 0, 1, \dots, m\}$ .

A Kähler current  $\omega$  is called a conical Kähler metric (or Hilbert Modular type) with angle  $2\pi\beta$ , ( $0 < \beta < 1$ ) along the divisor  $D$ , if  $\omega$  is smooth away from  $D$  and asymptotically equivalent along  $D$  to the model conic metric

$$\omega_\beta = \sqrt{-1} \left( \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right)$$

here  $(z_1, z_2, \dots, z_n)$  are local holomorphic coordinates and  $D = \{z_1 = 0\}$  locally. See[18]

For the log-Calabi-Yau fibration  $f : (X, D) \rightarrow Y$ , such that  $(X_t, D_t)$  are log Calabi-Yau varieties. If  $(X, \omega)$  be a Kähler variety with Poincar singularities then the semi-Ricci flat metric has  $\omega_{SRF}|_{X_t}$  is quasi-isometric with the following model which we call it **fibrewise Poincar singularities**.

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2 (\log |z_k|^2)^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left( \sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

We can define the same **fibrewise conical singularities**. and the semi-Ricci flat metric has  $\omega_{SRF}|_{X_t}$  is quasi-isometric with the following model

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left( \sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

In fact the previous remark will tell us that the semi Ricci flat metric  $\omega_{SRF}$  has pole singularities.

A pair  $(X, D)$  is log  $\mathbb{Q}$ -Fano if the  $\mathbb{Q}$ -Cartier divisor  $-(K_X + D)$  is ample. For a klt pair  $(X, D)$  with  $\kappa(K_X + D) = -\infty$ , according to the log minimal model program, there exists a birational map  $\phi : X \rightarrow Y$  and a morphism  $Y \rightarrow Z$  such that for  $D' = \phi_* D$ , the pair  $(Y_z, D'_z)$  is log  $\mathbb{Q}$ -Fano with Picard number  $\rho(Y_z) = 1$  for general  $z \in Z$ . In particular, log  $\mathbb{Q}$ -Fano pairs are the building blocks for pairs with negative Kodaira dimension.

Now, we consider fibrations with Kähler Einstein metrics with positive Ricci curvature and CM-line bundle. Let  $\pi : (X, D) \rightarrow B$  be a holomorphic submersion between compact Kähler manifolds of any dimensions, whose log fibers  $(X_s, D_s)$  and base have no no-zero holomorphic holomorphic vector fields and whose log fibers admit conical Kähler Einstein metrics with positive Ricci curvature on  $(X, D)$ . We give a sufficient topological condition that involves the CM-line bundle on  $B$ .

Let  $B$  be a normal variety such that  $K_B$  is  $\mathbb{Q}$ -Cartier, and  $f : X \rightarrow B$  a resolution of singularities. Then,

$$K_X = f^*(K_B) + \sum_i a_i E_i$$

where  $a_i \in \mathbb{Q}$  and the  $E_i$  are the irreducible exceptional divisors. Then the singularities of  $B$  are terminal, canonical, log terminal or log canonical if  $a_i > 0, \geq 0, > -1$  or  $\geq -1$ , respectively.

Firstly, we introduce the log CM-line bundle. Let  $\pi : (X, D) \rightarrow B$  be a holomorphic submersion between compact Kähler manifolds and let  $L + D$  be a relatively ample line bundle. Let  $\mathcal{Y} = (X_s, D_s)$  be a log fiber  $n = \dim X_s$  and let  $\eta$  denote the constant

$$\eta = \frac{nc_1(\mathcal{Y})c_1(L + D)|_{\mathcal{Y}}^{n-1}}{c_1(L + D)|_{\mathcal{Y}}^n}$$

Let  $K_{X/B}$  denote the relative canonical bundle. The log CM-line bundle is the virtual bundle and introduced by Tian [39] as follows

$$\mathcal{L}_{CM}^D = n + 1 \left( (K_{X/B} + D)^* - (K_{X/B} + D) \right) \otimes ((L + D) - (L + D)^*)^n - \eta \left( (L + D) - (L + D)^* \right)^{n+1}$$

For finding the first Chern class of log CM-line bundle, we need to the following Grothendieck-Riemann-Roch theorem.

Let  $X$  be a smooth quasi-projective scheme over a field and  $K_0(X)$  be the Grothendieck group of bounded complexes of coherent sheaves. Consider the Chern character as a functorial transformation

$$\text{ch} : K_0(X) \rightarrow A(X, \mathbb{Q})$$

, where  $A_d(X, \mathbb{Q})$  is the Chow group of cycles on  $X$  of dimension  $d$  modulo rational equivalence, tensored with the rational numbers. Now consider a proper morphism

$$f : X \rightarrow Y$$

between smooth quasi-projective schemes and a bounded complex of sheaves  $\mathcal{F}^\bullet$  on  $X$ . Let  $R$  be the right derived functor, then we have the following theorem of Grothendieck-Riemann-Roch.

**Theorem 1.1.** The Grothendieck-Riemann-Roch theorem relates the pushforward map

$$f_! = \sum (-1)^i R^i f_* : K_0(X) \rightarrow K_0(Y)$$

and the pushforward  $f_* : A(X) \rightarrow A(Y)$  by the formula

$$\text{ch}(f_! \mathcal{F}^\bullet) \text{td}(Y) = f_*(\text{ch}(\mathcal{F}^\bullet) \text{td}(X))$$

Here  $\text{td}(X)$  is the Todd genus of (the tangent bundle of)  $X$ . In fact, since the Todd genus is functorial and multiplicative in exact sequences, we can rewrite the Grothendieck-Riemann-Roch formula as

$$\text{ch}(f_! \mathcal{F}^\bullet) = f_*(\text{ch}(\mathcal{F}^\bullet) \text{td}(T_f))$$

where  $T_f$  is the relative tangent sheaf of  $f$ , defined as the element  $TX - f^*TY$  in  $K^0(X)$ . For example, when  $f$  is a smooth morphism,  $T_f$  is simply a vector bundle, known as the tangent bundle along the fibers of  $f$ .  $\square$

So by applying Grothendieck-Riemann-Roch theorem, the first Chern class of log CM-line bundle is

$$c_1(\mathcal{L}_{CM}) = 2^{n+1} \pi_* \left[ ((n+1)c_1(K_{X/B} + D) + \eta c_1(L + D)) c_1(L + D)^n \right]$$

We need to introduce log Weil-Petersson metric on the moduli space of conical Kähler-Einstein Fano varieties. We can introduce it by applying Deligne pairing. First we recall Deligne pairing here.[48]

Let  $\pi : X \rightarrow S$  be a projective morphism. Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let  $\det R\pi_* \mathcal{F}$  be the line bundle on  $S$ . If  $R^i \pi_* \mathcal{F}$  is locally free for any  $i$ , then

$$\det R\pi_* \mathcal{F} = \bigotimes_i (\det R^i \pi_* \mathcal{F})^{(-1)^i}$$

For any flat morphism  $\pi : X \rightarrow S$  of relative dimension one between normal integral schemes  $X$  and  $S$ , we denote the Deligne pairing by

$$\langle -, - \rangle : \text{Pic}(X) \times \text{Pic}(X) \rightarrow \text{Pic}(S)$$

In this case, the Deligne's pairing can be defined as follows. Let  $L_1, L_2 \in \text{Pic}(X)$ , then

$$\langle L_1, L_2 \rangle : \det R\pi_*(L_1 \otimes L_2) \otimes \det R\pi_*(L_1^{-1}) \otimes R\pi_*(L_2^{-1}) \otimes R\pi_* \mathcal{O}_X$$

In fact, Deligne pairings provide a method to construct a line bundle over a base  $S$  from line bundles over a fiber space  $X$ .

Suppose  $\pi : X \rightarrow S$  is a flat morphism of schemes of relative dimension  $n$ , i.e.,  $n = \dim X_s$ , and suppose  $L_i$  are line bundles over the fiber space  $X$ . Then the Deligne pairing  $\langle L_0, \dots, L_n \rangle_{X/S}$  is a line bundle over  $S$ . The sections are given formally by  $\langle s_0, \dots, s_n \rangle$ , where each  $s_i$  is a rational section of  $L_i$  and whose intersection of divisors is empty.

The transition functions between sections are defined

$$\langle s_0, \dots, f_j s_j, \dots, s_n \rangle = \mathcal{N}_f [\cap_{i \neq j} \text{div}(s_i)] \langle s_0, \dots, s_j, \dots, s_n \rangle$$

where  $X_p \cap_{i \neq j} \text{div}(s_i) = \sum n_k p_k$ , the formal sum of zeros and poles,  $\mathcal{N}_f [\cap_{i \neq j} \text{div}(s_i)]$  is the product  $\prod f(p_k)^{n_k}$ , where the  $p_i$  are the zeros and poles in the common intersection, and  $n_k$  the multiplicities.

Let  $\pi : (X, D) \rightarrow S$  be a projective family of canonically polarized varieties. Equip the relative canonical bundle  $K_{X/S} + D$  with the hermitian metric that is induced by the fiberwise Kähler-Einstein metrics. The log Weil-Petersson form is equal, up to a numerical factor, to the fiber integral

$$\omega_{WP}^D = \int_{X_s \setminus D_s} c_1(K_{X'/S})^{n+1} = \left( \int_{X_s \setminus D_s} |A|_{\omega_s}^2 \right) ds \wedge d\bar{s}$$

$A$  represents the Kodaira-Spencer class of the deformation. See [20]

In fact if we take  $\pi : (X, D) \rightarrow S$  be a projective family of canonically polarized varieties. Since, the Chern form of the metric on  $\langle L_0, \dots, L_n \rangle$  equals the fiber integral [20],

$$\int_{X/S} c_1(L_0, h_0) \wedge \dots \wedge c_1(L_n, h_n)$$

The curvature of the metric on the Deligne pairing  $\langle K_{X/S} + D, \dots, K_{X/S} + D \rangle$  given by the fiberwise Kähler-Einstein metric coincides with the generalized log Weil-Petersson form  $\omega_{WP}^D$  on  $S$ .

Take

$$\alpha = \frac{c_1(\mathcal{L}_{CM}^D)}{2^{n+1}(n+1)\pi_*(c_1(L+D)^n)} \in H^{1,1}(S)$$

In Song-Tian program when the log-Kodaira dimension  $\kappa(X, D)$  is  $-\infty$  then we don't have canonical model and by using Mori fibre space we can find canonical metric (with additional K-poly stability assumption on fibres), the following theorem give an answer for finding canonical metric by using Song-Tian program in the case when the log-Kodaira dimension  $\kappa(X, D)$  is  $-\infty$ . We can introduce the logarithmic Weil-Petersson metric on moduli space of log Fano varieties.

Let  $\mathcal{M}^n$  be the space of all  $n$ -dimensional Kähler-Einstein Fano manifolds, normalized so that the Kähler form  $\omega$  is in the class  $2\pi c_1(X)$ , modulo biholomorphic isometries. It is pre-compact in the Gromov-Hausdorff topology, see [40]. Donaldson and Sun [23], introduced the refined Gromov-Hausdorff compactification  $\overline{\mathcal{M}}^n$  of  $\mathcal{M}^n$  such that every point in the boundary  $\overline{\mathcal{M}}^n \setminus \mathcal{M}^n$  is naturally a  $\mathbb{Q}$ -Gorenstein smoothable  $\mathbb{Q}$ -Fano variety which admits a Kähler-Einstein metric. Recently a proper algebraic compactification  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  was constructed by Odaka, Chi Li, et al. [28],[30] for proving the projectivity of  $\overline{\mathcal{M}}$ .

There is a belief due to Tian saying that a moduli space with canonical metric is likely to be quasi-projective. Let  $W \subset \mathbb{C}^n$  be a domain, and  $\Theta$  a positive current of degree  $(q, q)$  on  $W$ . For a point  $p \in W$  one defines

$$\mathfrak{v}(\Theta, p, r) = \frac{1}{r^{2(n-q)}} \int_{|z-p|<r} \Theta(z) \wedge (dd^c|z|^2)^{n-q}$$

The Lelong number of  $\Theta$  at  $p$  is defined as

$$\mathfrak{v}(\Theta, p) = \lim_{r \rightarrow 0} \mathfrak{v}(\Theta, p, r)$$

Let  $\Theta$  be the curvature of singular hermitian metric  $h = e^{-u}$ , one has

$$\mathfrak{v}(\Theta, p) = \sup\{\lambda \geq 0 : u \leq \lambda \log(|z-p|^2) + O(1)\}$$

see [16]

We set  $K_{X/Y} = K_X \otimes \pi^* K_Y^{-1}$  and call it the relative canonical bundle of  $\pi : X \rightarrow Y$

**Definition 1.2.** Let  $X$  be a Kähler variety with  $\kappa(X) > 0$  then the **relative Kähler-Einstein metric** is defined as follows

$$Ric_{X/Y}^{h_{X/Y}^\omega}(\omega) = -\Phi\omega$$

where

$$Ric_{X/Y}^{h_{X/Y}^\omega}(\omega) = \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\omega^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m} \right)$$

and  $\omega_{can}$  is a canonical metric on  $Y = X_{can}$ . Where  $\Phi$  is the fiberwise constant function.

$$Ric_{X/Y}^{h_{X/Y}^{\omega_{SRF}}}(\omega) = \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m} \right) = \omega_{WP}$$

here  $\omega_{WP}$  is a Weil-Petersson metric.

Note that if  $\kappa(X) = -\infty$  then along Mori fibre space  $f : X \rightarrow Y$  we can define Relative Kähler-Einstein metric as

$$Ric_{X/Y}^{h_{X/Y}^\omega}(\omega) = \Phi\omega$$

when fibers and base are K-poly-stable, and we have fiberwise KE-stability □

**Remark 1:** From [28], by the same method we can show that, the log CM line bundle  $L_{CM}^D$  descends to a line bundle  $\Lambda_{CM}^D$  on the proper moduli space  $\overline{\mathcal{M}}^D$ . Moreover, there is a canonically defined continuous Hermitian metric  $h_{DP}^D$ , called as Deligne Hermitian metric on  $\Lambda_{CM}^D$  whose curvature form is a positive current  $\omega_{WP}^D$  and called logarithmic Weil-Petersson metric on compactified moduli space of Kähler-Einstein log Fano manifolds and it can be extended to canonical Weil-Petersson current  $\omega_{WP}^D$  on  $\mathcal{M}^D$ . Viehweg [29], showed that the moduli space of polarized manifolds with nef canonical line bundle is quasi-projective and by the same method of Chi Li [28],  $\Lambda_{CM}^D$  on the proper moduli space  $\overline{\mathcal{M}}^D$  is nef and big. Hence  $\mathcal{M}^D$  is quasi-projective. We can prove it by using Lelong number method also which is more simpler.

In fact, if we prove that the Lelong number of singular hermitian metric  $(L_{CM}, h_{WP})$  corresponding to Weil-Petersson current on virtual CM-bundle has vanishing Lelong number, then  $L_{CM}$  is nef and if we know the nefness of  $L_{CM}$  then due to a result of E.Viehweg, we can prove that the moduli space of Kahler-Einstein Fano manifolds  $M$  is quasi-projective and its compactification  $\bar{M}$  is projective. From Demailly's theorem, We know if  $(L, h)$  be a positive, singular hermitian line bundle, whose Lelong numbers vanish everywhere. Then  $L$  is nef. Gang Tian recently proved that, the CM-bundle  $L_{CM}$  is positive, So by the same theorem 2 and Proposition 6, of the paper, [42] we can show that the Weil-Petersson current on moduli space of Fano Kahler-Einstein varieties has zero Lelong number (in smooth case it is easy, for singular case (for log terminal singularities) we just need to show that the diameter of fibers are bounded).

Moreover, by using the same method of Theorem 2, in [42], hermitian metric corresponding to  $L_{CM}^D$  has vanishing Lelong number, and since from the recent result of Tian,  $L_{CM}^D$  is positive, hence  $L_{CM}^D$  is nef. So by the theorem of Viehweg [29],  $\mathcal{M}^D$  is quasi-projective. In fact this give a new analytical method to the recent result of [28].

We have

$$\omega_{WP}^D := -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h_{DP}$$

where locally  $h_{DP} = e^{-\Psi}$  and hence

$$\omega_{WP}^D := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \Psi$$

which  $\Psi$  called Deligne potential.

Moreover, we have

$$\omega_{WP} = Ric \left( \frac{\omega_{SKE}^k \wedge \|v^*\|_{(i^*h_{FS})^{\frac{1}{m}}}^{2/m} (v \wedge \bar{v})^{1/m}}{\|v^*\|_{(i^*h_{FS})^{\frac{1}{m}}}^{2/m} (v \wedge \bar{v})^{1/m} |S|^2} \right)$$

Here semi-Kähler-Einstein metric  $\omega_{SKE}$  is a  $(1, 1)$ -current with log-pole singularities such that its restriction on each fiber is Kahler-Einstein metric. Note that from Kodaira-Spencer theory, we always have such semi-Kähler-Einstein metric.

**Definition 1.3.** The null direction of fiberwise Kähler-Einstein metric  $\omega_{SKE}$  gives a foliation along Mori-fibre space  $\pi : X \rightarrow Y$  and we call it fiberwise Kähler-Einstein foliation and can be defined as follows

$$\mathcal{F} = \{\theta \in T_{X/Y} | \omega_{SKE}(\theta, \bar{\theta}) = 0\}$$

and along Mori-fibre space  $\pi : (X, D) \rightarrow Y$ , we can define the following foliation

$$\mathcal{F}' = \{\theta \in T_{X'/Y} | \omega_{SKE}^D(\theta, \bar{\theta}) = 0\}$$

where  $X' = X \setminus D$ . In fact from Theorem 0,9. the Weil-Petersson metric  $\omega_{WP}$  vanishes everywhere if and only if  $\mathcal{F} = T_{X/Y}$ . Note that such foliation may be fail to be as foliation in horizontal direction. But is is foliation in fiber direction.  $\square$

We have the following result due to Bedford-Kalka [49]

**Lemma:** Let  $\mathcal{L}$  be a leaf of  $f_*\mathcal{F}'$ , then  $\mathcal{L}$  is a closed complex submanifold and the leaf  $\mathcal{L}$  can be seen as fiber on the moduli map

$$\eta : \mathcal{Y} \rightarrow \mathcal{M}_{KE}^D$$

where  $\mathcal{M}_{KE}^D$  is the moduli space of Kahler-Einstein fibers with at worst log terminal singularities and

$$\mathcal{Y} = \{y \in Y_{reg} | (X_y, D_y) \text{ is Kawamata log terminal pair}\}$$

## 2 Main Theorems

Now by applying Tian-Li [22] method on Orbifold regularity of weak Kähler-Einstein metrics, we have the following theorem. We show that if general fibers and base are K-poly stable then there exists a generalized Kähler-Einstein metric on the total space which twisted with logarithmic Wil-Petersson metric and current of integration of special divisor, which is not exactly  $D$ . In fact along Minimal model  $\pi : X \rightarrow X_{can}$ , we can find such divisor  $N$  by Fujino-Mori's higher canonical bundle formula. But along Mori-fibre space I don't know such formula to apply it in next theorem.

Let  $f : X \rightarrow B$  be the surjective morphism of a normal projective variety  $X$  of dimension  $n = m + l$  to a nonsingular projective  $l$ -fold  $B$  such that:

- i)  $(X, D)$  is sub klt pair.
- ii) the generic fiber  $F$  of  $\pi$  is a geometrically irreducible variety with vanishing log Kodaira dimension. We fix the smallest  $b \in \mathbb{Z} > 0$  such that the

$$\pi_* \mathcal{O}_X(b(K_X + D)) \neq 0$$

The Fujino-Mori [25] log-canonical bundle formula for  $\pi : (X, D) \rightarrow B$  is

$$b(K_X + D) = \pi^*(bK_B + L_{(X,D)/B}^{ss}) + \sum_P s_P^D \pi^*(P) + B^D$$

where  $B^D$  is  $\mathbb{Q}$ -divisor on  $X$  such that  $\pi_* \mathcal{O}_X([iB^D]) = \mathcal{O}_B$  ( $\forall i > 0$ ). Here  $s_P^D := b(1 - t_P^D)$  where  $t_P^D$  is the log-canonical threshold of  $\pi^*P$  with respect to  $(X, D - B^D/b)$  over the generic point  $\eta_P$  of  $P$ . i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid (X, D - B^D/b + t\pi^*(P)) \text{ is sub log canonical over } \eta_P\}$$

Now if fibres  $(X_s, D_s)$  are log Calabi-Yau varieties and the base  $c_1(B) < 0$ , then we have

$$Ric(\omega_X) = -\omega_B + \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

Note that such pair of canonical metric is not unique.

Moreover, Let  $\pi : (X, D) \rightarrow B$  is a holomorphic submersion onto a compact Kähler manifold  $B$  with  $B$  be a Calabi-Yau manifold, log fibers  $(X_s, D_s)$  are log Calabi-Yau, and  $D$  is a simple normal crossing divisor in  $X$  with conic singularities. Then  $(X, D)$  admits a unique smooth metric  $\omega_B$  solving

$$Ric(\omega_{can}) = \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

as current where  $\omega_{can}$  has zero Lelong number and is good metric in the sense of Mumford.[45]

**Theorem 2.1.** Let  $\pi : (X, D) \rightarrow B$  be a holomorphic submersion between compact Kähler manifolds and  $B$  and central fiber is  $\mathbb{Q}$ -Fano K-poly stable variety with klt singularities. Let  $L + D$  be a relatively ample line bundle on  $(X, D)$  such that the restriction of  $c_1(L)$  to each fiber  $(X_s, D_s)$  admits a Kähler-Einstein metric. Suppose that  $\alpha - c_1(B) \geq 0$ . Then, for all sufficiently large  $r$ , the adiabatic class

$$\kappa_r = c_1(L + D) + r\pi^*\kappa_B$$

contains a Kähler Einstein metric which is twisted by log Weil-Petersson current and current of integration of special divisor  $N$  which is not  $D$  in general. In fact such current of integration must come from higher canonical bundle formula along Mori-fibre space(I don't know such formula at the moment).

$$Ric(\omega_{(X,D)}) = \omega_B + \omega_{WP}^D + (1 - \beta)[N]$$

If  $\alpha = c_1(B)$ ,  $\kappa_B$  is any Kähler class on the base, whilst if  $\alpha - c_1(B) > 0$ , then  $\kappa_B = \alpha - c_1(B)$ . □

**Proof.** Let  $B$  be any  $\mathbb{Q}$ -Fano variety with klt singularities. Assume  $p \in B^{orb}$  is a quotient singularity where  $B^{orb}$  is the orbifold locus of  $X$ . Then there exists a branched covering map  $\tilde{\mathcal{U}}_p \rightarrow \tilde{\mathcal{U}}_p/G \cong \mathcal{U}_p$  where  $\mathcal{U}_p$  is the small neighborhood of  $p$ . Now since  $B$  is  $\mathbb{Q}$ -Fano variety, we have an embedding  $i : B \rightarrow \mathbb{P}^n$  of linear system  $| -mK_B |$  for  $m \in \mathbb{N}$  sufficiently large and divisible. Now if  $h_{FS}$  be the Fubini Study hermitian metric, then

$$\omega = \sqrt{-1} \partial \bar{\partial} \log (i^* h_{FS})^{-\frac{1}{m}}$$

gives a smooth positive  $(1, 1)$ -form on regular part of  $B$  and we show it with  $B^{reg}$ . but the fact is that on the singular locus  $B^{sing}$  the form  $\omega$  in general is not canonically related to the local structure of  $B$ . So we must define new canonical measure for it. The following measure introduced by Bo Berndtsson,

$$\Omega = \|v^*\|_{(i^* h_{FS})^{\frac{1}{m}}}^{2/m} (v \wedge \bar{v})^{1/m}$$

Here  $v$  is any local generator of  $\mathcal{O}(m(K_B + D))$  and  $v^*$  is the dual generator of  $\mathcal{O}(-m(K_B + D))$ . Now consider the following Monge-Ampere equation on  $B$ .

$$(e^t \omega_{WP} + (1 + e^t) \omega_0 + \text{Rich} + \sqrt{-1} \partial \bar{\partial} \phi)^n = \frac{e^{-\phi + \Psi} \|v^*\|^{2/m} (v \wedge \bar{v})^{1/m}}{\|S\|^2} = e^{-\phi} \frac{\omega_{SKE}^s \wedge \|v^*\|^{2/m} (v \wedge \bar{v})^{1/m}}{\|v^*\|^{2/m} (i^* h_{FS})^{\frac{1}{m}} (v \wedge \bar{v})^{1/m} |S|^2}$$

Where  $\Psi$  is Deligne potential on log Tian's line bundle  $L_{CM}^D$  and  $\omega_{SKE}$  is called fiberwise Kähler-Einstein metric. Note that  $\text{diam}(X_s, \omega_s) \leq C$  if and only of central fiber be K-poly stable with klt singularities, so we can get  $C^0$ -estimate. Note that we are facing with three type Complex-Monge-Ampere equation, 1) MA equation on fiber direction, 2) MA equation on horizontal direction, and 3) MA foliation when fiber-wise Kahler-Einstein metric is zero, which the main difficulty for solving estimates is about Monge-Ampere equation corresponding to fiberwise Kähler-Einstein foliation. I don't know it yet. It is super difficult!

Take a resolution  $\pi : (\tilde{B}, \tilde{D}) \rightarrow (B, D)$  with a simple normal crossing exceptional divisor  $E = \pi^{-1}(B^{\text{sing}}) = \cup_{i=1}^r E_i \cup \cup_{j=1}^s F_j$  where by klt property we have  $a_i > 0$  and  $0 < b_j < 1$ . Then we have

$$K_{\tilde{B}} + \tilde{D} = \pi^*(K_B + D) + \sum_{i=1}^r a_i E_i - \sum_{j=1}^s b_j F_j$$

So by choosing a smooth Kähler metric  $\tilde{\omega}$  on  $\tilde{B}$ , there exists  $f \in C^\infty(\tilde{B})$  such that

$$\pi^*\left(\frac{\Omega_{X/B}}{\|S\|^2}\right) = e^f \frac{\prod_{i=1}^r \|\theta_i\|^{2a_i}}{\|\tilde{S}\|^2 \prod_{j=1}^s \|\sigma_j\|^{2b_j}} \tilde{\omega}^n$$

where  $\theta_i, \sigma_j$  and  $\tilde{S}$  are defining sections of  $E_i, F_j$  and  $\tilde{D}$  respectively. Then we can pull back our Monge-Ampere equation to  $(\tilde{B}, \tilde{D})$  and by taking  $\omega = e^t \omega_{WP} + (1 + e^t) \omega_0$

$$(\pi^* \omega + \text{Rich} + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{f - \psi + \Psi} \frac{\prod_{i=1}^r \|\theta_i\|^{2a_i}}{\|\tilde{S}\|^2 \prod_{j=1}^s \|\sigma_j\|^{2b_j}} \tilde{\omega}^n$$

Now by applying Theorem 0.8, if we take  $\psi_+ = f + \sum_i a_i \log |\theta_i|^2$  and  $\psi_- = \psi - \Psi + \sum_j b_j \log |\sigma_j|^2 + \sum_l \log c_l |\tilde{s}_l|^2$ , where  $\tilde{S} = \sum_l s_l \tilde{S}_l$  then we have

$$(\pi^* \omega + \text{Rich} + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{\psi_+ - \psi_-} \tilde{\omega}^n$$

Its easily can be checked they satisfy the quasi-plurisubharmonic condition:

$$\sqrt{-1} \partial \bar{\partial} \psi_+ \geq -C \tilde{\omega}^n, \quad \sqrt{-1} \partial \bar{\partial} \psi_- \geq -C \tilde{\omega}^n$$

for some uniform constant  $C > 0$ . By regularizing our Monge-Ampere equation away from singularities we have

$$(\omega_\epsilon + \epsilon \text{Rich} + \sqrt{-1} \partial \bar{\partial} \psi_\epsilon)^n = e^{\psi_+ - \psi_-} \tilde{\omega}_\epsilon^n$$

where  $\omega_\epsilon = \pi^* \omega - \epsilon \sqrt{-1} \partial \bar{\partial} |S_E|^2$  is a Kähler metric on  $\tilde{B} \setminus \tilde{D}$ . The Laplacian estimate for the solution  $\omega_\epsilon$  away from singularities is due to M. Paun [21] and Demailly-Pali [26] (a more simpler proof). So we have  $C^{1,\alpha}$  estimate for our solution. So, we get

$$\text{Ric}(\omega_X) = \omega_B + \omega_{WP}^D + (1 - \beta)[N]$$

.

**Remark 2:** A  $\mathbb{Q}$ -Fano variety  $X$  is K-stable if and only if it be K-stable with respect to the central fibre of test configuration. Hence  $X$  is a  $\mathbb{Q}$ -Fano K-stable if and only if

$$\int_{X_y} \|v^*\|^{2/m} (i^* h_{FS})^{\frac{1}{m}} (v \wedge \bar{v})^{1/m} \leq C$$

Here  $v$  is any local generator of  $\mathcal{O}(m(K_{X_y}))$  and  $v^*$  is the dual generator of  $\mathcal{O}(-m(K_{X_y}))$  and  $C$  is a constant and  $X_t$  is a general fibre with respect to test configuration.

We give the following conjecture about K-stability via Weil-Petersson geometry

**Conjecture:** A Fano variety  $X$  is K-stable if and only if 0 is at finite Weil-Petersson distance from  $C^0$  i.e.

$$d_{WP}(C^0, 0) < +\infty$$

where here  $C$  is just base of test configuration

Now we introduce relative Tian's alpha invariant along Fano fibration and Mori fibre space which corresponds to twisted K-stability.

Tian in [5], obtained a sufficient condition to get  $C^0$ -estimates for Monge-Ampere equation related to Kähler-Einstein metric for Fano varieties. It involves an invariant of the manifold and called the Tian's  $\alpha$ -invariant, that encodes the possible singularities of singular hermitian metrics with non negative curvature on  $K_M$ . The sufficient condition for the existence of Kähler-Einstein metric remain the same. One can also define the  $\alpha$ -invariant for any ample line bundle on a complex manifold  $M$ .

**Definition 2.2.** Let  $M$  be an  $n$ -dimensional compact Kähler manifold with an ample line bundle  $L$ . We fix  $\omega$  a Kähler metric in  $c_1(L)$ , and define Tians  $\alpha$ -invariant

$$\alpha(L) = \sup\{\alpha > 0 \mid \exists C > 0 \text{ with } \int_M e^{-\alpha(\varphi - \sup_M \varphi)} \omega^n \leq C\}$$

where we have assumed  $\varphi \in C^\infty$  and  $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$

□

Now we have the following remark as relative  $\alpha$ -invariant on the degeneration of Fano Kähler-Einstein metrics.

**Remark 3:** Let  $\pi : X \rightarrow B$  be a holomorphic submersion between compact Kähler manifolds whose fibers and the base admit no non-zero holomorphic vector fields and  $B$  is Fano variety. Let each Fano fiber  $X_s$  admits a Kähler-Einstein metric. Define the relative  $\alpha$ -invariant

$$\alpha(X/B) = \sup\{\alpha > 0 \mid \exists C > 0 \text{ with } \int_{X/B} e^{-\alpha(\varphi - \sup_M \varphi)} \frac{\omega_{SKE}^\kappa \wedge \|v^*\|_{(i^*h_{FS})}^{2/m} (v \wedge \bar{v})^{1/m}}{\|v^*\|_{(i^*h_{FS})}^{2/m} (v \wedge \bar{v})^{1/m} |S|^2} \leq C\}$$

and let

$$\alpha(X/B) > \frac{n}{n+1}$$

then we have relative Kähler Einstein metric

$$Ric_{X/B}(\omega) = \Phi\omega$$

where  $\Phi$  is the fiberwise constant function.

## 2.1 Relative Kähler Ricci soliton

In this subsection, we extend the notion of Kähler Ricci soliton to relative Kähler Ricci soliton along holomorphic fibre space  $\pi : X \rightarrow B$ . We think that the notion of relative Kähler Ricci soliton is more suitable for Song-Tian program along Mori fibre space or in general for Fano fibration.

A Kähler-Ricci soliton is one of the generalization of a Kähler-Einstein metric and closely related to the limiting behavior of the normalized Kähler-Ricci flow.

A Kähler metric  $h$  is called a Kähler-Ricci soliton if its Kähler form  $\omega_h$  satisfies equation

$$Ric(\omega_h) - \omega_h = L_X \omega_h = d(i_X \omega_h)$$

where  $Ric(\omega_h)$  is the Ricci form of  $h$  and  $L_X \omega_h$  denotes the Lie derivative of  $\omega_h$  along a holomorphic vector field  $X$  on  $M$ .

Since  $i_X \omega_h$  is  $(0, 1)$   $\bar{\partial}$ -closed form, by using Hodge theory,

$$i_X \omega_h = \alpha + \bar{\partial}\varphi$$

where  $\alpha$  is a  $(0, 1)$ -harmonic form and  $\varphi \in C^\infty(M, \mathbb{C})$  hence, we have

$$Ric(\omega_h) - \omega_h = \partial\bar{\partial}\varphi$$

we can find a smooth real-valued function  $\theta_X(\omega)$  such that

$$i_X \omega = \sqrt{-1}\partial\bar{\partial}\theta_X(\omega)$$

This means that if the Kähler metric  $h$  be a Kähler-Ricci soliton, then  $\omega_h \in c_1(M)$

**Theorem:** Let  $X$  and  $B$  be compact Kähler manifolds with holomorphic submersion  $\pi : X \rightarrow B$ , and let fibers  $\pi^{-1}(b) = X_b$  are K-poly stable Fano varieties and the Fano variety  $B$  admit Kähler Ricci soliton. Then there exists a generalized Kähler Ricci soliton

$$Ric(\omega_X) = \omega_B + \pi^* \omega_{WP} + L_X \omega_B$$

for some holomorphic vector field on  $X$ , where  $\omega_{WP}$  is a Weil-Petersson metric on moduli space of K-poly stable Fano fibres.

**Idea of Proof.** It is enough to solve the following Monge-Ampere equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{\Psi - \varphi - \theta_X - X(\varphi)} \omega^n$$

where  $\Psi$  is the Deligne potential on Tian's CM-line bundle  $L_{CM}$ . We need to show properness of the relative K-energy. We postpone it for future work.

Moreover, we have the same result on pair  $(X, D) \rightarrow B$ . i.e.

$$Ric(\omega_X) = \omega_B + \pi^* \omega_{WP}^D + L_X \omega_B + [N]$$

We define the relative Soliton-Kähler Ricci flow as

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) + \Phi \omega(t) + L_X \omega(t)$$

along holomorphic fiber space  $\pi : X \rightarrow Y$ , where here the relative Ricci form  $Ric_{X/Y, \omega}$  of  $\omega$  is defined by

$$Ric_{X/Y, \omega} = -\sqrt{-1} \partial \bar{\partial} \log(\omega^m \wedge \pi^* |dy_1 \wedge dy_2 \wedge \dots \wedge dy_k|^2)$$

where  $(y_1, \dots, y_k)$  is a local coordinate of  $Y$ .

Note that if fibers  $X_y$  be K-poly stable, base  $Y$  and central fiber admit Kähler Ricci soliton then relative Soliton-Kähler Ricci flow converges to

$$Ric(\omega_X) = \omega_B + \pi^* \omega_{WP} + L_X \omega$$

A relative Kähler metric  $g$  on holomorphic fibre space  $\pi : M \rightarrow B$  where  $M$  and  $B$  are compact Kähler manifolds is called a relative Kähler-Ricci soliton if there is a relative holomorphic vector field  $X$  on the relative tangent bundle  $T_{M/B}$  such that the relative Kähler form  $\omega_g$  of  $g$  satisfies

$$Ric_{M/B} \omega_g - \Phi \omega_g = L_X \omega_g$$

So we can extend the result of Tian-Zhu for future work. i.e. If we have already relative Kähler Ricci soliton  $\omega_{RKS}$ , then the solutions of the relative Kähler Ricci flow

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) + \Phi \omega(t)$$

converges to  $\omega_{RKS}$  in the sense of Cheeger-Gromov with additional assumption on the initial metric  $g_0$ .

We give a conjecture about invariance of plurigenera using K-stability and Kähler Ricci flow.

**Theorem 2.3.** (Siu [32]) Assume  $\pi : X \rightarrow B$  is smooth, and every  $X_t$  is of general type. Then the plurigenera  $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$  is independent of  $t \in B$  for any  $m$ .  $\square$

See [31, 33, 34] also for singular case.

**Conjecture:** Let  $\pi : X \rightarrow B$  is smooth, and every  $X_t$  is K-poly stable. Then the plurigenera  $P_m(X_t) = \dim H^0(X_t, -mK_{X_t})$  is independent of  $t \in B$  for any  $m$ .

Idea of proof. We can apply the relative Kähler Ricci flow method for it. In fact if we prove that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) + \Phi \omega(t)$$

has long time solution along Fano fibration such that the fibers are K-poly stable then we can get the invariance of plurigenera in the case of K-poly stability

**Conjecture:** The Fano variety  $X$  is K-poly stable if and only if for the proper holomorphic fibre space  $\pi : X \rightarrow \mathbb{D}$  which the Fano fibers have unique Kähler-Einstein metric with positive Ricci curvature, then the

fiberwise Kähler Einstein metric (Semi-Kähler Einstein metric)  $\omega_{SKE}$  be smooth and semi-positive. Note that if fibers are K-poly stable then by Schumacher and Berman [35, 36] result we have

$$-\Delta_{\omega_t} c(\omega_{SKE}) - c(\omega_{SKE}) = |A|_{\omega_t}^2$$

where  $A$  represents the Kodaira-Spencer class of the deformation and since  $\omega_{SKE}^{n+1} = c(\omega_{SKE})\omega_{SKE}^n ds \wedge d\bar{s}$  so  $c(\omega_{SKE})$  and  $\omega_{SKE}$  have the same sign. By the minimum principle  $\inf \omega_{SKE} < 0$ . But our conjecture says that the fibrewise Fano Kähler-Einstein metric  $\omega_{SKE}$  is smooth and semi-positive if and only if  $X$  be K-poly stable.

Chi Li in his thesis showed that an anti-canonically polarized Fano variety is K-stable if and only if it is K-stable with respect to test configurations with normal central fibre. So if the central fibre  $X_0$  admit Kähler-Einstein metric with positive Ricci curvature then along Mori-fiber space all the general fibres  $X_t$  admit Kähler-Einstein metric with positive Ricci curvature (we will prove it in this paper) and hence we can introduce fibrewise Kähler-Einstein metric  $\omega_{SKE}$ . See [35, 36]

Let  $L \rightarrow X$  be a holomorphic line bundle over a complex manifold  $X$  and fix an open cover  $X = \cup U_\alpha$  for which there exist local holomorphic frames  $e_\alpha : U_\alpha \rightarrow L$ . The transition functions  $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$  determine the Čech 1-cocycle  $\{(U_\alpha, g_{\alpha\beta})\}$ . If  $h$  is a singular Hermitian metric on  $L$  then  $h(e_\alpha, e_\alpha) = e^{-2\varphi_\alpha}$ , where the functions  $\varphi_\alpha \in L_{loc}^1(U_\alpha)$  are called the local weights of the metric  $h$ . We have  $\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$  on  $U_\alpha \cap U_\beta$  and the curvature of  $h$ ,

$$c_1(L, h)|_{U_\alpha} = dd^c \varphi_\alpha$$

is a well defined closed  $(1, 1)$  current on  $X$ .

One of the important example of singular hermitian metric is singular hermitian metric with algebraic singularities. Let  $m$  be a positive integer and  $\{S_i\}$  a finite number of global holomorphic sections of  $mL$ . Let  $\varphi$  be a  $C^\infty$ -function on  $X$ . Then

$$h := e^{-\varphi} \cdot \frac{1}{(\sum_i |S_i|^2)^{1/m}}$$

defines a singular hermitian metric on  $L$ . We call such a metric  $h$  a singular hermitian metric on  $L$  with algebraic singularities.

**Remark 4:** Fiberwise Kähler-Einstein metric  $\omega_{SKE}$  along Mori fiber space (when fibers are K-poly stable) has non-algebraic singularities. In fact such metric  $\omega_{SKE}$  has pole singularities

**Remark 5:** Let  $f: X \rightarrow Y$  be a surjective proper holomorphic fibre space such that  $X$  and  $Y$  are projective varieties and central fibre  $X_0$  is Calabi-Yau variety with canonical singularities, then all the fibres  $X_t$  are also Calabi-Yau varieties. Let for simplicity that  $Y$  is a smooth curve. Since  $X_0$  has canonical, then, we may assume that  $X$  is canonical. Nearby fibers are then also canonical.  $\mathcal{O}_X(K_X + X_0) \rightarrow \mathcal{O}_{X_0}(K_{X_0})$  is surjective and so if  $K_{X_0}$  is Cartier, then so is  $K_X$  on a neighborhood of  $X_0$ . We have that  $P_m(X_y)$  is deformation invariant for  $m \geq 1$  so in fact  $h^0(K_{X_y}) > 0$  for  $y \in Y$ . Since  $K_{X_0} \equiv 0$ , then  $K_{X_y} \equiv 0$  and so  $K_{X_y} \sim 0$

Note that if all of the fibres of the surjective proper holomorphic fibre space  $f: X \rightarrow Y$  are Calabi-Yau varieties, then the central fiber  $X_0$  may not be Calabi-Yau variety

**Remark 6:** Let  $f: X \rightarrow Y$  be a surjective proper holomorphic fibre space such that  $X$  and  $Y$  are projective varieties and central fibre  $X_0$  is of general type, then all the fibres  $X_t$  are also of general type varieties. Note that if base and fibers are of general type then the total space is of general type

**Remark 7:** Let  $f: X \rightarrow Y$  be a surjective proper holomorphic fibre space such that  $X$  and  $Y$  are projective varieties and central fibre  $X_0$  is K-poly stable with discrete Automorphism group, then all the fibres  $X_t$  are also K-poly stable. Note that if along Mori-fiber space, the general fibers and base space are of K-poly stable, then total space is also K-poly stable. Note that Mori fiber space is relative notion and we can assume that base is K-poly stable always.

Define

$$\text{Scal}(\omega) = \frac{n \text{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$$

and take a holomorphic submersion  $\pi: X \rightarrow D$  over some disc  $\mathbb{D}$ . It is enough to show smoothness of semi Ricci flat metric over a disc  $\mathbb{D}$ . Define a map  $F: \mathbb{D} \times C^\infty(\pi^{-1}(b)) \rightarrow C^\infty(\pi^{-1}(b))$  by

$$F(b, \rho) = \text{Scal}(\omega_0|_b + \sqrt{-1} \partial_b \bar{\partial}_b \rho)$$

We can extend  $F$  to a smooth map  $\mathbb{D} \times L_{k+4}^2(\pi^{-1}(b)) \rightarrow L_k^2(\pi^{-1}(b))$ . From the definition  $F(b, \rho_b)$  is constant. Recall that, if  $L$  denotes the linearization of  $\rho \rightarrow \text{Scal}(\omega_\rho)$ , then

$$L(\rho) = \mathcal{D}^* \mathcal{D}(\rho) + \nabla \text{Scal} \cdot \nabla \rho$$

where  $\mathcal{D}$  is defined by

$$\mathcal{D} = \bar{\partial}_0 \nabla : C^\infty \rightarrow \Omega^{0,1}(T)$$

and  $\mathcal{D}^*$  is  $L^2$  adjoint of  $\mathcal{D}$ . The linearisation of  $F$  with respect to  $\rho$  at  $b$  is given by

$$\mathcal{D}_b^* \mathcal{D}_b : C^\infty(\pi^{-1}(b)) \rightarrow C^\infty(\pi^{-1}(b))$$

Leading order term of  $L$  is  $\Delta^2$ . So  $L$  is elliptic and we can use the standard theory of elliptic partial differential equations. Since central fiber has discrete Automorphism, hence Kernel of such Laplace-Beltrami is zero, and hence  $\mathcal{D}_b^* \mathcal{D}_b$  is an isomorphism. By the implicit function theorem, the map  $b \rightarrow \rho_b$  is a smooth map

$$\mathbb{D} \rightarrow L_k^2(\pi^{-1}(b)) ; \forall k.$$

By Sobolev embedding, it is a smooth map  $D \rightarrow \mathbb{C}^r(\pi^{-1}(b))$  for any  $r$ . Hence cscK or Kähler-Einstein metric with positive Ricci curvature on the central fiber can be extended smoothly to general fibers. See [43, 44]

In final it is worth to mention that K-stability along special test configuration gives such foliation as we introduced and Tian, Chen-Sun-Donaldson didn't consider such Monge-Ampere foliation for their solution of Yau-Tian-Donaldson, conjecture.

## References

- [1] Jian Song; Gang Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Inventiones mathematicae. 170 (2007), no. 3, 609-653.
- [2] Jian Song; Gang Tian, *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. 25 (2012), no. 2, 303-353,
- [3] Tian, G. and Zhu, X., Uniqueness of Kähler-Ricci solitons, Acta Math. 184 (2000), no. 2, 271-305
- [4] Futaki, A., An obstruction to the existence of Einstein-Kähler metrics. Invent. Math., 73,437-443 (1983)
- [5] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with  $C_1(M) > 0$ . Invent. Math., 89(2):225-246, 1987.
- [6] X. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds. I, Approximation of metrics with cone singularities, Journal of the American Mathematical Society 28 (1), 183-197
- [7] X. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than  $2\pi$ , Journal of the American Mathematical Society 28 (1), 199-234
- [8] X. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds. III, Limits as cone angle approaches  $2\pi$  and completion of the main proof, Journal of the American Mathematical Society 28 (1), 235-278
- [9] G. Tian. K-stability and Kähler-Einstein metrics. arXiv:1211.4669
- [10] Odaka, The GIT stability of polarized varieties via discrepancy, Annals of Math. vol. 177, iss. 2, 645-661. (2013)
- [11] Jian Song, Gang Tian, The Kähler-Ricci flow through singularities, arXiv:0909.4898
- [12] H. Guenancia, M. Paun, Conic singularities metrics with prescribed Ricci curvature: the case of general cone angles along normal crossing divisors. Ann. Sci. c. Norm. Supr. (4) 46 (2013), no. 6, 879-916.
- [13] S.T. Yau, A General Schwarz Lemma for Kähler Manifolds, American Journal of Mathematics Vol. 100, No.1, 1978, 197-203
- [14] Jian Song, Riemannian geometry of Kähler-Einstein currents II: an analytic proof of Kawamata's base point free theorem, arXiv:1409.8374
- [15] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Monge-Ampere equations in big cohomology classes. Acta Mathematica 205 (2010), 199-262
- [16] H. Skoda. Sous-ensembles analytiques d'ordre fini ou infini dans  $\mathbb{C}^n$ . Bulletin de la Société Mathématique de France, 100:353-408, 1972.

- [17] J. P. Demailly and J. Kollar. Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. Ecole Norm. Sup.*, 34(4):525-556, 2001.
- [18] Frdric Campana, Henri Guenancia, Mihai Pun Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields, *Annales scientifiques de l'ENS* 46, fascicule 6 (2013), 879-916
- [19] Zsolt Patakfalvi and Chenyang Xu, Ampleness of the CM-line bundle on the moduli space of canonically polarized varieties, arXiv:1503.0866
- [20] Indranil Biswas, Georg Schumacher, Determinant bundle, Quillen metric, and Petersson-Weil form on moduli spaces, *GAFA, Geometric and Functional Analysis* 9, 226-256 (1999)
- [21] M. Paun: Regularity properties of the degenerate Monge-Ampere equations on compact Kahler manifolds. *Chin. Ann. Math. Ser. B*29(2008), no. 6, 623-630
- [22] Chi Li, Gang Tian, Orbifold regularity of weak Kähler-Einstein metrics, arXiv:1505.01925v1
- [23] S. Donaldson, S. Sun. Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry. *Acta Math.* 213 (2014), no.1, 63-106
- [24] Tian, G., Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* 130(1997), no. 1, 1-37
- [25] Fujino, Osamu; Mori, Shigefumi. A Canonical Bundle Formula. *J. Differential Geom.* 56 (2000), no. 1, 167-188
- [26] Demailly, Jean-Pierre; Pali, Nefton, Degenerate complex Monge-Ampere equations over compact Kähler manifolds. *Internat. J. Math.* 21 (2010), no. 3, 357405.
- [27] Tian, G.: Einstein metrics on Fano manifolds. *Metric and Differential Geomtry*, Proceeding of the 2008 conference celebrating J. Cheegers 65th birthday, edited by Dai et al., *Progress in Mathematics*, volume 239. Birkhauser, 2012
- [28] Chi Li, Xiaowei Wang, Chenyang Xu, Quasi-projectivity of the moduli space of smooth Kähler-Einstein Fano manifolds, arXiv:1502.06532
- [29] Eckart Viehweg, Quasi-projective moduli for polarized manifolds, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 30, Springer-Verlag, Berlin, 1995
- [30] Yuji Odaka, Compact moduli space of Kähler-Einstein Fano varieties , arXiv:1412.2972 (2014).
- [31] Noboru Nakayama, Invariance of the plurigenera of algebraic varieties under minimal model conjectures, *Topology* Vol.25, No.2, pp.237-251, 1986
- [32] Yum-Tong Siu, Invariance of plurigenera, *Invent. Math.* 134 (1998), no. 3, 661673.
- [33] Deformations of canonical singularities. *J. Amer. Math. Soc.*12 (1999), no. 1, 85-92
- [34] Takayama S., On the invariance and the lower semi-continuity of plurigenera of algebraic varieties, *J. Algebraic Geom.*16(2006) 1-18
- [35] Georg Schumacher, Positivity of relative canonical bundles and applications, *Inventiones Mathematicae*, October 2012, Volume 190, Issue 1, pp 1-56
- [36] Robert J. Berman, Relative Kähler-Ricci flows and their quantization, *Analysis and PDE*, Vol. 6 (2013), No. 1, 131180
- [37] S. Boucksom and H. Tsuji, Semipositivity of relative canonical bundles via Kähler-Ricci flows, preprint
- [38] Odaka, Yuji The GIT stability of polarized varieties via discrepancy. *Ann. of Math. (2)* 177 (2013), no. 2, 645-661.
- [39] Gang Tian, K-stability implies CM-stability, arXiv:1409.7836
- [40] Tian, G.: Einstein metrics on Fano manifolds. *Metric and Differential Geomtry*, Proceeding of the 2008 conference celebrating J. Cheegers 65th birthday, edited by Dai et al., *Progress in Mathematics*, volume 239. Birkhauser, 2012

- [41] Akira Fujiki and, Georg Schumacher, The Moduli Space of Extremal Compact Kähler Manifolds and Generalized Weil-Petersson Metrics, *Publ. Res. Inst. Math. Sci.* 26 (1990), 101-183.
- [42] Georg Schumacher and Hajime Tsuji, Quasi-projectivity of moduli spaces of polarized varieties, *Annals of Mathematics*, 159(2004), 597-639
- [43] Joel fine, Constant scalar curvature metrics on fibred complex surfaces, PhD thesis
- [44] LeBrun and S. R. Simanca. Extremal Kähler metrics and complex deformation theory. *Geom. Funct. Anal.*, 4(3):298-336, 1994.
- [45] Hassan Jolany, Analytical log Minimal Model Program via conical kähler Ricci flow: Song-Tian program, PhD thesis, Lille1 university
- [46] Jacopo Stoppa, Twisted constant scalar curvature Kähler metrics and Kähler slope stability, *J. Differential Geom.*, Volume 83, Number 3 (2009), 663-691.
- [47] J. Fine, Fibrations with constant scalar curvature Kähler metrics and the CM-line bundle. *Math. Res. Lett.* 14 (2007), no. 2, 239-247, .
- [48] J. Ross, J. Fine A note on Positivity of the CM line bundle *IMRN*, vol. 2006, Article ID 95875, 14 pages, 2006.
- [49] E. Bedford and M. Kalka, Foliations and Complex Monge-Ampere Equations, *Comm. Pure Appl. Math.* XXX (1977), 543-571.