Generalized Hitchin-Kobayashi correspondence and Weil-Petersson current

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In this paper we extend Hitchin-Kobayashi correspondence along holomorphic fibre space and we show that there exists a generalized Hermitian-Einstein metric which is twisted with Weil-Petersson current.

1 Hitchin-Kobayashi correspondence along holomorphic fibre space

Hitchin and Kobayashi conjectured that the stability of a holomorphic bundle should be related to the existence of Hermitian-Einstein metric. This conjecture solved by Uhlenbeck and Yau for compact Kähler manifolds[7]. The central conjecture in Kähler geometry is about finding canonical metric. Hermitian-Einstein metric is a canonical metric on a holomorphic vector bundle[8]. In this paper we apply Song-Tian program for finding generalized Hermitian-Einstein metric for holomorphic vector bundles. The main computation will appear later. This note is just an announcement.

Takemoto introduced the notion of of slope stability for a holomorphic vector bundle which is known as Mumford-Takemoto stability. We give brife introduction on it. Let (X, ω) be a compact Kähler manifold of complex dimension n. The degree of a torsion-free holomorphic vector bundle E is the integral of differential forms

$$\deg_\omega(E) = \int_X c_1(E) \wedge \omega^{n-1}$$

of the de Rham representative of the actual first Chern class of X wedge with copies of the given symplectic form. Here $c_1(E) := c_1(\det E)$ and the determinant of a vector bundle is the top exterior power of the vector bundle i.e. a line bundle $\det(V) = \wedge^{rk(V)}V$.

we can write also

$$\deg_\omega(E) = \int_X \frac{\sqrt{-1}}{2\pi} tr(F_h) \wedge \omega^{n-1}$$

where F_h is the curvature form of the Chern connection in E with respect to a smooth Hermitian metric h.

A holomorphic vector bundle $E \to X$ over a compact Kähler manifold (X, ω) is called stable if every coherent holomorphic subsheaf F of E satisfies

$$0 < \mathrm{rk}F < \mathrm{rk}E \longrightarrow \mu_\omega(F) < \mu_\omega(E)$$

where μ_ω is the ω-slope of the sheaf given by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\mathrm{rk}(E)}$$

We have the same notion of semi-stability, when we replace $<$ to $\leq$.

A Hermitian metric h in E is called a ω-Hermitian-Einstein metric if

$$\sqrt{-1}\Lambda_\omega F_h = \lambda_h id_E$$

with a real constant λ_h, where $\sqrt{-1}\Lambda_\omega$ is the contraction with ω, F_h is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle $(E; h)$ and id_E is the identity endomorphism of E.

Here Λ_ω be the adjoint operation such that

$$< \omega \wedge u, v >_g = < u, \Lambda_\omega v >_g$$

for $u \in A^{p,q}$, $v \in A^{p+1,q+1}$

If h is a ω-Hermitian-Einstein metric in E with Einstein factor λ_h, we have
d

$$\lambda_h = \frac{2\pi\mu_\omega(E)}{(n-1)!Vol_\omega(X)}$$

In the following theorem we explain Hitchin-Kobayashi correspondence about the existence of ω-Hermitian-Einstein metric in the holomorphic vector bundle E.

S. Kobayashi [5] proved that an irreducible Hermitian-Einstein vector bundle on a compact Kähler manifold is in fact stable. Donaldson [3] proved the existence of a Hermitian-Einstein metric in a stable holomorphic vector bundle on a projective-algebraic surface and later for projective-algebraic manifold of arbitrary dimension. Finally, in 1986, Uhlenbeck and Yau proved the theorem for any arbitrary compact Kähler manifold [7].

Theorem 1.1. Let E be a holomorphic vector bundle on a compact Kähler manifold $(X;\omega)$

I- If E is ω-stable, then there is a ω-Hermitian Einstein metric in E

II- If there is a ω-Hermitian Einstein metric in E , then E is ω-polystable in the sense that E is ω-semistable and is a direct sum

$$E = E_1 \oplus E_2 \oplus ... \oplus E_k$$

of ω-stable sub bundles E_k of E with $\mu_\omega(E_k) = \mu_\omega(E)$. In particular, if E is irreducible, then it is ω-stable.
□

Now if (X, E) do not admit Hermitian Einstein metric, the following theorem is useful for finding canonical metric on (X, E) which generalize the previous theorem

Theorem 1.2. Let $\pi : (X, E) \to S$ be a family of poly-stable vector bundles (X_s, E_s), such that X and S are compact Kähler manifolds, $E \to X$ is a poly stable holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

where ω_{WP} is a Weil-Petersson metric on moduli space of poly-stable vector bundles □

Idea of Proof. For the construction of Hermitian-Einstein metrics for stable bundles, we start with a fixed Hermitian metric H_0 and try to construct from it a Hermitian-Einstein metric by deforming H_0 through a one-parameter family of Hermitian metrics H_t $(0 < t < 1)$.

We can integrate the vector field and from any initial point get an integral curve given by $h = h(t)$, $0 < t < t_\infty$, where t_∞ is the maximum time-parameter value to which we can extend the integral curve. The integral curve with the initial point $h(0) = I$ is given by

$$\frac{\partial}{\partial t}h(t) = -(\Lambda F_{H(t)}^{(X,E)/S} - \lambda I)h(t)$$

where $H(t)$ is corresponds to hermitian metric of relative vector bundle $K_{X/S} + E$. Now by applying the same method of Donaldson we can get higher estimates for the solution of this flow. Moreover by using Biswas-Schumacher formula for moduli space of poly-stable vector bundles we have Weil-Petersson metric [**?**] and we get

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

.

Theorem 1.3. If h is a ω- generalized Hermitian-Einstein metric with Einstein factor λ,

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

we have

$$\lambda = \frac{2\pi\mu_\omega(E)}{(n-1)!vol_\omega(X)} - \frac{2\pi}{(n-1)!vol_\omega(X)} \int_X \omega_{WP} \wedge \omega^{n-1}$$

□

We have the same result for twisted Higgs bundles. Let X be a compact complex manifold and L an ample line bundle over X. A twisted higgs bundle over X is a pair (E, ϕ) consisting of a holomorphic vector bundle E over X and a holomorphic bundle morphism.

$$\phi : X \otimes E \to E$$

for some holomorphic vector bundle X(the twist). Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in R$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson-Uhlenbeck-Yau Theorem .

Unfortunately, the space of holomorphic bundles of fixed rank and fixed degree, up to isomorphism, is not a Hausdorff space. But, by using GIT one can construct the moduli space $\mathcal{M}(n, d)$ of stable bundles of fixed rank n and degree d, which has structure as an algebraic variety with canonical metric called Weil-Petersson metric. It is known that, when $(n, d) = 1$ for compact Riemann surface σ, the moduli space $\mathcal{M}(n, d)$ is a smooth projective algebraic variety of dimension $n^2(g-1) + 2$. Tian conjectured that the moduli space $\mathcal{M}(n, d)$ of stable vector bundles of rank n and degree d is smooth quasi-projective variety.

A vector subbundle F of E for which $\phi(F) \subset F \otimes K_X$ is said to be a ϕ-invariant subbundle of E. Stability for Higgs bundles is defined in terms of ϕ-invariant subbundles:

Definition 1.4. A Higg bundle (E, ϕ) is

- Stable if for each proper ϕ-invariant subbundle F one has $\mu(F) < \mu(E)$

- Semi-stable if for each proper ϕ-invariant sub-bundle F one has $\mu(F) < \mu(E)$

- Polystable if $(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2)... \oplus (E_r, \phi_r)$ where (E_i, ϕ_i) is stable with $\mu(E_i) = \mu(E)$ for all i.

$\square$

Simpson, started the study of Higgs bundles over compact Kähler manifolds of arbitrary dimension. He showed that a stable Higgs bundle admits a unique Hermitian-Yang-Mills connection. He constructed the moduli space of Higgs bundles over a complex projective manifold

Theorem 1.5. (Simpson[6]) (E, ϕ) is polystable if and only if E admits a hermitian metric h solving the Hitchin equations

$$\sqrt{-1}\Lambda F_h + c[\phi, \phi^*] = \lambda Id_E$$

where F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and λ is a topological constant.

$\square$

Theorem 1.6. Let $\pi : (X, E, \phi) \to S$ be a family of poly-stable twisted higg bundles (X_s, E_s, ϕ_s), such that X and S are compact Kähler manifolds, $E \to X$ is a poly stable holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} + c[\phi, \phi^*] = \lambda Id_E$$

where ω_{WP} is a Weil-Petersson metric on moduli space of poly-stable Higg vector bundles where such Weil-Petersson metric has been introduced by Schumacher-Biswas[10]

$$\omega_{WP} = \frac{1}{2} \int_{X/S} \text{tr}(\Omega \wedge \Omega) \wedge \frac{\omega_X^{n-1}}{(n-1)!} + \lambda \int_{X/S} tr(\sqrt{-1}\Omega) \frac{\omega_X^n}{n!} + \sqrt{-1}\partial\bar{\partial}\frac{1}{2} \int_{X/S} tr(\phi \wedge \phi^*) \frac{\omega_X^{n-1}}{(n-1)!}$$

where λ is determined by

$$\int_{X/S} tr(\sqrt{-1}\Omega) \frac{\omega_X^{n-1}}{(n-1)!} = \lambda \int_{X/S} \frac{\omega_X^n}{n!}$$

and Ω is the curvature form of (E, h)

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$$\lambda_h = \frac{2\pi\mu_\omega(E)}{(n-1)! Vol_\omega(X)}$$

In the following theorem we explain Hitchin-Kobayashi correspondence about the existence of ω-Hermitian-Einstein metric in the holomorphic vector bundle E.

Theorem 1.7. Let E be a holomorphic vector bundle on a compact Kähler manifold $(X; \omega)$

I- If E is ω-stable, then there is a ω-Hermitian Einstein metric in E

II- If there is a ω-Hermitian Einstein metric in E , then E is ω-polystable in the sense that E is ω-semistable and is a direct sum

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Now if (X, E) do not admit Hermitian Einstein metric, the following theorem is useful for finding canonical metric on (X, E) which generalize the previous theorem

Theorem 1.8. Let $\pi : (X, E) \to S$ be a family of poly-stable vector bundles (X_s, E_s), such that X and S are compact Kähler manifolds, $E \to X$ is a holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

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Theorem 1.9. If h is a ω- generalized Hermitian-Einstein metric with Einstein factor λ,

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we have

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$\square$

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Simpson, started the study of Higgs bundles over compact Kähler manifolds of arbitrary dimension. He showed that a stable Higgs bundle admits a unique Hermitian-Yang-Mills connection. He constructed the moduli space of Higgs bundles over a complex projective manifold

Theorem 1.11. (Simpson) (E, ϕ) is polystable if and only if E admits a hermitian metric h solving the Hitchin equations

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where F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and λ is a topological constant.

$\square$

Theorem 1.12. Let $\pi : (X, E, \phi) \to S$ be a family of poly-stable twisted higgs bundles (X_s, E_s, ϕ_s), such that X and S are compact Kähler manifolds, $E \to X$ is a holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

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and Ω is the curvature form of (E, h)

$\square$

In next theorem we show that log slope stability is an open condition. This means that if the central fibre $\mathcal{E}_0$ be slope stable then all the fibers $\mathcal{E}_b$, for all $b \in \mathcal{U}$, where $\mathcal{U}$ is open neighborhood of 0 are slope stable. An Irreducible vector bundle E on pair (X, D) is log slope stable if and only if it admit Hermitian-Einstein metric for $X \setminus D$, see [1, 2]

Theorem 1.13. Let $\mathcal{X}$ be a compact Kähler manifold and $\mathcal{D}$ be a simple normal crossing divisor on $\mathcal{X}$ and take $\pi : (\mathcal{X}, \mathcal{D}) \to B$, be a family of pairs of compact Kähler manifolds. Let $\mathcal{E}$ be a holomorphic vector bundle on $(\mathcal{X}, \mathcal{D})$ such that the central holomorphic vector bundle $\mathcal{E}_0$ admits irreducible Hermitian-Einstein metric metric. Then there exists a neighbourhood $\mathcal{U}$ of 0 in B such that for all $b \in \mathcal{U}$, $\mathcal{E}_b$ admits irreducible Hermitian-Einstein metric metrics. $\square$

Proof. The degree of a torsion-free holomorphic vector bundle $\mathcal{E}' = \mathcal{E}|_{\mathcal{X}'}$, where $\mathcal{X}' = \mathcal{X} \setminus \mathcal{D}$ is the integral of differential forms

$$\deg_\omega(\mathcal{E}') = \int_{\mathcal{X}} c_1(\mathcal{E}') \wedge \omega^{n-1}$$

of the de Rham representative of the actual first Chern class of $\mathcal{X}'$ wedge with copies of the given symplectic form. Here $c_1(\mathcal{E}') := c_1(\det \mathcal{E}')$.

A holomorphic vector bundle $\mathcal{E}' \to \mathcal{X}'$ over a compact Kähler manifold $(\mathcal{X}', \omega)$ is called stable if every coherent holomorphic subsheaf $\mathcal{F}$ of $\mathcal{E}'$ satisfies

$$0 < \text{rk}\mathcal{F} < \text{rk}\mathcal{E}' \longrightarrow \mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E}')$$

where μ_ω is the ω-slope of the sheaf given by

$$\mu_\omega^{(\mathcal{X}, \mathcal{D})}(\mathcal{E}') := \frac{\deg_\omega(\mathcal{E}')}{\text{rk}(\mathcal{E}')}$$

We apply the Siu's method for this proof. Let $A"(b) : \mathcal{A}^0(\mathcal{E}'_b) \to \mathcal{A}^{0,1}(\mathcal{E}'_b)$ be a family of connections on the fibres $\mathcal{E}'_b = \mathcal{E}_b|_{\mathcal{X}_b \setminus \mathcal{D}_b}$ and take $\Omega(A"(b))$ be the related curvature forms.

Take the map

$$\Phi : \mathbf{P}GL(\mathcal{E}') \times S \to \mathcal{S}(\mathcal{X}', End(\mathcal{E}'))$$

$$(f, b) \to \sqrt{-1}\Lambda_s\Omega(A"(b) + f^{-1}\bar{\partial}_{A"}f) - \lambda.id$$

where

$$T_{id}\mathbf{P}GL(\mathcal{E}') = \mathcal{S}(\mathcal{X}', End(\mathcal{E}')) = \left\{ \phi \in \mathcal{A}^0(\mathcal{X}', End(\mathcal{E}'))| \int \operatorname{tr}\phi gdv = 0 \right\}$$

and λ is the constant given by Irreducible Hermitian-Einstein metric for central fibre $\mathcal{E}'_0$. Note that since central fibre $\mathcal{E}'_0$ is simple, hence the first partial derivative of Φ is bijective. Moreover, the Laplace-Beltrami operator Δ_0 with respect $\mathcal{E}'_0$, by using this fact that $\mathcal{E}'_0$ is simple, the equation $\Delta_0 K = L$ always has solution for an endomorphisms L,and K of $\mathcal{E}'_0$. Hence $(f, b) \to \sqrt{-1}\Lambda_s\Omega(A''(b) + f^{-1}\bar{\partial}_{A''}f) - \lambda.id$ always has solution by implicit function theorem and by extending Φ using Sobolev embedding theorem, proof is complete.

References

[1] Y. T. Siu. Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics. Birkhäuser Basel, 1987.

[2] Matthias Stemmler, Stability and Hermitian-Einstein metrics for vector bundles on framed manifolds, International Journal of Mathematics, September 2012, Vol. 23, No. 09

[3] S. K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. (3) 50 (1985), 1–26.

[4] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles.Frontiers of the mathematical sciences: 1985 (New York, 1985). Comm. Pure Appl. Math. 39 (1986), no. S, suppl., S257–S293.

[5] S. Kobayashi, Curvature and stability of vector bundles, Proc. Japan Acad. Ser. A. Math. Sci., 58 (1982), 158–162.

[6] C. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Amer. Math. Soc. 1 (1988), 867–918.

[7] Uhlenbeck, K.; Yau, Shing-Tung, "On the existence of Hermitian–Yang–Mills connections in stable vector bundles", Communications on Pure and Applied Mathematics, 39: S257–S293

[8] Lübke, Martin; Teleman, Andrei, The Kobayashi-Hitchin correspondence, River Edge, NJ: World Scientific Publishing Co. Inc.(1995)

[9] Shoshichi Kobayashi, Differential geometry of complex vector bundles, Princeton University Press (1987)

[10] Indranil Biswas, Georg Schumacher, Geometry of the moduli space of Higgs bundles, Comm. Anal. Geom. 14 (2006), 765-793

[11] Indranil Biswas, Georg Schumacher, The Weil-Petersson current for moduli of vector bundles and applications to orbifolds, arXiv:1509.00304