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Generalized Hitchin-Kobayashi correspondence and Weil-Petersson current

Hassan Jolany

In this paper we extend Hitchin-Kobayashi correspondence along holomorphic fibre space and we show that there exists a generalized Hermitian-Einstein metric which is twisted with Weil-Petersson current.

1 Hitchin-Kobayashi correspondence along holomorphic fibre space

Hitchin and Kobayashi conjectured that the stability of a holomorphic bundle should be related to the existence of Hermitian-Einstein metric. This conjecture solved by Uhlenbeck and Yau for compact Kähler manifolds[7]. The central conjecture in Kähler geometry is about finding canonical metric. Hermitian-Einstein metric is a canonical metric on a holomorphic vector bundle[8]. In this paper we apply Song-Tian program for finding generalized Hermitian-Einstein metric for holomorphic vector bundles. The main computation will appear later. This note is just an announcement.

Takemoto introduced the notion of slope stability for a holomorphic vector bundle which is known as Mumford-Takemoto stability. We give brief introduction on it. Let (X, ω) be a compact Kähler manifold of complex dimension n . The degree of a torsion-free holomorphic vector bundle E is the integral of differential forms

$$\deg_{\omega}(E) = \int_X c_1(E) \wedge \omega^{n-1}$$

of the de Rham representative of the actual first Chern class of X wedge with copies of the given symplectic form. Here $c_1(E) := c_1(\det E)$ and the determinant of a vector bundle is the top exterior power of the vector bundle i.e. a line bundle $\det(V) = \wedge^{rk(V)} V$.

we can write also

$$\deg_{\omega}(E) = \int_X \frac{\sqrt{-1}}{2\pi} \text{tr}(F_h) \wedge \omega^{n-1}$$

where F_h is the curvature form of the Chern connection in E with respect to a smooth Hermitian metric h .

A holomorphic vector bundle $E \rightarrow X$ over a compact Kähler manifold (X, ω) is called stable if every coherent holomorphic subsheaf F of E satisfies

$$0 < \text{rk} F < \text{rk} E \longrightarrow \mu_{\omega}(F) < \mu_{\omega}(E)$$

where μ_{ω} is the ω -slope of the sheaf given by

$$\mu_{\omega}(E) = \frac{\deg_{\omega}(E)}{\text{rk}(E)}$$

We have the same notion of semi-stability, when we replace $<$ to \leq .

A Hermitian metric h in E is called a ω -Hermitian-Einstein metric if

$$\sqrt{-1}\Lambda_{\omega}F_h = \lambda_h id_E$$

with a real constant λ_h , where $\sqrt{-1}\Lambda_{\omega}$ is the contraction with ω , F_h is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle $(E; h)$ and id_E is the identity endomorphism of E .

Here Λ_{ω} be the adjoint operation such that

$$\langle \omega \wedge u, v \rangle_g = \langle u, \Lambda_{\omega}v \rangle_g$$

for $u \in A^{p,q}$, $v \in A^{p+1,q+1}$

If h is a ω -Hermitian-Einstein metric in E with Einstein factor λ_h , we have
d

$$\lambda_h = \frac{2\pi\mu_\omega(E)}{(n-1)!Vol_\omega(X)}$$

In the following theorem we explain Hitchin-Kobayashi correspondence about the existence of ω -Hermitian-Einstein metric in the holomorphic vector bundle E .

S. Kobayashi [5] proved that an irreducible Hermitian-Einstein vector bundle on a compact Kähler manifold is in fact stable. Donaldson [3] proved the existence of a Hermitian-Einstein metric in a stable holomorphic vector bundle on a projective-algebraic surface and later for projective-algebraic manifold of arbitrary dimension. Finally, in 1986, Uhlenbeck and Yau proved the theorem for any arbitrary compact Kähler manifold [7].

Theorem 1.1. Let E be a holomorphic vector bundle on a compact Kähler manifold $(X; \omega)$

I- If E is ω -stable, then there is a ω -Hermitian Einstein metric in E

II- If there is a ω -Hermitian Einstein metric in E , then E is ω -polystable in the sense that E is ω -semistable and is a direct sum

$$E = E_1 \oplus E_2 \oplus \dots \oplus E_k$$

of ω -stable sub bundles E_k of E with $\mu_\omega(E_k) = \mu_\omega(E)$. In particular, if E is irreducible, then it is ω -stable. \square

Now if (X, E) do not admit Hermitian Einstein metric, the following theorem is useful for finding canonical metric on (X, E) which generalize the previous theorem

Theorem 1.2. Let $\pi : (X, E) \rightarrow S$ be a family of poly-stable vector bundles (X_s, E_s) , such that X and S are compact Kähler manifolds, $E \rightarrow X$ is a poly stable holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

where ω_{WP} is a Weil-Petersson metric on moduli space of poly-stable vector bundles \square

Idea of Proof. For the construction of Hermitian-Einstein metrics for stable bundles, we start with a fixed Hermitian metric H_0 and try to construct from it a Hermitian-Einstein metric by deforming H_0 through a one-parameter family of Hermitian metrics H_t ($0 < t < 1$).

We can integrate the vector field and from any initial point get an integral curve given by $h = h(t)$, $0 < t < t_\infty$, where t_∞ is the maximum time-parameter value to which we can extend the integral curve. The integral curve with the initial point $h(0) = I$ is given by

$$\frac{\partial}{\partial t} h(t) = -(\Lambda_{F_{H(t)}^{(X,E)/S}} - \lambda I)h(t)$$

where $H(t)$ is corresponds to hermitian metric of relative vector bundle $K_{X/S} + E$. Now by applying the same method of Donaldson we can get higher estimates for the solution of this flow. Moreover by using Biswas-Schumacher formula for moduli space of poly-stable vector bundles we have Weil-Petersson metric [?] and we get

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

Theorem 1.3. If h is a ω -generalized Hermitian-Einstein metric with Einstein factor λ ,

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

we have

$$\lambda = \frac{2\pi\mu_\omega(E)}{(n-1)!vol_\omega(X)} - \frac{2\pi}{(n-1)!vol_\omega(X)} \int_X \omega_{WP} \wedge \omega^{n-1}$$

\square

We have the same result for twisted Higgs bundles. Let X be a compact complex manifold and L an ample line bundle over X . A twisted higgs bundle over X is a pair (E, ϕ) consisting of a holomorphic vector bundle E over X and a holomorphic bundle morphism.

$$\phi : X \otimes E \rightarrow E$$

for some holomorphic vector bundle X (the twist). Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in \mathbb{R}$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson-Uhlenbeck-Yau Theorem .

Unfortunately, the space of holomorphic bundles of fixed rank and fixed degree, up to isomorphism, is not a Hausdorff space. But, by using GIT one can construct the moduli space $\mathcal{M}(n, d)$ of stable bundles of fixed rank n and degree d , which has structure as an algebraic variety with canonical metric called Weil-Petersson metric. It is known that, when $(n, d) = 1$ for compact Riemann surface σ , the moduli space $\mathcal{M}(n, d)$ is a smooth projective algebraic variety of dimension $n^2(g - 1) + 2$. Tian conjectured that the moduli space $\mathcal{M}(n, d)$ of stable vector bundles of rank n and degree d is smooth quasi-projective variety.

A vector subbundle F of E for which $\phi(F) \subset F \otimes K_X$ is said to be a ϕ -invariant subbundle of E . Stability for Higgs bundles is defined in terms of ϕ -invariant subbundles:

Definition 1.4. A Higg bundle (E, ϕ) is

- Stable if for each proper ϕ -invariant subbundle F one has $\mu(F) < \mu(E)$
- Semi-stable if for each proper ϕ -invariant sub-bundle F one has $\mu(F) < \mu(E)$
- Polystable if $(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2) \dots \oplus (E_r, \phi_r)$ where (E_i, ϕ_i) is stable with $\mu(E_i) = \mu(E)$ for all i .

□

Simpson, started the study of Higgs bundles over compact Kähler manifolds of arbitrary dimension. He showed that a stable Higgs bundle admits a unique Hermitian-Yang-Mills connection. He constructed the moduli space of Higgs bundles over a complex projective manifold

Theorem 1.5. (Simpson[6]) (E, ϕ) is polystable if and only if E admits a hermitian metric h solving the Hitchin equations

$$\sqrt{-1}\Lambda F_h + c[\phi, \phi^*] = \lambda Id_E$$

where F_h denotes the curvature of h , $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and λ is a topological constant.

□

Theorem 1.6. Let $\pi : (X, E, \phi) \rightarrow S$ be a family of poly-stable twisted higg bundles (X_s, E_s, ϕ_s) , such that X and S are compact Kähler manifolds, $E \rightarrow X$ is a poly stable holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} + c[\phi, \phi^*] = \lambda Id_E$$

where ω_{WP} is a Weil-Petersson metric on moduli space of poly-stable Higg vector bundles where such Weil-Petersson metric has been introduced by Schumacher-Biswas[10]

$$\omega_{WP} = \frac{1}{2} \int_{X/S} \text{tr}(\Omega \wedge \Omega) \wedge \frac{\omega_X^{n-1}}{(n-1)!} + \lambda \int_{X/S} \text{tr}(\sqrt{-1}\Omega) \frac{\omega_X^n}{n!} + \sqrt{-1} \partial \bar{\partial} \frac{1}{2} \int_{X/S} \text{tr}(\phi \wedge \phi^*) \frac{\omega_X^{n-1}}{(n-1)!}$$

where λ is determined by

$$\int_{X/S} \text{tr}(\sqrt{-1}\Omega) \frac{\omega_X^{n-1}}{(n-1)!} = \lambda \int_{X/S} \frac{\omega_X^n}{n!}$$

and Ω is the curvature form of (E, h)

□

Takemoto introduced the notion of slope stability for a holomorphic vector bundle which is known as Mumford-Takemoto stability. We give brief introduction on it. Let (X, ω) be a compact Kähler manifold of complex dimension n . The degree of a torsion-free holomorphic vector bundle E is the integral of differential forms

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If h is a ω -Hermitian-Einstein metric in E with Einstein factor λ_h , we have

d

$$\lambda_h = \frac{2\pi\mu_{\omega}(E)}{(n-1)! \text{Vol}_{\omega}(X)}$$

In the following theorem we explain Hitchin-Kobayashi correspondence about the existence of ω -Hermitian-Einstein metric in the holomorphic vector bundle E .

Theorem 1.7. Let E be a holomorphic vector bundle on a compact Kähler manifold $(X; \omega)$

I- If E is ω -stable, then there is a ω -Hermitian Einstein metric in E

II- If there is a ω -Hermitian Einstein metric in E , then E is ω -polystable in the sense that E is ω -semistable and is a direct sum

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Now if (X, E) do not admit Hermitian Einstein metric, the following theorem is useful for finding canonical metric on (X, E) which generalize the previous theorem

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Theorem 1.9. If h is a ω -generalized Hermitian-Einstein metric with Einstein factor λ ,

$$\sqrt{-1}\Lambda_\omega F_h - \sqrt{-1}\Lambda_\omega \pi^* \omega_{WP} = \lambda Id_E$$

we have

$$\lambda = \frac{2\pi\mu_\omega(E)}{(n-1)!vol_\omega(X)} - \frac{2\pi}{(n-1)!vol_\omega(X)} \int_X \omega_{WP} \wedge \omega^{n-1}$$

□

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A vector subbundle F of E for which $\phi(F) \subset F \otimes K_X$ is said to be a ϕ -invariant subbundle of E . Stability for Higgs bundles is defined in terms of ϕ -invariant subbundles:

Definition 1.10. A Higgs bundle (E, ϕ) is

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- Semi-stable if for each proper ϕ -invariant sub-bundle F one has $\mu(F) < \mu(E)$
- Polystable if $(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2) \dots \oplus (E_r, \phi_r)$ where (E_i, ϕ_i) is stable with $\mu(E_i) = \mu(E)$ for all i .

□

Simpson, started the study of Higgs bundles over compact Kähler manifolds of arbitrary dimension. He showed that a stable Higgs bundle admits a unique Hermitian-Yang-Mills connection. He constructed the moduli space of Higgs bundles over a complex projective manifold

Theorem 1.11. (Simpson) (E, ϕ) is polystable if and only if E admits a hermitian metric h solving the Hitchin equations

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where F_h denotes the curvature of h , $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and λ is a topological constant. □

Theorem 1.12. Let $\pi : (X, E, \phi) \rightarrow S$ be a family of poly-stable twisted higg bundles (X_s, E_s, ϕ_s) , such that X and S are compact Kähler manifolds, $E \rightarrow X$ is a holomorphic vector bundle. Then there exists a generalized Hermitian Einstein metric on (X, E) as follows

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where λ is determined by

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and Ω is the curvature form of (E, h) □

In next theorem we show that log slope stability is an open condition. This means that if the central fibre \mathcal{E}_0 be slope stable then all the fibers \mathcal{E}_b , for all $b \in \mathcal{U}$, where \mathcal{U} is open neighborhood of 0 are slope stable. An Irreducible vector bundle E on pair (X, D) is log slope stable if and only if it admit Hermitian-Einstein metric for $X \setminus D$, see [1, 2]

Theorem 1.13. Let \mathcal{X} be a compact Kähler manifold and \mathcal{D} be a simple normal crossing divisor on \mathcal{X} and take $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow B$, be a family of pairs of compact Kähler manifolds. Let \mathcal{E} be a holomorphic vector bundle on $(\mathcal{X}, \mathcal{D})$ such that the central holomorphic vector bundle \mathcal{E}_0 admits irreducible Hermitian-Einstein metric. Then there exists a neighbourhood \mathcal{U} of 0 in B such that for all $b \in \mathcal{U}$, \mathcal{E}_b admits irreducible Hermitian-Einstein metric metrics. □

Proof. The degree of a torsion-free holomorphic vector bundle $\mathcal{E}' = \mathcal{E}|_{\mathcal{X}'}$, where $\mathcal{X}' = \mathcal{X} \setminus \mathcal{D}$ is the integral of differential forms

$$\text{deg}_\omega(\mathcal{E}') = \int_{\mathcal{X}} c_1(\mathcal{E}') \wedge \omega^{n-1}$$

of the de Rham representative of the actual first Chern class of \mathcal{X}' wedge with copies of the given symplectic form. Here $c_1(\mathcal{E}') := c_1(\det \mathcal{E}')$.

A holomorphic vector bundle $\mathcal{E}' \rightarrow \mathcal{X}'$ over a compact Kähler manifold (\mathcal{X}', ω) is called stable if every coherent holomorphic subsheaf \mathcal{F} of \mathcal{E}' satisfies

$$0 < \text{rk} \mathcal{F} < \text{rk} \mathcal{E}' \longrightarrow \mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E}')$$

where μ_ω is the ω -slope of the sheaf given by

$$\mu_\omega^{(\mathcal{X}, \mathcal{D})}(\mathcal{E}') := \frac{\text{deg}_\omega(\mathcal{E}')}{\text{rk}(\mathcal{E}')}$$

We apply the Siu's method for this proof. Let $A''(b) : \mathcal{A}^0(\mathcal{E}'_b) \rightarrow \mathcal{A}^{0,1}(\mathcal{E}'_b)$ be a family of connections on the fibres $\mathcal{E}'_b = \mathcal{E}'|_{\mathcal{X}'_b \setminus \mathcal{D}_b}$ and take $\Omega(A''(b))$ be the related curvature forms.

Take the map

$$\begin{aligned} \Phi : \mathbf{PGL}(\mathcal{E}') \times S &\rightarrow \mathcal{S}(\mathcal{X}', \text{End}(\mathcal{E}')) \\ (f, b) &\rightarrow \sqrt{-1}\Lambda_s \Omega(A''(b)) + f^{-1} \bar{\partial}_{A''} f - \lambda \cdot id \end{aligned}$$

where

$$T_{id}\mathbf{PGL}(\mathcal{E}') = \mathcal{S}(\mathcal{X}', \text{End}(\mathcal{E}')) = \left\{ \phi \in \mathcal{A}^0(\mathcal{X}', \text{End}(\mathcal{E}')) \mid \int \text{tr} \phi g dv = 0 \right\}$$

and λ is the constant given by Irreducible Hermitian-Einstein metric for central fibre \mathcal{E}'_0 . Note that since central fibre \mathcal{E}'_0 is simple, hence the first partial derivative of Φ is bijective. Moreover, the Laplace-Beltrami operator Δ_0 with respect \mathcal{E}'_0 , by using this fact that \mathcal{E}'_0 is simple, the equation $\Delta_0 K = L$ always has solution for an endomorphisms L , and K of \mathcal{E}'_0 . Hence $(f, b) \rightarrow \sqrt{-1}\Lambda_s \Omega(A^n(b) + f^{-1}\bar{\partial}_{A^n} f) - \lambda \cdot id$ always has solution by implicit function theorem and by extending Φ using Sobolev embedding theorem, proof is complete.

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