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► **To cite this version:**

Hassan Jolany. Relative Kähler-Einstein metric on Kähler varieties of positive Kodaira dimension. 2016. hal-01413754v1

HAL Id: hal-01413754

<https://hal.science/hal-01413754v1>

Preprint submitted on 10 Dec 2016 (v1), last revised 16 Sep 2017 (v5)

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Relative Kähler-Einstein metric on Kähler varieties of positive Kodaira dimension

Hassan Jolany

ABSTRACT

In this paper I have introduced a new notion of canonical metric. The notion of generalized Kähler-Einstein metric on the Kähler varieties with an intermediate Kodaira dimension is not suitable and we need to replace Kähler-Einstein metric (KE) to new notion of **Relative Kähler-Einstein metric** (RKE) for such varieties and its connection with fiberwise Calabi-Yau foliation

1. *Relative Kähler-Einstein metric*

Let X_0 be a projective variety with canonical line bundle $K \rightarrow X_0$ of Kodaira dimension

$$\kappa(X_0) = \limsup \frac{\log \dim H^0(X_0, K^{\otimes \ell})}{\log \ell}$$

This can be shown to coincide with the maximal complex dimension of the image of X_0 under pluri-canonical maps to complex projective space, so that $\kappa(X_0) \in \{-\infty, 0, 1, \dots, m\}$.

Lelong number: Let $W \subset \mathbb{C}^n$ be a domain, and Θ a positive current of degree (q, q) on W . For a point $p \in W$ one defines

$$\mathfrak{v}(\Theta, p, r) = \frac{1}{r^{2(n-q)}} \int_{|z-p|<r} \Theta(z) \wedge (dd^c|z|^2)^{n-q}$$

The Lelong number of Θ at p is defined as

$$\mathfrak{v}(\Theta, p) = \lim_{r \rightarrow 0} \mathfrak{v}(\Theta, p, r)$$

Let Θ be the curvature of singular hermitian metric $h = e^{-u}$, one has

$$\mathfrak{v}(\Theta, p) = \sup\{\lambda \geq 0 : u \leq \lambda \log(|z-p|^2) + O(1)\}$$

DEFINITION 1. Let $\pi : X \rightarrow Y$ be a holomorphic map of complex manifolds. A real d-closed $(1, 1)$ -form ω on X is said to be a relative Kähler form for π , if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function Ψ on W such that $\omega + \pi^*(\sqrt{-1}\partial\bar{\partial}\Psi)$ is a Kähler form on $\pi^{-1}(W)$. A morphism π is said to be Kähler, if there exists a relative Kähler form for π , and $\pi : X \rightarrow Y$ is said to be a Kähler fiber space, if π is proper, Kähler, and surjective with connected fibers.

We consider an effective holomorphic family of complex manifolds. This means we have a holomorphic map $\pi : X \rightarrow Y$ between complex manifolds such that

1. The rank of the Jacobian of π is equal to the dimension of Y everywhere.
 2. The fiber $X_t = \pi^{-1}(t)$ is connected for each $t \in Y$
 3. X_t is not biholomorphic to $X_{t'}$ for distinct points $t; t' \in B$.
- It is worth to mention that Kodaira showed that all fibers are dieomorphic to each other. The relative Kähler form is denoted by

$$\omega_{X/Y} = \sqrt{-1}g_{\alpha, \bar{\beta}}(z, s)dz^\alpha \wedge d\bar{z}^\beta$$

Moreover take $\omega_X = \sqrt{-1}\partial\bar{\partial} \log \det g_{\alpha, \bar{\beta}}(z, y)$ on the total space X . The fact is ω_X in general is not Kähler on total space and $\omega_X|_{X_y} = \omega_{X_y}$. More precisely $\omega_X = \omega_F + \omega_H$ where ω_F is a form along fiber direction and ω_H is a form along horizontal direction. ω_H may not be Kähler metric in general, but ω_F is Kähler metric. Now let ω be a relative Kähler form on X and $m := \dim X - \dim Y$, We define the relative Ricci form $Ric_{X/Y, \omega}$ of ω by

$$Ric_{X/Y, \omega} = -\sqrt{-1}\partial\bar{\partial} \log(\omega^m \wedge \pi^*|dy_1 \wedge dy_2 \wedge \dots \wedge dy_k|^2)$$

where (y_1, \dots, y_k) is a local coordinate of Y , where Y is a curve.

Let for family $\pi : \mathcal{X} \rightarrow Y$

$$\rho_{y_0} : T_{y_0}Y \rightarrow H^1(X, TX) = \mathcal{H}_{\bar{\sigma}}^{0,1}(TX)$$

be the Kodaira–Spencer map for the corresponding deformation of X over Y at the point $y_0 \in Y$ where $\mathcal{X}_{y_0} = X$

If $v \in T_{y_0}Y$ is a tangent vector, say $v = \frac{\partial}{\partial y} |_{y_0}$ and $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift to \mathcal{X} along X , then

$$\bar{\partial} \left(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha(z)}{\partial z^\beta} \frac{\partial}{\partial z^\alpha} dz^\beta$$

is a $\bar{\partial}$ -closed form on X , which represents $\rho_{y_0}(\partial/\partial y)$.

The Kodaira-Spencer map is induced as edge homomorphism by the short exact sequence

$$0 \rightarrow T_{X/Y} \rightarrow TX \rightarrow \pi^*T_Y \rightarrow 0$$

Weil-Petersson metric when fibers are Calabi-Yau manifolds can be defined as follows[6].

DEFINITION 2. Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. The local Kuranishi family of polarized Calabi-Yau manifolds $\mathcal{X} \rightarrow Y$ is smooth (unobstructed) by the Bogomolov-Tian-Todorov theorem. Let each fibers is a Calabi-Yau manifold. One can assign the unique (Ricci-flat) Yau metric $g(y)$ on X_y . The metric $g(y)$ induces a metric on $\wedge^{0,1}(TX)$. For $v, w \in T_y(Y)$, one then defines the Weil-Petersson metric on the base Y by

$$g_{WP}(v, w) = \int_X \langle \rho(v), \rho(w) \rangle_{g(y)}$$

1.1. Vafa-Yau’s semi Ricci-flat metric

The volume of fibers $\pi^{-1}(y) = X_y$ is a homological constant independent of y , and we assume that it is equal to 1. Since fibers are Calabi-Yau manifolds so $c_1(X_y) = 0$, hence there is a smooth function F_y such that $Ric(\omega_y) = \sqrt{-1}\partial\bar{\partial}F_y$ and $\int_{X_y} (e^{F_y} - 1)\omega_y^{n-m} = 0$. The function F_y vary smoothly in y . By Yau’s theorem there is a unique Ricci-flat Kähler metric $\omega_{SRF, y}$ on X_y cohomologous to ω_0 . So there is a smooth function ρ_y on $\pi^{-1}(y) = X_y$ such that $\omega_0|_{X_y} + \sqrt{-1}\partial\bar{\partial}\rho_y = \omega_{SRF, y}$ is the unique Ricci-flat Kähler metric on X_y . If we normalize

by $\int_{X_y} \rho_y \omega_0^n |_{X_y} = 0$ then ρ_y varies smoothly in y and defines a smooth function ρ on X and we let

$$\omega_{SRF} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho$$

which is called as Semi-Ricci Flat metric. Such Semi-Flat Calabi-Yau metrics were first constructed by Greene-Shapere-Vafa-Yau on surfaces [4]. More precisely, a closed real $(1, 1)$ -form ω_{SRF} on open set $U \subset X \setminus S$, (where S is proper analytic subvariety contains singular points of X) will be called semi-Ricci flat if its restriction to each fiber $X_y \cap U$ with $y \in f(U)$ be Ricci-flat. Notice that ω_{SRF} is semi-positive

For the log-Calabi-Yau fibration $f : (X, D) \rightarrow Y$, such that (X_t, D_t) are log Calabi-Yau varieties. If (X, ω) be a Kähler variety with Poincaré singularities then the semi-Ricci flat metric has $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model which we call it **fibrewise Poincaré singularities**.

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2 (\log |z_k|^2)^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left(\sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

We can define the same **fibrewise conical singularities**. and the semi-Ricci flat metric has $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left(\sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

In fact the previous remark will tell us that the semi Ricci flat metric ω_{SRF} has pole singularities with Poincare growth.

DEFINITION 3. Let X be a smooth projective variety with $\kappa(X) \geq 0$. Then for a sufficiently large $m > 0$, the complete linear system $|m!K_X|$ gives a rational fibration with connected fibers $f : X \dashrightarrow Y$. We call $f : X \dashrightarrow Y$ the Iitaka fibration of X . Iitaka fibration is unique in the sense of birational equivalence. We may assume that f is a morphism and Y is smooth. For Iitaka fibration f we have

1. For a general fiber F , $\kappa(F) = 0$ holds.
2. $\dim Y = \kappa(X)$.

Let X be a Kähler variety with an intermediate Kodaira dimension $\kappa(X) > 0$ then we have an Iitaka fibration $\pi : X \rightarrow Y = \text{Proj}R(X, K_X) = X_{can}$ such that fibers are Calabi-Yau varieties. We set $K_{X/Y} = K_X \otimes \pi^* K_Y^{-1}$ and call it the relative canonical bundle of $\pi : X \rightarrow Y$

DEFINITION 4. Let X be a Kähler variety with $\kappa(X) > 0$ then the **relative Kähler-Einstein metric** is defined as follows

$$Ric_{X/Y}^{h_{X/Y}^\omega}(\omega) = -\omega$$

where

$$Ric_{X/Y}^{h_{X/Y}^\omega}(\omega) = \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\omega^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m} \right)$$

and ω_{can} is a canonical metric on $Y = X_{can}$.

$$Ric_{X/Y}^{\omega_{SRF}}(\omega) = \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m} \right) = \omega_{WP}$$

here ω_{WP} is a Weil-Petersson metric[6].

Note that if $\kappa(X) = -\infty$ then along Mori fibre space $f : X \rightarrow Y$ we can define Relative Kähler-Einstein metric as

$$Ric_{X/Y}^{h_{X/Y}}(\omega) = \omega$$

when fibers and base are K-poly-stable, see[5]

Note that, if X be a Calabi-Yau variety and we have a holomorphic fibre space $\pi : X \rightarrow Y$, which fibres are Calabi-Yau varieties, then we have the relative Ricci flat metric $Ric_{X/Y}(\omega) = 0$, which turns out to be $Ric(\omega) = 0 + \pi^*(\omega'_Y)$ where $\omega'_Y = \omega_{WP}$, see[3]

Then by the definition of Relative Kähler metric $Ric(\omega) = -\omega + \pi^*(\omega'_Y)$ which $\omega'_Y = \omega_{WP}$ is Weil-Petersson metric (we can define Weil-Petersson metric completely on the base of Iitaka fibration) by using higher canonical bundle formula of Fujino-Mori

For the existence of Kähler-Einstein metric when our variety is of general type, we need to the nice deformation of Kähler-Ricci flow and for intermediate Kodaira dimension we need to work on relative version of Kähler Ricci flow. i.e

$$\frac{\partial \omega}{\partial t} = -Ric_{X/Y}(\omega) - \omega$$

take the reference metric as $\tilde{\omega}_t = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m})$ then the version of Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi_t)^n \wedge \pi^* \omega_{can}^m}{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m} - \phi_t$$

Take the relative canonical volume form $\Omega_{X/Y} = \frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m}$ and $\omega_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi_t$, then

$$\frac{\partial \omega_t}{\partial t} = \frac{\partial \tilde{\omega}_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial \phi_t}{\partial t}$$

By taking $\omega_\infty = -Ric(\Omega_{X/Y}) + \sqrt{-1}\partial\bar{\partial}\phi_\infty$ we obtain after using estimates

$$\log \frac{\omega_\infty^n}{\Omega_{X/Y}} - \phi_\infty = 0$$

By taking $-\sqrt{-1}\partial\bar{\partial}$ of both sides we get

$$Ric_{X/Y}(\omega_\infty) = -\omega_\infty$$

hence by the definition of relative Kähler-metric and higher canonical bundle formula we have the Song-Tian metric[1, 2]

$$Ric(\omega_\infty) = -\omega_\infty + \pi^*(\omega_{WP})$$

More explicitly on pair (X, D) where D is a snc divisor, we can write

$$Ric(\omega_{can}) = -\omega_{can} + \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

where B^D is \mathbb{Q} -divisor on X such that $\pi_* \mathcal{O}_X([iB^D]) = \mathcal{O}_B$ ($\forall i > 0$). Here $s_P^D := b(1 - t_P^D)$ where t_P^D is the log-canonical threshold of π^*P with respect to $(X, D - B^D/b)$ over the generic point η_P of P . i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid (X, D - B^D/b + t\pi^*(P)) \text{ is sub log canonical over } \eta_P\}$$

and ω_{can} has zero Lelong number

Remark:Note that the log semi-Ricci flat metric ω_{SRF}^D is not continuous in general. But if the central fiber has at worst canonical singularities and the central fiber (X_0, D_0) be itself as Calabi-Yau pair, then by open condition property of Kahler-Einstein metrics, semi-Ricci flat metric is smooth in an open Zariski subset.

Remark:So by applying the previous remark, the relative volume form

$$\Omega_{(X,D)/Y} = \frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m |S|^2}$$

is not smooth in general, where $S \in H^0(X, L_N)$ and N is a divisor which come from canonical bundle formula of Fujino-Mori. Note that Song-Tian measure is invariant under birational change

Now we try to extend the Relative Ricci flow to the fiberwise conical relative Ricci flow. We define the conical Relative Ricci flow on pair $\pi : (X, D) \rightarrow Y$ where D is a simple normal crossing divisor as follows

$$\frac{\partial \omega}{\partial t} = -Ric_{(X,D)/Y}(\omega) - \omega + [N]$$

where N is a divisor which come from canonical bundle formula of Fujino-Mori.

Take the reference metric as $\tilde{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) Ric(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m})$ then the conical relative Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\omega}_t + Ric(h_N) + \sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \pi^* \omega_{can}^m |S_N|^2}{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m} - \phi_t$$

Now we prove the C^0 -estimate for this relative Monge-Ampere equation. We use the following important lemma from Schumacher and also Cheeger-Yau,

LEMMA 1.1. *Suppose that the Ricci curvature of ω is bounded from below by negative constant -1 . Then there exists a strictly positive function $P_n(diam(X, D))$, depending on the dimension n of X and the diameter $diam(X, D)$ with the following property:*

Let $0 < \epsilon \leq 1$. If g is a continuous function and f is a solution of

$$(-\Delta_\omega + \epsilon)f = g,$$

then

$$f(z) \geq P_n(diam(X, D)) \int_X g dV_\omega$$

So along relative Kähler-Ricci flow we have $Ric(\omega) \geq -2\omega$ where ω is the solution of Kähler-Ricci flow. But if we restrict our relative Monge-Ampere equation to each fiber (X_s, D_s) , then we need diameter bound on the fibers, i.e.,

$$diam(X_s \setminus D_s, \omega_s) \leq C$$

But from recent result of Takayama(On Moderate Degenerations of Polarized Ricci-Flat Kähler Manifolds, J. Math. Sci. Univ. Tokyo, 22 (2015), 469–489) we know that we have

$$diam(X_s \setminus D_s, \omega_s) \leq 2 + D \int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega}_s}{|S_s|^2}$$

if and only if we have 1) central fiber $X_0 \setminus D_0$ has at worst canonical singularities and $K_{X_0} + D_0 = \mathcal{O}_{X_0}(D_0)$ which means the central fiber itself be log Calabi-Yau variety.

So this means that we have C^0 -estimate for relative Kähler-Ricci flow if and only if the central fiber be Calabi-Yau variety with at worst canonical singularities. Note that to get C^∞ -estimate we need just check that our reference metric is bounded. So it just remain to see that ω_{WP} is bounded. But when fibers are not smooth in general, Weil-Petersson metric is not bounded and Yoshikawa in Proposition 5.1 in [?] showed that under the some additional condition when central fiber X_0 is reduced and irreducible and has only canonical singularities we have

$$0 \leq \omega_{WP} \leq C \frac{\sqrt{-1} |s|^{2r} ds \wedge d\bar{s}}{|s|^2 (-\log |s|)^2}$$

So we can get easily by ancient method! the C^∞ -solution.

Note that the main difficulty of the solution of C^∞ for the solution of relative Kähler-Einstein metric is that the null direction Vafa-Yau semi Ricci flat metric ω_{SRF} gives a foliation along Iitaka fibration $\pi : X \rightarrow Y$ and we call it Song-Tian-Yau-Vafa foliation (shortly we call it STYV foliation) and can be defined as follows

$$\mathcal{F} = \{\theta \in TX | \omega_{SRF}(\theta, \bar{\theta}) = 0\}$$

and along log Iitaka fibration $\pi : (X, D) \rightarrow Y$, we can define the following foliation

$$\mathcal{F}' = \{\theta \in TX' | \omega_{SRF}^D(\theta, \bar{\theta}) = 0\}$$

where $X' = X \setminus D$. In fact the method of Song-Tian (only in fiber direction and they couldn't prove the estimates in horizontal direction which is the main part of computation) works when $\omega_{SRF} > 0$. More precisely, in null direction, the function φ satisfies in the complex Monge-Ampere equation

$$(e^{-t} \omega_{WP}^D + (1 - e^{-t}) \omega_0 + Ric(h_N) + \sqrt{-1} \sigma \bar{\sigma} \varphi)^\kappa = 0$$

gives rise to a foliation by X by complex sub-manifolds.

For the null direction we need to an extension of Monge-Ampere foliation method of Gang Tian in [18]. It will appear in my new paper [19]

A complex analytic space is a topological space such that each point has an open neighborhood homeomorphic to some zero set $V(f_1, \dots, f_k)$ of finitely many holomorphic functions in \mathbb{C}^n , in a way such that the transition maps (restricted to their appropriate domains) are biholomorphic functions.

Lemma: Fiberwise Calabi-Yau foliation

$$\mathcal{F}' = \{\theta \in TX' | \omega_{SRF}^D(\theta, \bar{\theta}) = 0\}$$

is complex analytic space and its leaves are also complex analytic spaces. Moreover the direct image $\pi_* \mathcal{F}'$ is complex analytic spaces on the log canonical model X_{can}^D

Lemma: Let \mathcal{L} be a leaf of $f_* \mathcal{F}'$, then \mathcal{L} is a closed complex submanifold and the leaf \mathcal{L} can be seen as fiber on the moduli map

$$\eta : \mathcal{Y} \rightarrow \mathcal{M}_{can}^D$$

where \mathcal{M}_{can}^D is the moduli space of log calabi-Yau fibers with at worst canonical singularities and

$$\mathcal{Y} = \{y \in Y_{reg} | (X_y, D_y) \text{ is Kawamata log terminal pair}\}$$

DEFINITION 5. Let $\pi : X \rightarrow B$ be a family of Kähler-Einstein varieties, then we introduce the new notion of stability and call it fiberwise KE-stability, if the Weil-Petersson distance

$d_{WP} < \infty$. Note when fibers are Calabi-Yau varieties, Takayama, by using Tian's Kähler-potential for Weil-Petersson metric for moduli space of Calabi-Yau varieties showed that Fiberwise KE-Stability is as same as when the central fiber is Calabi-Yau variety with at worst canonical singularities

So along canonical model $\pi : X \rightarrow X_{can}$ for mildly singular variety X , we have $Ric(\omega) = -\omega + \omega_{WP}$ if and only if our family of fibers be fiberwise KE-stable

Let $\pi : (X, D) \rightarrow B$ is a holomorphic submersion onto a compact Kähler manifold B with $c_1(K_B) < 0$ where the fibers are log Calabi-Yau manifolds and D is a simple normal crossing divisor in X . Let our family of fibers is fiberwise KE-stable. Then (X, D) admits a unique twisted Kähler-Einstein metric ω_B solving

$$Ric(\omega_B) = -\omega_B + \omega_{WP}^D + (1 - \beta)[N]$$

where ω_{WP} is the logarithmic Weil-Petersson form on the moduli space of log Calabi-Yau fibers and $[D]$ is the current of integration over D .

More precisely, we have

$$Ric(\omega_{can}) = -\omega_{can} + \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

where B^D is \mathbb{Q} -divisor on X such that $\pi_* \mathcal{O}_X([iB_+^D]) = \mathcal{O}_B$ ($\forall i > 0$). Here $s_P^D := b(1 - t_P^D)$ where t_P^D is the log-canonical threshold of π^*P with respect to $(X, D - B^D/b)$ over the generic point η_P of P . i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid (X, D - B^D/b + t\pi^*(P)) \text{ is sub log canonical over } \eta_P\}$$

and ω_{can} has zero Lelong number.

With cone angle $2\pi\beta$, ($0 < \beta < 1$) along the divisor D , where h is an Hermitian metric on line bundle corresponding to divisor N , i.e., L_N . This equation can be solved. Take, $\omega = \omega(t) = \omega_B + (1 - \beta)Ric(h) + \sqrt{-1}\partial\bar{\partial}v$ where $\omega_B = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m})$, by using Poincare-Lelong equation,

$$\sqrt{-1}\partial\bar{\partial} \log |s_N|_h^2 = -c_1(L_N, h) + [N]$$

we have

$$\begin{aligned} Ric(\omega) &= \\ &= -\sqrt{-1}\partial\bar{\partial} \log \omega^m \\ &= -\sqrt{-1}\partial\bar{\partial} \log \pi_* \Omega_{(X,D)/Y} - \sqrt{-1}\partial\bar{\partial}v - (1 - \beta)c_1([N], h) + (1 - \beta)\{N\} \end{aligned}$$

and

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial} \log \pi_* \Omega_{(X,D)/Y} + \sqrt{-1}\partial\bar{\partial}v &= \\ &= \sqrt{-1}\partial\bar{\partial} \log \pi_* \Omega_{(X,D)/Y} + \omega - \omega_B - Ric(h) \end{aligned}$$

Hence, by using

$$\omega_{WP}^D = \sqrt{-1} \partial \bar{\partial} \log \left(\frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m |S|^2} \right)$$

we get

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} \log \pi_* \Omega_{(X,D)/Y} + \sqrt{-1} \partial \bar{\partial} v &= \\ &= \omega - \omega_{WP}^D - (1 - \beta) c_1(N) \end{aligned}$$

So,

$$Ric(\omega) = -\omega + \omega_{WP}^D + (1 - \beta)[N]$$

which is equivalent with

$$Ric_{(X,D)/Y}(\omega) = -\omega + [N]$$

Uniqueness result of Relative Kähler-Einstein metric: Uniqueness of the solutions of relative Kahler Ricci flow along litaka fibration or $\pi : X \rightarrow X_{can}$ or along log canonical model $\pi : (X, D) \rightarrow X_{can}^D$. Let ϕ_0 and ψ_0 be ω -plurisubharmonic functions such that $\mathbf{v}(\phi_0, x) = 0$ for all $x \in X$, let ϕ_t , and ψ_t be the solutions of relative Kähler Ricci flow starting from ϕ_0 and ψ_0 , respectively. Then in [16] it has been proven that if $\phi_0 < \psi_0$ then $\phi_t < \psi_t$ for all t . In particular, the flow is unique. So from the deep result of Tsuji-Schumacher[15], it has been shown that Weil-Petersson metric has zero Lelong number on moduli space of Calabi-Yau varieties, and by the same method we can show that logarithmic Weil-Petersson metric has zero Lelong number on moduli space of log Calabi-Yau varieties, hence by taking the initial metric to be Weil-Petersson metric or logarithmic Weil-Petersson metric and since Weil-Petersson metric or logarithmic Weil-Petersson metric are Kahler and semi-positive hence we get the uniqueness of the solutions of relative Kähler Ricci flow.

The relation between the Existence of Zariski Decomposition and the Existence of Initial Kähler metric along relative Kähler Ricci flow:

Finding an initial Kähler metric ω_0 to run the Kähler Ricci flow is important. Along holomorphic fibration with Calabi-Yau fibres, finding such initial metric is a little bit mysterious. In fact, we show that how the existence of initial Kähler metric is related to finite generation of canonical ring along singularities.

Let $\pi : X \rightarrow Y$ be an litaka fibration of projective varieties X, Y , (possibly singular) then is there always the following decomposition

$$K_Y + \frac{1}{m!} \pi_* \mathcal{O}_X(m! K_{X/Y}) = P + N$$

where P is semiample and N is effective. The reason is that, If X is smooth projective variety, then as we mentioned before, the canonical ring $R(X, K_X)$ is finitely generated. We may thus assume that $R(X, kK_X)$ is generated in degree 1 for some $k > 0$. Passing to a log resolution of $|kK_X|$ we may assume that $|kK_X| = M + F$ where F is the fixed divisor and M is base point free and so M defines a morphism $f : X \rightarrow Y$ which is the litaka fibration. Thus $M = f^* O_Y(1)$ is semiample and F is effective.

In singular case, if X is log terminal. By using Fujino-Mori's higher canonical bundle formula, after resolving X' , we get a morphism $X' \rightarrow Y'$ and a klt pair $K_{Y'} + B_{Y'}$. The Y described above is the log canonical model of $K_{Y'} + B_{Y'}$ and so in fact (assuming as above that $Y' \rightarrow Y$ is a morphism), then $K_{Y'} + B_{Y'} \sim_{\mathbb{Q}} P + N$ where P is the pull-back of a rational multiple of $O_Y(1)$ and N is effective (the stable fixed divisor). If $Y' \rightarrow Y$ is not a morphism, then P will have a base locus corresponding to the indeterminacy locus of this map.

So the existence of Zariski decomposition is related to the finite generation of canonical ring (when X is smooth or log terminal). Now if such Zariski decomposition exists then, there exists a singular hermitian metric h , with semi-positive Ricci curvature $\sqrt{-1}\Theta_h$ on P , and it is enough to take the initial metric $\omega_0 = \sqrt{-1}\Theta_h + [N]$ or $\omega_0 = \sqrt{-1}\Theta_h + \sqrt{-1}\delta\delta\bar{\partial}\|S_N\|^{2\beta}$ along relative Kähler Ricci flow

$$\frac{\partial\omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

with log terminal singularities.

So when X, Y have at worst log terminal singularities (hence canonical ring is f.g and we have initial Kähler metric to run Kähler Ricci flow with starting metric ω_0) and central fibre is Calabi-Yau variety, and $-K_Y < 0$, then all the fibres are Calabi-Yau varieties and the relative Kähler-Ricci flow converges to ω which satisfies in

$$Ric(\omega) = -\omega + f^*\omega_{WP}$$

Remark: The fact is that the solutions of relative Kähler-Einstein metric or Song-Tian metric $Ric(\omega) = -\omega + f^*\omega_{WP}$ may not be C^∞ . In fact we have C^∞ of solutions if and only if the Song-Tian measure or Tian's Kähler potential be C^∞ . Now we explain that under some following algebraic condition we have C^∞ -solutions for

$$Ric(\omega) = -\omega + f^*\omega_{WP}$$

along Iitaka fibration. We recall the following Kawamata's theorem [17].

THEOREM 1.2. *Let $f : X \rightarrow B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$, $Q = \sum_l Q_l$, be normal crossing divisors on X and B , respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum_j d_j P_j$ be a \mathbb{Q} -divisor on X , where d_j may be positive, zero or negative, which satisfies the following conditions A,B,C:*

A) $D = D^h + D^v$ such that any irreducible component of D^h is mapped surjectively onto B by f , $f : \text{Supp}(D^h) \rightarrow B$ is relatively normal crossing over $B \setminus Q$, and $f(\text{Supp}(D^v)) \subset Q$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical)

B) $d_j < 1$ for all j

C) The natural homomorphism $\mathcal{O}_B \rightarrow f_*\mathcal{O}_X([-D])$ is surjective at the generic point of B .

D) $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B .

Let

$$\begin{aligned} f^*Q_l &= \sum_j w_{lj}P_j \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\ \delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\ \Delta &= \sum_l \delta_l Q_l. \\ M &= L - \Delta. \end{aligned}$$

Then M is nef.

The following theorem is straightforward from Kawamata's theorem

THEOREM 1.3. *Let $d_j < 1$ for all j be as above in Theorem 0.11, and fibers be log Calabi-Yau pairs, then*

$$\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2}$$

is continuous on a nonempty Zariski open subset of B .

Since the inverse of volume gives a singular hermitian line bundle, we have the following theorem from Theorem 0.11

THEOREM 1.4. *Let $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B and*

$$\begin{aligned} f^*Q_l &= \sum_j w_{lj} P_j \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\ \delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\ \Delta &= \sum_l \delta_l Q_l. \\ M &= L - \Delta. \end{aligned}$$

Then

$$\left(\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2} \right)^{-1}$$

is a continuous hermitian metric on the \mathbb{Q} -line bundle $K_B + \Delta$ when fibers are log Calabi-Yau pairs.

Conjecture: The twisted Kähler-Einstein metric $Ric(\omega) = -\omega + \alpha$ where α is a semi-positive current has unique solution if and only if α has zero Lelong number

THEOREM 1.5. *The maximal time existence T for the solutions of relative Kähler Ricci flow is*

$$T = \sup\{t \mid e^{-t}[\omega_0] + (1 - e^{-t})c_1(K_{X/Y} + D) \in \mathcal{K}((X, D)/Y)\}$$

where $\mathcal{K}((X, D)/Y)$ denote the relative Kähler cone of $f : (X, D) \rightarrow Y$

Now take we have holomorphic fibre space $f : X \rightarrow Y$ such that fibers and base are Fano K-poly stable, then we have the relative Kähler-Einstein metric

$$Ric_{X/Y}(\omega) = \omega$$

we need to work on relative version of Kähler Ricci flow. i.e

$$\frac{\partial \omega}{\partial t} = -Ric_{X/Y}(\omega) + \omega$$

take the reference metric as $\tilde{\omega}_t = e^t \omega_0 + (1 + e^t) Ric(\frac{(\int_{X/Y} \omega_{SKE}^{n+1}) \wedge \pi^* \omega_Y^m}{\pi^* \omega_Y^m})$ then the version of Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \pi^* \omega_Y^m}{(\int_{X/Y} \omega_{SKE}^{n+1}) \wedge \pi^* \omega_Y^m} + \phi_t$$

where ω_Y is the Kähler-Einstein metric corresponding to $Ric(\omega_Y) = \omega_Y$ and ω_{SKE} is the fiberwise Fano Kähler-Einstein metric.

In fact the relative volume form is $\Omega_{X/Y} = \frac{(\int_{X/Y} \omega_{SKE}^{n+1}) \wedge \pi^* \omega_Y^m}{\pi^* \omega_Y^m}$ and we have the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \frac{(\tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi_t)^n}{\Omega_{X/Y}} + \phi_t$$

Hence from $Ric_{X/Y}(\omega) = \omega$ by using the definition of relative Kähler metric we obtain $Ric(\omega) = \omega + f^*(\omega'_Y)$, for some ω'_Y on the base which this metric is correspond to canonical metric on moduli part of family of fibers, which is Weil-Petersson metric $\omega_{WP} = \int_{X/Y} c_1(K_{X/Y}, h)^{n+1}$. Moreover I think the equation of $Ric(\omega) = \omega + f^*(\omega_{WP})$ work when we have $0 < \int_{X/Y} \omega_{SKE}^{n+1} < C$ which in general ω_{SKE} may not be semi-positive and semi-positivity of fiberwise Fano Kähler-Einstein metric is correspond to K-poly stability of total space X (this is my conjecture). The same assumption must holds when fibers are of general types. See [5]

Remark: Note that we still don't know canonical bundle type formula along Mori-fiber space. So finding explicit Song-Tian type metric on pair (X, D) along Mori fiber space when base and fibers are K-poly stable is not known yet.

Conjecture: Let $\pi : X \rightarrow B$ is smooth, and every X_t is K-poly stable. Then the plurigenera $P_m(X_t) = \dim H^0(X_t, -mK_{X_t})$ is independent of $t \in B$ for any m .

Idea of proof. We can apply the relative Kähler Ricci flow method for it. In fact if we prove that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) + \omega(t)$$

has long time solution along Fano fibration such that the fibers are K-poly stable then we can get the invariance of plurigenera in the case of K-poly stability

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