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Canonical metric on moduli spaces of log Calabi-Yau varieties

Hassan Jolany

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Abstract

In this paper, we give a short proof of closed formula [9],[18] of logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties (if exists!) of conic and Poincare singularities and its connection with Bismut-Vergne localization formula. Moreover we give a relation between logarithmic Weil-Petersson metric and the logarithmic version of semi Ricci flat metric on the family of log Calabi-Yau pairs with conical singularities. In final we consider the semi-positivity of singular logarithmic Weil-Petersson metric on the moduli space of log-Calabi-Yau varieties. Moreover, we show that Song-Tian-Tsuji measure is bounded along Iitaka fibration if and only if central fiber has log terminal singularities and we consider the goodness of fiberwise Calabi-Yau metric in the sense of Mumford and goodness of singular Hermitian metric corresponding to Song-Tian-Tsuji measure.

Introduction

In this note we try to find the Kähler potential of logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties. We use the analysis of semi Ricci-flat metric introduced in [2]. Our method of proof is completely different from the proof of Candelas et al. and Tian. See [8],[9],

Historically, A. Weil introduced a Kähler metric for the Teichmuller space $\mathcal{T}_{g,n}$, the space of homotopy marked Riemann surfaces of genus g with n punctures and negative Euler characteristic. The Weil-Petersson metric measures the variations of the complex structure of R . The quotient of the Teichmuller space $\mathcal{T}_{g,n}$ by the action of the mapping class group is the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$. The Weil-Petersson metric is mapping class group invariant and descends to $\mathcal{M}_{g,n}$. A.Fujiki [14] considered Weil-Petersson metric on moduli space of general type Kähler-Einstein varieties. In this note we consider the logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties(if exists!).

We start we some elementary definitions of relative Kähler metric from Mourougane and Takayama [19].

Definition 1 Let $\pi : X \rightarrow Y$ be a holomorphic map of complex manifolds. A real d -closed $(1, 1)$ -form ω on X is said to be a relative Kähler form for π , if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function Ψ on W such that $\omega + \pi^*(\sqrt{-1}\partial\bar{\partial}\Psi)$ is a Kähler form on $\pi^{-1}(W)$. A morphism π is said to be Kähler, if there exists a relative Kähler form for π , and $\pi : X \rightarrow Y$ is said to be a Kähler fiber space, if π is proper, Kähler, and surjective with connected fibers.

We consider an effective holomorphic family of complex manifolds. This means we have a holomorphic map $\pi : X \rightarrow Y$ between complex manifolds such that

1. The rank of the Jacobian of π is equal to the dimension of Y everywhere.
2. The fiber $X_t = \pi^{-1}(t)$ is connected for each $t \in Y$
3. X_t is not biholomorphic to $X_{t'}$ for distinct points $t, t' \in B$.

It is worth to mention that Kodaira showed that all fibers are dieomorphic to each other.

For any effective holomorphic family of compact manifolds $\pi : X \rightarrow Y$ of dimension n with fibers X_y for $y \in Y$ the Calabi-Yau forms $\omega_{X/Y}$ depend differentiably on the parameter y . The relative Kähler form is denoted by

$$\omega_{X/Y} = \sqrt{-1}g_{\alpha,\bar{\beta}}(z, y)dz^\alpha \wedge d\bar{z}^\beta$$

Moreover take $\omega_X = \sqrt{-1}\partial\bar{\partial}\log\det g_{\alpha,\bar{\beta}}(z, y)$ on the total space X . The fact is ω_X in general is not Kähler on total space and $\omega_X|_{X_y} = \omega_{X_y}$. More precisely $\omega_X = \omega_F + \omega_H$ where ω_F is a form along fiber direction and ω_H is a form along horizontal direction. ω_H may not be Kähler metric in general, but ω_F is Kähler metric. Now let ω be a relative Kähler form on X and $m := \dim X - \dim Y$, We define the relative Ricci form $Ric_{X/Y,\omega}$ of ω by

$$Ric_{X/Y,\omega} = -\sqrt{-1}\partial\bar{\partial}\log(\omega^m \wedge \pi^*|dy_1 \wedge dy_2 \wedge \dots \wedge dy_k|^2)$$

where (y_1, \dots, y_k) is a local coordinate of Y . Here Y assumed to be a curve

Let for family $\pi : \mathcal{X} \rightarrow Y$

$$\rho_{y_0} : T_{y_0}Y \rightarrow H^1(X, TX) = \mathcal{H}_\sigma^{0,1}(TX)$$

be the Kodaira–Spencer map for the corresponding deformation of X over Y at the point $y_0 \in Y$ where $\mathcal{X}_{y_0} = X$

If $v \in T_{y_0}Y$ is a tangent vector, say $v = \frac{\partial}{\partial y}|_{y_0}$ and $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift to \mathcal{X} along X , then

$$\bar{\partial} \left(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha(z)}{\partial z^\beta} \frac{\partial}{\partial z^\alpha} dz^\beta$$

is a $\bar{\partial}$ -closed form on X , which represents $\rho_{y_0}(\partial/\partial y)$.

The Kodaira–Spencer map is induced as edge homomorphism by the short exact sequence

$$0 \rightarrow T_{X/Y} \rightarrow TX \rightarrow \pi^*TY \rightarrow 0$$

We briefly explain about the Weil-Petersson metric on moduli space of polarized Calabi-Yau manifolds. We study the moduli space of Calabi-Yau manifolds via the Weil-Petersson metric. We outline the important properties of such metrics here.

The Weil-Petersson metric is not complete metric in general but in the case of abelian varieties and $K3$ surfaces, the Weil-Petersson metric turns out to be equal to the Bergman metric of the Hermitian symmetric period domain, hence is in fact complete Kähler Einstein metric. Weil and Ahlfors showed that the Weil-Petersson metric is a Kähler metric. Ahlfors proved that it has negative holomorphic sectional, scalar, and Ricci curvatures. The quasi-projectivity of coarse moduli spaces of polarized Calabi-Yau manifolds in the category of separated analytic spaces (which also can be constructed in the category of Moishezon spaces) has been proved by Viehweg[23]. By using Bogomolov-Tian-Todorov theorem, these moduli spaces are smooth Kähler orbifolds equipped with the Weil-Petersson metrics. Let $X \rightarrow M$ be a family of polarized Calabi-Yau manifolds. Lu and Sun showed that the volume of the first Chern class with respect to the Weil-Petersson metric over the moduli space M is a rational number. Gang Tian (also Georg Schumacher) proved that the Weil-Petersson metric on moduli space of polarized Calabi-Yau manifolds is just pull back of Chern form of the tautological of $\mathbb{C}P^N$ restricted to period domain which is an open set of a quadric in $\mathbb{C}P^N$ and he showed that holomorphic sectional curvature is bounded away from zero. Let X be a compact projective Calabi-Yau manifold and let $f : X \rightarrow Y$ be an algebraic fiber space with Y an irreducible normal algebraic variety of lower dimension then Weil-Petersson metric measuring the change of complex structures of the fibers.

Now, consider a polarized Kähler manifolds $\mathcal{X} \rightarrow S$ with Kähler metrics $g(s)$ on \mathcal{X}_s . We can define a possibly degenerate hermitian metric G on S as follows: Take Kodaira-Spencer map

$$\rho : T_{S,s} \rightarrow H^1(X, T_X) \cong H_{\bar{\partial}}^{0,1}(T_X)$$

into harmonic forms with respect to $g(s)$; so for $v, w \in T_s(S)$, we may define

$$G(v, w) := \int_{\mathcal{X}_s} \langle \rho(v), \rho(w) \rangle_{g(s)}$$

When $\mathcal{X} \rightarrow S$ is a polarized Kähler-Einstein family and ρ is injective $G_{WP} := G$ is called the Weil-Petersson metric on S . Tian-Todorov, showed that if we take $\pi : \mathcal{X} \rightarrow S$, $\pi^{-1}(0) = X_0 = X$, $\pi^{-1}(s) = X_s$ be the family of X , then S is a non-singular complex analytic space such that

$$\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(X_s, T_{X_s})$$

Note that in general, if $f : X \rightarrow S$ be a smooth projective family over a complex manifold S . Then for every positive integer m ,

$$P_m(X_s) = \dim H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$$

is locally constant function on S .

It is worth to mention that the fibers X_s are diffeomorphic to each other and if fibers X_s be biholomorphic then π is holomorphic fiber bundle and Weil-Petersson metric is zero in this case in other words the Kodaira-Spencer maps

$$\rho : T_{S,s} \rightarrow H^1(X_s, T_{X_s}) \cong H_{\bar{\partial}}^{0,1}(T_{X_s})$$

are zero. In special case, let $\dim X_s = 1$, then the fibers are elliptic curves and π is holomorphic fiber bundle and hence the Weil-Petersson metric is zero. In general, the Weil-Petersson metric is semipositive definite on the moduli space of Calabi-Yau varieties. Note that Moduli space of varieties of general type has Weil-Petersson metric. The moduli space of K-stable varieties admit Weil-Petersson metric also.[27],[28]

Let us explain briefly how is organized this paper. In Section, Introduction, we provide a brief introduction on deformation theory of Kähler metrics, geometry of Weil-Petersson metrics and . In prove the main theorem of this paper about explicit formula for canonical metric on moduli space of log Calabi-Yau varieties by using fiberwise Calabi-Yau metric. We give a relation between relative volume form constructed by fiberwise Calabi-Yau metric and logarithmic Weil-Petersson metric. Moreover we talk about fiberwise Calabi-Yau foliation and extend such foliation for logarithmic case. In last part of this paper we talk about A relation between Invariance of plurigenera and positivity of logarithmic Weil-Petersson metric by using relative Kähler Ricci flow method (introduced by R.Berman) and also we show that Song-Tian-Tsuji measure is bounded if and only if the central fiber has log terminal singularities at worst.

Remark: Let $(E, \|\cdot\|)$ be the direct image bundle $f_*(K_{X'/S})$, where $X' = X \setminus D$, of relative canonical line bundle equipped with the L^2 metric $\|\cdot\|$. Then the fibre E_y is $H^0(X_y \setminus D_y, K_{X_y \setminus D_y})$. Since the pair (X_y, D_y) is Calabi-Yau pair, hence $H^0(X_y \setminus D_y, K_{X_y \setminus D_y})$ is a 1-dimensional vector space. This implies that E is a line bundle.

We give a new proof to the following theorem [9],[18].

Theorem 2 *Take holomorphic fiber space $\pi : X \rightarrow B$ and assume Ψ_y be any local non-vanishing holomorphic section of Hermitian line bundle $\pi_*(K_{X/B}^1)$, then the Weil-Petersson (1,1)-form on a small ball $N_r(y) \subset B$ can be written as*

$$\omega_{WP} = -\sqrt{-1}\partial_y\bar{\partial}_y \log \left((\sqrt{-1})^{n^2} \int_{X_y} (\Psi_y \wedge \bar{\Psi}_y)^{\frac{1}{r}} \right)$$

Note that ω_{WP} is globally defined on B

Now because we are in deal with Calabi-Yau pair (X, D) which $K_X + D$ is numerically trivial so we must introduce Log Weil-Petersson metrics instead Weil-Petersson metric. Here we introduce such metrics on moduli space of paired Calabi-Yau fibers (X_y, D_y) . Let $i : D \hookrightarrow X$ and $f : X \rightarrow Y$ be holomorphic mappings of complex manifolds such that i is a closed embedding and f as well

as $f \circ i$ are proper and smooth. Then a holomorphic family (X_y, D_y) are the fibers $X_y = f^{-1}(y)$ and $D_y = (f \circ i)^{-1}(y)$. Such family give rise to a fibered groupoid $p : \mathbf{F} \rightarrow \mathbf{A}$ from of category \mathbf{F} to the category of complex spaces with distinguished point in the sense of Grothendieck, [20]. There exists the moduli space of \mathcal{M} of such family because any (X_y, D_y) with trivial canonical bundle is non-uniruled. Now $X \setminus D$ is quasi-projective so we must deal with quasi-coordinate system instead of coordinate system. Let (X, D) be a Calabi-Yau pair and take $X' = X \setminus D$ equipped with quasi-coordinate system. We say that a tensor A on X' which are covariant of type (p, q) is quasi- $C^{k, \lambda}$ -tensor, if it is of class $C^{k, \lambda}$ with respect to quasi-coordinates. Now we construct the logarithmic version of Weil-Petersson metric on moduli space of paired Calabi-Yau fibers $f : (X, D) \rightarrow Y$.

Now, because we are in deal with singularities, so we use of $(1, 1)$ -current instead of $(1, 1)$ -forms which is singular version of forms. A current is a differential form with distribution coefficients. Let, give a definition of current here. We recall a singular metric h_{sing} on a Line bundle L which locally can be written as $h_{\text{sing}} = e^\phi h$ where h is a smooth metric, and ϕ is an integrable function. Then one can define locally the closed current $T_{L, h_{\text{sing}}}$ by the following formula

$$T_{L, h_{\text{sing}}} = \omega_{L, h} + \frac{1}{2i\pi} \partial \bar{\partial} \log \phi$$

The current Geometry is more complicated than symplectic geometry. For instance in general one can not perform the wedge product of currents due to this fact that one can not multiply the coefficients which are distributions and in general the product of two distributions is not well defined. However, in some special cases one can define the product of two currents. Here we mention the following important theorem about wedge product of two currents

One can simply defines the space of currents to be the dual of space of smooth forms, defined as forms on the regular part X_{reg} which, near X_{sing} , locally extend as smooth forms on an open set of \mathbb{C}^N in which X is locally embedded. A Kähler current on a compact complex space X is a closed positive current T of bidegree $(1, 1)$ which satisfies $T \geq \epsilon \omega$ for some $\epsilon > 0$ and some smooth positive hermitian form ω on X . In fact, This is a real closed current of type $(1, 1)$, that is a linear form on the space of compactly supported forms of degree $2n - 2$ on X , and $n = \dim X$. More precisely, Let $\mathcal{A}_c^{p, q}(X)$ denote the space of $C^\infty(p, q)$ forms of compact support on X with usual Fréchet space structure. The dual space $\mathcal{D}^{p, q}(X) := \mathcal{A}_c^{n-p, n-q}(X)^*$ is called the space of (p, q) -currents on M . The Linear operators $\partial : \mathcal{D}^{p, q}(X) \rightarrow \mathcal{D}^{p+1, q}(X)$ and $\bar{\partial} : \mathcal{D}^{p, q}(X) \rightarrow \mathcal{D}^{p, q+1}(X)$ is defined by

$$\partial T(\varphi) = (-1)^{p+q+1} T(\partial \varphi), \quad T \in \mathcal{D}^{p, q}(X), \quad \varphi \in \mathcal{A}_c^{n-p-1, n-q}(X)$$

and

$$\bar{\partial} T(\varphi) = (-1)^{p+q+1} T(\bar{\partial} \varphi), \quad T \in \mathcal{D}^{p, q}(X), \quad \varphi \in \mathcal{A}_c^{n-p, n-q-1}(X)$$

We set $d = \partial + \bar{\partial}$. $T \in \mathcal{D}^{p,q}(X)$ is called closed if $dT = 0$. $T \in \mathcal{D}^{p,p}(X)$ is called real if $T(\varphi) = T(\bar{\varphi})$ holds for all $\mathcal{A}_c^{n-p,n-q}(X)$. A real (p,p) -current T is called positive if $(\sqrt{-1})^{p(n-p)}T(\eta \wedge \bar{\eta}) \geq 0$ holds for all $\eta \in \mathcal{A}_c^{p,0}(X)$.

The topology on space of currents are so important. In fact the space of currents with weak topology is a Montel space, i.e., barrelled, locally convex, all bounded subsets are precompact which here barrelled topological vector space is Hausdorff topological vector space for which every barrelled set in the space is a neighbourhood for the zero vector.

Also because we use of push-forward and Pull back of a current and they can cont be defined in sense of forms, we need to introduce them. If $f : X \rightarrow Y$ be a holomorphic map between two compact Kähler manifolds then one can push-forward a current ω on X by duality setting

$$\langle f_*\omega, \eta \rangle := \langle \omega, f^*\eta \rangle$$

In general, given a current T on Y , it is not possible to define its pull-back by a holomorphic map. But it is possible to define pull-back of positive closed currents of bidegree $(1,1)$. We can writes such currents as $T = \theta + dd^c\varphi$ where $\theta \in T$ is a smooth form, and thus one define the pull-back of current T as follows

$$f^*T := f^*\theta + dd^c\varphi \circ f$$

Let X and Y be compact Kähler manifolds and let $f : X \rightarrow Y$ be the blow up of Y with smooth connected center Z and $\omega \in H^{1,1}(X, \mathbb{R})$. Demailly showed that

$$\omega = f^*f_*\omega + \lambda E$$

where E is the exceptional divisor and $\lambda \geq -v(\omega, Z)$ where $v(\omega, Z) = \inf_{x \in Z} v(\omega, x)$ and $v(\omega, x)$ is the Lelong number.

Note that if ω is Kähler then,

$$d\text{Vol}_{\omega_y}(X_y) = df_*(\omega^n) = f_*(d\omega^n) = 0$$

So, $\text{Vol}(X_y) = C$ for some constant $C > 0$ for every $y \in Y$ where $\pi^{-1}(y) = X_y$. Moreover direct image of volume form $f_*\omega_X^n = \sigma\omega_Y^m$ where $\sigma \in L^{1+\epsilon}$ for some positive constant ϵ

Theorem 3 *If T is a positive $(1,1)$ -current then locally one can find a plurisubharmonic function u such that*

$$\sqrt{-1}\partial\bar{\partial}u = T$$

Note that, if X be compact then there is no global plurisubharmonic function u [21].

Lelong number: Let $W \subset \mathbb{C}^n$ be a domain, and Θ a positive current of degree (q, q) on W . For a point $p \in W$ one defines

$$\mathfrak{v}(\Theta, p, r) = \frac{1}{r^{2(n-q)}} \int_{|z-p|<r} \Theta(z) \wedge (dd^c|z|^2)^{n-q}$$

The Lelong number of Θ at p is defined as

$$\mathfrak{v}(\Theta, p) = \lim_{r \rightarrow 0} \mathfrak{v}(\Theta, p, r)$$

Let Θ be the curvature of singular hermitian metric $h = e^{-u}$, one has

$$\mathfrak{v}(\Theta, p) = \sup\{\lambda \geq 0 : u \leq \lambda \log(|z-p|^2) + O(1)\}$$

see [26]

Main Theorem

Now we are ready to state our theorem. We must mention that The result of Tian, and Candelas et al. was on Polarized Calabi-Yau fibers and in this theorem we consider non-polarized fibers.

Theorem 4 *Let $\pi : X \rightarrow Y$ be a smooth family of compact Kähler manifolds with Calabi-Yau fibers. Then Weil-Petersson metric can be written as*

$$\omega_{WP} = -\sqrt{-1} \partial_y \bar{\partial}_y \log \int_{X_y} |\Omega_y|^2$$

where Ω_y is a holomorphic $(n, 0)$ -form on $\pi^{-1}(U)$, where U is a neighborhood of y

Proof: For prrof, We need to recall the Yau-Vafa semi Ricci flat metrics. Since fibers are Calabi-Yau varieties, so $c_1(X_y) = 0$, hence there is a smooth function F_y such that $Ric(\omega_y) = \sqrt{-1} \partial \bar{\partial} F_y$. The function F_y vary smoothly in y . By Yau's theorem, there is a unique Ricci-flat Kähler metric $\omega_{SRF,y}$ on X_y cohomologous to ω_0 where ω_0 is a Kähler metric attached to X . So there is a smooth function ρ_y on X_y such that $\omega_0|_{X_y} + \sqrt{-1} \partial \bar{\partial} \rho_y = \omega_{SRF,y}$ is the unique Ricci-flat Kähler metric on X_y . If we normalize ρ_y , then ρ_y varies smoothly in y and defines a smooth function ρ on X and we let

$$\omega_{SRF} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho$$

which is called as semi-Ricci flat metric. Robert Berman and Y.J.Choi independently showed that the semi-Ricci flat metric is semi positive along horizontal direction. Now for semi Ricci flat metric ω_{SRF} , we have

$$\omega_{SRF}^{n+1} = c(\omega_{SRF}) \cdot \omega_{SRF}^n dy \wedge d\bar{y}$$

Here $c(\omega_{SRF})$ is called a geodesic curvature of semi ω_{SRF} . Now from Berman and Choi formula, for $V \in T_y Y$, the following PDE holds on X_y

$$-\Delta_{\omega_{SRF}} c(\omega_{SRF})(V) = |\bar{\partial} V_{\omega_{SRF}}|_{\omega_{SRF}}^2 - \Theta_{V\bar{V}}(\pi_*(K_{X/Y}))$$

$\Theta_{V\bar{V}}$ is the Ricci curvature of direct image of relative line bundle (which is a line bundle, since fibers are Calabi-Yau manifolds). Now by integrating on both sides of this PDE, since

$$\int_X \Delta_{\omega_{SRF}} c(\omega_{SRF})(V) = 0$$

and from the definition of Weil-Petersson metric and this PDE we get $\pi^* \omega_{WP} = Ric(\pi_*(K_{X/Y}))$ and hence for some holomorphic $(n, 0)$ -form (as non vanishing holomorphic section on the direct image of relative line bundle, which is still line bundle, since fibres are Calabi-Yau varieties) Ω_y on $\pi^{-1}(U)$, where U is a neighborhood of y we have

$$Ric(\pi_*(K_{X/Y})) = -\sqrt{-1} \partial \bar{\partial} \log \|\Omega\|_y^2$$

From definition of pushforward of a current by duality, for any continuous function ψ on Y , we have

$$\int_Y \psi f_* \Omega = \int_X (f^* \psi) \Omega = \int_{y \in Y} \int_{\pi^{-1}(y)} (f^* \psi) \Omega$$

and hence on regular part of Y we have

$$\pi_* \Omega = \int_{\pi^{-1}(y)} \Omega$$

and so

$$\|\Omega\|_y^2 = \int_{\pi^{-1}(y)} |\Omega|_y^2$$

Hence

$$\omega_{WP} = -\sqrt{-1} \partial_y \bar{\partial}_y \log \int_{X_y} |\Omega_y|^2$$

and we obtain the desired result.

Logarithmic Weil-Petersson metric

Now we give a motivation that why the geometry of pair (X, D) must be interesting. The first one comes from algebraic geometry, in fact for deforming the cone angle we need to use the geometry of pair (X, D) . In the case of minimal general type manifold the canonical bundle of X , i.e., K_X is nef and we would like K_X to be ample and it is not possible in general and what we can do is that to add a small multiple of ample bundle $\frac{1}{m}A$, i.e., $K_X + \frac{1}{m}A$ and then we are dealing with the pair $(X, \frac{1}{m}H)$ which H is a generic section of it. The second one

is the works of Chen-Sun-Donaldson and Tian on existence of Kähler Einstein metrics for Fano varieties which they used of geometry of pair (X, D) for their proof .[34]

Now we explain Tian-Yau program to how to construct model metrics in general, like conical model metric, Poincare model metric, or Saper model metric, due to A.Fujiki's method

Let $\mathbb{C}^n = \mathbb{C}^n(z_1, \dots, z_n)$ be a complex Euclidian space for some $n > 0$. For a positive number ϵ with $0 < \epsilon < 1$ consider

$$\overline{X} = \overline{X}_\epsilon = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < \epsilon\}$$

Now, let $D_i = \{z_i = 0\}$ be the irreducible divisors and take $D = \sum_i D_i$ where

$$D = \{z \in \overline{X} \mid z_1 z_2 \dots z_k = 0\}$$

and take $X = \overline{X} \setminus D$. In polar coordinate we can write $z_i = r_i e^{i\theta_i}$. Let g be a Kähler metric on D such that the associated Kähler form ω is of the following form

$$\omega = \sqrt{-1} \sum_i \frac{1}{|dz_i|^2} dz_i \wedge d\bar{z}_i$$

Then the volume form dv associated to ω is written in the form;

$$dv = (\sqrt{-1})^n \prod_{i=1}^n \frac{1}{|dz_i|^2} \prod_i dz_i \wedge d\bar{z}_i, \quad v = \frac{1}{|dz_i|^2}$$

Let L be a (trivial) holomorphic line bundle defined on \overline{X} , with a generating holomorphic section S on \overline{X} . Fix a C^∞ hermitian metric h of L over X and denote by $|S|^2$ the square norm of S with respect to h . Assume the functions $|S|^2$ and $|dz_i|^2$ depend only on r_i , $1 \leq i \leq k$. Set

$$d(r_1, \dots, r_k) = |S|^2 \cdot v \cdot \prod_{1 \leq i \leq k} r_i$$

and further make the following three assumptions:

A1) The function d is of the form

$$d(r_1, \dots, r_k) = r_1^{c_1} \dots r_k^{c_k} (\log 1/r_1)^{b_1} \dots (\log 1/r_k)^{b_k} L(r_1, \dots, r_k)^t$$

where

$$L = L(r_1, \dots, r_k) = \sum_{i=1}^k \log 1/r_i$$

and c_i, b_j, t are real numbers with $t \geq 0$ such that $q_i = b_i + t \neq -1$ if c_i is an odd integer. We set $a_i = (c_i + 1)/2$ and denote by $[a_i]$ the largest integer which does not exceed a_i .

A2) If $1 \leq i \leq k$, then $|dz_i|^2$ is either of the following two forms;

$$|dz_i|^2(r) = r_i^2(\log 1/r_i)^2, \quad \text{or} \quad |dz_i|^2(r) = r_i^2 L^2$$

In fact, A2) implies that the Kähler metric g is (uniformly) complete along D .

A3) If $k+1 \leq i \leq n$, then $|dz_i|^{-2}$ is bounded (above) on X .

Now, we give some well-known examples of Tian-Yau-Fujiki picture, i.e, conical model metric, Poincare model metric, and Saper model metric.

A Kähler current ω is called a conical Kähler metric (or Hilbert Modular type) with angle $2\pi\beta$, ($0 < \beta < 1$) along the divisor D , if ω is smooth away from D and asymptotically equivalent along D to the model conic metric

$$\omega_\beta = \sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right)$$

here (z_1, z_2, \dots, z_n) are local holomorphic coordinates and $D = \{z_1 = 0\}$ locally.

After an appropriate -singular- change of coordinates, one can see that this model metric represents an Euclidean cone of total angle $\theta = 2\pi\beta$, whose model on \mathbb{R}^2 is the following metric: $d\theta^2 + \beta^2 dr^2$. The volume form V of a conical Kähler metric ω_D on the pair (X, D) has the form

$$V = \prod_j |S_j|^{2\beta_j - 2} e^f \omega^n$$

where $f \in C^0$.

This asymptotic behaviour of metrics can be translated to the second order asymptotic behaviour of their potentials

$$\omega_\beta = -\sqrt{-1} \partial \bar{\partial} \log e^{-u}$$

where $u = \frac{1}{2} \left(\frac{1}{\beta^2} |z_1|^{2\beta} + |z_2|^2 + \dots + |z_n|^2 \right)$.

Moreover, if we let $z = r e^{i\theta}$ and $\rho = r^\beta$ then the model metric in ω_β becomes

$$(d\rho + \sqrt{-1}\beta\rho d\theta) \wedge (d\rho - \sqrt{-1}\beta\rho d\theta) + \sum_{i>1} dz_i \wedge d\bar{z}_i$$

and if we set $\epsilon = e^{\sqrt{-1}\beta\theta} (d\rho + \sqrt{-1}\beta\rho d\theta)$ then the conical Kähler metric ω on $(X, (1-\beta)D)$ can be expressed as

$$\omega = \sqrt{-1} (f\epsilon \wedge \bar{\epsilon} + f_{\bar{j}}\epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j)$$

By the assumption on the asymptotic behaviour we mean there exists some coordinate chart in which the zero-th order asymptotic of the metric agrees with the model metric. In other words, there is a constant C , such that

$$\frac{1}{C} \omega_\beta \leq \omega \leq C \omega_\beta$$

In this note because we assume certain singularities for the Kähler manifold X we must design our Kähler Ricci flow such that our flow preserve singularities. Now fix a simple normal crossing divisor $D = \sum_i (1 - \beta_i) D_i$, where $\beta_i \in (0, 1)$ and simple normal crossing divisor D means that D_i 's are irreducible smooth divisors and for any $p \in \text{Supp}(D)$ lying in the intersection of exactly k divisors D_1, D_2, \dots, D_k , there exists a coordinate chart $(U_p, \{z_i\})$ containing p , such that $D_i|_{U_p} = \{z_i = 0\}$ for $i = 1, \dots, k$.

If $S_i \in H^0(X, \mathcal{O}_X(L_{D_i}))$ is the defining sections and h_i is hermitian metrics on the line bundle induced by D_i , then Donaldson showed that for sufficiently small $\epsilon_i > 0$, $\omega_i = \omega_0 + \epsilon_i \sqrt{-1} \partial \bar{\partial} |S_i|_{h_i}^{2\beta_i}$ gives a conic Kähler metric on $X \setminus \text{Supp}(D_i)$ with cone angle $2\pi\beta_i$ along divisor D_i and also if we set $\omega = \sum_{i=1}^N \omega_i$ then, ω is a smooth Kähler metric on $X \setminus \text{Supp}(D)$ and

$$\|S\|^{2(1-\beta)} = \prod_{i=1}^k \|S_i\|^{2(1-\beta)}$$

where $S \in H^0(X, \mathcal{O}(L_D))$. Moreover, ω is uniformly equivalent to the standard cone metric

$$\omega_p = \sum_{i=1}^k \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=k+1}^N \sqrt{-1} dz_i \wedge d\bar{z}_i$$

From Fujiki theory, $|dz_i|^2 = r_i^2$ for $1 \leq i \leq k$ and $|dz_j|^2 = 1$ for $k+1 \leq j \leq n$ so that A2) and A3) are again satisfied.

From now on for simplicity we write just "divisor D " instead "simple normal crossing divisor D ".

We give an example of varieties which have conical singularities. Consider a smooth geometric orbifold given by \mathbb{Q} -divisor

$$D = \sum_{j \in J} (1 - \frac{1}{m_j}) D_j$$

where $m_j \geq 2$ are positive integers and $\text{Supp} D = \cap_{j \in J} D_j$ is of normal crossings divisor. Let ω be any Kähler metric on X , let $C > 0$ be a real number and $s_j \in H^0(X, \mathcal{O}_X(D_j))$ be a section defining D_j . Consider the following expression

$$\omega_D = C\omega + \sqrt{-1} \sum_{j \in J} \partial \bar{\partial} |s_j|^{2/m_j}$$

If C is large enough, the above formula defines a closed positive $(1, 1)$ -current (smooth away from D). Moreover

$$\omega_D \geq \omega$$

in the sense of currents. Consider \mathbb{C}^n with the orbifold divisor given by the equation

$$\prod_{j=1}^n z_j^{1-1/m_j} = 0$$

(with eventually $m_j = 1$ for some j). The sections s_j are simply the coordinates z_j and a simple computation gives

$$\omega_D = \omega_{eucl} + \sqrt{-1} \sum_{j=1}^n \partial\bar{\partial}|z_j|^{2/m_j} = \omega_{eucl} + \sqrt{-1} \sum_{j=1}^n \frac{dz_j \wedge d\bar{z}_j}{m_j^2 |z_j|^{2(1-1/m_j)}}$$

Here we mention also metrics with non-conic singularities. We say a metric ω is of Poincare type, if it is quasi-isometric to

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right)$$

It is always possible to construct a Poincare metric on $M \setminus D$ by patching together local forms with C^∞ partitions of unity. Now, from Fujiki theory $|dz_i|^2 = r_i^2 (\log 1/r_i)^2$, $1 \leq i \leq k$ and $|dz_j|^2 = 1$, $k+1 \leq j \leq n$ so that A2) and A3) above are satisfied; we have

$$v = \prod_{1 \leq i \leq k} r_i^{-2} (\log 1/r_i)^{-2}$$

Let Ω_P be the volume form on $X \setminus D$, then, there exists a locally bounded positive continuous function $c(z)$ on polydisk \mathbb{D}^n such that

$$\Omega_P = c(z) \sqrt{-1} \left(\wedge_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \wedge_{i=k+1}^n dz_i \wedge d\bar{z}_i \right)$$

holds on $\mathbb{D}^n \cap (X \setminus D)$

Remark A: Note that if Ω_P be a volume form of Poincare growth on (X, D) , with X compact. If $c(z)$ be C^2 on \mathbb{D}^n , then $-Ric(\Omega_P)$ is of Poincare growth.

We say that ω is the homogeneous Poincare metric if its fundamental form ω_β is described locally in normal coordinates by the quasi-isometry

$$\omega_\beta = \sqrt{-1} \left(\frac{1}{(\log |z_1 z_2 \dots z_k|^2)^2} \sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2} + \sum_{i=1}^n dz_i \wedge d\bar{z}_i \right)$$

and we say ω has Ball Quotient singularities if it is quasi-isometric to

$$\omega_\beta = \sqrt{-1} \frac{dz_1 \wedge d\bar{z}_1}{(|z_1| \log(1/|z_1|))^2} + \sqrt{-1} \sum_{j=2}^n \frac{dz_j \wedge d\bar{z}_j}{\log 1/|z_1|}$$

It is called also Saper's distinguished metrics.

$$|dz_1|^2 = r_1^2 (\log 1/r_1)^2, \quad |dz_j|^2 = \log 1/r_1, \quad k+1 \leq j \leq n$$

so that A2) and A3) are satisfied; also we have the volume form as

$$v = r_1^{-2} (\log 1/r_1)^{-(n+1)}$$

If ω is the fundamental form of a metric on the compact manifold X , and ω_{sap} be the fundamental forms of Saper's distinguished metrics and $\omega_{P, hom}$ be the fundamental forms of homogeneous Poincare metric, on the noncompact manifold $M \setminus D$, then $\omega_{sap} + \omega$ and $\omega_{P, hom}$ are quasi-isometric.

For the log-Calabi-Yau fibration $f : (X, D) \rightarrow Y$, such that (X_t, D_t) are log Calabi-Yau varieties and central fiber (X_0, D_0) has simple normal crossing singularities, and if (X, ω) be a Kähler variety with Poincaré singularities then the semi Ricci flat metric ω_{SRF} has pole singularities with Poincare growth. So the semi-Ricci flat metric $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model which we call it fibrewise Poincaré singularities.

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2 (\log |z_k|^2)^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left(\sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

We can define the same fibrewise conical singularities and the semi-Ricci flat metric $\omega_{SRF}|_{X_t}$ is quasi-isometric with the following model

$$\frac{\sqrt{-1}}{\pi} \sum_{k=1}^n \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2} + \frac{\sqrt{-1}}{\pi} \frac{1}{(\log |t|^2 - \sum_{k=1}^n \log |z_k|^2)^2} \left(\sum_{k=1}^n \frac{dz_k}{z_k} \wedge \sum_{k=1}^n \frac{d\bar{z}_k}{\bar{z}_k} \right)$$

Note that if the base of fibration be smooth and of dimension one, then always by using base change and birational change we can assume the central fiber has simple normal crossing singularities due to following Mumford et al. theorem. Note that if we assume the general fibers are log-Calabi-Yau pairs, then we can assume the central fiber is Calabi-Yau pair after using semi-stable minimal model program [53].

Theorem (Grothendieck, Kempf, Knudsen, Mumford and Saint-Donat[54]) Let k be an algebraically closed field of characteristic 0 (e.g. $k = \mathbb{C}$). Let $f : X \rightarrow C$ be a surjective morphism from a k -variety X to a non-singular curve C and assume there exists a closed point $z \in C$ such that $f|_{X \setminus f^{-1}(z)} : X \setminus f^{-1}(z) \rightarrow C \setminus \{z\}$ is smooth. Then we find a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times_C C' & \xleftarrow{p} & X' \\ f \downarrow & & \downarrow & \swarrow f' & \\ C & \xleftarrow{\pi} & C' & & \end{array}$$

with the following properties

1. $\pi : C' \rightarrow C$ is a finite map, C' is a non-singular curve and $\pi^{-1}(z) = \{z'\}$.

2. p is projective and is an isomorphism over $C' \setminus \{z'\}$. X' is non-singular and $f'^{-1}(z')$ is a reduced divisor with simple normal crossings, i.e., we can write $f'^{-1}(z') = \sum_i E_i$ where the E_i are 1-codimensional subvarieties (i.e., locally they are defined by the vanishing of a single equation), which are smooth and, for all r , all the intersections $E_{i_1} \cap \dots \cap E_{i_r}$ are smooth and have codimension r .

Definition 5 A Kähler metric with cone singularities along D with cone angle $2\pi\beta$ is a smooth Kähler metric on $X \setminus D$ which satisfies the following conditions when we write $\omega_{sing} = \sum_{i,j} g_{i\bar{j}} \sqrt{-1} dz_i \wedge d\bar{z}_j$ in terms of the local holomorphic coordinates $(z_1; \dots; z_n)$ on a neighbourhood $U \subset X$ with $D \cap U = \{z_1 = 0\}$

1. $g_{1\bar{1}} = F|z_1|^{2\beta-2}$ for some strictly positive smooth bounded function F on $X \setminus D$
2. $g_{1\bar{j}} = g_{i\bar{1}} = O(|z_1|^{2\beta-1})$
3. $g_{i\bar{j}} = O(1)$ for $i, j \neq 1$

Now we shortly explain Donaldson's linear theory which is useful later in the definition of semi ricci flat metrics.

Definition 6 1) A function f is in $C^{\gamma,\beta}(X, D)$ if f is C^γ on $X \setminus D$, and locally near each point in D , f is C^γ in the coordinate $(\hat{\zeta} = \rho e^{i\theta} = z_1|z_1|^{\beta-1}, z_j)$.

2) A $(1,0)$ -form α is in $C^{\gamma,\beta}(X, D)$ if α is C^γ on $X \setminus D$ and locally near each point in D , we have $\alpha = f_1\epsilon + \sum_{j>1} f_j dz_j$ with $f_i \in C^{\gamma,\beta}$ for $1 \leq i \leq n$, and $f_1 \rightarrow 0$ as $z_1 \rightarrow 0$ where $\epsilon = e^{\sqrt{-1}\beta\theta}(d\rho + \sqrt{-1}\beta\rho d\theta)$

3) A $(1,1)$ -form ω is in $C^{\gamma,\beta}(X, D)$ if ω is C^γ on $X \setminus D$ and near each point in D we can write ω as

$$\omega = \sqrt{-1}(f\epsilon \wedge \bar{\epsilon} + f_{\bar{j}}\epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j)$$

such that $f, f_j, f_{\bar{j}}, f_{i\bar{j}} \in C^{\gamma,\beta}$, and $f_j, f_{\bar{j}} \rightarrow 0$ as $z_1 \rightarrow 0$

4) A function f is in $C^{2,\gamma,\beta}(X, D)$ if $f, \partial f, \partial\bar{\partial}f$ are all in $C^{\gamma,\beta}$

Fix a smooth metric ω_0 in $c_1(X)$, we define the space of admissible functions to be

$$\hat{C}(X, D) = C^{2,\gamma}(X) \cup \bigcup_{0 < \beta < 1} \left(\bigcup_{0 < \gamma < \beta^{-1} - 1} C^{2,\gamma,\beta}(X, D) \right)$$

and the space of admissible Kähler potentials to be

$$\hat{\mathcal{H}}(\omega_0) = \{\phi \in \hat{C}(X, D) \mid \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Note that

$$\mathcal{H}(\omega_0) \subset \hat{\mathcal{H}}(\omega_0) \subset \mathcal{PSH}(\omega_0) \cap L^\infty(X)$$

Where $\mathcal{PSH}(\omega_0) \cap L^\infty(X)$ is the space of bounded ω_0 -plurisubharmonic functions and

$$\mathcal{PSH}(\omega_0) = \{\phi \in L^1_{loc}(X) \mid \phi \text{ is u.s.c and } \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

The Ricci curvature of the Kählerian form ω_D on the pair (X, D) can be represented as:

$$Ric(\omega_D) = 2\pi \sum_j (1 - \beta_j)[D_j] + \theta + \sqrt{-1}\partial\bar{\partial}\psi$$

with $\psi \in C^0(X)$ and θ is closed smooth $(1, 1)$ -form.

We have also dd^c -lemma on $X = \bar{X} \setminus D$. Let Ω be a smooth closed $(1, 1)$ -form in the cohomology class $c_1(K_{\bar{X}}^{-1} \otimes L_D^{-1})$. Then for any $\epsilon > 0$ there exists an explicitly given complete Kähler metric g_ϵ on M such that

$$Ric(g_\epsilon) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f_\epsilon \text{ on } X$$

where f_ϵ is a smooth function on X that decays to the order of $O(\|S\|^\epsilon)$. Moreover, the Riemann curvature tensor $R(g_\epsilon)$ of the metric g_ϵ decays to the order of $O\left((-n \log \|S\|^2)^{-\frac{1}{n}}\right)$

Now we explain the logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau manifolds (if it exists. for special case of rational surfaces it has been proven that such moduli space exists). The logarithmic Weil-Petersson metric has pole singularities and we can introduce it also by elements of logarithmic Kodaira-Spencer tensors which represent elements of $H^1(X, \Omega_X^1(\log(D))^\vee)$. More precisely, Let X be a complex manifold, and $D \subset X$ a divisor and ω a holomorphic p -form on $X \setminus D$. If ω and $d\omega$ have a pole of order at most one along D , then ω is said to have a logarithmic pole along D . ω is also known as a logarithmic p -form. The logarithmic p -forms make up a subsheaf of the meromorphic p -forms on X with a pole along D , denoted

$$\Omega_X^p(\log D)$$

and for the simple normal crossing divisor $D = \{z_1 z_2 \dots z_k = 0\}$ we can write the stalk of $\Omega_X^1(\log D)$ at p as follows

$$\Omega_X^1(\log D)_p = \mathcal{O}_{X,p} \frac{dz_1}{z_1} \oplus \dots \oplus \mathcal{O}_{X,p} \frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p} dz_{k+1} \oplus \dots \oplus \mathcal{O}_{X,p} dz_n$$

Since, fibers are log Calabi-Yau manifolds and by recent result of Jeffres-Mazzeo-Rubinstein, we have Ricci flat metric on each fiber (X_y, D_y) and hence we can have log semi-Ricci flat metric and by the same method of previous theorem, the proof of Theorem 8 is straightforward.

Theorem 7 *Let (M, ω_0) be a compact Kähler manifold with $D \subset M$ a smooth divisor and suppose we have topological constraint condition $c_1(M) = (1 - \beta)[D]$*

where $\beta \in (0, 1]$ then there exists a conical Kähler Ricci flat metric with angle $2\pi\beta$ along D . This metric is unique in its Kähler class. This metric is polyhomogeneous; namely, the Kähler Ricci flat metric $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ admits a complete asymptotic expansion with smooth coefficients as $r \rightarrow 0$ of the form

$$\varphi(r, \theta, Z) \sim \sum_{j,k \geq 0} \sum_{l=0}^{N_{j,k}} a_{j,k,l}(\theta, Z) r^{j+k/\beta} (\log r)^l$$

where $r = |z_1|^\beta/\beta$ and $\theta = \arg z_1$ and with each $a_{j,k,l} \in C^\infty$. See page 150 of [48], about the same such estimate.

Now we can introduce Logarithmic semi Ricci flat metrics. The volume of fibers (X_y, D_y) are homological constant independent of y , and we assume that it is equal to 1. Since fibers are log Calabi-Yau varieties, so $c_1(X_y, D_y) = 0$, hence there is a smooth function F_y such that $Ric(\omega_y) = \sqrt{-1}\partial\bar{\partial}F_y$. The function F_y vary smoothly in y . By Jeffres-Mazzeo-Rubinstein's theorem [45], there is a unique conical Ricci-flat Kähler metric $\omega_{SRF,y}$ on $X_y \setminus D_y$ cohomologous to ω_0 . So there is a smooth function ρ_y on $X_y \setminus D_y$ such that $\omega_0|_{X_y \setminus D_y} + \sqrt{-1}\partial\bar{\partial}\rho_y = \omega_{SRF,y}$ is the unique Ricci-flat Kähler metric on $X_y \setminus D_y$. If we normalize ρ_y , then ρ_y varies smoothly in y and defines a smooth function ρ^D on $X \setminus D$ and we let

$$\omega_{SRF}^D = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho^D$$

which is called as Log Semi-Ricci Flat metric.

Let $f : X \setminus D \rightarrow S$, be a smooth family of quasi-projective Kähler manifolds. Let $x \in X \setminus D$, and $(\sigma, z_2, \dots, z_n, s^1, \dots, s^d)$, be a coordinate centered at x , where $(\sigma, z_2, \dots, z_n)$ is a local coordinate of a fixed fiber of f and (s^1, \dots, s^d) is a local coordinate of S , such that

$$f(\sigma, z_2, \dots, z_n, s^1, \dots, s^d) = (s^1, \dots, s^d)$$

Now consider a smooth form ω on $X \setminus D$, whose restriction to any fiber of f , is positive definite. Then ω can be written as

$$\begin{aligned} \omega(\sigma, z, s) = & \sqrt{-1}(\omega_{i\bar{j}} ds^i \wedge d\bar{s}^j + \omega_{i\bar{\beta}} ds^i \wedge d\bar{z}^\beta + \omega_{\alpha\bar{j}} dz^\alpha \wedge d\bar{s}^j + \omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \omega_\sigma d\sigma \wedge d\bar{s}^j \\ & + \omega_{i\bar{\sigma}} ds^i \wedge d\bar{\sigma} + \omega_{\sigma\bar{\sigma}} d\sigma \wedge d\bar{\sigma} + \omega_{\sigma\bar{j}} d\sigma \wedge d\bar{z}^j + \omega_{i\bar{\sigma}} dz^i \wedge d\bar{\sigma}) \end{aligned}$$

Since ω is positive definite on each fibre, hence

$$\sum_{\alpha, \beta=2} \omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \sum_{j=2} \omega_{\sigma\bar{j}} d\sigma \wedge d\bar{z}^j + \sum_{i=2} \omega_{i\bar{\sigma}} dz^i \wedge d\bar{\sigma}$$

gives a Kähler metric on each fiber $X_s \setminus D_s$. So

$$\det(\omega_{\lambda\bar{\eta}}^{-1}(\sigma, z, s)) = \det \begin{pmatrix} \omega_{\sigma\bar{\sigma}} & \omega_{\sigma\bar{2}} & \dots & \omega_{\sigma\bar{n}} \\ \omega_{2\bar{\sigma}} & \omega_{2\bar{2}} & \dots & \omega_{2\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n\bar{\sigma}} & \omega_{n\bar{2}} & \dots & \omega_{n\bar{n}} \end{pmatrix}^{-1}$$

gives a hermitian metric on the relative line bundle $K_{X'/S}$ and its Ricci curvature can be written as $\sqrt{-1}\partial\bar{\partial}\log\det\omega_{\lambda\bar{\eta}}(\sigma, z, s)$

Theorem 8 *By the same method we can introduce the logarithmic Weil-Petersson metric on $\pi : (X, D) \rightarrow Y$ with assuming fibers to be log Calabi-Yau manifolds and snc divisor D has conic singularities, then we have*

$$\omega_{WP}^D = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y \setminus D_y} \frac{\Omega_y \wedge \bar{\Omega}_y}{\|S_y\|^2}$$

where $S_y \in H^0(X_y, L_{D_y})$. Moreover, if ω has Poincare singularities along snc divisor D , we have the following formula for logarithmic Weil-Petersson metric

$$\omega_{WP}^D = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y \setminus D_y} \frac{\Omega_y \wedge \bar{\Omega}_y}{\|S_y\|^2 \log^2 \|S_y\|^2}$$

Now in next theorem we will find the relation between logarithmic Weil-Petersson metric and fiberwise Ricci flat metric.

We know from Fujino-Mori's canonical bundle formula, on log Iitaka fibration $\pi : (X, D) \rightarrow B$,

$$Ric(\omega_{can}) = -\omega_{can} + \omega_{WP}^D + \sum_P (b(1 - t_P^D))[\pi^*(P)] + [B^D]$$

where B^D is \mathbb{Q} -divisor on X such that $\pi_*\mathcal{O}_X([iB_+^D]) = \mathcal{O}_B$ ($\forall i > 0$). Here $s_P^D := b(1 - t_P^D)$ where t_P^D is the log-canonical threshold of π^*P with respect to $(X, D - B^D/b)$ over the generic point η_P of P . i.e.,

$$t_P^D := \max\{t \in \mathbb{R} \mid (X, D - B^D/b + t\pi^*(P)) \text{ is sub log canonical over } \eta_P\}$$

Theorem 9 *Let $\pi : (X, D) \rightarrow Y$ be a holomorphic family of log Calabi-Yau pairs (X_s, D_s) for the Kähler varieties X, Y . Then we have the following relation between logarithmic Weil-Petersson metric and fiberwise Ricci flat metric.*

$$\sqrt{-1}\partial\bar{\partial}\log\left(\frac{f^*\omega_Y^m \wedge (\omega_{SRF}^D)^{n-m}}{|S|^2}\right) = -f^*Ric(\omega_Y) + f^*\omega_{WP}^D$$

where $S \in H^0(X, \mathcal{O}(L_N))$, here N is a divisor which come from Fujino-Mori's canonical bundle formula(see [28] also)

Proof: Take $X' = X \setminus D$. Choose a local nonvanishing holomorphic section Ψ_y of $\pi_*(K_{X'/Y}^{\otimes l})$ with $y \in U \subset X'$. We define a smooth positive function on $\pi(U)$ by

$$u(y) = \frac{(\sqrt{-1})^{(n-m)^2} (\Psi_y \wedge \overline{\Psi_y})^{\frac{1}{l}}}{(\omega_{SRF}^D)^{n-m} |_{X_y \setminus D_y}}$$

But the Numerator and Denominator of u are Ricci flat volume forms on $X_y \setminus D_y$, so u is a constant function. Hence by integrating $u(y)(\omega_{SRF}^D)^{n-m} |_{X_y \setminus D_y}$ over $X_y \setminus D_y$ we see that

$$u(y) = \frac{(\sqrt{-1})^{(n-m)^2} \int_{X_y \setminus D_y} \frac{(\Psi_y \wedge \overline{\Psi_y})^{\frac{1}{l}}}{|S_y|^2}}{\int_{X_y \setminus D_y} \frac{(\omega_{SRF}^D)^{n-m}}{|S_y|^2}}$$

where $S_y \in H^0(X', \mathcal{O}(L_{D_y}))$.

But $y \mapsto \int_{X_y \setminus D_y} \frac{(\omega_{SRF}^D)^{n-m}}{|S_y|^2}$ is constant over Y . Hence the Logarithmic Weil-Petersson can be written as

$$-\sqrt{-1} \partial \bar{\partial} \log u = \omega_{WP}^D \quad (*)$$

Now, to finish the proof we can write $\Psi_y = F(\sigma, y, z)(d\sigma \wedge dz_2 \wedge \dots \wedge dz_{n-m})$ where F is holomorphic and non-zero. Hence by substituting Ψ_y in u and rewriting $\sqrt{-1} \partial \bar{\partial} \log(\frac{f^* \omega_Y^m \wedge (\omega_{SRF}^D)^{n-m}}{|S|^2})$ and using $(*)$ we get the desired result.

Remark B:Note that the log semi-Ricci flat metric ω_{SRF}^D is not continuous in general. But if the central fiber has at worst log canonical singularities then, semi-Ricci flat metric is smooth in an open Zariski subset.

Remark C:So by applying the previous remark, the relative volume form

$$\Omega_{(X,D)/Y} = \frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m |S|^2}$$

is not smooth in general, where $S \in H^0(X, L_N)$ and N is a divisor which come from canonical bundle formula of Fujino-Mori.

Now we try to extend the Relative Ricci flow to the fiberwise conical relative Ricci flow. We define the conical Relative Ricci flow on pair $\pi : (X, D) \rightarrow Y$ where D is a simple normal crossing divisor as follows

$$\frac{\partial \omega}{\partial t} = -Ric_{(X,D)/Y}(\omega) - \omega + [N]$$

where N is a divisor which come from canonical bundle formula of Fujino-Mori.

Take the reference metric as $\tilde{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) Ric(\frac{\omega_{SRF}^D \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m})$ then the conical relative Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\omega}_t + Ric(h_N) + \sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \pi^* \omega_{can}^m |S_N|^2}{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m} - \phi_t$$

With cone angle $2\pi\beta$, ($0 < \beta < 1$) along the divisor D , where h is an Hermitian metric on line bundle corresponding to divisor N , i.e., L_N . This equation can be solved. Take, $\omega = \omega(t) = \omega_B + (1 - \beta)Ric(h) + \sqrt{-1}\partial\bar{\partial}v$ where $\omega_B = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m})$, by using Poincare-Lelong equation,

$$\sqrt{-1}\partial\bar{\partial}\log|s_N|_h^2 = -c_1(L_N, h) + [N]$$

we have

$$\begin{aligned} Ric(\omega) &= \\ &= -\sqrt{-1}\partial\bar{\partial}\log\omega^m \\ &= -\sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} - \sqrt{-1}\partial\bar{\partial}v - (1 - \beta)c_1([N], h) + (1 - \beta)\{N\} \end{aligned}$$

and

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \sqrt{-1}\partial\bar{\partial}v &= \\ &= \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \omega - \omega_B - Ric(h) \end{aligned}$$

Hence, by using

$$\omega_{WP}^D = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m | S |^2}\right)$$

we get

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \sqrt{-1}\partial\bar{\partial}v &= \\ &= \omega - \omega_{WP}^D - (1 - \beta)c_1(N) \end{aligned}$$

So,

$$Ric(\omega) = -\omega + \omega_{WP}^D + (1 - \beta)[N]$$

which is equivalent with

$$Ric_{(X,D)/Y}(\omega) = -\omega + [N]$$

Now we prove the C^0 -estimate for this relative Monge-Ampere equation. We use the following important lemma from Schumacher and also Cheeger-Yau,

Lemma 10 *Suppose that the Ricci curvature of ω is bounded from below by negative constant -1 . Then there exists a strictly positive function $P_n(\text{diam}(X, D))$,*

depending on the dimension n of X and the diameter $\text{diam}(X, D)$ with the following property:

Let $0 < \epsilon \leq 1$. If g is a continuous function and f is a solution of

$$(-\Delta_\omega + \epsilon)f = g,$$

then

$$f(z) \geq P_n(\text{diam}(X, D)) \cdot \int_X g dV_\omega$$

So along relative Kähler-Ricci flow we have $\text{Ric}(\omega) \geq -2\omega$ where ω is the solution of Kähler-Ricci flow. But if we restrict our relative Monge-Ampere equation to each fiber (X_s, D_s) , then we need diameter bound on the fibers, i.e.,

$$\text{diam}(X_s \setminus D_s, \omega_s) \leq C$$

But from recent result of Takayama we know that

$$\text{diam}(X_s \setminus D_s, \omega_s) \leq 2 + D \int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega}_s}{|S_s|^2}$$

if and only if we have 1) central fiber $X_0 \setminus D_0$ has at worst canonical singularities and $K_{X_0} + D_0 = \mathcal{O}_{X_0}(D_0)$ which means the central fiber itself be log Calabi-Yau variety.

So this means that we have C^0 -estimate for relative Kähler-Ricci flow if and only if the central fiber be Calabi-Yau variety with at worst canonical singularities.

Remark D: Kähler potential of Weil-Petersson metric induces a singular Hermitian metric with semi-positive curvature current on the tautological quotient bundle over the projective-space bundle $\mathbb{P}(f_*(K_{X/B}))$.

Now we explain that under some algebraic condition the Kähler potential of Weil-Petersson metric on the moduli space of log Calabi-Yau pairs may be continuous (due to Tsuji-Kawamata). We recall the following Kawamata's theorem. [10]

Theorem 11 *Let $f : X \rightarrow B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$, $Q = \sum_l Q_l$, be normal crossing divisors on X and B , respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum_j d_j P_j$ be a \mathbb{Q} -divisor on X , where d_j may be positive, zero or negative, which satisfies the following conditions A, B, C:*

A) $D = D^h + D^v$ such that any irreducible component of D^h is mapped surjectively onto B by f , $f : \text{Supp}(D^h) \rightarrow B$ is relatively normal crossing over $B \setminus Q$, and $f(\text{Supp}(D^v)) \subset Q$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical)

B) $d_j < 1$ for all j

C) The natural homomorphism $\mathcal{O}_B \rightarrow f_* \mathcal{O}_X([-D])$ is surjective at the generic point of B .

D) $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B .
Let

$$\begin{aligned} f^*Q_l &= \sum_j w_{lj}P_j \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\ \delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\ \Delta &= \sum_l \delta_l Q_l. \\ M &= L - \Delta. \end{aligned}$$

Then M is nef.

The following theorem is straightforward from Kawamata's theorem

Theorem 12 Let $d_j < 1$ for all j be as above in Theorem 11, and fibers be log Calabi-Yau pairs, then

$$\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2}$$

is continuous on a nonempty Zariski open subset of B .

Since the inverse of volume gives a singular hermitian line bundle, we have the following theorem from Theorem 11

Theorem 13 Let $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B and

$$\begin{aligned} f^*Q_l &= \sum_j w_{lj}P_j \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\ \delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\ \Delta &= \sum_l \delta_l Q_l. \\ M &= L - \Delta. \end{aligned}$$

Then

$$\left(\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2} \right)^{-1}$$

is a continuous hermitian metric on the \mathbb{Q} -line bundle $K_B + \Delta$ when fibers are log Calabi-Yau pairs.

Remark E: Note that Yoshikawa[21], showed that when the base of Calabi-Yau fibration $f : X \rightarrow B$ is a disc and central fibre X_0 is reduced and irreducible and pair (X, X_0) has only canonical singularities then the Kähler potential of Weil-Petersson can be extended to a continuous Hermitian metric lying in the following class(due to Y.Kawamata also)

$$\mathcal{B}(B) = C^\infty(S) \oplus \bigoplus_{r \in \mathbb{Q} \cap (0,1]} \bigoplus_{k=0}^n |s|^{2r} (\log |s|)^k C^\infty(B)$$

In fact.

$$\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \bar{\Omega}_s}{|S_s|^2} = C (\log |t|)^m |t|^{2k} (1 + o(1))$$

where C is a constant [48], [41], [47].

Note that, If X_0 only has canonical singularities, or if X is smooth and X_0 only has isolated ordinary quadratic singularities, then if $\pi : X \rightarrow \mathbb{C}^*$ be a family of degeneration of of Calabi-Yau fibers. Then the L^2 -metric

$$\int_{X_s} \Omega_s \wedge \bar{\Omega}_s$$

is continuous. See Remark 2.10. of [52]

Now, such fiberwise integral are not C^∞ in general, and their smoothness correspond to Monodromy representation theory.

Smoothness of fiberwise integral of Calabi-Yau volume: Let X be a closed normal analytic subspace in some open subset U of \mathbb{C}^N with an isolated singularity. Take $f : X \rightarrow \Delta$ be a degeneration of smooth Calabi-Yau manifolds, then

$$s \rightarrow \int_{X_s} \Omega_s \wedge \bar{\Omega}_s \in C^\infty$$

if and only if the monodromy M acting on the cohomology of the Milnor fibre of f is the identity and the restriction map $j : H^n(X^*) \rightarrow H^n(F)^M$ is surjective, where $X^* = X \setminus \{0\}$ and M denotes monodromy acting on $H^n(F)$ and $H^n(F)^M$ is the M -invariant subgroup and F is the Milnor fiber at zero(see Corollary 6.2. [56]). In fact the C^∞ -smoothness of fiberwise Calabi-Yau volume ω_{SRF}^m must correspond to such information.

Remark F: Note that hermitian metric of semi Ricci flat metric ω_{SRF} is in the class of $\mathcal{B}(B)$ and is smooth if and only if we the central fiber has canonical singularities (due to diameter bound of fibers near to central fiber for obtaining C^0 -estimate)

Now, we can give a parallel definition of Tsuji's foliation for fiberwise Calabi-Yau metric.

Definition 14 *The null direction semi Ricci flat metric ω_{SRF} gives a foliation along Iitaka fibration $\pi : X \rightarrow Y$ and we call it fiberwise Calabi-Yau foliation*

and can be defined as follows

$$\mathcal{F} = \{\theta \in TX \mid \omega_{SRF}(\theta, \bar{\theta}) = 0\}$$

and along log Iitaka fibration $\pi : (X, D) \rightarrow Y$, we can define the following foliation

$$\mathcal{F}' = \{\theta \in TX' \mid \omega_{SRF}^D(\theta, \bar{\theta}) = 0\}$$

where $X' = X \setminus D$. In fact from Theorem 0.9. the Weil-Petersson metric ω_{WP} vanishes everywhere if and only if $\mathcal{F} = TX$

From Tsuji's work we have

Lemma: Let \mathcal{L} be a leaf of $f_*\mathcal{F}'$, then \mathcal{L} is a closed complex submanifold and the leaf \mathcal{L} can be seen as fiber on the moduli map

$$\eta : \mathcal{Y} \rightarrow \mathcal{M}_{CY}^D$$

where \mathcal{M}_{CY}^D is the moduli space of log Calabi-Yau fibers with at worst canonical singularities and

$$\mathcal{Y} = \{y \in Y_{reg} \mid (X_y, D_y) \text{ is Kawamata log terminal pair}\}$$

The following definition introduced by Tsuji

Definition 15 Let X be a compact complex manifold and let L be a line bundle on X . A singular Hermitian metric h on L is said to be an analytic Zariski decomposition (or shortly AZD), if the following hold.

1. the curvature Θ_h is a closed positive current.
2. for every $m \geq 0$, the natural inclusion

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(X, \mathcal{O}_X(mL))$$

is an isomorphism, where $\mathcal{I}(h^m)$ denotes the multiplier ideal sheaf of h^m .

Since the Weil-Petersson metric is semi-positive.

Remark G: The hermitian metric corresponding to Song-Tian measure is Analytic Zariski Decomposition., i.e.,

$$h = \left(\frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m \mid S|^2} \right)^{-1}$$

is AZD

Now we show that Song-Tian-Tsuji measure is bounded if and only if central fiber of Iitaka fibration has canonical singularities at worst. Note that Song-Tian just showed that such measure is in $L^{1,\epsilon}$ and they couldn't prove the boundedness of such measure.

Theorem 16 *Song-Tian-Tsuji measure*

$$\Omega_{X/Y} = \left(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m} \right)$$

on Iitaka fibration is bounded if and only if the central fiber X_0 has log terminal singularities.

Proof: R. Berman in [30], showed that for Iitaka fibration $F : X \rightarrow Y$ a canonical sequence of Bergman type measures

$$v_k = \int_{X^{N_k-1}} \mu^{(N_k)}$$

where

$$\mu^{(N_k)} = \frac{1}{Z_k} |S^{(k)}(z_1, \dots, z_{N_k})|^{2/k} dz_1 \wedge d\bar{z}_1 \dots dz_{N_k} \wedge d\bar{z}_{N_k}$$

and Z_k is the normalizing constant ensuring that $\mu^{(N_k)}$ is a probability measure, and

$$S^{(k)}(x_1, \dots, x_{N_k}) := \det \left(s_i^{(k)}(x_j) \right)_{1 \leq i, j \leq N_k}$$

where $s_i^{(k)}$ is a basis in $H^0(X, kK_X)$

then v_k converges weakly to Song-Tian measure

$$\mu_X = F^*(\omega_Y)^{\kappa(X)} \wedge \omega_{SRF}^{n-\kappa(X)}$$

But from Proposition 1.17 [17], ([16] also), we know that If K_{X_0} is a \mathbb{Q} -Cartier divisor then X_0 has log terminal singularities if and only if $\mathcal{L}_{X_0}^{2,m} = \mathcal{O}_{X_0}(m(K_{X_0}))$ for all $m \geq 1$ where $L^{2,m}$ is the sheaf of locally $2/m$ -integrable m -fold d -norms. So Song-Tian measure is bounded if and only if central fiber has log terminal singularities (in log Iitaka fibration case from Proposition 1.19 [17], we need to assume log canonical singularities for central fiber to obtain boundedness of Song-Tian measure).

Mumford in his celebrated paper [43] introduced the notion of good metric (which we call such metrics as Mumford metric) to extend the Chern-Weil theory to quasi-projective manifolds. First we recall Mumford metrics. For fixing our notation. Let \bar{E} be a holomorphic vector bundle of rank l over X and $E = \bar{E}|_X$ and h an Hermitian metric on E which may be singular near divisor D . Now cover a neighborhood of $D \subset X$ by finitely many polydiscs $\{U_\alpha = (\Delta^n, (z_1, \dots, z_n))\}$ such that $V_\alpha = U_\alpha \setminus D = (\Delta^*)^k \times \Delta^{n-k}$. Namely, $U_\alpha \cap D = \{z_1 \dots z_k = 0\}$. Let $U = \cup_\alpha U_\alpha$ and $V = \cup V_\alpha$ then on each V_α we have the local Poincaré metric.

Definition 17 Let ω be a smooth local p -form defined on V_α we say ω has Poincaré growth if there is a constant $C_\alpha > 0$ depending on ω such that

$$|\omega(t_1, t_2, \dots, t_p)|^2 \leq C_\alpha \prod_{i=1}^p \|t_i\|^2$$

for any point $z \in V_\alpha$ and $t_1, \dots, t_p \in T_z X$ where $\|\cdot\|$ is taken on Poincaré metric. We say ω is Mumford metric if both ω and $d\omega$ have Poincaré growth.

Now we recall the notion Hermitian Mumford metric h on the vector bundle E .

Definition 18 *An Hermitian metric h on E is good if for all $z \in V$, assuming $z \in V_\alpha$, and for all basis (e_1, \dots, e_n) of E over U_α , if we let $h_{i\bar{j}} = h(e_i, e_{\bar{j}})$, then*

$$I) |h_{i\bar{j}}|, (\det h)^{-1} \leq C (\sum_{i=1}^k \log |z_i|)^{2n} \text{ for some } C > 0$$

II) *The local 1-forms $(\partial h \cdot h^{-1})_{\alpha\lambda}$ are Mumford on V_α , namely the local connection and curvature forms of h have Poincaré growth.*

If h is a good metric on E , the Chern forms $c_i(E, h)$ are Mumford forms.

Theorem 19 *Fiberwise Calabi-Yau metric on Iitaka fibration and hermitian metric corresponding to Song-Tian-Tsuji measure (inverse of Song-Tian-Tsuji measure $h = \Omega_{X/Y}^{-1}$ gives a singular hermitian metric) is a good metric in the sense of Mumford, i.e, it is Mumford metric when central fiber has log terminal singularities at worst.*

Proof: From the Berman's formula in our proof of Theorem 16, and boundness of Song-Tian-Tsuji measure and using Theorem 30, Lemma 36 in [44], we get the desired result.

Remark: Fiberwise log Calabi-Yau metric on log Iitaka fibration and hermitian metric corresponding to Song-Tian-Tsuji measure (inverse of Song-Tian-Tsuji measure $h = \Omega_{(X,D)/Y}^{-1}$ gives a singular hermitian metric) is a good metric in the sense of Mumford, i.e, it is Mumford metric when central fiber has log canonical singularities at worst.

The following Theorem, gives a result to the goodness of canonical metric in the sense of Mumford, on moduli spaces of Calabi-Yau varieties with mild singularities in the sense of Minimal Model Program.

Theorem 20 *Since if a singular hermitian metric be a good metric in the sense of Mumford (Mumford metric), its first Chern current is also good metric in the sense of Mumford, hence from Theorem 9, the singular Weil-Petersson metric on moduli space of Calabi-Yau varieties is a good metric when central fiber has log terminal singularities at worst. Moreover if central fiber has log canonical singularities at worst, then the singular logarithmic Weil-Petersson metric on the moduli spaces of log Calabi-Yau varieties is a good metric in the sense of Mumford*

• **Weil-Petersson metric via Bismut-Vergne localization formula**

Let $\pi : M \rightarrow S$ be a submersion of smooth manifolds, with compact oriented fibres $X_s = \pi^{-1}(s)$. Let T be a torus acting smoothly on M , and preserving the fibers X_s . Fix $K \in \mathfrak{t}$. Let $X_{s,K} \subset X$ be the zero set of K^{X_s} where K^{X_s} is the corresponding vector fields on X_s . Then $X_{s,K}$ is a totally geodesic submanifold of X_s .

Let fibers X_s have vanishing first Chern class $c_1(X_s) = 0$. So we have the following formula of Weil-Petersson metric on moduli space of Calabi-Yau manifolds

$$\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \log \int_{X_s} \Omega_s \wedge \bar{\Omega}_s$$

So, from Bismut-Vergne localization formula [49],[50], [51] we have the following formula for Weil-Petersson metric

$$\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \log \int_{X_{s,K}} \frac{\Omega_s \wedge \bar{\Omega}_s}{e_K(N_{X_{s,K}/X_s}, \nabla^{N_{X_{s,K}/X_s}})} \text{ mod } d^S \Omega(S)$$

where the equivariant Euler class is as follows

$$e_K(N_{X_{s,K}/X_s}, \nabla^{N_{X_{s,K}/X_s}}) = \det^{\frac{1}{2}} \left[\frac{J_K + R^{N_{X_{s,K}/X_s}}}{2\pi} \right]$$

where J_K is an anti-symmetric parallel endomorphism of $N_{X_{s,K}/X_s}$, which is nondegenerate and $R^{N_{X_{s,K}/X_s}}$ is the curvature of any metric connection in $N_{X_{s,K}/X_s}$

In fact such Kähler potential of Weil-Petersson metric

$$\int_{X_{s,K}} \frac{\Omega_s \wedge \bar{\Omega}_s}{e_K(N_{X_{s,K}/X_s}, \nabla^{N_{X_{s,K}/X_s}})} \text{ mod } d^S \Omega(S)$$

is important to rewrite the relative volume form $\Omega_{M/S}$ by using such Kähler potential to get canonical metric on total space along Iitaka fibration (when \mathbb{T} be a torus acting smoothly on M , and preserving the fibers also) and reduce twisted version of Complex-Monge Ampere equation to modulo base space S (along Iitaka fibration or canonical model $X \rightarrow X_{can}$).

Now, if we assume the fibers (X_s, L_s) are polarized Calabi-Yau manifolds, then there exists $m_0 = m_0(n) > 0$, such that for any $m \geq m_0$, then we can embed $i_s : X_s \rightarrow \mathbb{P}(H^0(X_s, -m(K_{X_s} + L_s))^*)$. Now we choose an orthonormal basis $s_i(s)$ of $\mathbb{P}(H^0(X_s, -m(K_{X_s} + L_s))^*)$ with L^2 -inner product, then we can write

$$\Omega_{X_s} := \Omega_s = \left(\sum_{i=1}^{N_m} |s_i(s)|^2 \right)^{-\frac{1}{m}}$$

So we can rewrite the Weil-Petersson metric as

$$\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \log \int_{X_{s,K}} \frac{\left(\sum_{i=1}^{N_m} |s_i(s)|^2 \right)^{-\frac{1}{m}} \wedge \overline{\left(\sum_{i=1}^{N_m} |s_i(s)|^2 \right)^{-\frac{1}{m}} \otimes (ds \otimes d\bar{s})^{-1}}{e_K(N_{X_{s,K}/X_s}, \nabla^{N_{X_{s,K}/X_s}})} \pmod{d^S \Omega(S)}$$

1 Invariance of Plurigenera and positivity of logarithmic Weil-Petersson metric

Let X be a projective variety, for every positive integer m , the m -th plurigenus $P_m(X)$ is defined by

$$P_m(X) = \dim H^0(X, \mathcal{O}_X(mK_X))$$

The plurigenera are fundamental invariants under birational transformation of compact complex manifolds. It is an open conjecture that plurigenera is invariant under Kähler deformations. For non-Kähler manifolds we don't have the invariance of plurigenera. For projective varieties this conjecture has been solved.

We can solve Invariance of plurigenera by using algebraic geometric method and also by using complex analytical method.

By using Tsuji-Boucksom's method [37],[38], A complete solution for deformation of invariance of plurigenera is equivalent with the existence of singular hermitian metric h on K_X such that the curvature current Θ_h is semipositive and $h|_{X_t}$ is an AZD of K_{X_t} for every $t \in \Delta$. As soon as we construct such a metric h , the L^2 -extension theorem implies the invariance of the plurigenera.

In algebraic language invariance of plurigenera is equivalent as follows.

Algebraic language of Invariance of plurigenera say that any section mK_{X_0} extends to X , in other words, the restriction map

$$H^0(X, mK_X) \rightarrow H^0(X_0, mK_{X_0})$$

is surjective.

Takayama, showed that, if X_0 has at most terminal singularities, then X_t has terminal singularities at most for all $t \in B$. Moreover, If $\pi : X \rightarrow B$ is smooth and the "abundance conjecture" holds true for general X_t , then

$$P_m(X_t) = \dim H^0(X_t, mK_{X_t})$$

is independent of $t \in B$ for all m . N.Nakayama, showed that the invariance of plurigenera for smooth projective deformations can be derived if the minimal model program were completed for families immediately follow from the L^2 -extension theorem.

Now, we apply the relative Kähler-Ricci flow method which is the analytical version of MMP to solve invariance of plurigenera and give a method for

the problem of deformation of invariance of plurigenera in Kähler setting.(see the preprint about Semipositivity of relative canonical bundles via Kähler-Ricci flows, from Boucksom, S.; Tsuji, Hajime [37])

Let $\pi : X \rightarrow \Delta$ is smooth, and every X_t is of general type over open disc Δ . Consider the relative Kähler Ricci flow

$$\frac{\partial \omega}{\partial t} = -Ric_{X/\Delta}(\omega) - \omega$$

If such flow has solution, then we get the invariance of plurigenera. In fact if such flow has solution then $K_{X/\Delta}$ is pseudo-effective. In fact if such flow has solution then $e^{-t}\omega_0 + (1 - e^{-t})c_1(K_{X/\Delta})$ is a Kähler metric and since we take the initial metric to be positive, hence $c_1(K_{X/\Delta})$ is positive. Now since fibers are of general type then by using Yau's theorem there exists a Kähler-Einstein metric on each fiber. So we can introduce fiberwise Kähler Einstein metric ω_{SKE} which by using Schumacher and Paun result(a solution for such flow gives the semi-positivity of such fiberwise Kähler-Einstein metric. In fact to rewrite our relative Kähler Ricci flow to complex Monge-Ampere equation, we need to construct relative volume form by using such metric and relative volume form $\Omega_{X/\Delta} = \omega_{SKE}^m \wedge \pi^*(dt)$ must be semi-positive), ω_{SKE} is semi-positive and its corresponding hermitian metric is AZD, hence we can take a hermitian metric $h^{\omega_{SKE}}$ on K_X such that the curvature current $\Theta_{h^{\omega_{SKE}}}$ is semi-positive and $h^{\omega_{SKE}}|_{X_t}$ is an AZD of K_{X_t} for every $t \in \Delta$. and by the L^2 -extension theorem(it is still open for Kähler manifolds) we get the invariance of the plurigenera and we obtain Theorem of Siu.

In singular setting, we need to the following conjecture

Let $\pi : X \rightarrow \Delta$ is a family of projective varieties, and every X_t is of general type over open disc Δ . Then fiberwise Kähler-Einstein metric is semi-positive if and only if the central fiber X_0 has at worst canonical singularities.

If this Conjecture mentioned before, holds true, then we can have invariance of plurigenera of Kawamata result in singular setting. In fact to obtain a C^0 solution for relative Kähler Ricci flow we must have diameter bound of fibers X_t and this must be equivalent with canonical singularities of central fiber X_0 . A weak answer of this is the theorem of Donaldson-Sun [42]. A theorem of Donaldson-Sun states that if X_t are Kähler-Einstein metric with negative Ricci curvature with uniform diameter bound, then the central fiber is normal and Kawamata log terminal singularities at worst. In view of the moduli theory of canonically polarized varieties, limit of fibers should have canonical singularities.

Now we talk about semi-positivity of logarithmic-Weil-Petersson metric via Invariance of plurigenera.

Let $\pi : (X, D) \rightarrow Y$ be a smooth holomorphic fibre space whose fibres have pseudoeffective canonical bundles. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t) + [N]$$

be a relative Kähler Ricci flow that starts with (1,1) form $[\omega(t)] = e^{-t}\omega_0 +$

$(1 - e^{-t})\omega_{WP}^D$ and $X' = X \setminus D$, and here N is a divisor which come from Fujino-Mori's canonical bundle formula[28],[39]. From Tsuji's approach such flow has semi-positive solutions hence $\omega(t)$ and ω_0 is semi-positive, and hence the logarithmic Weil-Petersson metric ω_{WP}^D must be semi-positive $(1, 1)$ -Kähler form.

In fact the invariance of plurigenera holds true if and only if the solutions $\omega(t) = e^{-t}\omega_0 + (1 - e^{-t})\omega_{WP}^D$ are semi-positive(see the Analytical approach of Tsuji, Siu, Song-Tian). In fact an answer to this question leads to invariance of plurigenera in Kähler setting. If our family of fibers be fiberwise KE-stable, then invariance of plurigenera holds true from L^2 -extension theorem and also due to this fact that if the central fiber be psudo-effective, then all the general fibers are psudo-effective[4].

Theorem 21 (*L^2 -extension theorem[32]*) *Let X be a Stein manifold of dimension n , ψ a plurisubharmonic function on X and s a holomorphic function on X such that $ds \neq 0$ on every branch of $s^{-1}(0)$. We put $Y = s^{-1}(0)$ and $Y_0 = \{X \in Y; ds(x) \neq 0\}$ Let g be a holomorphic $(n - 1)$ -form on Y_0 with*

$$c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g} < \infty$$

where $c_k = (-1)^{k(k-1)/2}(\sqrt{-1})^k$ Then there exists a holomorphic n -form G on X such that $G(x) = g(x) \wedge ds(x)$ on Y_0 and

$$c_n \int_X e^{-\psi} (1 + |s|^2)^{-2} G \wedge \bar{G} < 1620\pi c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g}$$

Theorem 22 (*Y.T.Siu[40]*) *Assume $\pi : X \rightarrow B$ is smooth, and every X_t is of general type. Then the plurigenera $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for any m .*

After Siu, an ‘‘algebraic proof’’ is given, and applied to the deformation theory of certain type of singularities which appear in MMP by Kawamata.

Definition 23 *Let B be a normal variety such that K_B is \mathbb{Q} -Cartier, and $f : X \rightarrow B$ a resolution of singularities. Then,*

$$K_X = f^*(K_B) + \sum_i a_i E_i$$

where $a_i \in \mathbb{Q}$ and the E_i are the irreducible exceptional divisors. Then the singularities of B are terminal, canonical, log terminal or log canonical if $a_i > 0, \geq 0, > -1$ or ≥ -1 , respectively.

Theorem 24 (*Kawamata[6]*) *If X_0 has at most canonical singularities, then X_t has canonical singularities at most for all $t \in B$. Moreover, if all X_t are of general type and have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m*

Remark H: If along holomorphic fiber space $(X, D) \rightarrow B$ (with some stability condition on B) the fibers are of general type then to get

$Ric(\omega) = \lambda\omega + \omega_{WP}$ + additional term which come from higher canonical bundle formula

, (here Weil-Petersson metric is a metric on moduli space of fibers of general type) when fibers are singular and of general type then we must impose this assumption that the central fiber (X_0, D_0) must have canonical singularities and be of general type to obtain such result. ([27],[28])

Theorem 25 (Nakayama[5]) *If X_0 has at most terminal singularities, then X_t has terminal singularities at most for all $t \in B$. Moreover, If $\pi : X \rightarrow B$ is smooth and the “abundance conjecture” holds true for general X_t , then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m .*

Takayama, showed the following important theorem

Theorem 26 *Let all fibers $X_t = \pi^{-1}(t)$ have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all t*

Theorem 27 *Let $\pi : X \rightarrow Y$ be a proper smooth holomorphic fiber space of projective varieties such that all fibers X_y are of general type, then ω_{WP} is semi-positive*

Proof: Let $\pi : X \rightarrow Y$ be a smooth holomorphic fibre space whose fibres are of general type. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

be a Kähler ricci flow that starts with semi-positive Kähler form ω_0 (take it Weil-Petersson metric).

Then since Siu’s theorem holds true for invariance of plurigenera, so the pseudo-effectiveness of K_{X_0} gives the pseudo-effectiveness of K_{X_t} . The solutions of $\omega(t)$ are semi-positive. But by cohomological characterization we know that $[\omega(t)] = e^{-t}\omega_{WP} + (1 - e^{-t})[\omega_0]$ and since ω_0 and $\omega(t)$ are semi-positive, hence ω_{WP} is semi-positive. \square

We consider the semi-positivity of singular Weil-Petersson metric ω_{WP} in the sense of current.

Theorem 28 *Let $\pi : X \rightarrow Y$ be a proper holomorphic fiber space such that all fibers X_y are of general type and have at worst canonical singularities, then the Weil-Petersson metric ω_{WP} is semi-positive*

Proof. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

be a Kähler Ricci flow. Then since Kawamata's theorems say's that "If all fibers X_t are of general type and have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m " hence invariance of plurigenera hold's true, and the solutions of $\omega(t)$ are semi-positive by invariance of plurigenera. But by cohomological characterization we know that $[\omega(t)] = e^{-t}\omega_{WP} + (1 - e^{-t})[\omega_0]$ and since ω_0 and $\omega(t)$ are semi-positive, hence ω_{WP} is semi-positive. \square

Remark I: From Nakayama's theorem, if X_0 has at most terminal singularities, then X_t has terminal singularities at most for all $t \in B$. Moreover, If $\pi : X \rightarrow B$ is smooth and the "abundance conjecture" holds true for general X_t , then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m . So when fibers are of general type then the solutions of the relative Kähler Ricci flow $\omega(t)$ is semi-positive and hence by the same method of the proof of previous Theorem, the Weil-Petersson metric ω_{WP} is semi-positive on the moduli space of such families.

By the same method used in [35], [46], have the following result.

Remark J: Fiberwise Calabi-Yau metric ω_{SRF} and logarithmic Weil-Petersson metric on moduli spaces of log Calabi-Yau varieties have vanishing Lelong number.

Conjecture: : If a singular Hermitian metric on a Kähler manifold be a good metric in the sense of Mumford then it has vanishing Lelong number.

For compactification of the moduli spaces of polarized varieties Alexeev, and Kollar-Shepherd-Barron, [55] started a program by using new notion of moduli space of "stable family". They needed to use the new class of singularities, called semi-log canonical singularities.

Let X be an equidimensional algebraic variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective \mathbb{R} -divisor whose support does not contain any irreducible components of the conductor of X . The pair (X, Δ) is called a semi log canonical pair (an slc pair, for short) if

- (1) $K_X + \Delta$ is \mathbb{R} -Cartier;
- (2) (X^v, Θ) is log canonical, where $v : X^v \rightarrow X$ is the normalization and $K_{X^v} + \Theta = v^*(K_X + \Delta)$

Note that, the conductor \mathcal{C}_X of X is the subscheme defined by, $\text{cond}_X := \text{Hom}_{\mathcal{O}_X}(v_*\mathcal{O}_{X^v}, \mathcal{O}_X)$.

A morphism $f : X \rightarrow B$ is called a weakly stable family if it satisfies the following conditions:

1. f is flat and projective
2. $\omega_{X/B}$ is a relatively ample \mathbb{Q} -line bundle
3. X_b has semi log canonical singularities for all $b \in B$

A weakly stable family $f : X \rightarrow B$ is called a stable family if it satisfies Kollar's condition, that is, for any $m \in \mathbb{N}$

$$\omega_{X/B}^{[m]}|_{X_b} \cong \omega_{X_b}^{[m]}.$$

Note that, if the central fiber be Gorenstein and stable variety, then all general fibers are stable varieties, i.e, stability is an open condition

Conjecture: Weil-Petersson metric (or logarithmic Weil-Petersson metric) on stable family is semi-positive as current and such family has finite distance from zero i.e $d_{WP}(B, 0) < \infty$ when central fiber is stable variety also.

Moreover we predict the following conjecture holds true.

Conjecture: Let $f : X \rightarrow B$ is a stable family of polarized Calabi-Yau varieties, and let B is a smooth disc. then if the central fiber be stable variety as polarized Calabi-Yau variety, then we have following canonical metric on total space.

$$Ric(\omega) = -\omega + f^*(\omega_{WP})$$

Moreover, if we have such canonical metric then our family of fibers is stable.

We predict that if the base be singular with mild singularities of general type (for example $B = X_{can}$) then we have such canonical metric on the stable family

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