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A short proof of Tian's formula for logarithmic Weil-Petersson metric

Hassan Jolany

Abstract

In this short note, we give a short proof of Tian's formula[10] for Kähler potential of logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties (if exists!) of conic and Poincare singularities. Moreover we give a relation between logarithmic Weil-Petersson metric and the logarithmic version of Vafa-Yau semi Ricci flat metric on the family of log Calabi-Yau pairs with conical singularities. In final we consider the semi-positivity of singular logarithmic Weil-Petersson metric on the moduli space of log-Calabi-Yau varieties.

Introduction

In this note we try to find the Kähler potential of logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties. We use the analysis of semi Ricci-flat metric introduced by C.Vafa and S.T.Yau [6]. Our method of proof is completely different from the proof of G.Tian. See [1],[2],[3],[4],[5], [17], [18].

Historically, A. Weil introduced a Kähler metric for the Teichmuller space $\mathcal{T}_{g,n}$, the space of homotopy marked Riemann surfaces of genus g with n punctures and negative Euler characteristic. The Weil-Petersson metric measures the variations of the complex structure of R . The quotient of the Teichmuller space $\mathcal{T}_{g,n}$ by the action of the mapping class group is the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$. The Weil-Petersson metric is mapping class group invariant and descends to $\mathcal{M}_{g,n}$. Later, Tian considered Weil-Petersson metric on moduli space of polarized Calabi-Yau varieties, [10]. G.Schumacher and A.Fujiki [7] considered Weil-Petersson metric on moduli space of general type Kähler-Einstein varieties. In this note we consider the logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau varieties(if exists!). See [16]

We start we some elementary definitions.

Definition 0.1 *Let $\pi : X \rightarrow Y$ be a holomorphic map of complex manifolds. A real d -closed $(1, 1)$ -form ω on X is said to be a relative Kähler form for π , if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function Ψ on W such that $\omega + \pi^*(\sqrt{-1}\partial\bar{\partial}\Psi)$ is a Kähler form*

on $\pi^{-1}(W)$. A morphism π is said to be Kähler, if there exists a relative Kähler form for π , and $\pi : X \rightarrow Y$ is said to be a Kähler fiber space, if π is proper, Kähler, and surjective with connected fibers.

We consider an effective holomorphic family of complex manifolds. This means we have a holomorphic map $\pi : X \rightarrow Y$ between complex manifolds such that

1. The rank of the Jacobian of π is equal to the dimension of Y everywhere.
2. The fiber $X_t = \pi^{-1}(t)$ is connected for each $t \in Y$
3. X_t is not biholomorphic to $X_{t'}$ for distinct points $t, t' \in B$.

It is worth to mention that Kodaira showed that all fibers are dieomorphic to each other.

For any effective holomorphic family of compact manifolds $\pi : X \rightarrow Y$ of dimension n with fibers X_y for $y \in Y$ the Calabi-Yau forms $\omega_{X/Y}$ depend differentiably on the parameter y . The relative Kähler form is denoted by

$$\omega_{X/Y} = \sqrt{-1}g_{\alpha,\bar{\beta}}(z, y)dz^\alpha \wedge d\bar{z}^\beta$$

Moreover take $\omega_X = \sqrt{-1}\partial\bar{\partial} \log \det g_{\alpha,\bar{\beta}}(z, y)$ on the total space X . The fact is ω_X in general is not Kähler on total space and $\omega_X|_{X_y} = \omega_{X_y}$. More precisely $\omega_X = \omega_F + \omega_H$ where ω_F is a form along fiber direction and ω_H is a form along horizontal direction. ω_H may not be Kähler metric in general, but ω_F is Kähler metric. Now let ω be a relative Kähler form on X and $m := \dim X - \dim Y$, We define the relative Ricci form $Ric_{X/Y,\omega}$ of ω by

$$Ric_{X/Y,\omega} = -\sqrt{-1}\partial\bar{\partial} \log(\omega^m \wedge \pi^*|dy_1 \wedge dy_2 \wedge \dots \wedge dy_k|^2)$$

where (y_1, \dots, y_k) is a local coordinate of Y . Here Y assumed to be a curve

Let for family $\pi : \mathcal{X} \rightarrow Y$

$$\rho_{y_0} : T_{y_0}Y \rightarrow H^1(X, TX) = \mathcal{H}_\sigma^{0,1}(TX)$$

be the Kodaira–Spencer map for the corresponding deformation of X over Y at the point $y_0 \in Y$ where $\mathcal{X}_{y_0} = X$

If $v \in T_{y_0}Y$ is a tangent vector, say $v = \frac{\partial}{\partial y} |_{y_0}$ and $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift to \mathcal{X} along X , then

$$\bar{\partial} \left(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha(z)}{\partial z^\beta} \frac{\partial}{\partial z^\alpha} dz^\beta$$

is a $\bar{\partial}$ -closed form on X , which represents $\rho_{y_0}(\partial/\partial y)$.

The Kodaira–Spencer map is induced as edge homomorphism by the short exact sequence

$$0 \rightarrow T_{X/Y} \rightarrow TX \rightarrow \pi^*T_Y \rightarrow 0$$

We briefly explain about the Weil-Petersson metric on moduli space of polarized Calabi-Yau manifolds. We study the moduli space of Calabi-Yau manifolds via the Weil-Petersson metric. We outline the important properties of such metrics here.

The Weil-Petersson metric is not complete metric in general but in the case of abelian varieties and $K3$ surfaces, the Weil-Petersson metric turns out to be equal to the Bergman metric of the Hermitian symmetric period domain, hence is in fact complete Kähler Einstein metric. Weil and Ahlfors showed that the Weil-Petersson metric is a Kähler metric and later Tian gave a different proof for it. Ahlfors proved that it has negative holomorphic sectional, scalar, and Ricci curvatures. The quasi-projectivity of coarse moduli spaces of polarized Calabi-Yau manifolds in the category of separated analytic spaces (which also can be constructed in the category of Moishezon spaces) has been proved by Viehweg. By using Bogomolov-Tian-Todorov theorem [10], [23], these moduli spaces are smooth Kähler orbifolds equipped with the Weil-Petersson metrics. Let $X \rightarrow M$ be a family of polarized Calabi-Yau manifolds. Lu and Sun showed that the volume of the first Chern class with respect to the Weil-Petersson metric over the moduli space M is a rational number. Gang Tian proved that the Weil-Petersson metric on moduli space of polarized Calabi-Yau manifolds is just pull back of Chern form of the tautological of $\mathbb{C}P^N$ restricted to period domain which is an open set of a quadric in $\mathbb{C}P^N$ and he showed that holomorphic sectional curvature is bounded away from zero. Let X be a compact projective Calabi-Yau manifold and let $f : X \rightarrow Y$ be an algebraic fiber space with Y an irreducible normal algebraic variety of lower dimension then Weil-Petersson metric measuring the change of complex structures of the fibers.

Now, consider a polarized Kähler manifolds $\mathcal{X} \rightarrow S$ with Kähler metrics $g(s)$ on \mathcal{X}_s . We can define a possibly degenerate hermitian metric G on S as follows: Take Kodaira-Spencer map

$$\rho : T_{S,s} \rightarrow H^1(X, T_X) \cong H_{\bar{\partial}}^{0,1}(T_X)$$

into harmonic forms with respect to $g(s)$; so for $v, w \in T_s(S)$, we may define

$$G(v, w) := \int_{\mathcal{X}_s} \langle \rho(v), \rho(w) \rangle_{g(s)}$$

When $\mathcal{X} \rightarrow S$ is a polarized Kähler-Einstein family and ρ is injective $G_{WP} := G$ is called the Weil-Petersson metric on S . Tian-Todorov, showed that if we take $\pi : \mathcal{X} \rightarrow S$, $\pi^{-1}(0) = X_0 = X$, $\pi^{-1}(s) = X_s$ be the family of X , then S is a non-singular complex analytic space such that

$$\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(X_s, TX_s)$$

Note that in general, if $f : X \rightarrow S$ be a smooth projective family over a complex manifold S . Then for every positive integer m ,

$$P_m(X_s) = \dim H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$$

is locally constant function on S .

It is worth to mention that the fibers X_s are diffeomorphic to each other and if fibers X_s be biholomorphic then π is holomorphic fiber bundle and Weil-Petersson metric is zero in this case in other words the Kodaira-Spencer maps

$$\rho : T_{S,s} \rightarrow H^1(X_s, TX_s) \cong H_{\bar{\partial}}^{0,1}(TX_s)$$

are zero. In special case, let $\dim X_s = 1$, then the fibers are elliptic curves and π is holomorphic fiber bundle and hence the Weil-Petersson metric is zero. In general, the Weil-Petersson metric is semipositive definite on the moduli space of Calabi-Yau varieties. Note that Moduli space of varieties of general type has Weil-Petersson metric. The moduli space of K-stable varieties admit Weil-Petersson metric also.

Remark: Let $(E, \|\cdot\|)$ be the direct image bundle $f_*(K_{X'/S})$, where $X' = X \setminus D$, of relative canonical line bundle equipped with the L^2 metric $\|\cdot\|$. Then the fibre E_y is $H^0(X_y \setminus D_y, K_{X_y \setminus D_y})$. Since the pair (X_y, D_y) is Calabi-Yau pair, hence $H^0(X_y \setminus D_y, K_{X_y \setminus D_y})$ is a 1-dimensional vector space. This implies that E is a line bundle.

We give a new proof to the following theorem of Tian [16].

Theorem 0.2 (Tian's formula) *Take holomorphic fiber space $\pi : X \rightarrow B$ and assume Ψ_y be any local non-vanishing holomorphic section of Hermitian line bundle $\pi_*(K_{X/B}^L)$, then the Weil-Petersson (1,1)-form on a small ball $N_r(y) \subset B$ can be written as*

$$\omega_{WP} = -\sqrt{-1} \partial_y \bar{\partial}_y \log \left((\sqrt{-1})^{n^2} \int_{X_y} (\Psi_y \wedge \bar{\Psi}_y)^{\frac{1}{t}} \right)$$

Note that ω_{WP} is globally defined on B

Now because we are in deal with Calabi-Yau pair (X, D) which $K_X + D$ is numerically trivial so we must introduce Log Weil-Petersson metrics instead Weil-Petersson metric. Here we introduce such metrics on moduli space of paired Calabi-Yau fibers (X_y, D_y) . Let $i : D \hookrightarrow X$ and $f : X \rightarrow Y$ be holomorphic mappings of complex manifolds such that i is a closed embedding and f as well as $f \circ i$ are proper and smooth. Then a holomorphic family (X_y, D_y) are the fibers $X_y = f^{-1}(y)$ and $D_y = (f \circ i)^{-1}(y)$. Such family give rise to a fibered groupoid $p : \mathbf{F} \rightarrow \mathbf{A}$ from of category \mathbf{F} to the category of complex spaces with distinguished point in the sense of Grothendieck[16]. There exists the moduli space of \mathcal{M} of such family because any (X_y, D_y) with trivial canonical bundle is non-uniruled. Now $X \setminus D$ is quasi-projective so we must deal with quasi-coordinate system instead of coordinate system. Let (X, D) be a Calabi-Yau pair and take $X' = X \setminus D$ equipped with quasi-coordinate system. We say that a tensor A on X' which are covariant of type (p, q) is quasi- $C^{k, \lambda}$ -tensor, if it is of class $C^{k, \lambda}$ with respect to quasi-coordinates. Now we construct the logarithmic version of Weil-Petersson metric on moduli space of paired Calabi-Yau fibers $f : (X, D) \rightarrow Y$.

Now, because we are in deal with singularities, so we use of (1,1)-current instead of (1,1)-forms which is singular version of forms. A current is a differential form with distribution coefficients. Let, give a definition of current here. We recall a singular metric h_{sing} on a Line bundle L which locally can be written as $h_{\text{sing}} = e^\phi h$ where h is a smooth metric, and ϕ is an integrable function. Then one can define locally the closed current $T_{L, h_{\text{sing}}}$ by the following formula

$$T_{L,h_{\text{sing}}} = \omega_{L,h} + \frac{1}{2i\pi} \partial \bar{\partial} \log \phi$$

The current Geometry is more complicated than symplectic geometry. For instance in general one can not perform the wedge product of currents due to this fact that one can not multiply the coefficients which are distributions and in general the product of two distributions is not well defined. However, in some special cases one can define the product of two currents. Here we mention the following important theorem about wedge product of two currents

One can simply defines the space of currents to be the dual of space of smooth forms, defined as forms on the regular part X_{reg} which, near X_{sing} , locally extend as smooth forms on an open set of \mathbb{C}^N in which X is locally embedded. A Kähler current on a compact complex space X is a closed positive current T of bidegree $(1,1)$ which satisfies $T \geq \epsilon\omega$ for some $\epsilon > 0$ and some smooth positive hermitian form ω on X . In fact, This is a real closed current of type $(1,1)$, that is a linear form on the space of compactly supported forms of degree $2n - 2$ on X , and $n = \dim X$. More precisely, Let $\mathcal{A}_c^{p,q}(X)$ denote the space of $C^\infty(p,q)$ forms of compact support on X with usual Fréchet space structure. The dual space $\mathcal{D}^{p,q}(X) := \mathcal{A}_c^{n-p,n-q}(X)^*$ is called the space of (p,q) -currents on M . The Linear operators $\partial : \mathcal{D}^{p,q}(X) \rightarrow \mathcal{D}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{D}^{p,q}(X) \rightarrow \mathcal{D}^{p,q+1}(X)$ is defined by

$$\partial T(\varphi) = (-1)^{p+q+1} T(\partial\varphi), \quad T \in \mathcal{D}^{p,q}(X), \quad \varphi \in \mathcal{A}_c^{n-p-1,n-q}(X)$$

and

$$\bar{\partial} T(\varphi) = (-1)^{p+q+1} T(\bar{\partial}\varphi), \quad T \in \mathcal{D}^{p,q}(X), \quad \varphi \in \mathcal{A}_c^{n-p,n-q-1}(X)$$

We set $d = \partial + \bar{\partial}$. $T \in \mathcal{D}^{p,q}(X)$ is called closed if $dT = 0$. $T \in \mathcal{D}^{p,p}(X)$ is called real if $T(\varphi) = T(\bar{\varphi})$ holds for all $\mathcal{A}_c^{n-p,n-q}(X)$. A real (p,p) -current T is called positive if $(\sqrt{-1})^{p(n-p)} T(\eta \wedge \bar{\eta}) \geq 0$ holds for all $\eta \in \mathcal{A}_c^{p,0}(X)$.

The topology on space of currents are so important. In fact the space of currents with weak topology is a Montel space, i.e., barrelled, locally convex, all bounded subsets are precompact which here barrelled topological vector space is Hausdorff topological vector space for which every barrelled set in the space is a neighbourhood for the zero vector.

Also because we use of push-forward and Pull back of a current and they can cont be defined in sense of forms, we need to introduce them. If $f : X \rightarrow Y$ be a holomorphic map between two compact Kähler manifolds then one can push-forward a current ω on X by duality setting

$$\langle f_*\omega, \eta \rangle := \langle \omega, f^*\eta \rangle$$

In general, given a current T on Y , it is not possible to define its pull-back by a holomorphic map. But it is possible to define pull-back of positive closed currents of bidegree $(1,1)$. We can writes such currents as $T = \theta + dd^c\varphi$ where $\theta \in T$ is a smooth form, and thus one define the pull-back of current T as follows

$$f^*T := f^*\theta + dd^c\varphi \circ f$$

Let X and Y be compact Kähler manifolds and let $f : X \rightarrow Y$ be the blow up of Y with smooth connected center Z and $\omega \in H^{1,1}(X, \mathbb{R})$. Demailly showed that

$$\omega = f^*f_*\omega + \lambda E$$

where E is the exceptional divisor and $\lambda \geq -v(\omega, Z)$ where $v(\omega, Z) = \inf_{x \in Z} v(\omega, x)$ and $v(\omega, x)$ is the Lelong number.

As an application, the pushforward $f_*\Omega$ of a smooth nondegenerate volume form Ω on X with respect to the holomorphic map $\pi : X \rightarrow Y$ is defined as follows: From definition of pushforward of a current by duality, for any continuous function ψ on Y , we have

$$\int_Y \psi f_*\Omega = \int_X (f^*\psi)\Omega = \int_{y \in Y} \int_{\pi^{-1}(y)} (f^*\psi)\Omega$$

and hence on regular part of Y we have

$$\pi_*\Omega = \int_{\pi^{-1}(y)} \Omega \quad (*)$$

and hence

$$\|\Omega\|_Y^2 = \int_{\pi^{-1}(y)} \|\Omega\|_y^2$$

Note that if ω is Kähler then,

$$d\text{Vol}_{\omega_y}(X_y) = df_*(\omega^n) = f_*(d\omega^n) = 0$$

So, $\text{Vol}(X_y) = C$ for some constant $C > 0$ for every $y \in Y$ where $\pi^{-1}(y) = X_y$. See [1][2]. Moreover direct image of volume form $f_*\omega_X^n = \sigma\omega_Y^m$ where $\sigma \in L^{1+\epsilon}$ for some positive constant ϵ , see[1].

Theorem 0.3 *If T is a positive $(1,1)$ -current then it was proved in [?] that locally one can find a plurisubharmonic function u such that*

$$\sqrt{-1}\partial\bar{\partial}u = T$$

Note that, if X be compact then there is no global plurisubharmonic function u .

Main Theorem

Now we are ready to state our theorem. We must mention that The result of Tian was on Polarized Calabi-Yau fibers and in this theorem we consider non-polarized fibers.

Theorem 0.4 *Let $\pi : X \rightarrow Y$ be a smooth family of compact Kähler manifolds with Calabi-Yau fibers. Then Weil-Petersson metric can be written as*

$$\omega_{WP} = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y} |\Omega_y|^2$$

where Ω_y is a holomorphic $(n,0)$ -form on $\pi^{-1}(U)$, where U is a neighborhood of y

Proof: For proof, We need to recall the Yau-Vafa semi Ricci flat metrics. Since fibers are Calabi-Yau varieties, so $c_1(X_y) = 0$, hence there is a smooth function F_y such that $Ric(\omega_y) = \sqrt{-1}\partial\bar{\partial}F_y$. The function F_y vary smoothly in y . By Yau's theorem, there is a unique Ricci-flat Kähler metric $\omega_{SRF,y}$ on X_y cohomologous to ω_0 where ω_0 is a Kähler metric attached to X . So there is a smooth function ρ_y on X_y such that $\omega_0|_{X_y} + \sqrt{-1}\partial\bar{\partial}\rho_y = \omega_{SRF,y}$ is the unique Ricci-flat Kähler metric on X_y . If we normalize ρ_y , then ρ_y varies smoothly in y and defines a smooth function ρ on X and we let

$$\omega_{SRF} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho$$

which is called as semi-Ricci flat metric. Robert Berman and Y.J.Choi independently showed that the semi-Ricci flat metric is semi positive along horizontal direction. Now for semi Ricci flat metric ω_{SRF} , we have

$$\omega_{SRF}^{n+1} = c(\omega_{SRF})\omega_{SRF}^n dy \wedge d\bar{y}$$

Here $c(\omega_{SRF})$ is called a geodesic curvature of semi ω_{SRF} . Now from Berman and Choi formula, for $V \in T_y Y$, the following PDE holds on X_y

$$-\Delta_{\omega_{SRF}} c(\omega_{SRF})(V) = |\bar{\partial}V_{\omega_{SRF}}|_{\omega_{SRF}}^2 - \Theta_{V\bar{V}}(\pi_*(K_{X/Y}))$$

$\Theta_{V\bar{V}}$ is the Ricci curvature of direct image of relative line bundle(which is a line bundle, since fibers are Calabi Yau manifolds). Now by integrating on both sides of this PDE, since

$$\int_X \Delta_{\omega_{SRF}} c(\omega_{SRF})(V) = 0$$

and from the definition of Weil-Petersson metric and this PDE we get $\pi^*\omega_{WP} = Ric(\pi_*(K_{X/Y}))$ and hence for some holomorphic $(n,0)$ -form(as non vanishing holomorphic section on the direct image of relative line bundle, which is still line bundle, since fibres are Calabi-Yau varieties) Ω_y on $\pi^{-1}(U)$, where U is a neighborhood of y and by (*) we have

$$Ric(\pi_*(K_{X/Y})) = -\sqrt{-1}\partial\bar{\partial} \log \|\Omega\|_y^2$$

Hence

$$\omega_{WP} = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y} |\Omega_y|^2$$

and we obtain the desired result.

Logarithmic Weil-Petersson metric

Now we give a motivation that why the geometry of pair (X, D) must be interesting. The first one comes from algebraic geometry, in fact for deforming the cone angle we need to use of geometry of pair (X, D) . In the case of minimal general type manifold the canonical bundle of X , i.e., K_X is nef and we would like K_X to be ample and it is not possible in general and what we can do is that to add a small multiple of ample bundle $\frac{1}{m}A$, i.e., $K_X + \frac{1}{m}A$ and then we are deal with the pair $(X, \frac{1}{m}H)$ which H is a generic section of it. The second one is the works of Chen-Sun-Donaldson and Tian on existence of Kähler Einstein metrics for Fano varieties which they used of geometry of pair (X, D) for their proof .

Now we explain Tian-Yau program to how to construct model metrics in general, like conical model metric, Poincare model metric, or Saper model metric,....

Tian-Yau program:Let $\mathbb{C}^n = \mathbb{C}^n(z_1, \dots, z_n)$ be a complex Euclidian space for some $n > 0$. For a positive number ϵ with $0 < \epsilon < 1$ consider

$$\bar{X} = \bar{X}_\epsilon = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < \epsilon\}$$

Now, let $D_i = \{z_i = 0\}$ be the irreducible divisors and take $D = \sum_i D_i$ where

$$D = \{z \in \bar{X} \mid z_1 z_2 \dots z_k = 0\}$$

and take $X = \bar{X} \setminus D$. In polar coordinate we can write $z_i = r_i e^{i\theta_i}$. Let g be a Kähler metric on D such that the associated Kähler form ω is of the following form

$$\omega = \sqrt{-1} \sum_i \frac{1}{|dz_i|^2} dz_i \wedge d\bar{z}_i$$

Then the volume form dv associated to ω is written in the form;

$$dv = (\sqrt{-1})^n \prod_{i=1}^n \frac{1}{|dz_i|^2} \prod_i dz_i \wedge d\bar{z}_i, \quad v = \frac{1}{|dz_i|^2}$$

Let L be a (trivial) holomorphic line bundle defined on \bar{X} , with a generating holomorphic section S on \bar{X} . Fix a C^∞ hermitian metric h of L over X and denote by $|S|^2$ the square norm of S with respect to h . Assume the functions $|S|^2$ and $|dz_i|^2$ depend only on r_i , $1 \leq i \leq k$. Set

$$d(r_1, \dots, r_k) = |S|^2 \cdot v \cdot \prod_{1 \leq i \leq k} r_i$$

and further make the following three assumptions:

A1) The function d is of the form

$$d(r_1, \dots, r_k) = r_1^{c_1} \dots r_k^{c_k} (\log 1/r_1)^{b_1} \dots (\log 1/r_k)^{b_k} L(r_1, \dots, r_k)^t$$

where

$$L = L(r_1, \dots, r_k) = \sum_{i=1}^k \log 1/r_i$$

and c_i, b_j, t are real numbers with $t \geq 0$ such that $q_i = b_i + t \neq -1$ if c_i is an odd integer. We set $a_i = (c_i + 1)/2$ and denote by $[a_i]$ the largest integer which does not exceed a_i .

A2) If $1 \leq i \leq k$, then $|dz_i|^2$ is either of the following two forms;

$$|dz_i|^2(r) = r_i^2(\log 1/r_i)^2, \quad \text{or} \quad |dz_i|^2(r) = r_i^2 L^2$$

In fact, A2) implies that the Kähler metric g is (uniformly) complete along D .

A3) If $k + 1 \leq i \leq n$, then $|dz_i|^{-2}$ is bounded (above) on X .

Now, we give some well-known examples of Tian-Fujiki picture, i.e, conical model metric, Poincare model metric, and Saper model metric.

A Kähler current ω is called a conical Kähler metric (or Hilbert Modular type) with angle $2\pi\beta$, ($0 < \beta < 1$) along the divisor D , if ω is smooth away from D and asymptotically equivalent along D to the model conic metric

$$\omega_\beta = \sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right)$$

here (z_1, z_2, \dots, z_n) are local holomorphic coordinates and $D = \{z_1 = 0\}$ locally.

After an appropriate -singular- change of coordinates, one can see that this model metric represents an Euclidean cone of total angle $\theta = 2\pi\beta$, whose model on \mathbb{R}^2 is the following metric: $d\theta^2 + \beta^2 dr^2$. The volume form V of a conical Kähler metric ω_D on the pair (X, D) has the form

$$V = \prod_j |S_j|^{2\beta_j - 2} e^f \omega^n$$

where $f \in C^0$.

This asymptotic behaviour of metrics can be translated to the second order asymptotic behaviour of their potentials

$$\omega_\beta = -\sqrt{-1} \partial \bar{\partial} \log e^{-u}$$

where $u = \frac{1}{2} \left(\frac{1}{\beta^2} |z_1|^{2\beta} + |z_2|^2 + \dots + |z_n|^2 \right)$.

Moreover, if we let $z = r e^{i\theta}$ and $\rho = r^\beta$ then the model metric in ω_β becomes

$$(d\rho + \sqrt{-1}\beta\rho d\theta) \wedge (d\rho - \sqrt{-1}\beta\rho d\theta) + \sum_{i>1} dz_i \wedge d\bar{z}_i$$

and if we set $\epsilon = e^{\sqrt{-1}\beta\theta} (d\rho + \sqrt{-1}\beta\rho d\theta)$ then the conical Kähler metric ω on $(X, (1 - \beta)D)$ can be expressed as

$$\omega = \sqrt{-1} (f\epsilon \wedge \bar{\epsilon} + f_{\bar{j}}\epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j)$$

By the assumption on the asymptotic behaviour we mean there exists some coordinate chart in which the zero-th order asymptotic of the metric agrees with the model metric. In other words, there is a constant C , such that

$$\frac{1}{C}\omega_\beta \leq \omega \leq C\omega_\beta$$

In this note because we assume certain singularities for the Kähler manifold X we must design our Kähler Ricci flow such that our flow preserve singularities. Now fix a simple normal crossing divisor $D = \sum_i (1 - \beta_i)D_i$, where $\beta_i \in (0, 1)$ and simple normal crossing divisor D means that D_i 's are irreducible smooth divisors and for any $p \in \text{Supp}(D)$ lying in the intersection of exactly k divisors D_1, D_2, \dots, D_k , there exists a coordinate chart $(U_p, \{z_i\})$ containing p , such that $D_i|_{U_p} = \{z_i = 0\}$ for $i = 1, \dots, k$.

If $S_i \in H^0(X, \mathcal{O}_X(L_{D_i}))$ is the defining sections and h_i is hermitian metrics on the line bundle induced by D_i , then Donaldson showed that for sufficiently small $\epsilon_i > 0$, $\omega_i = \omega_0 + \epsilon_i \sqrt{-1} \partial \bar{\partial} |S_i|_{h_i}^{2\beta_i}$ gives a conic Kähler metric on $X \setminus \text{Supp}(D_i)$ with cone angle $2\pi\beta_i$ along divisor D_i and also if we set $\omega = \sum_{i=1}^N \omega_i$ then, ω is a smooth Kähler metric on $X \setminus \text{Supp}(D)$ and

$$\|S\|^{2(1-\beta)} = \prod_{i=1}^k \|S_i\|^{2(1-\beta)}$$

where $S \in H^0(X, \mathcal{O}(L_D))$. Moreover, ω is uniformly equivalent to the standard cone metric

$$\omega_p = \sum_{i=1}^k \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=k+1}^N \sqrt{-1} dz_i \wedge d\bar{z}_i$$

From Tian-Fujiki theory, $|dz_i|^2 = r_i^2$ for $1 \leq i \leq k$ and $|dz_j|^2 = 1$ for $k+1 \leq j \leq n$ so that A2) and A3) are again satisfied.

From now on for simplicity we write just "divisor D " instead "simple normal crossing divisor D ".

We give an example of varieties which have conical singularities. Consider a smooth geometric orbifold given by \mathbb{Q} -divisor

$$D = \sum_{j \in J} (1 - \frac{1}{m_j}) D_j$$

where $m_j \geq 2$ are positive integers and $\text{Supp} D = \bigcap_{j \in J} D_j$ is of normal crossings divisor. Let ω be any Kähler metric on X , let $C > 0$ be a real number and $s_j \in H^0(X, \mathcal{O}_X(D_j))$ be a section defining D_j . Consider the following expression

$$\omega_D = C\omega + \sqrt{-1} \sum_{j \in J} \partial \bar{\partial} |s_j|^{2/m_j}$$

If C is large enough, the above formula defines a closed positive $(1, 1)$ -current (smooth away from D). Moreover

$$\omega_D \geq \omega$$

in the sense of currents. Consider \mathbb{C}^n with the orbifold divisor given by the equation

$$\prod_{j=1}^n z_j^{1-1/m_j} = 0$$

(with eventually $m_j = 1$ for some j). The sections s_j are simply the coordinates z_j and a simple computation gives

$$\omega_D = \omega_{eucl} + \sqrt{-1} \sum_{j=1}^n \partial\bar{\partial} |z_j|^{2/m_j} = \omega_{eucl} + \sqrt{-1} \sum_{j=1}^n \frac{dz_j \wedge d\bar{z}_j}{m_j^2 |z_j|^{2(1-1/m_j)}}$$

Here we mention also metrics with non-conic singularities. We say a metric ω is of Poincare type, if it is quasi-isometric to

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right)$$

It is always possible to construct a Poincare metric on $M \setminus D$ by patching together local forms with C^∞ partitions of unity. Now, from Tian-Fujiki theory $|dz_i|^2 = r_i^2 (\log 1/r_i)^2$, $1 \leq i \leq k$ and $|dz_j|^2 = 1$, $k+1 \leq j \leq n$ so that A2) and A3) above are satisfied; we have

$$v = \prod_{1 \leq i \leq k} r_i^{-2} (\log 1/r_i)^{-2}$$

Let Ω_P be the volume form on $X \setminus D$, then, there exists a locally bounded positive continuous function $c(z)$ on polydisk \mathbb{D}^n such that

$$\Omega_P = c(z) \sqrt{-1} \left(\wedge_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \wedge_{i=k+1}^n dz_i \wedge d\bar{z}_i \right)$$

holds on $\mathbb{D}^n \cap (X \setminus D)$

Remark: Note that if Ω_P be a volume form of Poincare growth on (X, D) , with X compact. If $c(z)$ be C^2 on \mathbb{D}^n , then $-Ric(\Omega_P)$ is of Poincare growth.

We say that ω is the homogeneous Poincare metric if its fundamental form ω_β is described locally in normal coordinates by the quasi-isometry

$$\omega_\beta = \sqrt{-1} \left(\frac{1}{(\log |z_1 z_2 \dots z_k|^2)^2} \sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2} + \sum_{i=1}^n dz_i \wedge d\bar{z}_i \right)$$

and we say ω has Ball Quotient singularities if it is quasi-isometric to

$$\omega_\beta = \sqrt{-1} \frac{dz_1 \wedge d\bar{z}_1}{(|z_1| \log(1/|z_1|))^2} + \sqrt{-1} \sum_{j=2}^n \frac{dz_j \wedge d\bar{z}_j}{\log 1/|z_1|}$$

It is called also Saper's distinguished metrics.

$$|dz_1|^2 = r_1^2 (\log 1/r_1)^2, \quad |dz_j|^2 = \log 1/r_1, \quad k+1 \leq j \leq n$$

so that A2) and A3) are satisfied; also we have the volume form as

$$v = r_1^{-2} (\log 1/r_1)^{-(n+1)}$$

If ω is the fundamental form of a metric on the compact manifold X , and ω_{sap} be the fundamental forms of Saper's distinguished metrics and $\omega_{P,hom}$ be the fundamental forms of homogeneous Poincare metric, on the noncompact manifold $M \setminus D$, then $\omega_{sap} + \omega$ and $\omega_{P,hom}$ are quasi-isometric.

Definition 0.5 *A Kähler metric with cone singularities along D with cone angle $2\pi\beta$ is a smooth Kähler metric on $X \setminus D$ which satisfies the following conditions when we write $\omega_{sing} = \sum_{i,j} g_{i\bar{j}} \sqrt{-1} dz_i \wedge d\bar{z}_j$ in terms of the local holomorphic coordinates $(z_1; \dots; z_n)$ on a neighbourhood $U \subset X$ with $D \cap U = \{z_1 = 0\}$*

1. $g_{1\bar{1}} = F|z_1|^{2\beta-2}$ for some strictly positive smooth bounded function F on $X \setminus D$
2. $g_{1\bar{j}} = g_{i\bar{1}} = O(|z_1|^{2\beta-1})$
3. $g_{i\bar{j}} = O(1)$ for $i, j \neq 1$

Now we shortly explain Donaldson's linear theory which is useful later in the definition of logarithmic Vafa-Yau's semi ricci flat metrics.

Definition 0.6 1) *A function f is in $C^{\gamma,\beta}(X, D)$ if f is C^γ on $X \setminus D$, and locally near each point in D , f is C^γ in the coordinate $(\hat{\zeta} = \rho e^{i\theta} = z_1|z_1|^{\beta-1}, z_j)$.*

2) *A $(1,0)$ -form α is in $C^{\gamma,\beta}(X, D)$ if α is C^γ on $X \setminus D$ and locally near each point in D , we have $\alpha = f_1 \epsilon + \sum_{j>1} f_j dz_j$ with $f_i \in C^{\gamma,\beta}$ for $1 \leq i \leq n$, and $f_1 \rightarrow 0$ as $z_1 \rightarrow 0$ where $\epsilon = e^{\sqrt{-1}\beta\theta} (d\rho + \sqrt{-1}\beta\rho d\theta)$*

3) *A $(1,1)$ -form ω is in $C^{\gamma,\beta}(X, D)$ if ω is C^γ on $X \setminus D$ and near each point in D we can write ω as*

$$\omega = \sqrt{-1} (f \epsilon \wedge \bar{\epsilon} + f_{\bar{j}} \epsilon \wedge d\bar{z}_j + f_j dz_j \wedge \bar{\epsilon} + f_{i\bar{j}} dz_i \wedge d\bar{z}_j)$$

such that $f, f_j, f_{\bar{j}}, f_{i\bar{j}} \in C^{\gamma,\beta}$, and $f_j, f_{\bar{j}} \rightarrow 0$ as $z_1 \rightarrow 0$

4) *A function f is in $C^{2,\gamma,\beta}(X, D)$ if $f, \partial f, \partial\bar{\partial} f$ are all in $C^{\gamma,\beta}$*

Fix a smooth metric ω_0 in $c_1(X)$, we define the space of admissible functions to be

$$\hat{C}(X, D) = C^{2,\gamma}(X) \cup \bigcup_{0 < \beta < 1} \left(\bigcup_{0 < \gamma < \beta^{-1} - 1} C^{2,\gamma,\beta}(X, D) \right)$$

and the space of admissible Kähler potentials to be

$$\hat{\mathcal{H}}(\omega_0) = \{\phi \in \hat{C}(X, D) \mid \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Note that

$$\mathcal{H}(\omega_0) \subset \hat{\mathcal{H}}(\omega_0) \subset \mathcal{PSH}(\omega_0) \cap L^\infty(X)$$

Where $\mathcal{PSH}(\omega_0) \cap L^\infty(X)$ is the space of bounded ω_0 -plurisubharmonic functions and

$$\mathcal{PSH}(\omega_0) = \{\phi \in L^1_{loc}(X) \mid \phi \text{ is u.s.c and } \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

The Ricci curvature of the Kählerian form ω_D on the pair (X, D) can be represented as:

$$Ric(\omega_D) = 2\pi \sum_j (1 - \beta_j)[D_j] + \theta + \sqrt{-1}\partial\bar{\partial}\psi$$

with $\psi \in C^0(X)$ and θ is closed smooth $(1, 1)$ -form.

We have also dd^c -lemma on $X = \bar{X} \setminus D$. Let Ω be a smooth closed $(1, 1)$ -form in the cohomology class $c_1(K_{\bar{X}}^{-1} \otimes L_D^{-1})$. Then for any $\epsilon > 0$ there exists an explicitly given complete Kähler metric g_ϵ on M such that

$$Ric(g_\epsilon) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f_\epsilon \text{ on } X$$

where f_ϵ is a smooth function on X that decays to the order of $O(\|S\|^\epsilon)$. Moreover, the Riemann curvature tensor $R(g_\epsilon)$ of the metric g_ϵ decays to the order of $O\left((-n \log \|S\|^2)^{-\frac{1}{n}}\right)$

Now we explain the logarithmic Weil-Petersson metric on moduli space of log Calabi-Yau manifolds (if it exists. for special case of rational surfaces it has been proven that such moduli space exists). The logarithmic Weil-Petersson metric has pole singularities [7] and we can introduce it also by elements of logarithmic Kodaira-Spencer tensors which represent elements of $H^1(X, \Omega_X^1(\log(D))^\vee)$. More precisely, Let X be a complex manifold, and $D \subset X$ a divisor and ω a holomorphic p -form on $X \setminus D$. If ω and $d\omega$ have a pole of order at most one along D , then ω is said to have a logarithmic pole along D . ω is also known as a logarithmic p -form. The logarithmic p -forms make up a subsheaf of the meromorphic p -forms on X with a pole along D , denoted

$$\Omega_X^p(\log D)$$

and for the simple normal crossing divisor $D = \{z_1 z_2 \dots z_k = 0\}$ we can write the stalk of $\Omega_X^1(\log D)$ at p as follows

$$\Omega_X^1(\log D)_p = \mathcal{O}_{X,p} \frac{dz_1}{z_1} \oplus \dots \oplus \mathcal{O}_{X,p} \frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p} dz_{k+1} \oplus \dots \oplus \mathcal{O}_{X,p} dz_n$$

Since, fibers are log Calabi-Yau manifolds and by recent result of Jeffres-Mazzeo-Rubinstein [9], we have Ricci flat metric on each fiber (X_y, D_y) and hence we can have log semi-Ricci flat metric and by the same method of previous theorem, the proof of Theorem 0.8 is straightforward.

Theorem 0.7 *Let (M, ω_0) be a compact Kähler manifold with $D \subset M$ a smooth divisor and suppose we have topological constraint condition $c_1(M) = (1 - \beta)[D]$ where $\beta \in (0, 1]$ then there exists a conical Kähler Ricci flat metric with angle $2\pi\beta$ along D . This metric is unique in its Kähler class. This metric is polyhomogeneous; namely, the Kähler Ricci flat metric $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ admits a complete asymptotic expansion with smooth coefficients as $r \rightarrow 0$ of the form*

$$\varphi(r, \theta, Z) \sim \sum_{j,k \geq 0} \sum_{l=0}^{N_{j,k}} a_{j,k,l}(\theta, Z) r^{j+k/\beta} (\log r)^l$$

where $r = |z_1|^\beta / \beta$ and $\theta = \arg z_1$ and with each $a_{j,k,l} \in C^\infty$

Now we can introduce Logarithmic Yau-Vafa semi Ricci flat metrics. The volume of fibers (X_y, D_y) are homological constant independent of y , and we assume that it is equal to 1. Since fibers are log Calabi-Yau varieties, so $c_1(X_y, D_y) = 0$, hence there is a smooth function F_y such that $Ric(\omega_y) = \sqrt{-1}\partial\bar{\partial}F_y$. The function F_y vary smoothly in y . By Jeffres-Mazzeo-Rubinstein's theorem, there is a unique conical Ricci-flat Kähler metric $\omega_{SRF,y}$ on $X_y \setminus D_y$ cohomologous to ω_0 . So there is a smooth function ρ_y on $X_y \setminus D_y$ such that $\omega_0|_{X_y \setminus D_y} + \sqrt{-1}\partial\bar{\partial}\rho_y = \omega_{SRF,y}$ is the unique Ricci-flat Kähler metric on $X_y \setminus D_y$. If we normalize ρ_y , then ρ_y varies smoothly in y and defines a smooth function ρ^D on $X \setminus D$ and we let

$$\omega_{SRF}^D = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho^D$$

which is called as Log Semi-Ricci Flat metric.

Let $f : X \setminus D \rightarrow S$, be a smooth family of quasi-projective Kähler manifolds. Let $x \in X \setminus D$, and $(\sigma, z_2, \dots, z_n, s^1, \dots, s^d)$, be a coordinate centered at x , where $(\sigma, z_2, \dots, z_n)$ is a local coordinate of a fixed fiber of f and (s^1, \dots, s^d) is a local coordinate of S , such that

$$f(\sigma, z_2, \dots, z_n, s^1, \dots, s^d) = (s^1, \dots, s^d)$$

Now consider a smooth form ω on $X \setminus D$, whose restriction to any fiber of f , is positive definite. Then ω can be written as

$$\begin{aligned} \omega(\sigma, z, s) = & \sqrt{-1}(\omega_{i\bar{j}} ds^i \wedge d\bar{s}^j + \omega_{i\bar{\beta}} ds^i \wedge d\bar{z}^\beta + \omega_{\alpha\bar{j}} dz^\alpha \wedge d\bar{s}^j + \omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \omega_\sigma d\sigma \wedge d\bar{s}^j \\ & + \omega_{i\bar{\sigma}} ds^i \wedge d\bar{\sigma} + \omega_{\sigma\bar{\sigma}} d\sigma \wedge d\bar{\sigma} + \omega_{\sigma\bar{j}} d\sigma \wedge d\bar{z}^j + \omega_{i\bar{\sigma}} dz^i \wedge d\bar{\sigma}) \end{aligned}$$

Since ω is positive definite on each fibre, hence

$$\sum_{\alpha, \beta=2} \omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \omega_{\sigma\bar{\sigma}} d\sigma \wedge d\bar{\sigma} + \sum_{j=2} \omega_{\sigma\bar{j}} d\sigma \wedge d\bar{z}^j + \sum_{i=2} \omega_{i\bar{\sigma}} dz^i \wedge d\bar{\sigma}$$

gives a Kähler metric on each fiber $X_s \setminus D_s$. So

$$\det(\omega_{\lambda\bar{\eta}}^{-1}(\sigma, z, s)) = \det \begin{pmatrix} \omega_{\sigma\bar{\sigma}} & \omega_{\sigma\bar{2}} & \dots & \omega_{\sigma\bar{n}} \\ \omega_{2\bar{\sigma}} & \omega_{2\bar{2}} & \dots & \omega_{2\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n\bar{\sigma}} & \omega_{n\bar{2}} & \dots & \omega_{n\bar{n}} \end{pmatrix}^{-1}$$

gives a hermitian metric on the relative line bundle $K_{X'/S}$ and its Ricci curvature can be written as $\sqrt{-1}\partial\bar{\partial} \log \det \omega_{\lambda\bar{\eta}}(\sigma, z, s)$

Theorem 0.8 *By the same method we can introduce the logarithmic Weil-Petersson metric on $\pi : (X, D) \rightarrow Y$ with assuming fibers to be log Calabi-Yau manifolds and snc divisor D has conic singularities, then we have*

$$\omega_{WP}^D = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y \setminus D_y} \frac{\Omega_y \wedge \bar{\Omega}_y}{\|S_y\|^2}$$

where $S_y \in H^0(X_y, L_{D_y})$. Moreover, if ω has Poincare singularities along snc divisor D , we have the following formula for logarithmic Weil-Petersson metric

$$\omega_{WP}^D = -\sqrt{-1}\partial_y\bar{\partial}_y \log \int_{X_y \setminus D_y} \frac{\Omega_y \wedge \bar{\Omega}_y}{\|S_y\|^2 \log^2 \|S_y\|^2}$$

Now in next theorem we will find the relation between logarithmic Weil-Petersson metric and fiberwise Ricci flat metric which can be considered as the logarithmic version of Song-Tian formula [1, 2].

Theorem 0.9 *Let $\pi : (X, D) \rightarrow Y$ be a holomorphic family of log Calabi-Yau pairs (X_s, D_s) for the Kähler varieties X, Y . Then we have the following relation between logarithmic Weil-Petersson metric and fiberwise Ricci flat metric.*

$$\sqrt{-1}\partial\bar{\partial} \log \left(\frac{f^* \omega_Y^m \wedge (\omega_{SRF}^D)^{n-m}}{|S|^2} \right) = -f^* Ric(\omega_Y) + f^* \omega_{WP}^D$$

where $S \in H^0(X, \mathcal{O}(L_N))$, here N is a divisor which come from Fujino-Mori's canonical bundle formula

Proof: Take $X' = X \setminus D$. Choose a local nonvanishing holomorphic section Ψ_y of $\pi_*(K_{X'/Y}^{\otimes l})$ with $y \in U \subset X'$. We define a smooth positive function on $\pi(U)$ by

$$u(y) = \frac{(\sqrt{-1})^{(n-m)^2} (\Psi_y \wedge \bar{\Psi}_y)^{\frac{1}{l}}}{(\omega_{SRF}^D)^{n-m} |_{X_y \setminus D_y}}$$

But the Numerator and Denominator of u are Ricci flat volume forms on $X_y \setminus D_y$, so u is a constant function. Hence by integrating $u(y)(\omega_{SRF}^D)^{n-m} |_{X_y \setminus D_y}$ over $X_y \setminus D_y$ we see that

$$u(y) = \frac{(\sqrt{-1})^{(n-m)^2} \int_{X_y \setminus D_y} \frac{(\Psi_y \wedge \bar{\Psi}_y)}{|S_y|^2} \frac{1}{|S_y|^2}}{\int_{X_y \setminus D_y} \frac{(\omega_{SRF}^D)^{n-m}}{|S_y|^2}}$$

where $S_y \in H^0(X', \mathcal{O}(L_{D_y}))$.

But $y \mapsto \int_{X_y \setminus D_y} \frac{(\omega_{SRF}^D)^{n-m}}{|S_y|^2}$ is constant over Y . Hence the Logarithmic Weil-Petersson can be written as

$$-\sqrt{-1} \partial \bar{\partial} \log u = \omega_{WP}^D \quad (*)$$

Now, to finish the proof we can write $\Psi_y = F(\sigma, y, z)(d\sigma \wedge dz_2 \wedge \dots \wedge dz_{n-m})$ where F is holomorphic and non-zero. Hence by substituting Ψ_y in u and rewriting $\sqrt{-1} \partial \bar{\partial} \log(\frac{f^* \omega_Y^m \wedge (\omega_{SRF}^D)^{n-m}}{|S|^2})$ and using $(*)$ we get the desired result.

Remark:Note that the log semi-Ricci flat metric ω_{SRF}^D is not continuous in general. But if the central fiber has at worst canonical singularities and the central fiber (X_0, D_0) be itself as Calabi-Yau pair, then by open condition property of Kahler-Einstein metrics, semi-Ricci flat metric is smooth in an open Zariski subset.

Remark:So by applying the previous remark, the relative volume form

$$\Omega_{(X,D)/Y} = \frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m |S|^2}$$

is not smooth in general, where $S \in H^0(X, L_N)$ and N is a divisor which come from canonical bundle formula of Fujino-Mori.

Now we try to extend the Relative Ricci flow to the fiberwise conical relative Ricci flow. We define the conical Relative Ricci flow on pair $\pi : (X, D) \rightarrow Y$ where D is a simple normal crossing divisor as follows

$$\frac{\partial \omega}{\partial t} = -Ric_{(X,D)/Y}(\omega) - \omega + [N]$$

where N is a divisor which come from canonical bundle formula of Fujino-Mori.

Take the reference metric as $\tilde{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) Ric(\frac{\omega_{SRF}^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m})$ then the conical relative Kähler Ricci flow is equivalent to the following relative Monge-Ampere equation

$$\frac{\partial \phi_t}{\partial t} = \log \frac{(\tilde{\omega}_t + Ric(h_N) + \sqrt{-1} \partial \bar{\partial} \phi_t)^n \wedge \pi^* \omega_{can}^m |S_N|^2}{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m} - \phi_t$$

With cone angle $2\pi\beta$, ($0 < \beta < 1$) along the divisor D , where h is an Hermitian metric on line bundle corresponding to divisor N , i.e., L_N . This

equation can be solved. Take, $\omega = \omega(t) = \omega_B + (1 - \beta)Ric(h) + \sqrt{-1}\partial\bar{\partial}v$ where $\omega_B = e^{-t}\omega_0 + (1 - e^{-t})Ric(\frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m})$, by using Poincare-Lelong equation,

$$\sqrt{-1}\partial\bar{\partial}\log|s_N|_h^2 = -c_1(L_N, h) + [N]$$

we have

$$\begin{aligned} Ric(\omega) &= \\ &= -\sqrt{-1}\partial\bar{\partial}\log\omega^m \\ &= -\sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} - \sqrt{-1}\partial\bar{\partial}v - (1 - \beta)c_1([N], h) + (1 - \beta)\{N\} \end{aligned}$$

and

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \sqrt{-1}\partial\bar{\partial}v &= \\ &= \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \omega - \omega_B - Ric(h) \end{aligned}$$

Hence, by using

$$\omega_{WP}^D = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{(\omega_{SRF}^D)^n \wedge \pi^*\omega_{can}^m}{\pi^*\omega_{can}^m |S|^2}\right)$$

we get

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log\pi_*\Omega_{(X,D)/Y} + \sqrt{-1}\partial\bar{\partial}v &= \\ &= \omega - \omega_{WP}^D - (1 - \beta)c_1(N) \end{aligned}$$

So,

$$Ric(\omega) = -\omega + \omega_{WP}^D + (1 - \beta)[N]$$

which is equivalent with

$$Ric_{(X,D)/Y}(\omega) = -\omega + [N]$$

Now we prove the C^0 -estimate for this relative Monge-Ampere equation. We use the following important lemma from Schumacher and also Cheeger-Yau,

Lemma 0.10 *Suppose that the Ricci curvature of ω is bounded from below by negative constant -1 . Then there exists a strictly positive function $P_n(\text{diam}(X, D))$, depending on the dimension n of X and the diameter $\text{diam}(X, D)$ with the following property:*

Let $0 < \epsilon \leq 1$. If g is a continuous function and f is a solution of

$$(-\Delta_\omega + \epsilon)f = g,$$

then

$$f(z) \geq P_n(\text{diam}(X, D)) \cdot \int_X g dV_\omega$$

So along relative Kähler-Ricci flow we have $\text{Ric}(\omega) \geq -2\omega$ where ω is the solution of Kähler-Ricci flow. But if we restrict our relative Monge-Ampere equation to each fiber (X_s, D_s) , then we need diameter bound on the fibers, i.e.,

$$\text{diam}(X_s \setminus D_s, \omega_s) \leq C$$

But from recent result of Takayama (On Moderate Degenerations of Polarized Ricci-Flat Kähler Manifolds, J. Math. Sci. Univ. Tokyo, 22 (2015), 469–489) we know that we have

$$\text{diam}(X_s \setminus D_s, \omega_s) \leq 2 + D \int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega}_s}{|S_s|^2}$$

if and only if we have 1) central fiber $X_0 \setminus D_0$ has at worst canonical singularities and $K_{X_0} + D_0 = \mathcal{O}_{X_0}(D_0)$ which means the central fiber itself be log Calabi-Yau variety.

So this means that we have C^0 -estimate for relative Kähler-Ricci flow if and only if the central fiber be Calabi-Yau variety with at worst canonical singularities.

Remark: Tian's Kähler potential induces a singular Hermitian metric with semi-positive curvature current on the tautological quotient bundle over the projective-space bundle $\mathbb{P}(f_*(K_{X/B}))$.

Now we explain that under some algebraic condition the Tian's Kähler potential on the moduli space of log Calabi-Yau pairs may be continuous. We recall the following Kawamata's theorem. [19]

Theorem 0.11 *Let $f : X \rightarrow B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$, $Q = \sum_l Q_l$, be normal crossing divisors on X and B , respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum_j d_j P_j$ be a \mathbb{Q} -divisor on X , where d_j may be positive, zero or negative, which satisfies the following conditions A, B, C:*

A) $D = D^h + D^v$ such that any irreducible component of D^h is mapped surjectively onto B by f , $f : \text{Supp}(D^h) \rightarrow B$ is relatively normal crossing over $B \setminus Q$, and $f(\text{Supp}(D^v)) \subset Q$. An irreducible component of D^h (resp. D^v) is called horizontal (resp. vertical)

B) $d_j < 1$ for all j

C) The natural homomorphism $\mathcal{O}_B \rightarrow f_* \mathcal{O}_X([-D])$ is surjective at the generic point of B .

D) $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B .

Let

$$\begin{aligned}
f^*Q_l &= \sum_j w_{lj}P_j \\
\bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\
\delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\
\Delta &= \sum_l \delta_l Q_l. \\
M &= L - \Delta.
\end{aligned}$$

Then M is nef.

The following theorem is straightforward from Kawamata's theorem

Theorem 0.12 *Let $d_j < 1$ for all j be as above in Theorem 0.11, and fibers be log Calabi-Yau pairs, then*

$$\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2}$$

is continuous on a nonempty Zariski open subset of B .

Since the inverse of volume gives a singular hermitian line bundle, we have the following theorem from Theorem 0.11

Theorem 0.13 *Let $K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some \mathbb{Q} -divisor L on B and*

$$\begin{aligned}
f^*Q_l &= \sum_j w_{lj}P_j \\
\bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}}, \text{ if } f(P_j) = Q_l \\
\delta_l &= \max\{\bar{d}_j; f(P_j) = Q_l\}. \\
\Delta &= \sum_l \delta_l Q_l. \\
M &= L - \Delta.
\end{aligned}$$

Then

$$\left(\int_{X_s \setminus D_s} (-1)^{n^2/2} \frac{\Omega_s \wedge \overline{\Omega_s}}{|S_s|^2} \right)^{-1}$$

is a continuous hermitian metric on the \mathbb{Q} -line bundle $K_B + \Delta$ when fibers are log Calabi-Yau pairs.

Remark: Note that Yoshikawa [24], showed that when the base of Calabi-Yau fibration $f : X \rightarrow B$ is a disc and central fibre X_0 is reduced and irreducible and pair (X, X_0) has only canonical singularities then Tian's Kähler potential can be extended to a continuous Hermitian metric lying in the following class

$$\mathcal{B}(B) = C^\infty(S) \oplus \bigoplus_{r \in \mathbb{Q} \cap (0,1]} \bigoplus_{k=0}^n |s|^{2r} (\log |s|)^k C^\infty(B)$$

Remark: Note that hermitian metric of Yau-Vafa semi Ricci flat metric ω_{SRF} is in the class of $\mathcal{B}(B)$

Definition 0.14 *The null direction Vafa-Yau semi Ricci flat metric ω_{SRF} gives a foliation along Itaka fibration $\pi : X \rightarrow Y$ and we call it fiberwise Calabi-Yau foliation and can be defined as follows*

$$\mathcal{F} = \{\theta \in TX | \omega_{SRF}(\theta, \bar{\theta}) = 0\}$$

and along log Itaka fibration $\pi : (X, D) \rightarrow Y$, we can define the following foliation

$$\mathcal{F}' = \{\theta \in TX' | \omega_{SRF}^D(\theta, \bar{\theta}) = 0\}$$

where $X' = X \setminus D$. In fact from Theorem 0.9. the Weil-Petersson metric ω_{WP} vanishes everywhere if and only if $\mathcal{F} = TX$

Lemma: Let \mathcal{L} be a leaf of $f_*\mathcal{F}'$, then \mathcal{L} is a closed complex submanifold and the leaf \mathcal{L} can be seen as fiber on the moduli map

$$\eta : \mathcal{Y} \rightarrow \mathcal{M}_{CY}^D$$

where \mathcal{M}_{CY}^D is the moduli space of log calabi-Yau fibers with at worst canonical singularities and

$$\mathcal{Y} = \{y \in Y_{reg} | (X_y, D_y) \text{ is Kawamata log terminal pair}\}$$

The following definition introduced by Tsuji

Definition 0.15 *Let X be a compact complex manifold and let L be a line bundle on X . A singular Hermitian metric h on L is said to be an analytic Zariski decomposition (or shortly AZD), if the following hold.*

1. the curvature Θ_h is a closed positive current.
2. for every $m \geq 0$, the natural inclusion

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(X, \mathcal{O}_X(mL))$$

is an isomorphism, where $\mathcal{I}(h^m)$ denotes the multiplier ideal sheaf of h^m .

Since the Weil-Petersson metric is semi-positive.

Remark: The hermitian metric corresponding to Song-Tian measure is Analytic Zariski Decomposition., i.e.,

$$h = \left(\frac{(\omega_{SRF}^D)^n \wedge \pi^* \omega_{can}^m}{\pi^* \omega_{can}^m |S|^2} \right)^{-1}$$

is AZD

1 Invariance of Plurigenera and positivity of logarithmic Weil-Petersson metric

Now we talk about semi-positivity of logarithmic-Weil-Petersson metric via Invariance of plurigenera.

Let $\pi : (X, D) \rightarrow Y$ be a smooth holomorphic fibre space whose fibres have pseudoeffective canonical bundles. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X'/Y}(\omega(t)) - \omega(t) + [N]$$

be a relative Kähler ricci flow that starts with (1,1) form $[\omega(t)] = e^{-t}\omega_0 + (1 - e^{-t})\omega_{WP}^D$ and $X' = X \setminus D$, and here N is a divisor which come from Fujino-Mori's canonical bundle formula. From song-Tian approach such flow has semi-positive solutions hence $\omega(t)$ and ω_0 is semi-positive, and hence the logarithmic Weil-Petersson metric ω_{WP}^D must be semi-positive (1,1)-Kähler form.

In fact the invariance of plurigenera holds true if and only if the solutions $\omega(t) = e^{-t}\omega_0 + (1 - e^{-t})\omega_{WP}^D$ are semi-positive(see the Analytical approach of Tsuji, Siu, Song-Tian). In fact an answer to this question leads to invariance of plurigenera in Kähler setting. Thanks to Song-Tian program. If our family of fibers be fiberwise KE-stable, then invariance of plurigenera holds true from L^2 -extension theorem and also due to this fact that if the central fiber be pseudo-effective, then all the general fibers are pseudo-effective[11].

Theorem 1.1 (*L^2 -extension theorem*) *Let X be a Stein manifold of dimension n , ψ a plurisubharmonic function on X and s a holomorphic function on X such that $ds \neq 0$ on every branch of $s^{-1}(0)$. We put $Y = s^{-1}(0)$ and $Y_0 = \{X \in Y; ds(x) \neq 0\}$ Let g be a holomorphic $(n-1)$ -form on Y_0 with*

$$c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g} < \infty$$

where $c_k = (-1)^{k(k-1)/2}(\sqrt{-1})^k$ Then there exists a holomorphic n -form G on X such that $G(x) = g(x) \wedge ds(x)$ on Y_0 and

$$c_n \int_X e^{-\psi} (1 + |s|^2)^{-2} G \wedge \bar{G} < 1620\pi c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g}$$

Theorem 1.2 (*Siu [13]*) *Assume $\pi : X \rightarrow B$ is smooth, and every X_t is of general type. Then the plurigenera $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for any m .*

After Siu, an ‘‘algebraic proof’’ is given, and applied to the deformation theory of certain type of singularities which appear in MMP by Kawamata.

Definition 1.3 Let B be a normal variety such that K_B is \mathbb{Q} -Cartier, and $f : X \rightarrow B$ a resolution of singularities. Then,

$$K_X = f^*(K_B) + \sum_i a_i E_i$$

where $a_i \in \mathbb{Q}$ and the E_i are the irreducible exceptional divisors. Then the singularities of B are terminal, canonical, log terminal or log canonical if $a_i > 0, \geq 0, > -1$ or ≥ -1 , respectively.

Theorem 1.4 (Kawamata[14]) If X_0 has at most canonical singularities, then X_t has canonical singularities at most for all $t \in B$. Moreover, if all X_t are of general type and have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m

Remark: If along holomorphic fiber space $(X, D) \rightarrow B$ (with some stability condition on B) the fibers are of general type then to get

$Ric(\omega) = \lambda\omega + \omega_{WP} + \text{additional term which come from higher canonical bundle formula}$

, (here Weil-Petersson metric is a metric on moduli space of fibers of general type) when fibers are singular and of general type then we must impose this assumption that the central fiber (X_0, D_0) must have canonical singularities and be of general type to obtain such result.

Theorem 1.5 (Nakayama[12]) If X_0 has at most terminal singularities, then X_t has terminal singularities at most for all $t \in B$. Moreover, If $\pi : X \rightarrow B$ is smooth and the ‘‘abundance conjecture’’ holds true for general X_t , then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m .

Takayama, showed the following important theorem

Theorem 1.6 Let all fibers $X_t = \pi^{-1}(t)$ have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all t

Theorem 1.7 Let $\pi : X \rightarrow Y$ be a proper smooth holomorphic fiber space of projective varieties such that all fibers X_y are of general type, then ω_{WP} is semi-positive

Proof. Let $\pi : X \rightarrow Y$ be a smooth holomorphic fibre space whose fibres are of general type. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

be a Kähler ricci flow that starts with semi-positive Kähler form ω_0 (take it Weil-Petersson metric).

Then since Siu’s therems holds true for invariance of plurigenera, so the pseudo-effectiveness of K_{X_0} gives the pseudo-effectiveness of K_{X_t} . The solutions of $\omega(t)$ are semi-positive. But by cohomological characterization we know

that $[\omega(t)] = e^{-t}\omega_{WP} + (1 - e^{-t})[\omega_0]$ and since ω_0 and $\omega(t)$ are semi-positive, hence ω_{WP} is semi-positive. \square

We consider the semi-positivity of singular Weil-Petersson metric ω_{WP} in the sense of current.

Theorem 1.8 *Let $\pi : X \rightarrow Y$ be a proper holomorphic fiber space such that all fibers X_y are of general type and have at worst canonical singularities, then the Weil-Petersson metric ω_{WP} is semi-positive*

Proof. Suppose that

$$\frac{\partial \omega(t)}{\partial t} = -Ric_{X/Y}(\omega(t)) - \omega(t)$$

be a Kähler Ricci flow. Then since Kawamata's theorem says that "If all fibers X_t are of general type and have canonical singularities at most, then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m " hence invariance of plurigenera holds true, and the solutions of $\omega(t)$ are semi-positive by invariance of plurigenera. But by cohomological characterization we know that $[\omega(t)] = e^{-t}\omega_{WP} + (1 - e^{-t})[\omega_0]$ and since ω_0 and $\omega(t)$ are semi-positive, hence ω_{WP} is semi-positive. \square

Remark: From Nakayama's theorem, if X_0 has at most terminal singularities, then X_t has terminal singularities at most for all $t \in B$. Moreover, if $\pi : X \rightarrow B$ is smooth and the "abundance conjecture" holds true for general X_t , then $P_m(X_t) = \dim H^0(X_t, mK_{X_t})$ is independent of $t \in B$ for all m . So when fibers are of general type then the solutions of the relative Kähler Ricci flow $\omega(t)$ is semi-positive and hence by the same method of the proof of previous Theorem, the Weil-Petersson metric ω_{WP} is semi-positive on the moduli space of such families.

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