Regularity of Wiener functionals under a Hörmander type condition of order one

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Abstract. We study the local existence and regularity of the density of the law of a functional on the Wiener space which satisfies a criterion that generalizes the Hörmander condition of order one (that is, involving the first order Lie brackets) for diffusion processes.

Keywords: Malliavin calculus; local integration by parts formulas; total variation distance; variance of the Brownian path.

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1 Introduction

Hörmander's theorem gives sufficient non degeneracy assumptions under which the law of a diffusion process is absolutely continuous with respect to the Lebesgue measure and has a smooth density. This condition involves the coefficients of the diffusion process as well as the Lie brackets up to an arbitrary order. The aim of this paper is to give a partial generalization of this result to general functionals on the Wiener space. We give in this framework a condition corresponding to the first order Hörmander condition - we mean the condition which says that the coefficients and the first Lie brackets span the space. Roughly speaking our regularity criterion is as follows. Let F be a functional on the Wiener space associated to a Brownian motion $W = (W^1, ..., W^d)$. We denote by D^i the Malliavin derivative with respect to W^i and, for some $T > 0$, we define

$$\lambda(T) = \inf_{|\xi|=1} \Big(\sum_{i=1}^{d} \langle D^i_T F, \xi \rangle^2 + \sum_{i,j=1}^{d} \langle D^i_T D^j_T F - D^j_T D^i_T F, \xi \rangle^2 \Big) \tag{1.1}$$

We fix x and we suppose that there exist $r, \lambda > 0$ such that

$$1_{\{|F-x|\leq r\}}(\lambda(T) - \lambda) \geq 0 \qquad \text{a.s.} \tag{1.2}$$

Notice that, since $s \mapsto D_s F$ is defined as an element of $L^2([0,T])$, the quantity $D_T F$ in (1.1) makes no sense. So, we will replace it by $\frac{1}{\delta} \int_{T-\delta}^{T} \mathbb{E}_{T,\delta}(D_s F) ds$ for small values of δ, where $\mathbb{E}_{T,\delta}$ denotes a suitable conditional expectation (see (2.3) for details). Then, we actually replace (1.2) with an asymptotic variant (see next Remark 2.2).

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So, we assume that F is five times differentiable in Malliavin sense (actually in a slightly stronger sense) and that the above non degeneracy condition holds for some $T > 0$. Then we prove that the restriction of the law of F to $B_{r/2}(x)$ is absolutely continuous and has a smooth density.
The analysis of the Malliavin covariance matrix under the non degeneracy hypothesis (1.2) is based on an estimate concerning the variance of the Brownian path. This is done by using its Laplace transform, which has been studied by Donati-Martin and Yor [5]. We employ also another important argument, which is the regularity criterion for the law of a random variable given in [2]: it allows one to use integration by parts formulas in an "asymptotic way".
The main result is Theorem 2.1, and Section 2 is devoted to its proof, for which we use results on the variance of the Brownian path which are postponed to Appendix A. In Section 3 we illustrate the result with an example from diffusion processes with coefficients which may depend on the path of the process.
At our knowledge there are not many results concerning general vectors on the Wiener space - except of course the celebrated criterion given by Malliavin and the Bouleau Hirsh criterion for the absolute continuity. Another criterion proved by Kusuoka in [6] and further generalized by Nourdin and Poly [11] and Nualart, Nourdin and Poly [12] concerns vectors living in a finite number of chaoses. All these criterions suppose that the determinant of the Malliavin covariance matrix is non null in a more or less strong sense - but give no hint about the possible analysis of this condition. This remains to be checked using ad hoc methods in each particular example. So the main progress in our paper is to give a rather general condition under which the above mentioned determinant behaves well.

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2 Existence and smoothness of the local density

Let us recall some notations from Malliavin calculus (we refer to Nualart [10] or Ikeda and Watanabe [7]). We work on a probability space $(\Omega, \mathcal{F}, P)$ with a d dimensional Brownian motion $W = (W^1, ..., W^d)$ and we denote by $\mathcal{F}_t$ the standard filtration associated to W. We fix a time-horizon $T_0 > 0$ and we denote by $\mathbb{D}^{k,p}$ the space of the functionals on the Wiener space which are k times differentiable in L^p in Malliavin sense on the time interval $[0, T_0]$ and we put $\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}$. For a multi index $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, ..., d\}^k$ and a functional $F \in \mathbb{D}^{k,p}$ we denote $D^\alpha F = (D^\alpha_{s_1,...,s_k} F)_{s_1,...,s_k \in [0,T_0]}$ with $D^\alpha_{s_1,...,s_k} F = D^{\alpha_k}_{s_k} ... D^{\alpha_1}_{s_1} F$. Moreover, for $|\alpha| = k$ we define the norms

$$|D^\alpha F|^p_{L^p[0,T_0]^k} := \int_{[0,T_0]^k} \left|D^\alpha_{s_1,...,s_k} F\right|^p ds_1, ..., ds_k \quad \text{and} \tag{2.1}$$

$$\|F\|_{k,p} = \|F\|_p + \sum_{r=1}^{k} \sum_{|\alpha|=r} \mathbb{E}(|D^\alpha F|^p_{L^2[0,T_0]^r})^{1/p}.$$

If $F = (F^1, ..., F^n)$, we set

$$|D^\alpha F|^p_{L^p[0,T_0]^k} = \sum_{i=1}^{n} \left|D^\alpha F^i\right|^p_{L^p[0,T_0]^k} \quad \text{and} \quad \|F\|_{k,p} = \sum_{i=1}^{n} \left\|F^i\right\|_{k,p}.$$

Moreover we will use the following seminorms:

$$
\begin{aligned}
|||F|||_{k,p,q} &= \sum_{r=3}^{k} \sum_{|\alpha|=r} \mathbb{E}(|D^{\alpha}F|^{p}_{L^{q}[0,T_0]^{r}})^{1/p} \\
&= \sum_{r=3}^{k} \sum_{|\alpha|=r} \mathbb{E}\Big(\Big(\int_{[0,T_0]^{r}} \big|D^{\alpha}_{s_1,\dots,s_k}F\big|^{q} \, ds_1 ... ds_r \Big)^{p/q} \Big)^{1/p}.
\end{aligned}
$$

Notice that $|||\cdot|||_{k,p,q}$ does not take into account $\|F\|_p$ and the norm of the first two derivatives. Moreover, for $q = 2$ we find out the usual norms but if $q > 2$ the control given by $|||F|||_{k,q,p}$ (on the derivatives of order larger or equal to three) is stronger than the one given by $||F||_{k,p}$. We define the spaces

$$
\mathbb{D}^{k,p} = \{F : \|F\|_{k,p} < \infty\}, \qquad \mathbb{D}^{k,p,q} = \mathbb{D}^{k,p} \cap \{F : |||F|||_{k,p,q} < \infty\}.
$$

Clearly $\mathbb{D}^{k,p,q} \subset \mathbb{D}^{k,p}$ for $q > 2$ and for $q = 2$ we have equality. We also denote

$$
\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}, \qquad \mathbb{D}^{k,\infty,q} = \cap_{p\geq 1}\mathbb{D}^{k,p,q}, \qquad \mathbb{D}^{k,\infty,\infty} = \cap_{p\geq 1} \cap_{q\geq 2} \mathbb{D}^{k,p,q} \tag{2.2}
$$

For $s < t$ we denote

$$
\mathcal{F}^{t}_{s} = \mathcal{F}_s \vee \sigma(W_u - W_t, u \geq t) = \sigma(W_v, v \leq s) \vee \sigma(W_u - W_t, u \geq t).
$$

Now, for a fixed instant $T \in (0, T_0]$, we denote by $\mathbb{E}_{T,\delta}$ the conditional expectation with respect to $\mathcal{F}^{T}_{T-\delta}$ that is

$$
\mathbb{E}_{T,\delta}(\Theta) = \mathbb{E}(\Theta \mid \mathcal{F}^{T}_{T-\delta}). \tag{2.3}
$$

We will use the following slight extension of the Clark-Ocone formula: for $F \in \mathbb{D}^{1,2}$ and for $0 \leq \delta < T$ one has

$$
F = \mathbb{E}_{T,\delta}(F) + \sum_{i=1}^{d} \int_{T-\delta}^{T} \mathbb{E}_{T,T-s}(D^{i}_{s}F) dW^{i}_{s}. \tag{2.4}
$$

(2.4) is immediate for simple functionals, and then can be straightforwardly generalized to functionals in $\mathbb{D}^{1,2}$.
For $\delta \in (0, T)$, we consider a family of random vectors

$$
a(T,\delta) = (a_i(T,\delta), a_{k,j}(T,\delta))_{i,k,j=1,\dots,d}
$$

and we assume that $a(T,\delta)$ is $\mathcal{F}^{T}_{T-\delta}$ measurable. We denote

$$
\begin{aligned}
&[a]_{i,j}(T,\delta) = a_{i,j}(T,\delta) - a_{j,i}(T,\delta), \\
&\overline{a}(T,\delta) = \Big(\sum_{i=1}^{d} |a_i(T,\delta)|^2 + \sum_{i,j=1}^{d} |a_{i,j}(T,\delta)|^2 \Big)^{1/2}, \\
&\lambda(T,\delta) = \inf_{|\xi|=1} \Big(\sum_{i=1}^{d} \langle a_i(T,\delta), \xi \rangle^2 + \sum_{i,j=1}^{d} \langle [a]_{i,j}(T,\delta), \xi \rangle^2 \Big).
\end{aligned} \tag{2.5}
$$

For $p \geq 1, \alpha > 0, 0 < \delta < T$ we define

$$\begin{aligned}\varepsilon_{\alpha,p,T,\delta}(a,F) := &\sum_{i=1}^{d} \Big(\mathbb{E}\Big(\Big(\frac{1}{\delta}\int_{T-\delta}^{T} \Big|\frac{\mathbb{E}_{T,\delta}(D_s^i F) - a_i(T,\delta)}{\delta^{\frac{1}{2}+\alpha}}\Big|^2 ds\Big)^p\Big)\Big)^{1/2p} + \\ &+ \sum_{i,j=1}^{d} \Big(\frac{1}{\delta^2}\int_{T-\delta}^{T}\int_{T-\delta}^{s_1} \mathbb{E}\Big(\Big|\frac{\mathbb{E}_{T,\delta}(D_{s_2}^j D_{s_1}^i F) - a_{i,j}(T,\delta)}{\delta^{\alpha/2}}\Big|^{2p}\Big) ds_1 ds_2\Big)^{1/2p}.\end{aligned} \tag{2.6}$$

Our main result is the following.

Theorem 2.1 *Let $F = (F^1, .., F^n)$ be $\mathcal{F}_{T_0}$-measurable with $F^i \in \mathbb{D}^{2,\infty}, i = 1, ..., n$. We fix $y \in \mathbb{R}^n$ and $r > 0$ and we suppose that there exists $\alpha, \lambda_* > 0, \gamma \in [0, \frac{1}{2})$, a time $T \in (0, T_0]$ and a family $a(T,\delta) = (a_i(T,\delta), a_{i,j}(T,\delta))_{i,j=1,...,d}$, of $\mathcal{F}_{T-\delta}^T$ measurable vectors such that for every $p \geq 1$*

$$\begin{aligned} &i) \qquad \limsup_{\delta\to 0} \varepsilon_{\alpha,p,T,\delta}(a,F) < \infty, \\ &ii) \qquad \limsup_{\delta\to 0} \delta^{p\gamma}\mathbb{E}(\overline{a}^p(T,\delta)) < \infty, \\ &iii) \qquad \limsup_{\delta\to 0} \delta^{-p}\mathbb{P}(\{|F-y| \leq r\} \cap \{\lambda(T,\delta) < \lambda_*\}) < \infty. \end{aligned} \tag{2.7}$$

Then the following statements hold.

***A**. Suppose that $F^i \in \cup_{p>6}\mathbb{D}^{5,\infty,p}, i = 1, ..., n$. Then the law of F on $B_{r/2}(y) := \{x : |x-y| < r/2\}$ is absolutely continuous with respect to the Lebesgue measure. We denote by p_F the density of the law.*

***B**. Suppose that for some $k \geq 5$ one has $F^i \in \mathbb{D}^{k,\infty,\infty}, i = 1, ..., n$. Then*

$$p_F \in \cap_{p\geq 1} W^{k-5,p}(B_{r/2}(y)).$$

Remark 2.2 *Morally, $D_T^i F \sim \frac{1}{\delta}\int_{T-\delta}^{T} \mathbb{E}_{T,\delta}(D_s^i F)ds$. Then, condition i) in (2.7) says that we may replace $D_T^i F$ by $a_i(T,\delta)$, and we have a precise control of the error. The same for $D_T^i D_t^j F$, which is replaced by $a_{i,j}(T,\delta)$. Then, iii) in (2.7) gives the asymptotic non-degeneracy condition in terms of $\lambda(T,\delta)$, which is associated to $a(T,\delta)$.*

Remark 2.3 *Notice that we may ask the non-degeneracy condition iii) in (2.7) to hold in any intermediary time $T \in (0, T_0]$ and not only for $T = T_0$ (we thank to E. Pardoux for a remark in this sense).*

The proof is postponed to Section 2.4. We first need to state some preliminary results.

2.1 A short discussion on the proof of Theorem 2.1

Let us give the main ideas and the strategy we are going to use to prove Theorem 2.1.
We will look to the law of F under $\mathbb{P}_U$ where U is a localization random variable for the set $\{|F-y| \leq r\}$. We want to prove that this law is absolutely continuous with respect to the Lebesgue measure - this implies that the law of F restricted to $\{|F-y| \leq r\}$ is absolutely continuous (and this is our aim). In order to do it we proceed as follows: for each $\delta > 0$

we construct some localization random variables U_δ in such a way that on the set $\{U_\delta \neq 0\}$ the random variable F has nice properties - this means that we may control the Malliavin derivatives and the Malliavin covariance matrix of F on the set $\{U_\delta \neq 0\}$. This allows us to build integration by parts formulas for F under $\mathbb{P}_{U_\delta}$. The L^p norms of the weights which appear in these integration by parts formulas blow up as $\delta \to 0$ but we have a sufficiently precise control of the rate of the blow up. On the other hand we will estimate the total variation distance between the law of F under $\mathbb{P}_U$ and under $\mathbb{P}_{U_\delta}$. We prove that this distance goes to zero as $\delta \to 0$ and we obtain sufficiently precise estimates of the rate of convergence. Then we use Theorem 2.13 in [2], that we recall here in next Theorem 2.10, which guarantees that if one may achieve a good equilibrium between the rate of the blow up and the rate of convergence to zero, then one obtains a density for the limit law.
It worth to stress that the strategy employed here is slightly different from the usual one. In fact, in next (2.8) we decompose F as $F = \mathbb{E}_{T,\delta}(F) + Z_\delta(a) + R_\delta$ and one would expect that we approximate F by $\mathbb{E}_{T,\delta}(F) + Z_\delta(a)$. But we do not proceed in this way. We keep all the time the same random variable F (which includes R_δ) but we change the probability measure under which we work in order to have a good localization: we replace $\mathbb{P}_U$ by $\mathbb{P}_{U_\delta}$. The decomposition $F = \mathbb{E}_{T,\delta}(F) + Z_\delta(a) + R_\delta$ is not used in order to produce the approximation $\mathbb{E}_{T,\delta}(F) + Z_\delta(a)$ but just to analyze the properties for F itself under different localizations given in $\mathbb{P}_{U_\delta}$. As we will see soon, such a decomposition appears as a Taylor expansion of order one in which $Z_\delta(a)$ represents the principal term and R_δ is a reminder in the sense that it is small on the set $\{U_\delta \neq 0\}$.

2.2 Preliminary results

Let $F \in \mathbb{D}^{4,2}$. Using twice Clark Ocone formula (2.4), we obtain

$$F - \mathbb{E}_{T,\delta}(F) = Z_\delta(a) + R_\delta(F) \tag{2.8}$$

with

$$Z_\delta(a) = \sum_{i=1}^d a_i(T,\delta)(W_T^i - W_{T-\delta}^i) + \sum_{i,j=1}^d a_{i,j}(T,\delta) \int_{T-\delta}^T (W_s^i - W_{T-\delta}^i) dW_s^j \tag{2.9}$$

and $R_\delta(F) = R'_\delta(F) + R''_\delta(F)$ with

$$\begin{aligned} R'_\delta(F) =& \sum_{i=1}^d \int_{T-\delta}^T (\mathbb{E}_{T,\delta}(D_s^i F) - a_i(T,\delta)) dW_s^i \\ &+ \sum_{i,j=1}^d \int_{T-\delta}^T \int_{T-\delta}^{s_1} (\mathbb{E}_{T,\delta}(D_{s_2}^j D_{s_1}^i F) - a_{i,j}(T,\delta)) dW_{s_2}^j dW_{s_1}^i \\ R''_\delta(F) =& \sum_{i,j,k=1}^d \int_{T-\delta}^T \int_{T-\delta}^{s_1} \int_{T-\delta}^{s_2} \mathbb{E}_{T,T-s_3}(D_{s_3}^k D_{s_2}^j D_{s_1}^i F) dW_{s_3}^k dW_{s_2}^j dW_{s_1}^i \end{aligned} \tag{2.10}$$

Since T and δ are fixed we will use in the following shorter notation

$$a_i = a_i(T,\delta), \qquad a_{i,j} = a_{i,j}(T,\delta), \qquad \overline{a} = \overline{a}(T,\delta).$$

We will use the Malliavin calculus restricted to W_s, $s \in [T-\delta, T]$. Straightforward computations give

$$D_s^j Z_\delta(a) = a_j + \sum_{i \neq j} [a]_{i,j}(W_s^i - W_{T-\delta}^i) + r_j, \text{ with } r_j = \sum_{i=1}^d a_{i,j}(W_T^i - W_{T-\delta}^i). \tag{2.11}$$

We denote

$$\begin{gathered} q_1(W) = |W_T - W_{T-\delta}|, \qquad q_2(W) = \frac{1}{\delta}\int_{T-\delta}^T |W_s - W_{T-\delta}|^2 \, ds, \\ G_\delta = \int_{T-\delta}^T |D_s R_\delta|^2 \, ds \end{gathered} \tag{2.12}$$

and we define

$$\Lambda_{T,\delta} = \left\{ q_1(W) \leq \frac{1}{8\overline{a}} \frac{\sqrt{\lambda_*}}{d} \right\} \cap \left\{ q_2(W) \leq 1 \right\} \cap \left\{ G_\delta \leq \frac{\lambda_*}{34}\delta^2 \right\} \cap \left\{ \lambda(T,\delta) \geq \lambda_* \right\} \tag{2.13}$$

We set $\sigma_{F,T,\delta}$ as the Malliavin covariance matrix of F associated to the Malliavin derivatives restricted to W_s, $s \in [T-\delta, T]$ that is

$$\sigma_F^{i,j} = \sigma_{F,T,\delta}^{i,j} = \int_{T-\delta}^T \left\langle D_s F^i, D_s F^j \right\rangle ds, \quad i,j = 1, \ldots, n. \tag{2.14}$$

The main step of the proof is the following estimate. It is based on an analysis of the variance of the Brownian path, which is done in Appendix A.

Lemma 2.4 *Let $F = (F^1, ..., F^n)$ with $F^i \in \mathbb{D}^{4,2}$. Let $0 \leq \delta < T$ be fixed and $\mathbb{E}_{T,\delta}$ be defined in (2.3). Then for every $p \geq 1$*

$$\mathbb{E}_{T,\delta}(1_{\Lambda_{T,\delta}} (\det \sigma_{F,T,\delta})^{-p}) \leq \frac{C_{n,p}}{\lambda_*^{pn} \delta^{2pn}} \tag{2.15}$$

with

$$C_{n,p} = 2\Gamma(p) \int_{\mathbb{R}^n} |\xi|^{n(2p-1)} \, e^{-\frac{1}{34}|\xi|^2} d\xi.$$

Proof. By using Lemma 7-29, pg 92 in [4], for every $n \times n$ dimensional and non negative defined matrix σ one has

$$(\det \sigma)^{-p} \leq \Gamma(p) \int |\xi|^{n(2p-1)} \, e^{-\langle \sigma\xi, \xi \rangle} d\xi,$$

so that

$$\mathbb{E}_{T,\delta}((\det \sigma_F)^{-p} 1_{\Lambda_{T,\delta}}) \leq \Gamma(p) \int |\xi|^{n(2p-1)} \, \mathbb{E}_{T,\delta}(1_{\Lambda_{T,\delta}} e^{-\langle \sigma_F \xi, \xi \rangle}) d\xi.$$

Since $\Lambda_{T,\delta} \subset \{G_\delta \leq \frac{\lambda_*}{34}\delta^2\}$ we have

$$\begin{aligned} \langle \sigma_F \xi, \xi \rangle &\geq \frac{1}{2} \left\langle \sigma_{Z_\delta(a)} \xi, \xi \right\rangle - \langle G_\delta \xi, \xi \rangle \geq \frac{1}{2} \left\langle \sigma_{Z_\delta(a)} \xi, \xi \right\rangle - G_\delta \, |\xi|^2 \\ &\geq \frac{1}{2} \left\langle \sigma_{Z_\delta(a)} \xi, \xi \right\rangle - \frac{\lambda_*}{34}\delta^2 \, |\xi|^2 \end{aligned}$$

so that

$$\mathbb{E}_{T,\delta}((\det \sigma_F)^{-p} 1_{\Lambda_{T,\delta}}) \leq \Gamma(p) \int |\xi|^{n(2p-1)} e^{\frac{\lambda_*}{34}\delta^2|\xi|^2} \mathbb{E}_{T,\delta}(1_{\Lambda_{T,\delta}} e^{-\left\langle \sigma_{Z_\delta(a)}\xi,\xi\right\rangle}) d\xi.$$

We fix $\xi \in \mathbb{R}^n$ and we choose $j = j(\xi)$ such that

$$\langle a_j, \xi\rangle^2 + \sum_{i\neq j} \langle [a]_{i,j}, \xi\rangle^2 \geq \frac{\lambda_*}{d} |\xi|^2 .$$

This is possible because we are on the set $\Lambda_{T,\delta} \subset \{\lambda(T,\delta) \geq \lambda_*\}$. Then by (2.11)

$$\begin{aligned}\left\langle \sigma_{Z_\delta(a)}\xi,\xi\right\rangle &= \int_{T-\delta}^T \left\langle D_s^j Z_\delta(a), \xi\right\rangle^2 ds \\ &= \int_{T-\delta}^T \Big(\langle a_j,\xi\rangle + \langle r_j,\xi\rangle + \sum_{i\neq j} \langle [a]_{i,j},\xi\rangle \, (W_s^i - W_{T-\delta}^i)\Big)^2 ds.\end{aligned}$$

We define

$$\beta_j^2(\xi) = \sum_{i\neq j} \langle [a]_{i,j},\xi\rangle^2$$

and for $\beta_j^2(\xi) > 0$,

$$b_s(j,\xi) = \frac{1}{\beta_j(\xi)} \sum_{i\neq j} \langle [a]_{i,j},\xi\rangle \, (W_{T-\delta+s}^i - W_{T-\delta}^i).$$

Notice that $b(j,\xi)$ is a Brownian motion under $\mathbb{P}_{T,\delta}$. We also set $b_s(j,\xi) = 0$ in the case $\beta_j^2(\xi) = 0$. Then the previous equality reads

$$\left\langle \sigma_{Z_\delta(a)}\xi,\xi\right\rangle = \int_0^\delta (\langle a_j,\xi\rangle + \langle r_j,\xi\rangle + \beta_j(\xi) b_s(j,\xi))^2 \, ds.$$

We use now Lemma A.1 in Appendix A with $\alpha = \langle a_j,\xi\rangle, \beta = \beta_j(\xi), r = \langle r_j,\xi\rangle$ and $b_s = b_s(j,\xi)$. We have to check that the assumptions there are verified. Using Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\frac{1}{\delta}\Big| \int_0^\delta b_s(j,\xi) ds\Big| &\leq \Big(\frac{1}{\delta}\int_0^\delta |b_s(\xi)|^2 \, ds\Big)^{1/2} \\ &\leq \Big(\frac{1}{\delta}\int_0^\delta |W_{T-\delta+s} - W_{T-\delta}|^2 \, ds\Big)^{1/2} = \sqrt{q_2(W)} \leq 1.\end{aligned}$$

Moreover, since $\alpha^2 + \beta^2 \geq \frac{\lambda_*}{d}|\xi|^2$ we have

$$|r|^2 \leq |r_j|^2 |\xi|^2 \leq d\overline{a}^2 q_1^2(W) |\xi|^2 \leq \frac{1}{64}\frac{\lambda_*}{d}|\xi|^2 \leq \frac{1}{64}(\alpha^2+\beta^2).$$

So the hypothesis are verified: by using (A.3) we obtain

$$\mathbb{E}_{T,\delta}(1_{\Lambda_{T,\delta}} e^{-\left\langle \sigma_{Z_\delta(a)}\xi,\xi\right\rangle}) \leq 2e^{-\frac{\delta^2}{17}(|\alpha|^2+|\beta|^2)} \leq 2e^{-\frac{\delta^2\lambda_*}{17d}|\xi|^2}.$$

We come back and we obtain

$$\begin{aligned}\mathbb{E}_{T,\delta}((\det\sigma_F)^{-p}1_{\Lambda_{T,\delta}}) &\leq 2\Gamma(p)\int|\xi|^{n(2p-1)}\,e^{\frac{\lambda_*}{34d}\delta^2|\xi|^2}e^{-\frac{\delta^2\lambda_*}{17d}|\xi|^2}d\xi\\ &= 2\Gamma(p)\int|\xi|^{n(2p-1)}\,e^{-\frac{\delta^2\lambda_*}{34}|\xi|^2}d\xi\\ &= \frac{C_{n,p}}{\lambda_*^{pn}\delta^{2pn}}\end{aligned}$$

where the last equality easily follows by a change of variable . □

We also need the following estimate.

Lemma 2.5 *Suppose that (2.7) i) holds and let G_δ be defined as in* (2.12).

A. *If $F^i\in\cup_{p>6}\mathbb{D}^{4,\infty,p}, i=1,...,n$, there exists $\varepsilon>0$ such that*

$$\limsup_{\delta\to 0}\,\delta^{-\varepsilon}\,\mathbb{P}(G_\delta\geq\delta^2)<\infty. \tag{2.16}$$

B. *If $F^i\in\mathbb{D}^{4,\infty,\infty}, i=1,...,n$ then (2.16) holds for every $\varepsilon>0$.*

Proof. **A.** Let $F\in(\mathbb{D}^{4,\infty,p})^n$ for some $p>6$. We recall that R'_δ and R''_δ are defined in (2.10). We write $R'_\delta(F)=\sum_{i=1}^d r_\delta^i+\sum_{i,j=1}^d r_\delta^{i,j}$ and $R''_\delta=\sum_{i,j,k=1}^d r_\delta^{i,j,k}$, with

$$\begin{aligned}r_\delta^i &= \int_{T-\delta}^T(\mathbb{E}_{T,\delta}(D_s^iF)-a_i(T,\delta))dW_s^i,\\ r_\delta^{i,j} &= \int_{T-\delta}^T\int_{T-\delta}^{s_1}(\mathbb{E}_{T,\delta}(D_{s_2}^jD_{s_1}^iF)-a_{i,j}(T,\delta))dW_{s_2}^jdW_{s_1}^i,\\ r_\delta^{i,j,k} &= \int_{T-\delta}^T\int_{T-\delta}^{s_1}\int_{T-\delta}^{s_2}\mathbb{E}_{T,T-s_3}(D_{s_3}^kD_{s_2}^jD_{s_1}^iF)dW_{s_3}^kdW_{s_2}^jdW_{s_1}^i.\end{aligned}$$

Step 1. We estimate $G_\delta^i=\int_{T-\delta}^T\left|D_s^ir_\delta^i\right|^2ds$. For $s\in[T-\delta,T]$ we have $D_s^ir_\delta^i=\mathbb{E}_{T,\delta}(D_s^iF)-a_i(T,\delta)$ so

$$G_\delta^i=\int_{T-\delta}^T\left|\mathbb{E}_{T,\delta}(D_s^iF)-a_i(T,\delta)\right|^2ds.$$

It follows that

$$\begin{aligned}\frac{1}{\delta^\varepsilon}\mathbb{P}(G_\delta^i\geq\delta^2) &\leq \frac{1}{\delta^\varepsilon}\delta^{-2p}\left\|G_\delta^i\right\|_p^p=\frac{1}{\delta^\varepsilon}\mathbb{E}\Big(\Big|\delta^{-1}\int_{T-\delta}^T\left|\frac{\mathbb{E}_{T,\delta}(D_s^iF)-a_i(T,\delta)}{\delta^{1/2}}\right|^2ds\Big|^p\Big)\\ &\leq \frac{1}{\delta^\varepsilon}\times\delta^{2\alpha p}\varepsilon^{2p}_{\alpha,p,T,\delta}(a,F)\end{aligned}$$

and consequently, by our hypothesis (2.7) i), this term satisfies (2.16) for every $\varepsilon>0$ (it suffices to take p sufficiently large).

Step 2. We estimate $G_\delta^{i,j}=\sum_{\ell=1}^d\int_{T-\delta}^T\left|D_s^\ell r_\delta^{i,j}\right|^2ds$. We have

$$\begin{aligned}D_s^\ell r_\delta^{i,j} =& 1_{i=\ell}\int_{T-\delta}^s(\mathbb{E}_{T,\delta}(D_{s_2}^jD_s^iF)-a_{i,j}(T,\delta))dW_{s_2}^j+\\ &+1_{j=\ell}\int_s^T(\mathbb{E}_{T,\delta}(D_s^pD_{s_1}^iF)-a_{i,p}(T,\delta))dW_{s_1}^i=:1_{i=\ell}u_s^{i,j}+1_{j=\ell}v_s^{i,j}.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\Big(\Big|\int_{T-\delta}^{T}\big|u_s^{i,j}\big|^2\,ds\Big|^p\Big) &\le \delta^{p-1}\int_{T-\delta}^{T}\mathbb{E}\big(\big|u_s^{i,j}\big|^{2p}\big)ds\\ &\le C\delta^{p-1}\int_{T-\delta}^{T}\mathbb{E}\Big(\Big|\int_{T-\delta}^{s}(\mathbb{E}_{T,\delta}(D^j_{s_2}D^i_sF)-a_{i,j}(T,\delta))^2ds_2\Big|^p\Big)ds\\ &\le C\delta^{2p-2}\int_{T-\delta}^{T}\int_{T-\delta}^{s}\mathbb{E}\big(\big|\mathbb{E}_{T,\delta}(D^j_{s_2}D^i_sF)-a_{i,j}(T,\delta)\big|^{2p}\big)ds_2ds\\ &= C\delta^{2p+\alpha p}\frac{1}{\delta^2}\int_{T-\delta}^{T}\int_{T-\delta}^{T}\mathbb{E}\Big(\Big|\frac{\mathbb{E}_{T,\delta}(D^j_{s_2}D^i_{s_1}F)-a_{i,j}(T,\delta)}{\delta^{\alpha/2}}\Big|^{2p}\Big)ds_1ds_2\\ &\le C\delta^{2p+\alpha p}\varepsilon^{2p}_{\alpha,p,T,\delta}(a,F).\end{aligned}$$

Using Chebyshev inequality we obtain

$$\mathbb{P}\Big(\int_{T-\delta}^{T}\big|u_s^{i,j}\big|^2\,ds\ge\delta^2\Big)\le C\delta^{-2p}\delta^{2p+\alpha p}\varepsilon^{2p}_{\alpha,p,T,\delta}(a,F)=C\delta^{\alpha p}\varepsilon^{2p}_{\alpha,p,T,\delta}(a,F).$$

which by (2.7) i), satisfies (2.16) for every $\varepsilon>0$. For $v_s^{i,j}$ the argument is the same.
Step 3. We estimate $G^{i,j,k}_\delta=\sum_{\ell=1}^d\int_{T-\delta}^T|D^\ell_sr^{i,j,k}_\delta|^2ds$. We have

$$\begin{aligned}D^\ell_sr^{i,j,k}_\delta =&1_{i=\ell}\int_{T-\delta}^{s}\int_{T-\delta}^{s_2}\mathbb{E}_{T,T-s_3}(D^k_{s_3}D^j_{s_2}D^i_sF)dW^k_{s_3}dW^j_{s_2}+\\ &+1_{j=\ell}\int_{T-\delta}^{T}\int_{T-\delta}^{s_2}1_{s<s_1}\mathbb{E}_{T,T-s_3}(D^k_{s_3}D^j_sD^i_{s_1}F)dW^k_{s_3}dW^i_{s_1}+\\ &+1_{k=\ell}\int_{T-\delta}^{T}\int_{T-\delta}^{s_1}1_{s<s_2}\mathbb{E}_{T,T-s}(D^k_sD^j_{s_2}D^i_{s_1}F)dW^j_{s_2}dW^i_{s_1}+\\ &+\int_{T-\delta}^{T}\int_{T-\delta}^{s_1}\int_{T-\delta}^{s_2}1_{s<s_3}D^\ell_s\mathbb{E}_{T,T-s_3}(D^k_{s_3}D^j_{s_2}D^i_{s_1}F)dW^k_{s_3}dW^j_{s_2}dW^i_{s_1}\\ =:&1_{i=\ell}u^{i,j,k}_s+1_{j=\ell}v^{i,j,k}_s+1_{k=\ell}w^{i,j,k}_s+z^{i,j,k,\ell}_s.\end{aligned}$$

By using Hölder and Burkholder inequality as in step 1, one obtains

$$\begin{aligned}\mathbb{E}\Big(\Big|\int_{T-\delta}^{T}\Big|u_s^{i,j,k}\Big|^2ds\Big|^p\Big)&\le\delta^{3p-3}\int_{T-\delta}^{T}\int_{T-\delta}^{T}\int_{T-\delta}^{T}\mathbb{E}\big(|D^k_{s_3}D^j_{s_2}D^i_sF|^{2p}\big)ds_3ds_2ds\\ &\le\delta^{3p-3}|||F|||^{2p}_{3,2p,2p}.\end{aligned}$$

An identical bound holds for $\mathbb{E}(|\int_{T-\delta}^T|v^{i,j,k}_s|^2ds|^p)$ and $\mathbb{E}(|\int_{T-\delta}^T|w^{i,j,k}_s|^2ds|^p)$. As for $z^{i,j,k,\ell}$, one more further integral appears, so we get $\mathbb{E}(|\int_{T-\delta}^T|z^{i,j,k,\ell}_s|^2ds|^p)\le\delta^{4p-4}|||F|||^{2p}_{4,2p,2p}$. By summarizing, we get

$$\mathbb{E}\Big(\Big|\int_{T-\delta}^{T}\big|D_sR''_\delta\big|^2\,ds\Big|^p\Big)\le\delta^{3p-3}|||F|||^{2p}_{4,2p,2p}$$

so that for every $p>1$

$$\mathbb{P}\Big(\int_{T-\delta}^{T}\Big|D^\ell_sR''_\delta\Big|^2\,ds\ge\delta^2\Big)\le C\delta^{-2p}\delta^{3p-3}|||F|||^{2p}_{4,2p,2p}=C\delta^{p-3}|||F|||^{2p}_{4,2p,2p}.$$

Suppose first that $F^i \in \cup_{p>6}\mathbb{D}^{4,\infty,p}$. Then we may find $p > 3$ such that $|||F|||_{4,2p,2p} < \infty$ and consequently the above quantity is upper bounded by $C\delta^{p-3}$. This means that (2.16) holds for $\varepsilon < p-3$. If $F^i \in \mathbb{D}^{4,\infty,\infty}$ then we may take p arbitrary large and so we obtain (2.16) for every $\varepsilon > 0$.
□

We will also need the following property for G_δ.

Lemma 2.6 *If $F \in \mathbb{D}^{k+1,2p}$ then*

$$\|G_\delta\|_{k,p} \leq C\big(\|F\|^2_{k+1,2p} + \delta\|\overline{a}(T,\delta)\|^2_{4p}\big),$$

where C denotes a constant depending on k, p, d only.

Proof. For $G \in (\mathbb{D}^{k,p})^n$, we set $|D^{(k)}G| = \sum_{\ell=0}^k \sum_{|\gamma|=\ell} |D^\gamma G|^2$, where, for $|\gamma| = \ell$,

$$|D^\gamma G|^2 = \int_{[0,T]^\ell} |D^\gamma_{s_1\dots s_\ell} G|^2 ds_1 \cdots ds_\ell,$$

that is $|D^\gamma G|$ is the one given in (2.2) with $p = 2$. Here, the case $|\gamma| = 0$, that is $\gamma = \emptyset$, reduces to the original random variable: $D^\emptyset G = G$ and $|D^{(0)}G| = |G|$.
In the following, we let C denote a positive constant, independent of δ and the random variables we are going to write. And we let C vary from line to line.
We take $G_\delta = \int_{t-\delta}^T |D_s R_\delta|^2 ds$ and we first prove the following (deterministic) estimate: there exists a constant C depending on k and d such that

$$|D^{(k)}G_\delta| \leq C|D^{(k+1)}R_\delta|^2. \tag{2.17}$$

For $k = 0$, this is trivial. Consider $k = 1$. One has $D^i_u G_\delta = \sum_{\ell=1}^d \int_{t-\delta}^T 2D^\ell_s R_\delta D^i_u D^\ell_s R_\delta ds$, so that, by using the CauchySchwarz inequality, we get

$$\begin{aligned}
|DG_\delta|^2 &\leq 4\sum_{i,\ell=1}^d \int_{T-\delta}^T \Big| \int_{T-\delta}^T 2D^\ell_s R_\delta D^i_u D^\ell_s R_\delta ds\Big|^2 du \\
&\leq 4\sum_{i,\ell=1}^d \int_{T-\delta}^T du \int_{T-\delta}^T 2|D^\ell_s R_\delta|^2 ds \int_{T-\delta}^T 2|D^i_u D^\ell_s R_\delta|^2 ds \\
&\leq C|D^{(1)}R_\delta|^2 |D^{(2)}R_\delta|^2 \leq C|D^{(2)}R_\delta|^4
\end{aligned}$$

and (2.17) holds for $k = 1$. For $k \geq 2$, we use the following straightforward formula: if α denotes a multi-index of length k, then

$$D^\alpha G_\delta = \sum_{\ell=1}^d \int_{T-\delta}^T \Big(2D^\ell_s R_\delta D^\alpha R_\delta + \sum_{\beta\in\mathcal{P}_\alpha} D^\beta D^\ell_s R_\delta D^{\alpha\setminus\beta} D^\ell_s R_\delta\Big) ds,$$

where $\mathcal{P}_\alpha$ is the set of the non empty multi indeces β which are a subset of α and $\alpha \setminus \beta$ stands for the multi index of length $|\alpha| - |\beta|$ given by eliminating from α the entries of β. By using the above formula and the CauchySchwarz inequality, one easily gets

$$\begin{aligned}
\int_{[T-\delta,T]^k} |D^\alpha_{s_1,\dots,s_k} G_\delta|^2 ds_1 \cdots ds_k &\leq C\Big(|D^{(1)}R_\delta|^2 |D^{(k)}R_\delta|^2 + \sum_{r=1}^k |D^{(r+1)}R_\delta|^2 |D^{(k-r+1)}R_\delta|^2\Big) \\
&\leq C|D^{(k+1)}R_\delta|^4
\end{aligned}$$

and (2.17) follows. Passing to expectation in (2.17), it follows that

$$\|G_\delta\|_{k,p} \leq C\|R_\delta\|^2_{k+1,2p}$$

and by recalling that $R_\delta = F - \mathbb{E}_{T,\delta}(F) - Z_\delta(a)$, we obtain

$$\|G_\delta\|_{k,p} \leq C\big(\|F\|^2_{k+1,2p} + \|Z_\delta(a)\|^2_{k+1,2p}\big).$$

From (2.9), by using Hölder's inequality we get

$$\begin{aligned}
\|Z_\delta(a)\|_{k+1,2p} \leq & \sum_{i=1}^d \|a_i(T,\delta)\|_{4p}\|W^i_T - W^i_{T-\delta}\|_{k+1,4p} \\
& + \sum_{i,j=1}^d \|a_{i,j}(T,\delta)\|_{4p}\Big\| \int_{T-\delta}^T (W^i_s - W^i_{T-\delta})dW^j_s\Big\|_{k+1,4p} \\
& \leq C\|\overline{a}(T,\delta)\|_{4p}\delta^{1/2} + C\|\overline{a}(T,\delta)\|_{4p}\delta \\
& \leq C\delta^{1/2}\,\|\overline{a}(T,\delta)\|_{4p},
\end{aligned}$$

and the statement follows. □

Remark 2.7 *If hypothesis* (2.7) *ii) holds then* $\limsup_{\delta\to 0}\delta\|\overline{a}(T,\delta)\|^2_{4p} = 0$ *because in this case one takes* $\gamma < 1/2$*, so that for* $F \in (\mathbb{D}^{k+1,2p})^n$ *one has*

$$\sup_{\delta>0}\|G_\delta\|_{k,p} < \infty.$$

2.3 Localization

We will use a localization argument from [2] that we recall here. We consider a random variable U taking values in $[0,1]$ and we denote

$$d\mathbb{P}_U = U d\mathbb{P}. \tag{2.18}$$

This is a non negative measure (but generally not a probability measure - one must divide with $\mathbb{E}(U)$ to get a probability measure). We denote

$$\begin{aligned}
&\|F\|_{U,p} := \mathbb{E}_U(|F|^p)^{1/p} = \mathbb{E}(|F|^p\,U)^{1/p} \quad \text{and} \\
&\|F\|_{U,k,p} := \|F\|_{U,p} + \sum_{r=1}^k \sum_{|\alpha|=r} \mathbb{E}_U(|D^\alpha F|^p_{L^2[0,T_0]^r})^{1/p}.
\end{aligned} \tag{2.19}$$

Clearly $\|F\|_{U,k,p} \leq \|F\|_{k,p}$. For a random variable $F \in (\mathbb{D}^{1,2})^n$ we denote

$$\sigma_{U,F}(p) = \mathbb{E}_U((\det\sigma_F)^{-p})^{1/p}. \tag{2.20}$$

We assume that $U \in \mathbb{D}^{1,\infty}$ and that for every $p \geq 1$

$$m_p(U) := \mathbb{E}_U(|D\ln U|^p) < \infty. \tag{2.21}$$

In Lemma 2.1 in [1] we have proved the following:

Lemma 2.8 *Assume that (2.21) holds. Let $F \in (\mathbb{D}^{2,\infty})^n$ be such that $\det \sigma_F \neq 0$ on the set $\{U \neq 0\}$. We denote $\widehat{\sigma}_F$ the inverse of σ_F and we assume that $\sigma_{U,F}(p) < \infty$ for every $p \in \mathbb{N}$. Then for every $V \in \mathbb{D}^{1,\infty}$ and every $f \in C_b^\infty(\mathbb{R}^n)$ one has*

$$\mathbb{E}_U(\partial_i f(F)V) = \mathbb{E}_U(f(F)H_{i,U}(F,V)) \tag{2.22}$$

with

$$H_{i,U}(F,V) = \sum_{j=1}^n \left(V\widehat{\sigma}_F^{j,i} LF^j - \left\langle D(V\widehat{\sigma}_F^{j,i}), DF^j \right\rangle - V\widehat{\sigma}_F^{j,i} \left\langle D(\ln U), DF^j \right\rangle \right). \tag{2.23}$$

Suppose that $\ln U \in \mathbb{D}^{k,\infty}$. Iterating (2.22) one obtains for a multi index $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, ..., n\}^k$

$$\mathbb{E}_U(\partial_\alpha f(F)V) = \mathbb{E}_U(f(F)H_{\alpha,U}(F,V)), \text{ with } H_{\alpha,U}(F,V) = H_{\alpha_k,U}(F, H_{\underline{\alpha},U}(F,V)) , \tag{2.24}$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_{k-1})$.

We will use this result with a localization random variable U constructed in the following way. For $a \in (0,1)$ we define $\psi_a : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi_a(x) = 1_{[0,a)}(x) + 1_{[a,2a)}(x) \exp\Big(1 - \frac{a^2}{a^2 - (x-a)^2}\Big). \tag{2.25}$$

Then for every multi index α and every $p \in \mathbb{N}$ there exists a universal constant $C_{\alpha,p}$ such that

$$\sup_{x \in \mathbb{R}_+} \psi_a(x) \left|\partial_\alpha \ln \psi_a(x)\right|^p \leq \frac{C_{\alpha,p}}{a^{p|\alpha|}}. \tag{2.26}$$

Let $a_i > 0$ and $Q_i \in \mathbb{D}^{1,p}, i = 1, ..., l$ and $U = \prod_{i=1}^l \psi_{a_i}(Q_i)$. As an easy consequence of (2.26) we obtain the following estimates

$$m_p(U) \leq C \sum_{i=1}^l \frac{1}{a_i^p} \left\|Q_i\right\|_{U,1,p}^p \leq C \sum_{i=1}^l \frac{1}{a_i^p} \left\|Q_i\right\|_{1,p}^p \tag{2.27}$$

where C is a universal constant. And moreover, for every $k, p \in \mathbb{N}$ there exists a universal constant C such that

$$\left\|\ln U\right\|_{U,k,p} \leq C \sum_{i=1}^l \frac{1}{a_i^k} \left\|Q_i\right\|_{k,p}. \tag{2.28}$$

The function ψ_a is suited for localization around zero. In order to localize far from zero we have to use the following alternative version:

$$\phi_a(x) = 1_{[a,\infty)}(x) + 1_{[a/2,a)}(x) \exp\Big(1 - \frac{a^2}{(2x-a)^2}\Big). \tag{2.29}$$

The property (2.26) holds for ϕ_a as well. And if one employs both ψ_{a_i} and ϕ_{a_i} in the construction of U, that is if one sets

$$U = \prod_{i=1}^l \psi_{a_i}(Q_i) \times \prod_{j=1}^{l'} \phi_{a_{l+j}}(Q_{l+j}), \tag{2.30}$$

both properties (2.27) and (2.28) hold again. Then we have the following estimate.

Lemma 2.9 *Let $k, l, l' \in \mathbb{N}$, $Q_i \in \mathbb{D}^{k+1,\infty}, i = 1, ..., l+l'$ and set U as in (2.30). Consider also some $F \in (\mathbb{D}^{k+1,\infty})^n$. Then for every $p \geq 1$ there exist some universal constants $C > 0$ and $p' > p$ (depending on k, n, p only) such that for every multi index α with $|\alpha| \leq k$ one has*

$$\|H_{\alpha,U}(F,1)\|_{U,p} \leq C\big(1 + \sigma_{U,F}(p')^{k+1}\big)\Big(1 + \sum_{i=1}^{l+l'} \frac{1}{a_i^k} \|Q_i\|_{k,p'}\Big)\big(1 + \|F\|_{k+1,p'}^{2nk}\big).$$

Proof. For $G \in (\mathbb{D}^{r,p})^n$, let $|D^{(r)}G| = \sum_{\ell=0}^r \sum_{|\gamma|=\ell} |D^\gamma G|^2$ as in the proof of Lemma 2.6. Then the following deterministic estimate for the Malliavin weights holds:

$$\begin{aligned} |H_{\alpha,U}(F,V)| \leq \; & C\Big(\sum_{r=0}^k |D^{(r)}V|\Big) \times \Big(1 + \sum_{r=1}^k |D^{(r)} \ln U|\Big) \times \\ & \times \big(1 + |\det \sigma_F|^{-(k+1)}\big) \times \\ & \times \Big(1 + \sum_{r=1}^{k+1} |D^{(r)}F| + \sum_{r=0}^{k-1} |D^{(r)}LF|\Big)^{2nk}. \end{aligned} \tag{2.31}$$

The proof of (2.31) is straightforward, although non trivial, and can be found in the preprint version of the present paper, see [3]. The statement now easily follows by applying to the r.h.s. of (2.31) the Hölder inequality and the Meyer inequality $\|LF\|_{U,r,p} \leq \|LF\|_{r,p} \leq C\|F\|_{r+2,p}$. □

We finally recall the result in Theorem 2.13 from [2], on which the proof of Theorem 2.1 is based.

Consider a random variable F, a probability measure $\mathbb{Q}$ and a family of probabilities $\mathbb{Q}_\delta$, $\delta > 0$. We denote by μ the law of F under $\mathbb{Q}$ and by μ_δ the law of F under $\mathbb{Q}_\delta$. In the following, we will take $\mathbb{Q} = \mathbb{P}_U$ and $\mathbb{Q}_\delta = \mathbb{P}_{U_\delta}$ as given in (2.18), where U and U_δ are both of the form (2.30). Actually, $\mathbb{P}_U$ and $\mathbb{P}_{U_\delta}$ are not probability measures but they are both finite with total mass less or equal to 1, and this is enough.

We let $\mathbb{E}_\mathbb{Q}$ and $\mathbb{E}_{\mathbb{Q}_\delta}$ denote expectation under $\mathbb{Q}$ and $\mathbb{Q}_\delta$ respectively.

Fix $\delta > 0$. For $m \in \mathbb{N}_*$ and $p \geq 1$, we say that $F \in \mathcal{R}_{m,p}(\mathbb{Q}_\delta)$ if for every multi index α with $|\alpha| \leq m$ there exists a random variable $H_{\alpha,\delta}$ such that the following abstract integration by parts formula holds:

$$\mathbb{E}_{\mathbb{Q}_\delta}(\partial_\alpha f(F)) = \mathbb{E}_{\mathbb{Q}_\delta}(f(F)H_{\alpha,\delta}) \qquad \forall f \in C_c^\infty, \quad \text{with } \mathbb{E}_{\mathbb{Q}_\delta}(|H_{\alpha,\delta}|^p) < \infty. \tag{2.32}$$

By using Theorem 2.13 **A** in [2] with $m = 1$ and $k = 0$, we have

Theorem 2.10 *Let $q \in \mathbb{N}$ and $p > 1$ be fixed and let $r_n = 2(n+1)$. Let $F \in \cap_{\delta>0}\mathcal{R}_{q+3,r_n}(\mathbb{Q}_\delta)$. Suppose that there exist $\theta > 0$, $C \geq 1$ and $\eta > \frac{q+n/p_*}{2}$, with p_* the conjugate of p, such that one has*

$$\limsup_{\delta\to 0} \Big(\mathbb{E}_{\mathbb{Q}_\delta}(|F|^{r_n})^{1/r_n} + \sum_{|\alpha|\leq q+3} \delta^{\theta|\alpha|}\mathbb{E}_{\mathbb{Q}_\delta}(|H_{\alpha,\delta}|^{r_n})^{1/r_n}\Big) < \infty, \tag{2.33}$$

$$d_0(\mu, \mu_\delta) \leq C\delta^{\eta\theta n^2(q+3)}, \tag{2.34}$$

where d_0 denotes the total variation distance, that is $d_0(\mu,\nu) = \sup\{|\int f d\mu - \int f d\nu| : \|f\|_\infty \leq 1\}$. Then μ is absolutely continuous and has a density $p_F \in W^{q,p}(\mathbb{R}^n)$.

Proof. Let us first notice that Theorem 2.13 in [2] concerns a family of r.v.'s F_δ, $\delta > 0$, and it is assumed that all these random variables F_δ are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. But this is just for simplicity of notations. In fact the statement concerns just the law of $(F_\delta, H_\alpha(F_\delta, 1), |\alpha| \leq 2m + q + 1)$, where $H_\alpha(F_\delta, 1)$ are the weights in the integration by parts formulas for F_δ. So we may assume that each F_δ is defined on a different probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{Q}_\delta)$. In our case, we take $F_\delta = F$ for each δ, we work on the space $(\Omega, \mathcal{F}, \mathbb{Q}_\delta)$ and we have $H_\alpha(F_\delta, 1) = H_{\alpha,\delta}$. We then apply Theorem 2.13 in [2] with $m = 1$ and $k = 0$. (2.33) immediately gives that $\sup_\delta \mathbb{E}_{\mathbb{Q}_\delta}(|F|^{n+3}) < \infty$ because $2(n+1) \geq n+3$. Moreover, in view of (2.39) in [2], the quantity $T_{q+3,2(n+1)}(F_\delta)$ in the statement of Theorem 2.13 therein can be upper bounded by

$$S_{q+3,2(n+1)}(\delta) := \mathbb{E}_{\mathbb{Q}_\delta}(|F|^{r_n})^{1/r_n} + \sum_{|\alpha| \leq q+3} \mathbb{E}_{\mathbb{Q}_\delta}(|H_{\alpha,\delta}|^{r_n})^{1/r_n}.$$

As an immediate consequence of (2.33) and (2.34), all the requirements in Theorem 2.13 in [2] hold, and the statement follows. □

2.4 Proof of Theorem 2.1

We are now ready to prove our main result.

Proof of Theorem 2.1. Step 1: construction of the localization r.v.'s U and U_δ. We consider the functions $\psi = \psi_{1/2}$ and $\phi = \phi_2$ defined in (2.25) and (2.29) with $a = \frac{1}{2}$ and $a = 2$ respectively. We recall that in hypothesis (2.7) *ii)* some $\gamma < \frac{1}{2}$ is considered. We denote $\lambda = \frac{1}{3}(\frac{1}{2} - \gamma)$. Recall that $q_i(W), i = 1, 2$ are defined in (2.12). Then we define

$$Q_0 = r^{-1}\,|F - y|\,, \qquad Q_1 = \frac{68d^3}{\lambda_* \delta^2} G_\delta, \qquad Q_2 = \delta^{-(\gamma+2\lambda)} q_1(W),$$
$$Q_3 = q_2(W), \qquad Q_4 = \delta^{\gamma+\lambda}\overline{a}, \qquad Q_5 = \frac{\lambda(T,\delta)}{\lambda_*}$$

and we set

$$U = \psi(Q_0), \qquad U_\delta = \prod_{i=0}^{4} \psi(Q_i) \times \phi(Q_5).$$

Step 2: construction and estimate of the weights $H_{\alpha,\delta}$ (defined in (2.32)) under $\mathbb{P}_{U_\delta}$. We fix $k \in \mathbb{N}_*$ and we assume that $F \in (\mathbb{D}^{k+3,\infty,\infty})^n$.
Notice that for $\delta^\lambda \leq \frac{1}{8d}\sqrt{\lambda_*}$, on the set $\{U_\delta \neq 0\}$ we have

$$\overline{a}(T,\delta) q_1(W) = \big(\delta^{\gamma+\lambda}\overline{a}(T,\delta)\big)\big(\delta^{-(\gamma+2\lambda)} q_1(W)\big) \times \delta^\lambda \leq \delta^\lambda \leq \frac{1}{8}\,\frac{\sqrt{\lambda_*}}{d}.$$

The other restriction required in $\Lambda_{T,\delta}$ (see (2.13) for the definition) are easy to check. So, we obtain

$$\{U_\delta \neq 0\} \subset \{|F - y| \leq r\} \cap \Lambda_{T,\delta}.$$

Then, by using Lemma 2.4 we have

$$\mathbb{E}_{T,\delta}\big(1_{\{U_\delta \neq 0\}} (\det \sigma_{F,T,\delta})^{-p}\big) \leq \frac{C_{n,p}}{\lambda_*^{np} \delta^{2np}} \tag{2.35}$$

where $\sigma_{F,T,\delta}$ is given in (2.14).
We use the Malliavin calculus with respect to $W_s - W_{T-\delta}, s \in (T-\delta, T)$. So, we denote with L_δ the Ornstein Uhlenbeck operator with respect to $W_s - W_{T-\delta}, s \in (T-\delta, T)$ and with $\langle g, f\rangle_\delta$ the scalar product in $L^2[T-\delta, T]$. So, $\sigma_{F,T,\delta}$ is the Malliavin covariance matrix of F w.r.t. this partial calculus. We set, as usual, $\widehat{\sigma}_{F,T,\delta}$ the inverse of $\sigma_{F,T,\delta}$ and we set

$$H_{i,U_\delta}(F,V) := \sum_{j=1}^{n} (V\widehat{\sigma}^{j,i}_{F,T,\delta} L_\delta F^j - \Big\langle D(V\widehat{\sigma}^{j,i}_{F,T,\delta}), DF^j \Big\rangle_\delta - V\widehat{\sigma}^{j,i}_{F,T,\delta} \big\langle D(\ln U_\delta), DF^j \big\rangle_\delta).$$

Then (2.22) reads

$$\mathbb{E}_{U_\delta}(\partial_i f(F)V) = \mathbb{E}_{U_\delta}(H_{i,U_\delta}(f,V)).$$

By iteration, for a multi index $\alpha \in \{1,\dots,n\}^k$ we have

$$\mathbb{E}_{U_\delta}(\partial_\alpha f(F)V) = \mathbb{E}_{U^\delta}(H_{\alpha,U_\delta}(f,V)),$$

where $H_{\alpha,U_\delta}(f,V) = H_{\alpha_h,U_\delta}(f, H_{(\alpha_1,\dots,\alpha_{k-1}),U_\delta}(f,V))$. And by using Lemma 2.9, we can find $C>0$ and $p'>1$ such that

$$\|H_{\alpha,U_\delta}(F,1)\|_{U_\delta,p} \le C\big(1+\sigma_{U_\delta,F}(p')^{k+1}\big)\Big(1+\sum_{i=1}^{5}\|Q_i\|_{k,p'}\Big)\big(1+\|F\|^{2nk}_{k+1,p'}\big)$$

with

$$\sigma_{U_\delta,F}(p)^p = \mathbb{E}_{U_\delta}((\det\sigma_{F,T,\delta})^{-p}) = \mathbb{E}(U_\delta(\det\sigma_{F,T,\delta})^{-p}).$$

Since $0 \le U_\delta \le 1_{U_\delta \neq 0}$, and by using estimate (2.35) we get

$$\sigma_{U_\delta,F}(p)^p \le \mathbb{E}(1_{\Lambda_{T,\delta}}(\det\sigma_{F,T,\delta})^{-p}) = \mathbb{E}(\mathbb{E}_{T,\delta}(1_{\Lambda_{T,\delta}}(\det\sigma_{F,T,\delta})^{-p})) \le \frac{C}{\lambda_*^{np}\delta^{2np}}.$$

Moreover, by applying Remark 2.7 we obtain $\sum_{i=0}^{5}\|Q_i\|_{k+1,p'} \le C\delta^{-2}$. So, we conclude that if $|\alpha| \le k$ then

$$\begin{aligned}\|H_{\alpha,U_\delta}(F,1)\|_{U_\delta,p} &\le \frac{C}{\delta^{2n(k+1)+2}}\big(1+\|F\|^{2nk}_{k+1,p'}\big)\\ &\le \frac{C}{\delta^{\theta k}}\big(1+\|F\|^{2nk}_{k+1,p'}\big) \quad \text{with} \quad \theta = 4n+2 \end{aligned} \tag{2.36}$$

where C is a universal constant depending on n, k (recall that $k \ge 1$) and λ_*.

Step 3: estimate of the total variation distance. We recall that for two non negative finite measures μ, ν the total variation distance is defined by

$$d_0(\mu,\nu) = \sup\Big\{\Big|\int f d\mu - \int f d\nu\Big| : \|f\|_\infty \le 1\Big\}.$$

We consider the measures μ and μ_δ defined by

$$\int f d\mu = \mathbb{E}_U(f(F)), \qquad \int f d\mu_\delta = \mathbb{E}_{U_\delta}(f(F)),$$

so that $d_0(\mu,\mu_\delta) \le \mathbb{E}(|U-U_\delta|)$. Therefore, we have

$$\begin{aligned} d_0(\mu,\mu_\delta) \le & \mathbb{P}\Big(G_\delta \ge \frac{\lambda_* \delta^2}{68 d^3}\Big) + \mathbb{P}\big(\,|W_T - W_{T-\delta}| \ge \delta^{\frac{1}{2}-\lambda}\big) + \\ & + \mathbb{P}\Big(\sum_{j=1}^d \int_{T-\delta}^T \Big|W_s^j - W_{T-\delta}^j\Big|^2 ds \ge \delta\Big) + \\ & + \mathbb{P}\big(\overline{a}(T,\delta) \ge \delta^{-(\gamma+\lambda)}\big) + \mathbb{P}\big(\{|F-y| \le r\} \cap \{\lambda(T,\delta) < \lambda_*\}\big) \\ =: & \sum_{i=1}^5 \epsilon_i(\delta). \end{aligned}$$

For every $r \ge 1$, by using Chebychev's inequality we obtain $\epsilon_2(\delta) \le C\delta^{r(\frac{1}{2}-\gamma)}$ and in a similar way, for every $r \ge 1$ then $\epsilon_3(\delta) \le C\delta^{r/2}$. By (2.7) ii)

$$\epsilon_4(\delta) \le C\delta^{r(\gamma+\lambda)} \mathbb{E}(\overline{a}^r(T,\delta)) \le C\delta^{r\lambda}$$

and by (2.7) iii) $\epsilon_5(\delta) \le C\delta^r$ for every $r \ge 1$. We conclude that for every $\varepsilon \ge 1$,

$$\limsup_{\delta\to 0} \delta^{-\varepsilon} \epsilon_i(\delta) = 0 \quad \text{ for every } \varepsilon > 0 \text{ and } i = 2,3,4,5.$$

The behavior of $\epsilon_1(\delta)$ is given by Lemma 2.5: if $F \in \cup_{p>6}(\mathbb{D}^{4,\infty,p})^n$ then there exists $\varepsilon > 0$ such that $\limsup_{\delta\to 0} \delta^{-\varepsilon}\epsilon_1(\delta) = 0$ and if $F \in (\mathbb{D}^4,\infty,\infty)$ then $\limsup_{\delta\to 0} \delta^{-\varepsilon}\epsilon_1(\delta) = 0$ for every $\varepsilon > 0$. Therefore, we get

$$\begin{aligned} &(i) \quad F \in \cup_{p>6}(\mathbb{D}^{4,\infty,p})^n \Rightarrow \exists\ \varepsilon > 0 \text{ such that } \limsup_{\delta\to 0} \delta^{-\varepsilon} d_0(\mu,\mu_\delta) = 0; \\ &(ii) \quad F \in (\mathbb{D}^{4,\infty,\infty})^n \Rightarrow \forall\ \varepsilon > 0 \text{ then } \limsup_{\delta\to 0} \delta^{-\varepsilon} d_0(\mu,\mu_\delta) = 0. \end{aligned} \tag{2.37}$$

Step 4: conclusions. We first prove part **A** of Theorem 2.1. Since $F \in \cup_{p>6}(\mathbb{D}^{5,\infty,p})^n$, we have that (2.37) (i) holds. We apply now Theorem 2.10 with $q = 0$, $\mathbb{Q} = \mathbb{P}_U$ and $\mathbb{Q}_\delta = \mathbb{P}_{U_\delta}$. By using (2.36), (2.33) holds with $\theta = 4n+2$. Now, we choose $p > 1$ sufficiently close to 1 such that

$$\Big(1 - \frac{1}{p}\Big) \times 3n^3(4n+2) < \varepsilon.$$

So, taking $\eta = \frac{n}{p_*}$ we get $\eta > \frac{n/p_*}{2}$ and $3\eta\theta n^2 < \varepsilon$ and by using (2.37) (i) we have that hypothesis (2.34) holds. Then, by applying Theorem 2.10, we conclude that $\mu(dx) = f(x)dx$ and $f \in L^p(\mathbb{R}^n)$.

We prove now **B** of Theorem 2.1. As before, (2.33) holds with $\theta = 4n+2$. Moreover, by (2.37) (ii), we get that (2.34) holds for every choice of $p > 1$ and of $\eta > \frac{q+n/p_*}{2}$. So, the only restriction in the application of Theorem 2.10 is that $F \in \cap_{\delta>0}\mathcal{R}_{q+3,2(n+1)}(\mathbb{Q}_\delta)$. But in order to have this, we need that each component of F is k-times differentiable in Malliavin sense with $k \ge (q+3)+2 = q+5$, that is $q \le k-5$. And we apply Theorem 2.10 with $q = k-5$, giving the result. □

3 An example from diffusion processes

We consider the N dimensional diffusion process

$$dX_t = \sum_{j=1}^{d} \sigma_j(X_t) dW_t^j + b(X_t)dt. \tag{3.1}$$

We assume that $\sigma_j, b \in C_b^\infty(\mathbb{R}^N)$. In particular $X_T^i \in \cap_{m=1}^\infty \mathbb{D}^{m,\infty,\infty}$ (see Nualart [10]).
Our aim is to study the regularity of $\overline{X}_T = (X_T^1, ..., X_T^n)$ with $n \leq N$. One may consider $\overline{X}_t$ as the solution of an equation with coefficients depending on the past. We introduce some notation. For a function $f : \mathbb{R}^N \to \mathbb{R}^N$ we denote $\overline{f} = (f^1, ..., f^n)$ and for $x = (x_1, ..., x_N) \in \mathbb{R}^N$ we denote $\overline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\widehat{x} = (x_{n+1}, ..., x_N) \in \mathbb{R}^{N-n}$. And for $\overline{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\widehat{x} = (x_{n+1}, ..., x_N) \in \mathbb{R}^{N-n}$ we denote $(\overline{x}, \widehat{x}) = (x_1, ..., x_n, x_{n+1}, ..., x_N) \in \mathbb{R}^N$. We define

$$\Lambda_{\widehat{x},\xi}(\overline{x}) = \sum_{j=1}^{d} \langle \overline{\sigma}_j(\overline{x}, \widehat{x}), \xi \rangle^2 + \sum_{j,p=1}^{d} \left\langle \overline{[\sigma_j, \sigma_p]}(\overline{x}, \widehat{x}), \xi \right\rangle^2 \qquad \text{and}$$
$$\Lambda(\overline{x}) = \inf_{\widehat{x} \in \mathbb{R}^{N-n}} \inf_{|\xi|=1} \Lambda_{\widehat{x},\xi}(\overline{x}).$$

Proposition 3.1 *We assume that $\sigma_j, b \in C_b^\infty(\mathbb{R}^N)$ and consider a point $\overline{x}_0 \in \mathbb{R}^n$ such that $\Lambda(\overline{x}_0) > 0$. Then there exists some $r > 0$ such that the restriction of the law of $\overline{X}_T$ to $B_r(\overline{x}_0)$ is absolutely continuous and has an infinitely differentiable density on this ball.*

Remark 3.2 *Other types of dependence on the past may be considered. For example equations with delay (see e.g. Mohammed [9]) or interacting particle systems (see e.g. Löcherbach [8]). For simplicity, we treat here the model given by the first n components of the N-dimensional diffusion in* (3.1).

Proof. We consider $a_j, a_{j,p}, j, p = 1, ..., d$ defined by

$$a_j(T, \delta) = \overline{\sigma}(X_{T-\delta}), \qquad a_{j,p}(T, \delta) = \sum_{k=1}^{N} \sigma_j^k(X_{T-\delta}) \partial_k \overline{\sigma}_p(X_{T-\delta}).$$

Notice that $[a]_{j,p}(T, \delta) = \overline{[\sigma_j, \sigma_p]}(X_{T-\delta})$ so that, with the notation in (2.5), we have $\lambda(T, \delta) \geq \Lambda(\overline{X}_{T-\delta})$.
Since the derivatives of σ_j are uniformly bounded one has $\left|\Lambda_{\widehat{x},\xi}(\overline{x}) - \Lambda_{\widehat{x},\xi}(\overline{x}')\right| \leq C \left|\overline{x} - \overline{x}'\right|$ for some C depending on $\|\sigma\|_\infty + \|\nabla\sigma\|_\infty$. So we may find $r > 0$ such that $\Lambda(\overline{x}) \geq \frac{1}{2}\Lambda(\overline{x}_0)$ for $\overline{x} \in B_{2r}(\overline{x}_0)$. It follows that $\lambda(T, \delta) \geq \frac{1}{2}\Lambda(\overline{x}_0)$ for $\overline{X}_{T-\delta} \in B_{2r}(\overline{x}_0)$. Then

$$\mathbb{P}(\{\left|\overline{X}_T - \overline{x}_0\right| < r\} \cap \{\lambda(T, \delta) < \frac{1}{4}\Lambda(\overline{x}_0)\}) \leq \mathbb{P}(\left|\overline{X}_T - \overline{X}_{T-\delta}\right| > r) \leq Ce^{-r^2/C'\delta}$$

which proves that the hypothesis (2.7), iii) holds true. Since σ_j are bounded the hypothesis (2.7), ii) holds true also. Let us check (2.7), i). We compute

$$D_s^j \overline{X}_T = \overline{\sigma}_j(X_s) + \sum_{p=1}^{d} \int_s^T \nabla \overline{\sigma}_p(X_r) D_s^j X_r dW_r^p + \int_s^T \nabla \overline{b}(X_r) D_s^j X_r dr.$$

So for $T-\delta \le s \le T$ we have

$$\mathbb{E}_{T,\delta}(D_s^j \overline{X}_T) = \mathbb{E}_{T,\delta}(\overline{\sigma}_j(X_s)) + \int_s^T \mathbb{E}_{T-\delta}(\nabla \overline{b}(X_r) D_s^j X_r) dr = a_j(T,\delta) + R_\delta^j(s)$$

with

$$R_\delta^j(s) = \mathbb{E}_{T,\delta}(\overline{\sigma}_j(X_s) - \overline{\sigma}_j(X_{T-\delta})) + \int_s^T \mathbb{E}_{T,\delta}(\nabla \overline{b}(X_r) D_s^j X_r) dr.$$

With L denoting the infinitesimal generator associated to the diffusion (3.1), one has

$$\overline{\sigma}_j(X_s) - \overline{\sigma}_j(X_{T-\delta}) = \sum_{k=1}^d \int_{T-\delta}^s \nabla \overline{\sigma}_j(X_u) \sigma_k(X_u) dW_u^k + \int_{T-\delta}^s L\overline{\sigma}_j(X_u) du,$$

so that

$$R_\delta^j(s) = \int_{T-\delta}^s \mathbb{E}_{T,\delta}(L\overline{\sigma}_j(X_u)) du + \int_s^T \mathbb{E}_{T,\delta}(\nabla \overline{b}(X_r) D_s^j X_r) dr.$$

Standard computations show that $\mathbb{E}(|R_\delta^j(s)|^{2p}) \le C\delta^{2p}$ for any $s \in [T-\delta, T]$, so that

$$\begin{aligned}\mathbb{E}\Big(\Big|\frac{1}{\delta}\int_{T-\delta}^T \Big|\frac{\mathbb{E}_{T,\delta}(D_s^j \overline{X}_T) - a_j(T,\delta)}{\delta^{\frac{1}{2}+\alpha}}\Big|^2 ds\Big|^p\Big) &\le \frac{1}{\delta}\int_{T-\delta}^T \mathbb{E}\big(\big|\delta^{-(\frac{1}{2}+\alpha)} R_\delta^j(s)\big|^{2p}\big) ds \\ &\le C\delta^{2p(\frac{1}{2}-\alpha)}.\end{aligned}$$

We fix $T-\delta \le s_2 \le s_1 \le T$ and we compute the second order derivatives:

$$\begin{aligned}\mathbb{E}_{T,\delta}(D_{s_2}^p D_{s_1}^j \overline{X}_T) &= \mathbb{E}_{T,\delta}(\nabla \overline{\sigma}_j(X_{s_2}) D_{s_2}^p \overline{X}_{s_1}) + \sum_{k,l=1}^d \int_{s_1}^T \mathbb{E}_{T,\delta}(\partial_k \partial_l \overline{b}(X_r) D_{s_2}^p \overline{X}_r^l D_{s_1}^j \overline{X}_r^k) dr \\ &+ \sum_{k=1}^d \int_{s_1}^T \mathbb{E}_{T,\delta}(\partial_k \overline{b}(X_r) D_{s_2}^p D_{s_1}^j \overline{X}_r^k) dr = a_{p,j}(T,\delta) + R_\delta^{p,j}(s_1,s_2)\end{aligned}$$

with

$$\begin{aligned}R_\delta^{p,j} &= \mathbb{E}_{T,\delta}(\nabla \overline{\sigma}_j(X_{s_2}) D_{s_2}^p \overline{X}_{s_1} - \nabla \overline{\sigma}_j(X_{T-\delta}) \sigma(X_{T-\delta})) \\ &+ \sum_{k,l=1}^d \int_{s_1}^T \mathbb{E}_{T,\delta}(\partial_k \partial_l \overline{b}(X_r) D_{s_2}^p \overline{X}_r^l D_{s_1}^j \overline{X}_r^k) dr + \sum_{k=1}^d \int_{s_1}^T \mathbb{E}_{T,\delta}(\partial_k \overline{b}(X_r) D_{s_2}^p D_{s_1}^j \overline{X}_r^k) dr.\end{aligned}$$

Similarly as before, one has $\mathbb{E}(|R_\delta^{p,j}(s_1,s_2)|^{2p}) \le C\delta^{2p}$ so that

$$\begin{aligned}&\frac{1}{\delta^2}\int_{T-\delta}^T \int_{T-\delta}^{s_1} \mathbb{E}\Big(\Big|\frac{\mathbb{E}_{T,\delta}(D_{s_2}^p D_{s_1}^j \overline{X}_T) - a_{p,j}(T,\delta)}{\delta^{\alpha/2}}\Big|^{2p}\Big) ds_2 ds_1 = \\ &\qquad = \frac{1}{\delta^2}\int_{T-\delta}^T \int_{T-\delta}^{s_1} \mathbb{E}\big(\big|\delta^{-\alpha/2} R_\delta^{p,j}(s_1,s_2)\big|^{2p}\big) ds_2 ds_1 \le C\delta^{2p(1-\alpha/2)}.\end{aligned}$$

We conclude that for $\alpha \le \frac{1}{2}$ we have $\varepsilon_{\alpha,p,\delta}(a, \overline{X}_T) \le C$ so that the hypothesis (2.7) $i)$ is verified. The statement now follows by applying Theorem 2.1. $\square$

A The variance lemma

In [5] (see (1.f), p. 183) one gives the explicit expression of the Laplace transform of the variance of the Brownian path on $(0,1)$. More precisely let B be an one dimensional Brownian motion and let

$$V(B) = \int_0^1 \Big(B_s - \int_0^1 B_r dr\Big)^2 ds. \tag{A.1}$$

Then

$$\mathbb{E}(e^{-\lambda V(B)}) = \frac{2\lambda}{\sinh 2\lambda}, \qquad \lambda > 0. \tag{A.2}$$

As an easy consequence we obtain the following estimate:

Lemma A.1 *On a probability space we consider a one dimensional Brownian motion b and a random variable r. We also consider two real numbers α, β and $\delta > 0$ and we denote $A_\delta = \{r^2 \leq \frac{1}{32}(\alpha^2+\beta^2)\} \cap \{\Big|\frac{1}{\delta}\int_0^\delta b_s ds\Big| \leq 1\}$. Then*

$$\mathbb{E}(1_{A_\delta} \exp(-\int_0^\delta (r+\alpha+\beta b_s)^2 ds)) \leq 2\exp(-\frac{\delta^2}{17}(\alpha^2+\beta^2)). \tag{A.3}$$

Proof. We consider the probability measure $\mu_\delta(ds) = \delta^{-1} 1_{(0,\delta)}(s) ds$, so that

$$\int_0^\delta (r+\alpha+\beta b_s)^2 ds = \delta \int (r+\alpha+\beta b_s)^2 d\mu_\delta(s).$$

Setting

$$V_{\mu_\delta}(b) = \int (b_s - \int b_r d\mu_\delta(r))^2 d\mu_\delta(r),$$

it is easy to check that

$$\int (r+\alpha+\beta b_s)^2 d\mu_\delta(s) = \Big(\int (r+\alpha+\beta b_s) d\mu_\delta(s)\Big)^2 + \beta^2 V_{\mu_\delta}(b) \tag{A.4}$$

and

$$V_{\mu_\delta}(b) = \delta V(B) \qquad with \qquad B_t = \delta^{-1/2} b_{t\delta}. \tag{A.5}$$

We consider two cases. Suppose first that $|\alpha| \geq 4|\beta|$. On the set A_δ we have $2|\alpha| \geq |\alpha| + |\beta| \geq 8|r|$ and $\big|\int b_s d\mu_\delta(s)\big| \leq 1$ so we obtain

$$\begin{aligned}\Big|r+\alpha+\beta\int b_s d\mu_\delta(s)\Big| &\geq |\alpha| - |r| - |\beta| \Big|\int b_s d\mu_\delta(s)\Big| \geq |\alpha| - |r| - |\beta| \\ &\geq \frac{1}{2}|\alpha| \geq \frac{1}{4}(|\alpha|+|\beta|).\end{aligned}$$

Using (A.4) this gives

$$\begin{aligned}\int_0^\delta (r+\alpha+\beta b_s)^2 ds &\geq \delta \Big(\int (r+\alpha+\beta b_s) d\mu_\delta(s)\Big)^2 \\ &\geq \frac{\delta}{16}(|\alpha|+|\beta|)^2 \geq \frac{\delta}{16}(\alpha^2+\beta^2) \\ &\geq \frac{\delta^2}{17}(\alpha^2+\beta^2).\end{aligned}$$

Suppose now that $|\alpha| < 4\,|\beta|$. Then using (A.4) we can write

$$\int_0^\delta (r+\alpha+\beta b_s)^2 ds \geq \delta\beta^2 V_{\mu_\delta}(b) = \delta^2\beta^2 V(B) \geq \frac{\delta^2}{17}(\alpha^2+\beta^2)V(B)$$

Then we have

$$\mathbb{E}(1_{A_\delta} e^{-\int_0^\delta (r+\alpha+\beta b_s)^2 ds}) \leq 1_{\{|\alpha|\geq 4|\beta|\}} e^{-\frac{\delta^2}{17}(\alpha^2+\beta^2)} + 1_{\{|\alpha|>4|\beta|\}} \mathbb{E}(e^{-\frac{\delta^2}{17}(\alpha^2+\beta^2)V(B)})$$

and by using (A.2) and the estimate $\frac{2\lambda}{\sinh(2\lambda)} \leq 2\lambda e^{-2\lambda} \leq 2e^{-\lambda}$, we get

$$\mathbb{E}(1_{A_\delta} e^{-\int_0^\delta (r+\alpha+\beta b_s)^2 ds}) \leq 1_{\{|\alpha|\geq 4|\beta|\}} e^{-\frac{\delta^2}{17}(\alpha^2+\beta^2)} + 1_{\{|\alpha|>4|\beta|\}} 2e^{-\frac{\delta^2}{17}(\alpha^2+\beta^2)}$$

and the statement follows.
□

B Proof of inequality (2.31)

Let us briefly recall the notations we are going to use. For $r \in \mathbb{N}$ and a multi index $\beta \in \{1,\dots,d\}^r$, if $F \in (\mathbb{D}^{r,\infty})^n$ we set

$$|D^\beta F|^2 = \int_{[0,T]^r} |D^\beta_{s_1\dots s_r} F|^2 ds_1\cdots ds_r = \sum_{j=1}^d \int_{[0,T]^r} |D^\beta_{s_1\dots s_r} F^j|^2 ds_1\cdots ds_r \text{ and}$$

$$|D^{(r)}F|^2 = \sum_{|\beta|=r} |D^\beta F|^2.$$

For the sake of completeness, we allow $\beta = \emptyset$, or equivalently $|\beta| = 0$: we set

$$D^\beta F = F \quad \text{and} \quad |D^{(0)}F|^2 = |F|^2.$$

Moreover, $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2([0,T],dt)$, so that for $F, G \in (\mathbb{D}^{1,\infty})^n$,

$$\langle DG, DF\rangle = \sum_{i=1}^d \int_{[0,T]} D^i_s G D^i_s F ds = \sum_{i=1}^d \sum_{j=1}^n \int_{[0,T]} D^i_s G^j D^i_s F^j ds$$

and $|DF|^2 = \langle DF, DF\rangle = |D^{(1)}F|^2$.
For F taking values in $\mathbb{R}^n$, V in $\mathbb{R}$ and α multi index of length k in $\{1,\dots,n\}$, let $H_{\alpha,U}(F,V)$ denote the weight in (2.24), that is the weight from the integration by parts formula of order k of F w.r.t. V localized through U. The appendix is devoted to the proof of the following

Proposition B.1 *For $\ell = 0,1,\dots$, let $\beta \in \{1,\dots,d\}^\ell$ (the case $\ell = 0$ referring to $\beta = \emptyset$) and for $k = 1,2,\dots$, let $\alpha \in \{1,\dots,n\}^k$ be a multi index of length k. On the set $\{U > 0\}$, let $V, \ln U \in \mathbb{D}^{k+\ell,\infty}$ and let $F \in (\mathbb{D}^{k+\ell+1,\infty})^n$ be such that the associated Malliavin covariance matrix σ_F is invertible. Then, on the set $\{U > 0\}$ the following estimate holds:*

$$\begin{aligned}|D^\beta H_{\alpha,U}(F,V)| \leq & C\Big(\sum_{r=0}^{k+\ell} |D^{(r)}V|\Big) \times \Big(1 + \sum_{r=1}^{k+\ell} |D^{(r)} \ln U|\Big) \times \\ & \times \big(1 + |\det \sigma_F|^{-(k+\ell+1)}\big) \times \\ & \times \Big(1 + \sum_{r=1}^{k+\ell+1} |D^{(r)}F| + \sum_{r=0}^{k+\ell-1} |D^{(r)}LF|\Big)^{2(k+\ell)n},\end{aligned}$$

C being a positive constant depending on β and α but independent of U, F and V.

As a consequence, taking $\beta = \emptyset$ one gets that (2.31) holds. The proof of Proposition B.1 requires some preliminary estimates.

Lemma B.2 *Let $r \in \mathbb{N}$ and $\gamma \in \{1, \dots, d\}^r$ be a multi index of length r. Then for every $F, G \in (\mathbb{D}^{r+1,\infty})^n$ the following statements hold:*

$$|D^\gamma \langle DG, DF\rangle| \le 2\Big(\sum_{\ell=1}^{r+1} |D^{(\ell)}G|\Big)\Big(\sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big), \tag{B.1}$$

$$|D^\gamma \sigma_F| \le c\Big(\sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big)^2 \tag{B.2}$$

$$|D^\gamma \det \sigma_F| \le c_r\Big(\sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big)^{2n} \tag{B.3}$$

$$|D^\gamma (\det \sigma_F)^{-1}| \le c_r\Big(1 + |\det \sigma_F|^{-(r+1)}\Big)\Big(1 + \sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big)^{2nr} \tag{B.4}$$

$$|D^\gamma \widehat{\sigma}_F| \le c_r\Big(1 + |\det \sigma_F|^{-(r+1)}\Big)\Big(1 + \sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big)^{2n(r+1)-2} \tag{B.5}$$

Here, c and c_r denote suitable positive constants, possibly depending on r but universal w.r.t. the choice of F and/or G.

Proof. *Proof of* (B.1). One has

$$D^\gamma_{s_1,\dots,s_r}\langle DG, DF\rangle = \langle D^\gamma_{s_1,\dots,s_r} D.G, D.F\rangle + \langle D.G, D^\gamma_{s_1,\dots,s_r} D.F\rangle$$

so that by the Cauchy-Schwartz inequality we get

$$|D^\gamma_{s_1,\dots,s_r}\langle DG, DF\rangle| \le |D^\gamma_{s_1,\dots,s_r} D.G||DF| + |DG||D^\gamma_{s_1,\dots,s_r} D.F|.$$

By noticing that $|D^\gamma_{s_1,\dots,s_r} D.G|^2 = \int_{[0,T]} |D^\gamma_{s_1,\dots,s_r} D_s G|^2 ds$, the statement follows.

Proof of (B.2). Since $|D^\gamma \sigma_F^{ij}| = |D^\gamma \langle DF^i, DF^j\rangle|$, the result follows from (B.1).

Proof of (B.3). Recall that $\det \sigma_F = \sum_{\rho \in \mathcal{P}_n} \sigma_F^{1\rho_1} \cdots \sigma_F^{n\rho_n}$, where $\mathcal{P}_n$ is the set of all permutations of $(1, \dots, n)$. For γ multi index in $\{1, \dots, d\}$ with $|\gamma| = r$ we set $s_\gamma \in \mathbb{R}^r$ as $s_\gamma = (s_{\gamma_1}, \dots, s_{\gamma_r})$. Then, we can write

$$D^\gamma_{s_\gamma} \det \sigma_F = \sum_{\rho \in \mathcal{P}_n} D^\gamma\big(\sigma_F^{1\rho_1} \cdots \sigma_F^{n\rho_n}\big) = \sum_{\rho \in \mathcal{P}_n} \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_\gamma} D^{\beta_1}_{s_{\beta_1}} \sigma_F^{1\rho_1} \cdots D^{\beta_n}_{s_{\beta_n}} \sigma_F^{n\rho_n}$$

where "$\beta_1, \dots, \beta_n \in \mathcal{A}_\gamma$" means that $\beta_1, \dots, \beta_n$ is a partition of γ running through the list of all of the "blocks" of γ. We use now (B.3) and we obtain

$$|D^\gamma \det \sigma_F| \le \sum_{\rho \in \mathcal{P}_n} \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_\gamma} |D^{\beta_1} \sigma_F^{1\rho_1}| \cdots |D^{\beta_n} \sigma_F^{n\rho_n}| \le c\Big(\sum_{\ell=1}^{r+1} |D^{(\ell)}F|\Big)^{2n}.$$

Proof of (B.4). We set again $s_\gamma = (s_{\gamma_1}, \dots, s_{\gamma_r})$. For $f \in C^r$ we can write

$$D^\gamma_{s_\gamma} f(G) = \sum_{\ell=1}^{r} f^{(\ell)}(G) \sum_{\beta_1,\dots,\beta_\ell \in \mathcal{B}_\gamma} D^{\beta_1}_{s_{\beta_1}} G \cdots D^{\beta_\ell}_{s_{\beta_\ell}} G$$

where "$\beta_1, \dots, \beta_\ell \in \mathcal{B}_\gamma$" means that $\beta_1, \dots, \beta_\ell$ are non empty multi indexes of γ running through the list of all of the (non empty) "blocks" of γ. Then, it follows that

$$|D^\gamma f(G)| \le \sum_{\ell=1}^{r} |f^{(\ell)}(G)| \sum_{\beta_1,\dots,\beta_\ell \in \mathcal{B}_\gamma} |D^{\beta_1} G| \cdots |D^{\beta_\ell} G| \le c \max_{1\le\ell\le r} |f^{(\ell)}(G)| \left(1 + \sum_{j=1}^{r} |D^{(j)} G|\right)^r$$

because $1 \le |\beta_i| \le |\gamma| = r$. We consider now $f(x) = 1/x$ and $G = \det \sigma_F$. Here, $|f^{(\ell)}(x)| = \ell! x^{-(\ell+1)}$. So, by noticing that for $\ell \le r$

$$1 + |\det \sigma_F|^{-(\ell+1)} \le 2(1 \vee |\det \sigma_F|^{-1})^{\ell+1} \le 2(1 \vee |\det \sigma_F|^{-1})^{r+1} \le 2(1 + |\det \sigma_F|^{-(r+1)})$$

and by using (B.3) one gets the result.

Proof of (B.5). We set $\tilde\sigma_F$ as the matrix of cofactors, so that $\widehat\sigma^{ij}_F = (-1)^{i+j} (\det \sigma)^{-1} \tilde\sigma^{ji}_F$. Then,

$$D^\gamma_{s_\gamma} \widehat\sigma^{ij}_F = (-1)^{i+j} \sum_{\beta_1,\beta_2 \in \mathcal{A}_\gamma} D^{\beta_1}_{s_{\beta_1}} (\det \sigma_F^{-1}) D^{\beta_2}_{s_{\beta_2}} \tilde\sigma^{ji}_F$$

where we say that $\beta_1, \beta_2 \in \mathcal{A}_\gamma$ iff β_1, β_2 is a partition of γ. By recalling that $\tilde\sigma^{ji}_F$ is the determinant of the sub-matrix of σ_F obtaining by deleting the jth row and the ith column of σ, we can apply (B.3) to $D^{\beta_2} \tilde\sigma^{ji}_F$. And by using (B.4) for $D^{\beta_1}(\det \sigma_F^{-1})$, (B.5) immediately holds. □

We are now ready for the

Proof of Proposition B.1. We first consider the case $|\alpha| = 1$. Here, we use a reduced notation and we write

$$H_U(F,V) = V \widehat\sigma_F L F - \widehat\sigma_F \langle DV, DF\rangle - V \langle D\widehat\sigma_F, DF\rangle - V \widehat\sigma_F \langle D \ln U, DF\rangle$$

(recall that V is always one dimensional, while F takes values in $\mathbb{R}^n$), so that $H_{i,U}(F,V)$ is the ith entry of the random vector $H_U(F,V)$. For a multi index β with $|\beta| = \ell$ we have

$$\begin{aligned} D^\beta_{s_\beta} H_U(F,V) = &\sum_{\gamma_1,\gamma_2,\gamma_3 \in \mathcal{A}^3_\beta} D^{\gamma_1}_{s_{\gamma_1}} V D^{\gamma_2}_{s_{\gamma_2}} \widehat\sigma_F D^{\gamma_3}_{s_{\gamma_3}} L F - \sum_{\gamma_1,\gamma_2 \in \mathcal{A}^2_\beta} D^{\gamma_1}_{s_{\gamma_1}} \widehat\sigma_F D^{\gamma_2}_{s_{\gamma_2}} \langle DV, DF\rangle + \\ &- \sum_{\gamma_1,\gamma_2 \in \mathcal{A}^2_\beta} D^{\gamma_1}_{s_{\gamma_1}} V D^{\gamma_2}_{s_{\gamma_2}} \langle D\widehat\sigma_F, DF\rangle + \\ &- \sum_{\gamma_1,\gamma_2\gamma_3 \in \mathcal{A}^3_\beta} D^{\gamma_1}_{s_{\gamma_1}} V D^{\gamma_2}_{s_{\gamma_2}} \widehat\sigma_F D^{\gamma_3}_{s_{\gamma_3}} \langle D \ln U, DF\rangle \end{aligned}$$

where the condition "$\gamma_1, \dots, \gamma_i \in \mathcal{A}^i_\beta$" means that $\gamma_1, \dots, \gamma_i$ is a partition of β given by i subsets. We set now

$$\mathcal{H}_k(G) = \sum_{r=0}^{k} |D^{(r)} G|.$$

Then, for a suitable constant C (independent of V, F and U) that can vary from line to line, we can write

$$|D^\beta H_U(F,V)| \leq C\Big(\mathcal{H}_\ell(V)\mathcal{H}_\ell(\widehat{\sigma}_F)\mathcal{H}_\ell(LF) + \mathcal{H}_\ell(\widehat{\sigma}_F)\mathcal{H}_\ell(\langle DV, DF\rangle) + \\ + \mathcal{H}_\ell(V)\mathcal{H}_\ell(\langle D\widehat{\sigma}_F, DF\rangle) + \mathcal{H}_\ell(V)\mathcal{H}_\ell(\widehat{\sigma}_F)\mathcal{H}_k(\langle D\ln U, DF\rangle)\Big)$$

We estimate the above terms by using Lemma B.2:

• from (B.5) one has

$$\mathcal{H}_\ell(\widehat{\sigma}_F) \leq C\big(1 + |\det \sigma_F|^{-(\ell+1)}\big)\Big(1 + \sum_{i=1}^{\ell+1} |D^{(i)}F|\Big)^{2n(\ell+1)};$$

• from (B.1) one has

$$\mathcal{H}_\ell(\langle DV, DF\rangle) \leq C\sum_{i=1}^{\ell+1} |D^{(i)}V| \times \sum_{i=1}^{\ell+1} |D^{(i)}F| \leq C\sum_{i=1}^{\ell+1} |D^{(i)}V| \times \Big(1 + \sum_{i=1}^{\ell+1} |D^{(i)}F|\Big);$$

• from (B.1) and (B.5) one has

$$\mathcal{H}_\ell(\langle D\widehat{\sigma}_F, DF\rangle) \leq C\sum_{i=1}^{\ell+1} |D^{(i)}\widehat{\sigma}_F| \times \sum_{i=1}^{\ell+1} |D^{(i)}F| \\ \leq C\big(1 + |\det \sigma_F|^{-(\ell+2)}\big)\Big(1 + \sum_{i=1}^{\ell+2} |D^{(i)}F|\Big)^{2n(\ell+1)};$$

• from (B.1) one has

$$\mathcal{H}_\ell(\langle D\ln U, DF\rangle) \leq C\sum_{i=1}^{\ell+1} |D^{(i)}\ln U| \times \sum_{i=1}^{\ell+1} |D^{(i)}F|.$$

So, by inserting the above estimates we get the result for $|\alpha| = 1$. The case $|\alpha| = k > 1$ now easily follows by induction. □

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