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# PINNING MODEL IN RANDOM CORRELATED ENVIRONMENT: APPEARANCE OF AN *INFINITE DISORDER* REGIME

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ABSTRACT. We study the influence of a correlated disorder on the localization phase transition in the pinning model [17]. When correlations are strong enough, an *infinite disorder* regime arises: large and frequent attractive regions appear in the environment. We present here a pinning model in random binary ( $\{-1, 1\}$ -valued) environment. Defining *infinite disorder* via the requirement that the probability of the occurrence of a large attractive region is *sub-exponential* in its size, we prove that it coincides with the fact that the critical point is equal to its minimal possible value, namely  $h_c(\beta) = -\beta$ . We also stress that in the infinite disorder regime, the phase transition is smoother than in the homogeneous case, *whatever the critical exponent of the homogeneous model is*: disorder is therefore *always relevant*. We illustrate these results with the example of an environment based on the sign of a Gaussian correlated sequence, in which we show that the phase transition is of infinite order in presence of infinite disorder. Our results contrast with results known in the literature, in particular in the case of an IID disorder, where the question of the influence of disorder on the critical properties is answered via the so-called Harris criterion, and where a conventional relevance/irrelevance picture holds.

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## 1. INTRODUCTION

**1.1. Physical motivations.** In the study of critical phenomena, a fundamental question is that of the influence of a quenched randomness on the physical properties of a system when approaching criticality. One compares the disordered and non-disordered systems: if the features of the phase transition (essentially the critical exponents) are changed by the presence of arbitrarily small randomness, disorder is said to be *relevant*. The Harris criterion (see [22]) gives a prediction for the influence of disorder for  $d$ -dimensional disordered systems, when randomness has short-range correlations: disorder is irrelevant if  $d\nu^{\text{pur}} > 2$ , where  $\nu^{\text{pur}}$  is the correlation length critical exponent of the homogeneous model. When disorder presents long-range correlations, one invokes the Weinrib-Halperin prediction [32]. With correlations between two couplings at  $i$  and  $j$  decaying like  $|i - j|^{-a}$ ,  $a > 0$ , one should have that the condition for disorder irrelevance becomes  $\min(d, a)\nu^{\text{pur}} > 2$ : the Harris prediction is changed only if  $a < d$ .

When dealing with the question of the influence of disorder on critical properties, the effect of rare regions with atypical disorder reveals to be crucial, cf. [31]. To simplify the statements, if the rare attractive regions are too spread out, their effect on the system remains bounded when considering larger and larger length scale: in the renormalization group language, the fixed point is at a finite randomness. One then lies in what we call the *conventional* regime, where Harris' and Weinrib-Halperin's ideas should be applicable. On the other hand, if atypical regions occur very frequently, their impact on the system

increases without limit when taking larger and larger length scale: the system is said to be governed by an infinite-randomness fixed point, and one lies in the *infinite disorder* regime. Infinite disorder features have been studied for various systems, essentially in random quantum Ising model (starting with Fisher [14, 15]), we refer to [23] for a review.

In the mathematical literature, one class of models has been given much attention lately regarding the question of disorder relevance/irrelevance: the disordered pinning model (see [10, 17, 19]), that models the adsorption of a polymer on a wall or a defect line. Its advantage is that one can play on  $\nu^{\text{pur}}$  as a parameter, to cover the whole range of the relevance/irrelevance picture (in the conventional regime). In the IID case, the Harris criterion has been proven, thanks to a series of papers [1, 2, 8, 11, 20, 21, 24, 29]. More recently, the case of long-range correlated disorder has also been attacked, mostly in the case of Gaussian correlated disorder [3, 5, 27], with correlation decay exponent  $a > 0$ ; and some steps were made towards the Weinrib-Halperin criterion in the case  $a > 1$  (*i.e.* when the Harris criterion should be unchanged). Infinite disorder arises (although it was not mentioned in these terms) in the case  $a < 1$ : the phase transition has been shown to disappear, precisely because of large disorder fluctuations, as in Class C of [31].

In this article, we focus on the appearance of an *infinite disorder* regime for pinning models. With a choice of non-Gaussian (bounded) environment and strong enough correlations, we show that the relevance/irrelevance picture is drastically modified, and we present some of the unusual characteristics of the critical behavior of the system, similar to the smearing of the phase transition described in Class C of [31].

**1.2. Definition of the pinning model.** Let  $\tau := (\tau_i)_{i \geq 0}$  be a renewal process, with law  $\mathbf{P}$ :  $\tau$  is a sequence with  $\tau_0 = 0$   $\mathbf{P}$ -a.s., and such that the variables  $(\tau_i - \tau_{i-1})_{i \in \mathbb{N}}$  are IID with support in  $\mathbb{N}$ , with common law that is called inter-arrival distribution.

**Assumption 1.1.** *We make the assumption that the renewal process is recurrent, that is  $\mathbf{P}(\tau_1 = +\infty) = 0$ . We assume that the inter-arrival distribution, denoted by  $K(\cdot)$ , satisfies*

$$K(n) := \mathbf{P}(\tau_1 = n) \stackrel{n \rightarrow \infty}{\sim} (1 + o(1)) \frac{c_K}{n^{1+\alpha}}, \quad (1.1)$$

for some  $\alpha > 0$  and  $c_K > 0$ . We also note for convenience  $\bar{K}(n) := \mathbf{P}(\tau_1 > n)$ .

The set  $\tau := \{\tau_0, \tau_1, \dots\}$  (we make here a slight abuse of notations) can be thought as the set of contact points between a polymer and a defect line. For example, if  $S$  is a random walk on  $\mathbb{Z}^d$ , the graph of the directed random walk  $(k, S_k)_{k \in [0, N]}$  represents a 1-dimensional polymer chain living in a  $(1+d)$ -dimensional space, and interacting with the defect line  $\mathbb{N} \times \{0\}$ . Then, the set of return times to 0 of the random walk  $S$  is a renewal process. In the case of a  $d$ -dimensional simple random walk, the inter-arrival distribution is known to satisfy the asymptotic (1.1) (up to the aperiodicity condition, which is easily overcome): one has  $\alpha = 1/2$  for  $d = 1$ ,  $\alpha = 0$  (with logarithmic correction) if  $d = 2$ , and  $\alpha = d/2 - 1$  if  $d \geq 3$ , see [13]. Also, considering a recurrent renewal results in no loss of generality, as seen in a moment.

**1.2.1. The disordered model.** Let us consider a random sequence  $\omega = \{\omega_i\}_{i \in \mathbb{N}}$  whose law, denoted  $\mathbb{P}$ , is ergodic and such that  $\mathbb{E}[|\omega_1|] < +\infty$ . Given the sequence  $\omega$  (the environment) and parameters  $h \in \mathbb{R}, \beta \geq 0$ , we define the *quenched* polymer measure with *free boundary condition*. It is a Gibbs transformation of the law  $\mathbf{P}$ , up to length  $N$ :

$$\frac{d\mathbf{P}_{N,h}^{\omega,\beta}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}^{\omega,\beta}} \exp\left(\sum_{n=1}^N (h + \beta\omega_n)\delta_n\right), \quad (1.2)$$

with the notation  $\delta_n := \mathbf{1}_{\{n \in \tau\}}$ . The quantity  $Z_{N,h}^{\omega,\beta} := \mathbf{E} \left[ \exp \left( \sum_{n=1}^N (h + \beta \omega_n) \delta_n \right) \right]$  is used to normalize  $\mathbf{P}_{N,h}^{\omega,\beta}$  to a probability measure, and is called the *partition function* of the disordered system. This polymer measure corresponds to giving an energy reward (or penalty, depending on its sign) to the trajectory of the renewal when it touches the defect line, at the times  $\{\tau_i\}_{i \in \mathbb{N}}$ .

Note that if the renewal were transient, one could reduce to a recurrent renewal  $\tilde{\tau}$ , by considering the inter-arrival distribution  $\tilde{K}(n) = \tilde{\mathbf{P}}(\tilde{\tau}_1 = n) := K(n)/\mathbf{P}(\tau_1 < +\infty)$ , resulting only in a translation of the parameter  $h \mapsto h + \log \mathbf{P}(\tau_1 = +\infty)$ .

**Proposition 1.2.** *One defines the quenched free energy of the system*

$$\mathbf{F}(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,h}^{\omega,\beta} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E} \log Z_{N,h}^{\omega,\beta}, \quad (1.3)$$

which exists and is  $\mathbb{P}$ -a.s. constant. The map  $h \mapsto \mathbf{F}(\beta, h)$  is convex, non-negative and non-decreasing. Thus there exists a (quenched) critical point  $h_c(\beta)$ , possibly infinite, for which one has that  $\mathbf{F}(\beta, h) > 0$  if and only if  $h > h_c(\beta)$ .

This is a classical result for pinning models (see [17, Ch. 4]), and we do not prove it here. The free energy, or energy per monomer, carries physical information on the system: it is easy to see that  $\frac{\partial \mathbf{F}}{\partial h}$  (when it exists) is the asymptotic density of contacts under  $\mathbf{P}_{N,h}^{\omega,\beta}$ . Therefore, a phase transition occurs at the *quenched* critical point  $h_c(\beta)$ , from a *delocalized phase* for  $h < h_c(\beta)$ , ( $\mathbf{F}(\beta, h) = 0$ , null density of contacts), to a *localized phase* for  $h > h_c(\beta)$  ( $\mathbf{F}(\beta, h) > 0$ , positive density of contacts).

We also define the *annealed* system, that is often compared with the disordered system: the *annealed* partition function is  $Z_{N,h,\beta}^a := \mathbf{E}[Z_{N,h}^{\omega,\beta}]$ , and the *annealed* free energy is  $\mathbf{F}^a(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbf{E} Z_{N,h}^{\omega,\beta}$ . One also has an *annealed* critical point  $h_c^a(\beta)$  that separates phases where  $\mathbf{F}^a(\beta, h) = 0$  and where  $\mathbf{F}^a(\beta, h) > 0$ . It is straightforward to obtain from the Jensen inequality that  $\mathbf{F}^a(\beta, h) \geq \mathbf{F}(\beta, h)$ , which gives  $h_c^a(\beta) \leq h_c(\beta)$ . When the inequality is strict, we have an indication that disorder is relevant.

**1.2.2. Reminder on the homogeneous pinning model.** Let us consider the homogeneous (or pure) model, and its partition function, that we denote  $Z_{N,h}^{\text{pur}} := \mathbf{E} \left[ \exp \left( h \sum_{n=1}^N \delta_n \right) \right]$ . The particularity of the homogeneous pinning model is that it is exactly solvable, see [16].

**Proposition 1.3** (Critical behavior of the homogeneous model). *Under Assumption 1.1, one has that  $h_c := h_c(0) = 0$ . The behavior of the free energy (denoted  $\mathbf{F}(h)$  for simplicity of the statements) for  $h$  close to  $h_c = 0$  is*

$$\mathbf{F}(0, h) =: \mathbf{F}(h) \underset{h \searrow 0}{\sim} \begin{cases} \left( \frac{\alpha}{\Gamma(1-\alpha)c_K} \right)^{1/\alpha} h^{1/\alpha} & \text{if } \alpha < 1, \\ \frac{1}{c_K} |\log h|^{-1} h & \text{if } \alpha = 1, \\ \left( \sum_{n \in \mathbb{N}} n K(n) \right)^{-1} h & \text{if } \alpha > 1. \end{cases} \quad (1.4)$$

The critical exponent of the pure free energy is therefore  $\nu^{\text{pur}} = 1 \vee 1/\alpha$  (we note  $a \vee b = \max(a, b)$  and  $a \wedge b = \min(a, b)$ ), leaving aside the log factor in the case  $\alpha = 1$ . Let us mention that  $\nu^{\text{pur}}$  is also the critical exponent of the correlation length, cf. [18]. In the sequel, we do not treat the case  $\alpha = 1$ , but only to avoid too many technicalities (this case is not fundamentally different).

### 1.3. Review of the known results.

1.3.1. *Case of an IID environment.* The picture of disorder relevance/irrelevance is now mathematically understood, the marginal case  $\nu^{\text{pur}} = 2$  ( $\alpha = 1/2$ ) being also settled. We collect the results (cf. [19] for an overview), predicted by the Harris criterion:

- If  $\alpha < 1/2$ , disorder is *irrelevant*: for  $\beta$  small enough, one has  $h_c(\beta) = h_c^{\text{a}}(\beta)$ , and the order of the (disordered) phase transition is the same as for the pure system, see [1, 6, 24, 30].

- If  $\alpha \geq 1/2$ , disorder is *relevant*: one has that  $h_c(\beta) > h_c^{\text{a}}(\beta)$  for all  $\beta > 0$ , see [2, 6, 11, 20] (bounds on the gap between the critical points are also given). Moreover, the order of the (disordered) phase transition is shown to be at least 2 [21], showing disorder relevance when  $\nu^{\text{pur}} < 2$  ( $\alpha > 1/2$ ).

1.3.2. *Case of a correlated environment.* The first type of correlated environment to be considered is the Gaussian one:  $\omega = \{\omega_n\}_{n \in \mathbb{N}}$  is a centered Gaussian stationary sequence, with covariance function  $\mathbb{E}[\omega_i \omega_{i+n}] =: \rho_n$ . One considers long-range, power-law decaying correlations:  $\rho_n \stackrel{n \rightarrow \infty}{\sim} c n^{-a}$ , for some  $a > 0$  and some constant  $c > 0$ . Thanks to the Gaussian structure of the correlations, one is able to compute the annealed partition function,

$$\mathbb{E} \left[ Z_{N,h}^{\omega,\beta} \right] = \mathbf{E} \left[ \exp \left( h \sum_{i=1}^N \delta_i + \frac{\beta^2}{2} \sum_{1 \leq i,j \leq N} \rho_{|j-i|} \delta_i \delta_j \right) \right]. \quad (1.5)$$

Note that in the case of an IID environment, the annealed system is just the homogeneous pinning model with parameter  $h + \frac{\beta^2}{2} \mathbb{E}[\omega_1^2]$ . With correlations, the annealed model is much more difficult to solve. In [3] and [27], the authors show that if  $a > 2$ , then the annealed critical exponent is equal to  $\nu^{\text{pur}}$ . In [3], it is also proven that when  $a > 1$ , the phase transition is of order at least 2, proving disorder relevance when  $\alpha > 1/2$  ( $\nu^{\text{pur}} < 2$ ), as predicted. In [5], the authors treat the hierarchical version of this model, and prove the Weinrib-Halperin prediction in the case  $a > 1$ , both in terms of critical points and of critical exponents. The case  $a < 1$  appears to be more problematic. Both in the standard and the hierarchical version of the model ([3] and [5]), the annealed free energy is infinite, and there is no phase transition anymore for the disordered system ( $h_c(\beta) = -\infty$ ). This phenomenon is due to the fact that when  $a < 1$ , there are large and frequent regions in the environment that are arbitrarily attractive, which is characteristic of the infinite disorder regime (see Definition 1.4).

In [4], the authors construct an ad-hoc binary environment: it has blocks of 0 and of  $-1$ 's, where the sizes of the different blocks are independent, with power-law tail distribution, of exponent  $\vartheta > 1$  (the case  $\vartheta < 1$  being trivial). It is proven that for all  $\beta > 0$ , the critical point is equal to its minimal possible value  $h_c(\beta) = 0$ . The sharp critical behavior of the free energy is also given, the critical exponent being  $\nu^{\text{que}} = \vartheta \nu^{\text{pur}} > \nu^{\text{pur}}$  (with explicit logarithmic corrections): disorder is relevant irrespective of the value of  $\nu^{\text{pur}}$ . Moreover, one remarks from this example that the presence of infinite disorder is not characterized in terms of the power-law decay exponent of the two point correlations function (which is the central quantity for the Weinrib-Halperin prediction).

1.4. **Outline of the results.** In this paper, we consider a bounded (correlated) environment, and we actually focus on the choice of a binary environment,  $\omega \in \{-1, +1\}^{\mathbb{N}}$ , which is also assumed to be stationary and ergodic. We now define precisely what we mean by *infinite disorder*. It is characterized by the fact that favorable regions are very large and frequent: the distance between two attractive regions (*i.e.* constituted of only  $+1$ ) of size larger than  $n$  is subexponential in  $n$ . From the ergodicity of the environment, it is enough

to consider the exponential decay of  $\mathcal{T}_1(n)$ , the distance to the origin of the first attractive region of size larger than  $n$ .

**Definition 1.4.** If  $\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{T}_1(n) = 0$   $\mathbb{P}$ -p.s., then we say that one has *infinite disorder*. Moreover, *infinite disorder* is characterized by

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{T}_1(n) = 0 \quad \mathbb{P} - p.s. \Leftrightarrow \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\omega_1 = +1, \dots, \omega_n = +1) = 0. \quad (1.6)$$

The presence of infinite disorder is then read in terms of subexponential decay of the probability to have a completely attractive environment of size  $n$ . We prove (1.6) in Section 3, see Lemma 3.1. The question that we address here is to determine how the critical properties of the phase transition are modified in the infinite disorder regime, with respect to the pure case. Since

$$Z_{N,h}^{\omega,\beta} = \mathbf{E} \left[ e^{\sum_{i=1}^N (\beta\omega_i + h)\delta_i} \right] \leq \mathbf{E} \left[ e^{(h+\beta) \sum_{i=1}^N \delta_i} \right] = Z_{N,h+\beta}^{\text{pur}}, \quad (1.7)$$

one has  $\mathbf{F}(\beta, h) \leq \mathbf{F}(0, h + \beta)$  and  $h_c(\beta) \geq -\beta$  (since  $h_c(0) = 0$ ). Our first theorem then identifies the appearance of a infinite disorder regime by the fact that the critical point is equal to its minimal possible value,  $-\beta$ .

**Theorem 1.5.** *Under the assumption that correlations are non-increasing (in the sense of Assumption 2.2), then the following criterion holds:*

$$h_c(\beta) = -\beta \text{ for every } \beta > 0 \text{ if and only if one lies in the infinite disorder regime.}$$

A more precise statement is made in Theorem 2.3. We guess (see Conjecture 2.8) that this result is true even without Assumption 2.2 (remark that Theorem 2.4 below says that the implication “infinite disorder implies  $h_c(\beta) = -\beta$ ” does not require such assumption). As far as the features of the phase transition are concerned, Theorem 1.6 yields that in presence of infinite disorder, the quenched free energy has always a smoother critical behavior than that of the pure case: disorder is relevant irrespective of the value of  $\nu^{\text{pur}}$ , and is said to be *strongly relevant*.

**Theorem 1.6.** *If  $h_c(\beta) = -\beta$  (and in particular in presence of infinite disorder), then for all  $\beta > 0$ , one has*

$$\mathbf{F}(\beta, -\beta + u) \stackrel{u \searrow 0}{\asymp} o(\mathbf{F}(0, u)). \quad (1.8)$$

We also give bounds on the free energy in a very general setting, see Proposition 2.7. To complete the picture, we present a natural example, the *Gaussian signs* environment: the  $\{-1, 1\}$ -valued sequence  $\omega$  is simply based on the sign of a correlated Gaussian sequence (with power-law decaying correlations, of decay exponent  $a$ ). Infinite disorder regime appears when  $a < 1$ , and the phase transition is then of infinite order.

In Section 2, we present some useful notations, and expose our main results, that we emphasize thanks to the Gaussian signs example. We comment these results in Section 3. In Section 4, we discuss the annealed model: it exhibits an unconventional behavior in the infinite disorder regime, and enables us to prove parts of our results. In Section 5, we prove the remaining results, giving lower bounds and upper bounds on the free energy. In Appendix, we prove some of the Gaussian estimates needed for the Gaussian signs example, and Lemma B.1 on the homogeneous model.



## 2. MAIN RESULTS

**2.1. First notations and results.** We consider a sequence  $\omega = \{\omega_i\}_{i \geq -1}$  (we choose  $i \geq -1$  instead of  $i \in \mathbb{N}$  for notation convenience, see the following definitions), and we assume that  $\omega$  is ergodic and  $\{-1, 1\}$ -valued (we note abusively  $\omega \in \{-1, 1\}^{\mathbb{N}}$ ). We also take  $\omega$  non-trivial, in the sense that  $\mathbb{P}(\omega_1 = +1) > 0$  and  $\mathbb{P}(\omega_1 = -1) > 0$ .

Our environment is then composed of favorable and unfavorable regions, whether  $\omega_i = +1$  or  $\omega_i = -1$ . For every set of indices  $\mathcal{E} = \{i_1, \dots, i_n\}$ , we define the event

$$\mathfrak{F}_{\mathcal{E}} = \mathfrak{F}_{\{i_1, \dots, i_n\}} := \{\omega_{i_1} = +1, \dots, \omega_{i_n} = +1\} \quad (2.1)$$

that the environment is attractive at sites  $i_1, \dots, i_n$ . With Definition 1.4, one characterizes infinite disorder by the subexponential decay of  $\mathbb{P}(\mathfrak{F}_{[1, n]})$ .

Given the environment  $\omega = \{\omega_i\}_{i \geq -1}$ , we condition it to have  $\omega_{-1} = -1$ ,  $\omega_0 = +1$  (which has positive probability, so that the free energy is not affected). We then define the sequences  $(T_n)_{n \geq 0}$  and  $(\xi_n)_{n \geq 1}$  iteratively, setting  $T_0 := 0$ , and for all  $n \geq 1$

$$\begin{aligned} T_n &:= \inf\{i > T_{n-1} ; \omega_{i+1} \neq \omega_i\}, \\ \xi_n &:= T_n - T_{n-1} \end{aligned} \quad (2.2)$$

Thus our system is cut into segments of size  $\xi_n$ , on which  $\omega$  is constant valued, equal alternatively to  $+1$  and to  $-1$  (we write  $\omega \equiv +1$  and  $\omega \equiv -1$ ). The choice of conditioning to  $\omega_{-1} = -1, \omega_0 = +1$  enables us to identify the blocks with odd indices  $(T_{2j}, T_{2j+1}]$  for  $j \geq 0$  (therefore of size  $\xi_{2j+1}$ ), as the attractive ones.

**Remark 2.1.** The ergodicity and non triviality of the sequence  $\omega$  implies that  $\mathbb{E}[\xi_1] < +\infty$  and  $\mathbb{E}[\xi_2] < +\infty$ . Indeed, one considers the (ergodic) sequence  $\bar{\omega} = \{\bar{\omega}_n\}_{n \geq 0}$  defined by  $\bar{\omega}_n := (\omega_{n-1}, \omega_n)$  for all  $n \geq 0$ . This sequence is, according to our notations, conditioned to start with  $\bar{\omega}_0 = (-1, 1)$ . Then in [28, Ch.I.2.c], the *generalized renewal process* of the set  $\{(-1, 1)\}$  is defined as the sequence of indices  $\{k \geq 0, \bar{\omega}_k = (-1, 1)\} = \{T_{2j}\}_{j \geq 0}$ , and the *return-time process* to the set  $\{(-1, 1)\}$  (of positive measure) is defined as the sequence  $\{T_{2j} - T_{2(j-1)}\}_{j \in \mathbb{N}} = \{\xi_{2j-1} + \xi_{2j}\}_{j \in \mathbb{N}}$ . It is shown that the return-time process is ergodic (see [28, Th.I.2.19]), and [28, Eq. (14) Ch.I.2] states that the first return time to  $(-1, 1)$  (*i.e.*  $T_2 = \xi_1 + \xi_2$ ) has expectation  $\mathbb{P}(\omega_1 = -1, \omega_2 = 1)^{-1} < +\infty$ .

We now introduce a notion of “good” block: we call “*A-block*” a segment  $(T_i, T_{i+1}]$  on which  $\omega \equiv +1$  (take  $i$  even), and whose size  $\xi_{i+1}$  is larger than  $A$ . We define

$$\mathcal{T}_1(A) = \inf\{T_{2i+1} ; \xi_{2i+1} \geq A\} \quad (2.3)$$

the position of the first *A-block*, and iteratively,  $\mathcal{T}_k(A)$  the position of the  $k^{\text{th}}$  *A-block*,

$$\mathcal{T}_k(A) := \inf\{T_{2i+1} > \mathcal{T}_{k-1}(A) ; \xi_{2i+1} \geq A\}. \quad (2.4)$$

The quantities  $\mathcal{T}_k(A) - \mathcal{T}_{k-1}(A)$  represent the distances between the rare large attractive regions of length at least  $A$ , and that is why the subexponential decay (in  $A$ ) of  $\mathcal{T}_k(A) - \mathcal{T}_{k-1}(A)$  is used to characterize the presence of infinite disorder (recall Definition 1.4). We regroup in Figure 1 the above notations, *i.e.* the decomposition of our environment  $\omega$  into elementary blocks  $(T_{i-1}, T_i]_{i \in \mathbb{N}}$  of size  $\xi_i$ , and for  $A > 0$  fixed, into meta-blocks  $(\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]_{k \in \mathbb{N}}$ .

We stress that the quantity  $\frac{1}{T_N} \sum_{n=1}^N \mathbf{1}_{\{n \text{ is odd}, \xi_n \geq A\}}$  represents the density of *A-blocks* in a system of size  $T_N$ . Using Birkhoff’s Ergodic Theorem (cf. [25, Chap. 2]), one gets that the (asymptotic) density of *A-blocks* is then equal to  $\mathbb{E}[T_2]^{-1} \mathbb{P}(\xi_1 \geq A)$ . Since one has  $\frac{\mathcal{T}_N(A)}{N} = \left( \frac{1}{\mathcal{T}_N(A)} \sum_{n=1}^{\gamma_N(A)} \mathbf{1}_{\{n \text{ is odd}, \xi_n \geq A\}} \right)^{-1}$ , where  $\gamma_N(A)$  is the index such that





We point out that the condition in point (2) is weaker than assuming absence of infinite disorder plus Assumption 2.2 as was done in Theorem 2.3. Indeed,

**Lemma 2.5.** *Under Assumption 2.2, if one has that  $\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathfrak{F}_{\llbracket 1, n \rrbracket}) > 0$ , then there exists a constant  $c > 0$  such that for any sequence of indices  $i_1 < i_2 < \dots$  one has*

$$\mathbb{P}(\mathfrak{F}_{\{i_1, \dots, i_n\}}) \leq e^{-cn}, \quad \text{for all } n \geq 1. \quad (2.6)$$

**Proof** For all indices  $i_1 < \dots < i_n$ , one has, thanks to repeated use of Assumption 2.2, that  $\mathbb{P}(\mathfrak{F}_{\{i_1, \dots, i_n\}}) \leq \mathbb{P}(\mathfrak{F}_{\llbracket 1, n \rrbracket})$ , thanks to Assumption 2.2. From this, one gets that  $\mathbb{P}(\mathfrak{F}_{\{i_1, \dots, i_n\}}) \leq \mathbb{P}(\mathfrak{F}_{\llbracket 1, n \rrbracket}) \leq e^{-cn}$ , which gives the conclusion of Lemma 2.5.  $\square$

Point (1) follows from Proposition 2.7 below. Point (2) of Theorem 2.4 is proven in Section 4.2, via bounds on the annealed model.

### 2.3. Bounds on the free energy.

**Definition 2.6.** One defines

$$\varepsilon(x) := \inf_{n \leq x} -\frac{1}{n} \log \mathbb{P}(\xi_1 \geq n), \quad (2.7)$$

which is non-increasing: one can therefore note  $\varepsilon^{-1}$ , its generalized inverse, defined by  $\varepsilon^{-1}(u) := \sup\{x, \varepsilon(x) \geq u\}$ .

Before we state the result, keep in mind that  $\mathbb{P}(\xi_1 \geq n) \stackrel{n \rightarrow \infty}{\asymp} o(1/n)$  since  $\mathbb{P}(\xi_1 \geq n)$  is decreasing and summable (because  $\mathbb{E}[\xi_1] < \infty$ ).

**Proposition 2.7. 1. Lower bound.** *If  $\lim_{x \rightarrow \infty} \varepsilon(x) = 0$  (infinite disorder), then  $\varepsilon^{-1}$  is defined on a neighborhood of 0, and one has  $\lim_{u \rightarrow 0} \varepsilon^{-1}(u) = +\infty$ . Then there exist two constants  $c_0, c'_0 > 0$  (that do not depend on  $\beta$ ), such that for all  $u \in (0, 1)$  and all  $\beta > 0$  one has*

$$\mathbf{F}(\beta, -\beta + u) \geq c'_0 A_u \mathbb{P}(\xi_1 \geq A_u) \mathbf{F}(u), \quad (2.8)$$

where we defined  $A_u := \varepsilon^{-1}(c_0 \mathbf{F}(u))$ , that goes to infinity as  $u$  goes to 0. In particular one has  $h_c(\beta) = -\beta$ , and the r.h.s. of (2.8) is  $o(\mathbf{F}(u))$  when  $u$  goes to 0 (since  $A_u \mathbb{P}(\xi_1 \geq A_u)$  goes to 0).

**2. Upper bounds.** *Whether  $h_c(\beta) = -\beta$  or not, one also has the following upper bound on the free energy: there exist constants  $C_1, c_1, \mathbf{c} > 0$  (that do not depend on  $\beta$ ), such that for all  $\beta \in (0, 1)$  and  $u \in (0, c_1 \beta)$  one has*

$$\mathbf{F}(\beta, -\beta + u) \leq C_1 \beta^{1-\nu^{\text{pur}}} \mathbf{F}(u) \mathbb{E} \left[ \frac{T_{L(u)}}{L(u)} \mathbf{1}_{\{T_{L(u)} > 4\mathbb{E}[T_2]L(u)\}} \right], \quad (2.9)$$

where  $L(u) := \lfloor \mathbf{c} \beta^{\nu^{\text{pur}}} \mathbf{F}(u)^{-1} \rfloor$ .

One has another bound, easier to handle: for all  $u \in (0, c_1 \beta)$ , one has

$$\mathbf{F}(\beta, -\beta + u) \leq C_1 u \mathbb{E} [\xi_1 \mathbf{1}_{\{\xi_1 > \mathbf{c} \beta u^{-1}\}}]. \quad (2.10)$$

This Proposition gives simple and fruitful bounds and can be applied to many types of environment. We give possible applications in the sequel, in particular Theorem 2.9. It is also used to prove Theorem 1.6 (see Section 3.3).

To prove the lower bound (see Section 5.1), we use a localization strategy. We focus on the contribution of the trajectories that target an attractive region of size  $A = A_u := \varepsilon^{-1}(c_0 \mathbf{F}(u))$ . The energetic reward one gets on this region is  $e^{\mathbf{F}(u)A_u}$ , and the entropic cost is approximatively  $K(\mathbb{P}(\xi_1 \geq A_u)) \approx e^{-cA_u \varepsilon(A_u)}$  (the first segment of size  $A_u$  being at

mean distance  $\mathbb{P}(\xi_1 \geq A_u)^{-1} \approx e^{-A_u \varepsilon(A_u)}$  from the origin). Then our (optimized) choice of  $A_u$  gives that  $\mathbf{F}(u)A_u > -cA_u \varepsilon(A_u)$ . Targeting the first region where the energetic gain overcomes the entropic cost of doing such a long jump is therefore a good localization strategy. The intuition that these are the only strategies contributing to the free energy is confirmed by the (partly heuristic) reasoning of Section 5.3, thanks to a multiscale coarse-graining argument, and allows us to formulate the conjecture:

**Conjecture 2.8.** *One has  $h_c(\beta) = -\beta \forall \beta > 0$  if and only if one lies in the infinite disorder regime.*

When using a simpler (and amenable) coarse-graining procedure, we get the upper bound (2.9), see Section 5.2.3.

**2.4. Case of the Gaussian signs environment.** Let  $W := (W_n)_{n \geq 0}$  be a centered normalized stationary Gaussian process whose law is denoted  $\mathbb{P}$ , and with correlation function  $\rho_n := \mathbb{E}[\omega_i \omega_{i+n}]$  (it does not depend on  $i$  because of the stationarity),  $\rho_0 = 1$ . We also assume that correlations are non-negative and are power-law decaying: there exist some  $a > 0$  and some constant  $c > 0$  such that the correlation function  $(\rho_n)_{n \geq 0}$  verifies

$$\rho_k \stackrel{k \rightarrow \infty}{\sim} ck^{-a}, \quad \text{and } \rho_k \geq 0 \text{ for all } k \geq 0. \quad (2.11)$$

It is natural from the Gaussian sequence  $\{W_n\}_{n \in \mathbb{N}}$ , to define the environment  $\omega$  with values in  $\{-1, 1\}$  by

$$\omega_i := \mathbf{1}_{\{W_i \geq 0\}} - \mathbf{1}_{\{W_i < 0\}}. \quad (2.12)$$

We refer to this choice as the *Gaussian signs* environment. The sequence  $W$  is ergodic since  $|\rho_k| \stackrel{k \rightarrow \infty}{\rightarrow} 0$  (see [9, Ch.14 §2, Th.2]). Therefore,  $\omega$  is also ergodic, so that  $\mathbf{F}(\beta, h)$  exists, and one can apply Theorem 2.3 and Proposition 2.7. We do that in Section 3.4, proving the following result.

**Theorem 2.9.** *For the Gaussian signs environment defined above, with Assumption (2.11), one has:*

- *If  $a < 1$ , then  $h_c(\beta) = -\beta$  for all  $\beta > 0$ . There exists some constant  $c_2 > 0$ , such that for all  $u \in (0, 1)$  and  $\beta > 0$  one has*

$$\mathbf{F}(\beta, -\beta + u) \geq \exp\left(-c_2 |\log u|^{1/(1-a)} \mathbf{F}(u)^{-a/(1-a)}\right). \quad (2.13)$$

*There exist constants  $c_3, c'_3 > 0$ , such that for all  $\beta \in (0, 1)$  and  $u \in (0, c_3\beta)$  one has*

$$\mathbf{F}(\beta, -\beta + u) \leq \exp(-c'_3 u^{-a}) \quad (2.14)$$

- *If  $a > 1$ , one has some  $c_\beta > 0$  such that  $\mathbf{F}^a(\beta, h) \leq \mathbf{F}(h + (1 - c_\beta)\beta)$  for all  $\beta > 0$ , and in particular  $h_c(\beta) \geq h_c^a(\beta) > -\beta$ .*

*The previous bounds give a threshold leading from  $h_c(\beta) > -\beta$  for  $a > 1$  to  $h_c(\beta) = -\beta$  for  $a < 1$ . Moreover, if one has  $\rho_{k+1} \leq \rho_k$  for all  $k \geq 0$ , then one has the following criterion:*

$$h_c(\beta) = -\beta \iff \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(W_1 \geq 0, \dots, W_n \geq 0) = 0 \quad (\text{infinite disorder}). \quad (2.15)$$

Note that in the Gaussian signs environment, if the correlation decay exponent is  $a < 1$ , then the phase transition is of infinite order. This stresses that disorder is strongly relevant in that case, in a substantial way. It is the first example we are aware of in the pinning model framework where the presence of disorder makes the phase transition of infinite order (leaving aside the case  $\alpha = 0$  where one already has that  $\nu^{\text{pur}} = \infty$ ).

### 3. GENERAL COMMENTS ON THE RESULTS

**3.1. Reduction to a  $\{-1,1\}$ -valued environment.** Our choice of a  $\{-1,1\}$ -valued environment is made essentially to simplify notations, and to restrict the question on the critical point to decide whether  $h_c(\beta) = -\beta$  or  $h_c(\beta) > -\beta$ . We now explain why this choice is actually not restrictive, and how general environments can be treated with the techniques used in this paper.

To a general bounded sequence  $\omega$ , we associate a binary sequence, that keeps track of the distribution of "favorable" and "unfavorable" regions. We note  $\mathbf{M} := \text{ess sup}(\omega_1)$  ( $\mathbf{M} < +\infty$ ), we take  $\eta$  small, and consider  $I_\eta = [\mathbf{M} - \eta, \mathbf{M}]$  a neighborhood of  $\mathbf{M}$ . We now define the sequence  $w^{(\eta)} = (w_i^{(\eta)})_{i \in \mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$  the following way:

$$w_i^{(\eta)} := \mathbf{1}_{\{\omega \in I_\eta\}} - \mathbf{1}_{\{\omega_i \notin I_\eta\}}. \quad (3.1)$$

Then all the results presented can be dealt via the sequence  $w^{(\eta)}$ , studying its properties for arbitrary  $\eta$ . For example, one can translate Theorem 2.3 into the following assertion:

If the sequence  $w^{(\eta)}$  verifies Assumption 2.2 for every (small)  $\eta > 0$ , then one has

$$h_c(\beta) = -\mathbf{M}\beta \quad \text{for all } \beta > 0$$

$$\Leftrightarrow \text{for all } \eta > 0, \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(w_i^{(\eta)} = 1 \text{ for all } 1 \leq i \leq n) = 0. \quad (3.2)$$

One also remarks that  $\omega_n \leq \mathbf{M} - \eta/2 + \frac{\eta}{2} w_n^{(\eta)}$ , so that  $Z_{N,h}^{\omega,\beta} \leq Z_{N,h+\beta\mathbf{M}-\beta\eta/2}^{w^{(\eta)},\beta\eta/2}$ . The upper bounds one gets on the free energy in Proposition 2.7 can therefore also be generalized. In particular, one obtains that whenever  $h_c(\beta) = -\mathbf{M}\beta$ , disorder is relevant for any value of  $\nu^{\text{pur}}$ . For the lower bounds, one simply uses the bound  $\omega_n \geq (\mathbf{M} - \eta) \mathbf{1}_{\{w_n^{(\eta)} = +1\}}$ , since only favorable regions are used in the proofs of our results.

### 3.2. Discussion on the criterion in Theorem 2.4.

**Lemma 3.1.** *The following conditions are equivalent, and characterize infinite disorder:*

- (a)  $\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathfrak{F}_{[1,n]}) = 0$  ;
- (b)  $\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{T}_1(n) = 0$   $\mathbb{P}$ -a.s. ;
- (c) for all  $\delta > 0$ ,  $A_\delta(\omega) < +\infty$   $\mathbb{P}$ -a.s., where we define for any  $\delta > 0$

$$A_\delta^{(\omega)} := \inf \{A : \log \mathcal{T}_1(A) < \delta A\}. \quad (3.3)$$

We introduced  $A_\delta^{(\omega)}$  because, with the right choice of  $\delta = \mathbf{F}(u)/(1 + \alpha)$ ,  $\mathcal{T}_1(A_\delta^\omega)$  is the first position where the energetic gain on the  $A_\delta^\omega$ -block,  $A_\delta^\omega \mathbf{F}(u)$ , overcomes the entropic cost of targeting it,  $-(1 + \alpha) \log \mathcal{T}_1(A_\delta^\omega)$ . Therefore,  $A_\delta^\omega = +\infty$  would mean that the energetic gain on an  $A$ -block can never balance the entropic cost of targeting it, bringing some support to Conjecture 2.8.

**Proof** Jensen's inequality gives that  $\mathbb{E}[\log \mathcal{T}_1(n)] \leq \log \mathbb{E}[\mathcal{T}_1(n)]$ , which together with (2.5) shows that (a) implies  $\liminf_{n \rightarrow \infty} \mathbb{E}[\frac{1}{n} \log \mathcal{T}_1(n)] = 0$ . Then, a direct application of Fatou's Lemma gives  $\mathbb{E}[\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{T}_1(n)] = 0$ , and (b). Using the definition of  $A_\delta(\omega)$ , (b) then directly implies that, for every fixed  $\delta > 0$ ,  $A_\delta(\omega) < +\infty$   $\mathbb{P}$ -a.s.

On the other hand, if  $A_\delta(\omega) < +\infty$   $\mathbb{P}$ -a.s. for all  $\delta > 0$ , then it means that for any  $p \in \mathbb{N}$ , there exists  $\mathbb{P}$ -a.s. some finite  $A$  such that  $\frac{1}{A} \log \mathcal{T}_1(A) \leq 1/p$ . It shows that  $\liminf_{A \rightarrow \infty} \frac{1}{A} \log \mathcal{T}_1(A) \leq 1/p$   $\mathbb{P}$ -a.s. for any  $p \in \mathbb{N}$ , which gives  $\liminf_{A \rightarrow \infty} \frac{1}{A} \log \mathcal{T}_1(A) = 0$   $\mathbb{P}$ -a.s.

We are left to show that (b) implies (a). We actually prove a stronger statement:

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathfrak{F}_{\llbracket 1, n \rrbracket}) > 0 \Rightarrow \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{T}_1(n) > 0 \quad \mathbb{P} - a.s. \quad (3.4)$$

Indeed, if  $\mathbb{P}(\mathfrak{F}_{\llbracket 1, n \rrbracket}) = \mathbb{P}(\xi_1 \geq n) \leq e^{-\delta n}$  for some  $\delta > 0$ , then one has that

$$\mathbb{P}(\mathcal{T}_1(n) \leq e^{\delta n/2}) \leq \mathbb{P}(\exists k \leq e^{\delta n/2}, \xi_{2k+1} \geq n) \leq e^{\delta n/2} \mathbb{P}(\xi_1 \geq n) \leq e^{-\delta n/2}, \quad (3.5)$$

where we first used that there are at most  $\mathcal{T}_1(n)$  segments  $(T_{2k}, T_{2k+1}]$  in a system of size  $\mathcal{T}_1(n)$ , and then a union bound. From (3.5) and using Borel-Cantelli Lemma, one gets that  $\mathbb{P}$ -a.s.,  $\frac{1}{n} \log \mathcal{T}_1(n) \leq \delta/2$  happens only a finite number of time, meaning that  $\liminf \frac{1}{n} \log \mathcal{T}_1(n) > \delta/2$ .  $\square$

**3.3. Comments on the bounds of Proposition 2.7.** We first stress that Proposition 2.7, and in particular the upper bound (2.9) on the free energy, implies Theorem 1.6. Indeed, thanks to the Ergodic Theorem, one has that  $\frac{T_n}{n} \xrightarrow{n \rightarrow \infty} \frac{1}{2} \mathbb{E}[T_2]$   $\mathbb{P}$ -a.s. It is then standard to obtain that  $\limsup_{n \rightarrow \infty} \mathbb{E} \left[ \frac{T_n}{n} \mathbf{1}_{\{T_n \geq 4\mathbb{E}[T_2]n\}} \right] = 0$ , which directly gives that  $\mathbb{F}(\beta, -\beta + u) \stackrel{u \searrow 0}{\asymp} o(\mathbb{F}(u))$  from (2.9). Moreover, in the case  $\alpha > 1$ , where  $\mathbb{F}(u) \stackrel{u \searrow 0}{\asymp} cst.u$ , the bound (2.10) gives more directly that  $\mathbb{F}(\beta, -\beta + u) \stackrel{u \searrow 0}{\asymp} o(\mathbb{F}(u))$ .

One is also able to get, in many cases, a very simple bound on the free energy, knowing only the behavior of  $\mathbb{P}(\xi_1 \geq A)$ . Indeed, a small computation gives that

$$\mathbb{E} [\xi_1 \mathbf{1}_{\{\xi_1 \geq A\}}] = \sum_{n \geq A} n (\mathbb{P}(\xi_1 \geq n) - \mathbb{P}(\xi_1 \geq n+1)) = A\mathbb{P}(\xi_1 \geq A) + \sum_{n > A} \mathbb{P}(\xi_1 \geq n). \quad (3.6)$$

If  $\mathbb{P}(\xi_1 \geq A)$  decays sufficiently fast (essentially faster than  $A^{-(1+\varepsilon)}$ , one has to treat each case cautiously), one gets that  $\mathbb{E} [\xi_1 \mathbf{1}_{\{\xi_1 \geq A\}}] \leq cA\mathbb{P}(\xi_1 \geq A)$ . From (2.10) one obtains

$$\mathbb{F}(\beta, -\beta + u) \leq C' \mathbb{P}(\xi_1 \geq cu^{-1}), \quad (3.7)$$

which gives a very explicit (but rough) upper bound.

**3.4. Properties of the Gaussian signs environment.** The following proposition, proven in Appendix A, corresponds to the estimate of  $\mathbb{P}(\xi_1 \geq n)$  for the Gaussian signs environment. Together with Proposition 2.7, it gives (2.13)-(2.14).

**Proposition 3.2.** *We make the assumption (2.11) on the Gaussian sequence  $W$ .*

- If  $a < 1$ , there exist two constants  $c, c' > 0$ , such that for every  $n \in \mathbb{N}$  one has

$$\mathbb{P}(W_i \geq 0; \forall i \in \{1, \dots, n\}) \geq e^{-cn^a \log n}. \quad (3.8)$$

Moreover, for all subsequences  $1 \leq i_1 < \dots < i_n$ , one has

$$\mathbb{P}(W_{i_j} \geq 0; \forall j \in \{1, \dots, n\}) \leq e^{-c'n^a}. \quad (3.9)$$

- If  $a > 1$ , there exists some constant  $c'' > 0$  such that for all subsequences  $1 \leq i_1 < \dots < i_n$  one has

$$\mathbb{P}(W_{i_j} \geq 0; \forall j \in \{1, \dots, n\}) \leq e^{-c''n}. \quad (3.10)$$

Let us mention that in the case  $a < 1$ , for a rather particular choice of the Gaussian covariance structure, [7, Th.1.1] gives a much sharper result than ours. More precisely, in the case where the covariances of  $(W_n)_{n \in \mathbb{N}}$  are given by the Green function of some transient random walk on  $\mathbb{Z}^d$  (one can construct such a random walk in a way that  $\rho_n \sim c_a n^{-a}$ , with some explicit constant  $c_a$ , see [7]), one has

$$\lim_{n \rightarrow \infty} -\frac{1}{n^a \log n} \log \mathbb{P}(W_i \geq 0; \forall i \in \{1, \dots, n\}) = C_a, \quad (3.11)$$

where the constant  $C_a$  is explicit. If we had an estimate like (3.11) in Proposition 3.2 for  $a < 1$ , we would get a slightly more precise upper bound in Theorem 2.9. As we do not hunt for the sharp behavior in Theorem 2.9, we are satisfied with Proposition 3.2 which is valid with very little assumptions on the correlation structure.

**Remark 3.3.** The case  $a = 1$  is more problematic because it is a marginal case, and our proofs would adapt to this case, giving

$$e^{-cn/\log n} \geq \mathbb{P}(W_i \geq 0; \forall i \in \{1, \dots, n\}) \geq e^{-cn}, \quad (3.12)$$

which does not allow us to make any conclusion.

**Proof of Theorem 2.9** If  $a < 1$ , Proposition 3.2 gives that  $\mathbb{P}(\xi_1 \geq n) \geq e^{-c_2 n^a \log n}$ . One has  $\varepsilon(x) \leq c' x^{a-1} \log x$ , and thus there is a constant  $c$  such that  $\varepsilon^{-1}(t) \leq c(t^{-1} |\log t|)^{1/(1-a)}$ . Defining  $A_u := \varepsilon^{-1}(c_0 \mathbf{F}(u))$  as in Proposition 2.7, one gets  $A_u \leq c(\mathbf{F}(u)^{-1} |\log u|)^{1/(1-a)}$  (recall that  $\mathbf{F}(u)$  is of polynomial order). One concludes using again Proposition 3.2, which gives that

$$\mathbb{P}(\xi_1 \geq A_u) \geq \exp\left(-c |\log u|^{1/(1-a)} \mathbf{F}(u)^{-a/(1-a)}\right), \quad (3.13)$$

that combined with (2.8) brings (2.13). Moreover, Proposition 3.2 also implies that  $\mathbb{P}(\xi_1 \geq n) \leq e^{-c_2 n^a}$  so that (2.14) follows directly from the bound (3.7). In the case  $a > 1$ , the conclusion follows from Proposition 3.2, that gives the condition (2) in Theorem 2.4.

To get the criterion (2.15) we have to show that the Gaussian signs environment satisfies Assumption 2.2 if the correlation function is non-increasing. Indeed, let us consider two sets of indices  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , with  $\max \mathcal{E}_1 < \min \mathcal{E}_2$ . We notice that, if  $\rho_{k+1} \leq \rho_k$  for all  $k \geq 0$ , then the covariances of the Gaussian vector  $X = (W_i)_{i \in \mathcal{E}_1 \cup \theta^k \mathcal{E}_2}$  are all smaller or equal than those of  $Y = (W_i)_{i \in \mathcal{E}_1 \cup \theta^k \mathcal{E}_2}$ , for any  $k$ . Then one uses a convenient equality due to Piterbarg [26].

**Proposition 3.4.** *Let  $X = (X_1, \dots, X_n)$  and  $Y = (Y_1, \dots, Y_n)$  be two independent families of centered Gaussian random variables. Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^1$  be a function with bounded second derivatives. Then*

$$\mathbb{E}[g(X)] - \mathbb{E}[g(Y)] = \frac{1}{2} \sum_{i,j=1}^n (\mathbb{E}[X_i X_j] - \mathbb{E}[Y_i Y_j]) \int_0^1 \frac{\partial^2 g}{\partial x_i \partial x_j} ((1-t)^{1/2} X + t^{1/2} Y) dt.$$

We approximate the indicator function  $\mathbf{1}_{\{x \geq 0\}}$  by a twice differentiable non-decreasing function  $f$ , and  $\mathbf{1}_{\{x_i \geq 0, \forall i \in \{1, \dots, n\}\}}$  by  $g(x_1, \dots, x_n) = \prod_{i=1}^n f(x_i)$ , we have in particular that  $\frac{\partial^2 g}{\partial x_i \partial x_j} \geq 0$  for every  $i \neq j$ . Since in our case  $\mathbb{E}[X_i X_j] \leq \mathbb{E}[Y_i Y_j]$  for all  $i, j \in \{1, \dots, n\}$  (with equality for  $i = j$ ), we get that  $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ . In the end, one obtains that  $\mathbb{P}(\mathfrak{F}_{\mathcal{E}_1 \cup \theta^k \mathcal{E}_2}) \geq \mathbb{P}(\mathfrak{F}_{\mathcal{E}_1 \cup \theta^{k+1} \mathcal{E}_2})$  for all  $k \geq 0$ .  $\square$

We stress that one is also able to estimate the covariances of the Gaussian signs environment: they have the same decay exponent as the correlation function  $\rho_n$ : thanks to a standard Gaussian computation, one gets that  $\text{Cov}(\omega_i, \omega_{i+k}) \stackrel{k \rightarrow \infty}{\sim} \frac{2\rho_k}{\pi}$ .

## 4. ANNEALED ESTIMATES, PROOF OF PROPOSITION 4.1

We now comment on the annealed model, that exhibits a trivial behavior in the infinite disorder regime.

**Proposition 4.1.** *Under Assumption 2.2, one has the following criterion*

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathfrak{F}_{[1,n]}) = 0 \iff \mathbf{F}^a(\beta, h) = \mathbf{F}(0, h + \beta) \text{ for all } h \in \mathbb{R}, \beta \geq 0. \quad (4.1)$$

**4.1. Triviality of the annealed system in the infinite disorder regime.** If one has that  $\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\xi_1 \geq n) = 0$  (infinite disorder), the average behavior of the partition function is actually dominated by the very large fluctuations in the disorder: the annealed model is trivial. Indeed, imposing all  $\omega_i$ 's to be equal to +1 in a system of size  $N$ , one obtains the bound  $\mathbb{E}Z_{N,h}^{\omega,\beta} \geq \mathbb{P}(\xi_1 \geq N)Z_{N,h+\beta}^{\text{pur}}$ . Thus, we readily have

$$\mathbf{F}^a(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}Z_{N,h}^{\omega,\beta} \geq \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\xi_1 \geq N) + \mathbf{F}(0, h + \beta) = \mathbf{F}(0, h + \beta). \quad (4.2)$$

Since one has the other trivial bound  $Z_{n,h}^{\omega,\beta} \leq Z_{n,h+\beta}^{\text{pur}}$ , one gets  $\mathbf{F}^a(\beta, h) = \mathbf{F}(0, h + \beta)$ , which gives the first part of Proposition 4.1 (the other implication is proven by Proposition 4.2).

The bound  $\mathbf{F}(\beta, h) \leq \mathbf{F}^a(\beta, h)$  therefore gives no more information than the trivial one  $\mathbf{F}(\beta, h) \leq \mathbf{F}(0, h + \beta)$ .

**On the correlation lengths.** In the IID case,  $\mathbf{F}(\beta, h)^{-1}$  is  $\mathbb{P}$ -a.s. equal to the exponential decay rate of the two-point correlation function  $\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i \delta_{i+k}) - \mathbf{E}_{N,h}^{\omega,\beta}(\delta_i) \mathbf{E}_{N,h}^{\omega,\beta}(\delta_{i+k})$  when  $k \rightarrow \infty$ , as proven in [29, Th.3.5] (when  $\mathbf{P}$  is the law of the return times to the origin of the simple random walk). Therefore,  $\mathbf{F}(\beta, h)^{-1}$  is the *quenched* correlation length in the IID case. One also defines the usual *quenched-averaged* correlation length, *i.e.* the inverse of the exponential decay rate of  $\mathbb{E}[\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i \delta_{i+k}) - \mathbf{E}_{N,h}^{\omega,\beta}(\delta_i) \mathbf{E}_{N,h}^{\omega,\beta}(\delta_{i+k})]$ . In the IID case, it is shown to be equal to  $\mu(\beta, h)^{-1}$  [29, Th.3.5] (under the particular assumptions already mentioned), where

$$\mu(\beta, h) := - \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[ \frac{1}{Z_{N,h}^{\omega,\beta}} \right]. \quad (4.3)$$

We believe that this correlation length(s) interpretation is still valid in the correlated framework. One easily gets from Jensen inequality that  $\mu(\beta, h) \leq \mathbf{F}(\beta, h)$ , and in the IID framework one actually has that  $c_\beta \mathbf{F}(\beta, h)^2 < \mu(\beta, h) < \mathbf{F}(\beta, h)$  for  $h > h_c(\beta)$  (a better lower bound is given in [29, Th.3.3]). It means that the *quenched* and *quenched-averaged* correlation lengths diverge at the same critical point, namely  $h_c(\beta)$ .

We now stress that this picture is fully changed in the infinite disorder regime. If we assume that the distribution of +1's and -1's are symmetric, then one also has that  $\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\xi_2 \geq n) = 0$ , and the average in (4.3) is dominated by the rare repulsive regions. One has

$$\frac{1}{N} \log \mathbb{E} \left[ \frac{1}{Z_{N,h}^{\omega,\beta}} \right] \geq \frac{1}{N} \log \mathbb{P}(\xi_1 = 1, \xi_2 \geq N) - \frac{1}{N} \log Z_{N,h-\beta}^{\text{pur}}, \quad (4.4)$$

which directly give that  $\mu(\beta, h) \leq \mathbf{F}(0, h - \beta)$  by letting  $N$  go to infinity. The trivial bound  $Z_{N,h}^{\omega,\beta} \geq Z_{N,h-\beta}^{\text{pur}}$  then gives that  $\mu(\beta, h) = \mathbf{F}(0, h - \beta)$ .

Therefore, in the infinite disorder regime, the *quenched* correlation length  $\mathbf{F}(\beta, h)^{-1}$  diverges at  $h_c(\beta) = -\beta$  with a critical exponent larger than  $\nu^{\text{pur}}$  and possibly infinite (see Theorems 1.6-2.9). On the other hand, the *quenched-average* correlation length



$\mu(\beta, h)^{-1} = \mathbf{F}(0, h - \beta)^{-1}$  diverges at  $+\beta$ , with critical exponent  $\nu^{\text{pur}}$ . This stresses again the unconventional behavior of the system in presence of infinite disorder, where  $\mathbf{F}(\beta, h)$  and  $\mu(\beta, h)$  have different critical points, and where  $\mathbf{F}(\beta, h)$  has a larger critical exponent than  $\mu(\beta, h)$ .

**4.2. Annealed bounds in the conventional regime.** The following result shows that in the *conventional* regime, the annealed bound  $\mathbf{F}(\beta, h) \leq \mathbf{F}^{\text{a}}(\beta, h)$  is not trivial: bounds on the annealed free energy are fruitful, in particular to show that  $h_c(\beta) > -\beta$ .

**Proposition 4.2.** *If there exists a constant  $c_0 > 0$  such that for all indices  $1 \leq i_1 < \dots < i_m$  one has  $\mathbb{P}(\mathfrak{F}_{\{i_1, \dots, i_m\}}) \leq e^{-c_0 m}$ , then for all  $\beta > 0$  one has*

$$\mathbf{F}^{\text{a}}(\beta, h) \leq \mathbf{F}(0, h + (1 - c_\beta)\beta) \quad \text{with } c_\beta := -\frac{1}{\beta} \log(1 - (1 - e^{-c_0})(1 - e^{-\beta})) > 0. \quad (4.5)$$

In particular  $h_c(\beta) \geq h_c^{\text{a}}(\beta) \geq -(1 - c_\beta)\beta > -\beta$  for all  $\beta > 0$ . Note that the constant  $c_\beta$  is uniformly bounded away from 0 for  $\beta \in (0, 1)$ .

This result gives the second part of Proposition 4.1 (since the condition in Proposition 4.2 is stronger than Assumption 2.2, recall Lemma 2.5) and of Theorem 2.4.

**Proof** First, we use a simple bound  $\omega_n \leq \tilde{\omega}_n := \omega_n \mathbf{1}_{\{\omega_n = +1\}}$ , so that one has  $Z_{N,h}^{\omega, \beta} \leq Z_{N,h}^{\tilde{\omega}, \beta}$ . Then, as  $\tilde{\omega}_n \in \{0, 1\}$ , we can expand  $e^{\beta \sum_{n=1}^N \tilde{\omega}_n \delta_n}$ , thanks to the following binomial expansion

$$e^{\beta \sum_{n=1}^N \tilde{\omega}_n \delta_n} = (e^\beta - 1 + 1)^{\sum_{n=1}^N \tilde{\omega}_n \delta_n} = \sum_{m=0}^N (e^\beta - 1)^m \sum_{1 \leq i_1 < \dots < i_m \leq N} \prod_{k=1}^m \tilde{\omega}_{i_k} \delta_{i_k}. \quad (4.6)$$

Thus, using the condition  $\mathbb{P}(\mathfrak{F}_{\{i_1, \dots, i_m\}}) \leq e^{-c_0 m}$ , one gets

$$\begin{aligned} \mathbb{E} \left[ e^{\beta \sum_{n=1}^N \tilde{\omega}_n \delta_n} \right] &= \sum_{m=0}^N (e^\beta - 1)^m \sum_{1 \leq i_1 < \dots < i_m \leq N} \mathbb{P}(\omega_{i_1} = 1, \dots, \omega_{i_m} = 1) \prod_{k=1}^m \delta_{i_k} \\ &\leq \sum_{m=0}^N (e^\beta - 1)^m e^{-c_0 m} \sum_{1 \leq i_1 < \dots < i_m \leq N} \prod_{k=1}^m \delta_{i_k} = (e^{-c_0}(e^\beta - 1) + 1)^{\sum_{n=1}^N \delta_n}. \end{aligned} \quad (4.7)$$

Defining  $c_\beta := -\frac{1}{\beta} \log(1 - (1 - e^{-c_0})(1 - e^{-\beta}))$  such that  $e^{-c_0}(e^\beta - 1) + 1 = e^{(1 - c_\beta)\beta}$ , one finally obtains that for all  $\beta > 0$

$$\mathbb{E} Z_{N,h}^{\omega, \beta} \leq \mathbf{E} \left[ e^{(h + (1 - c_\beta)\beta) \sum_{n=1}^N \delta_n} \right] = Z_{N, h + (1 - c_\beta)\beta}^{\text{pur}}, \quad (4.8)$$

which gives the results.  $\square$

## 5. BOUNDS ON THE FREE ENERGY, PROOF OF PROPOSITION 2.7

**5.1. Lower Bound on the free energy.** First of all, let us define the *pinned* partition function, where trajectories are constrained to return to 0 at their endpoint  $N$ :  $Z_{N,h}^{\omega, \beta, \text{pin}} := \mathbf{E} \left[ \exp \left( \sum_{n=1}^N (h + \beta \omega_n) \delta_n \right) \mathbf{1}_{\{N \in \tau\}} \right]$ . The main advantage of the *pinned* partition function is that it has a (ergodic) supermultiplicative property:

$$Z_{N+M, h}^{\omega, \beta, \text{pin}} \geq \mathbf{E} \left[ e^{\sum_{n=1}^N (h + \beta \omega_n) \delta_n} \mathbf{1}_{\{N \in \tau\}} \mathbf{1}_{\{N+M \in \tau\}} \right] = Z_{N, h}^{\omega, \beta, \text{pin}} \times Z_{M, h}^{\theta^N \omega, \beta, \text{pin}}, \quad (5.1)$$

where  $\theta$  is the shift operator:  $\theta^p \omega = (\omega_{i+p})_{i \in \mathbb{N}}$ . This translate into a super-additive property for the log of the pinned partition function. We also introduce the notation used in [4] of the partition functions (with free or pinned boundary condition) over a given segment  $[a, b]$ ,  $a, b \in \mathbb{N}$ ,  $a < b$ :

$$Z_{[a,b],h}^{\omega,\beta} := e^{\beta\omega_a+h} Z_{(b-a),h}^{\theta^a \omega,\beta}, \quad \text{and} \quad Z_{[a,b],h}^{\omega,\beta,\text{pin}} := Z_{(b-a),h}^{\theta^a \omega,\beta,\text{pin}}. \quad (5.2)$$

We mention that as  $\omega$  is ergodic (thus stationary),  $Z_{(b-a),h}^{\theta^a \omega,\beta}$  has the same law as  $Z_{(b-a),h}^{\omega,\beta}$ . With notations (5.2), the super-additive property of the log of the pinned partition function extends to partition functions over segments: for any non-negative integers  $a < b < c$ , one has

$$\log Z_{[a,c],h}^{\omega,\beta,\text{pin}} \geq \log Z_{[a,b],h}^{\omega,\beta,\text{pin}} + \log Z_{[b,c],h}^{\omega,\beta,\text{pin}}. \quad (5.3)$$

5.1.1. *General lower bound.* We work with the pinned partition function, and we recall that, as far as the free energy is concerned, this is equivalent to working with the partition function with “free” boundary condition (see [17, Rem. 1.2]). To get a lower bound on the free energy, we use a classical technique, that is to find a strategy of localization for the polymer, aiming only at some particular blocks. We give a very general lower bound, valid for any strategy, where trajectories are ”aiming” at particularly favorable stretches in the environment. We divide the environment in a sequence of blocks  $(\mathbf{T}_j, \mathbf{T}_{j+1}]_{j \geq 0}$  (we take for example  $\mathbf{T}_j$  for among the  $T_i$ ’s). The super-additivity of the logarithm of the partition function (see (5.3)) gives

$$\log Z_{\mathbf{T}_N,h}^{\omega,\beta,\text{pin}} \geq \sum_{j=1}^N \log Z_{[\mathbf{T}_{j-1}, \mathbf{T}_j],h}^{\omega,\beta,\text{pin}}, \quad (5.4)$$

where we used notation (5.2), and that  $\mathbf{T}_0 = 0$

One natural choice is actually to define the sequence  $(\mathbf{T}_j)_{j \geq 0}$  iteratively:  $\mathbf{T}_0 = 0$ ,  $\mathbf{T}_1 = \mathbf{T}_1^{h,\beta}(\omega)$  is the first time when a ”favorable” region ends (for example,  $\mathcal{T}_1(A)$  for some  $A$ ), and then for any  $k \geq 1$ , define  $\mathbf{T}_{k+1} = \mathbf{T}_1(\theta^{\mathbf{T}_k} \omega)$ . Then, the increments  $(\mathbf{T}_k - \mathbf{T}_{k-1})_{k \geq 1}$  form an ergodic sequence [28, Th.I.2.19], and have the same law as  $\mathbf{T}_1$  ( $\mathbf{T}_k$  can be thought as *return times* in terms of the sequences  $(\omega_n)_{n \geq 0}$  or  $(T_n - T_{n-1})_{n \geq 1}$ ).

For instance, one can take  $\mathbf{T}_k = T_{kN_0}$  for some fixed  $N_0$ , or  $\mathcal{T}_k(A)$  for some (deterministic or random) number  $A$ . This iterative definition allows us to choose any initial strategy (finding the first favorable region), and then to repeat this strategy all along the environment. In that case, and provided that  $\mathbb{E}[\mathbf{T}_1] < \infty$ , (5.4) gives

$$\begin{aligned} \mathbb{F}(\beta, h) &= \lim_{N \rightarrow \infty} \frac{1}{\mathbf{T}_N} \log Z_{\mathbf{T}_N,h}^{\omega,\beta,\text{pin}} \\ &\geq \lim_{N \rightarrow \infty} \frac{N}{\mathbf{T}_N} \frac{1}{N} \sum_{j=1}^N \log Z_{[\mathbf{T}_{j-1}, \mathbf{T}_j],h}^{\omega,\beta,\text{pin}} = \frac{1}{\mathbb{E}[\mathbf{T}_1]} \mathbb{E} \left[ \log Z_{\mathbf{T}_1,h}^{\omega,\beta,\text{pin}} \right], \end{aligned} \quad (5.5)$$

where we used twice Birkhoff’s Ergodic Theorem for the last equality.

From now on, we write  $h := -\beta + u$ , since we study the behavior of the free energy as  $h$  goes to  $-\beta$  ( $\beta$  being a fixed constant). Thanks to the bound (5.5), one is only left with picking some  $\mathbf{T}_1$  wisely, and building a strategy of localization on the segment  $[0, \mathbf{T}_1]$ . We take  $\mathbf{T}_1$  the first position where some large attractive region appears, and target directly this region. Namely, one chooses  $\mathbf{T}_1 = \mathcal{T}_1(\mathbf{A})$  for some  $\mathbf{A} := \mathbf{A}(\beta, u, \omega)$  (that has yet to be

chosen), and target directly the last  $\mathbf{A}$ -block, where  $\omega \equiv +1$  (recall notations of Section 2.1). This leads to the lower bound  $Z_{\mathcal{T}_1(\mathbf{A}), -\beta+u}^{\omega, \beta, \text{pin}} \geq K(\mathcal{T}_1(\mathbf{A}) - \mathbf{A}) Z_{\mathbf{A}, u}^{\text{pur}, \text{pin}}$ .

Now, one uses [4, Lem. 2.1] that gives  $Z_{n, u}^{\text{pur}, \text{pin}} \geq Cn^{-1}e^{nF(u)}$  for all  $u \in (0, 1)$  and all  $n \in \mathbb{N}$ . Together with the assumption on  $K(\cdot)$  that yields that there is a constant  $c$  such that  $K(n) \geq cn^{-(1+\alpha)}$ , one finally gets

$$\log Z_{\mathcal{T}_1(\mathbf{A}), -\beta+u}^{\omega, \beta, \text{pin}} \geq -(1+\alpha) \log \mathcal{T}_1(\mathbf{A}) + \mathbf{A}F(u) - \log \mathbf{A} - C. \quad (5.6)$$

Then, using that  $\mathcal{T}_1(\mathbf{A}) \geq \mathbf{A}$  (and  $\mathbf{A}$  is also chosen larger than 2) and recalling (5.5), one ends up with

$$F(\beta, -\beta + u) \geq \frac{1}{\mathbb{E}[\mathcal{T}_1(\mathbf{A})]} (\mathbb{E}[\mathbf{A}]F(u) - \mathbf{C}\mathbb{E}[\log \mathcal{T}_1(\mathbf{A})]), \quad \text{for all } u \in (0, 1), \quad (5.7)$$

with  $\mathbf{C}$  a given constant depending only on the law of the renewal. The choice of the strategy is now reduced to the choice of  $\mathbf{A} = \mathbf{A}(u, \omega)$  (we do not have to optimize on  $\beta$  since none of the constants depend on  $\beta$ ). To get localization, *i.e.*  $F(\beta, h) > 0$ ,  $\mathbf{A}$  must be such that  $\mathbb{E}[\mathcal{T}_1(\mathbf{A})]$  is finite, and also such that  $\mathbb{E}[\mathbf{A}]F(u) > \mathbf{C}\mathbb{E}[\log \mathcal{T}_1(\mathbf{A})]$ .

5.1.2. *Choice of the strategy.* We now take  $\mathbf{A} := A_u$  non-random, chosen in a moment. One gets from (5.7) that

$$\begin{aligned} F(\beta, -\beta + u) &\geq \frac{1}{\mathbb{E}[\mathcal{T}_1(A_u)]} A_u \left( F(u) - \mathbf{C} \frac{1}{A_u} \mathbb{E}[\log \mathcal{T}_1(A_u)] \right) \\ &\geq \text{cst. } A_u \mathbb{P}(\xi_1 \geq A_u) \left( F(u) - \mathbf{C} \left( -\frac{1}{A_u} \log \mathbb{P}(\xi_1 \geq A_u) \right) \right), \end{aligned} \quad (5.8)$$

where we used Jensen's inequality to have that  $\mathbb{E}[\log \mathcal{T}_1(A_u)] \leq \log \mathbb{E}[\mathcal{T}_1(A_u)]$ , equation (2.5) to get  $\mathbb{E}[\mathcal{T}_1(A_u)] = \text{cst.} \mathbb{P}(\xi_1 \geq A_u)^{-1} \geq \mathbb{P}(\xi_1 \geq A_u)^{-1}$ . and the definition (2.7) of  $\varepsilon$ . Then, in (5.8) one chooses  $A_u := \varepsilon^{-1} \left( \frac{F(u)}{2\mathbf{C}} \right)$ , to finally get (2.8).

**Remark 5.1.** We now comment on another natural strategy: one would choose  $\mathcal{T}_1(\mathbf{A})$  the first position where the energetic gain on the  $\mathbf{A}$ -block,  $\mathbf{A}F(u)$ , compensates the entropic cost of targeting it,  $-\mathbf{C} \log \mathcal{T}_1(\mathbf{A})$ . Recall the notation (3.3): for every  $\delta > 0$   $A_\delta^{(\omega)} := \inf \{A, \log \mathcal{T}_1(A) < \delta A\}$ . It is then natural to take  $\mathbf{A} := A_{F(u)/\mathbf{C}}^{(\omega)}$  (which is random), since  $A_{F(u)/\mathbf{C}}^{(\omega)} F(u) - \mathbf{C} \log \mathcal{T}_1(A_{F(u)/\mathbf{C}}^{(\omega)}) > 0$ . One has however to be careful to show that this is also a good localization strategy, because the condition  $\mathbb{E}[\mathcal{T}_1(A_{F(u)/\mathbf{C}}^{(\omega)})] < \infty$  in (5.7) is not guaranteed here.

5.2. **Upper bound on the free energy.** In this Section, we work with the partition function with free boundary condition.

5.2.1. *The coarse-graining argument.* We cut the system into segments  $(\mathbf{T}_{k-1}, \mathbf{T}_k]$ , we estimate the contribution of the different segments separately, and then identify the ones that could actually contribute to the free energy. We give the following coarse-graining lemma, that enables us to do so.

**Lemma 5.2.** *For any increasing sequence of integers  $0 = \mathbf{T}_0 < \mathbf{T}_1 < \mathbf{T}_2 < \dots$ , and every  $N \in \mathbb{N}$ , one has*

$$Z_{\mathbf{T}_N, h}^{\omega, \beta} \leq \prod_{k=1}^N \max_{y \in [0, \mathbf{T}_N]} \left[ \sum_{t=1}^{\mathbf{T}_k - \mathbf{T}_{k-1}} \frac{K(y+t)}{\bar{K}(y)} Z_{[\mathbf{T}_{k-1}+t, \mathbf{T}_k], h}^{\omega, \beta} + \frac{\bar{K}(y + \mathbf{T}_k - \mathbf{T}_{k-1})}{\bar{K}(y)} \right]. \quad (5.9)$$

One also gets the rougher bound

$$Z_{\mathbf{T}_N, h}^{\omega, \beta} \leq \prod_{k=1}^N \left[ \left( \max_{x \in (\mathbf{T}_{k-1}, \mathbf{T}_k]} Z_{[x, T_i], h}^{\omega, \beta} \right) \vee 1 \right]. \quad (5.10)$$

We repeat here that a natural choice, that we make, is to define the sequence  $(\mathbf{T}_k)_{j \geq 0}$  iteratively:  $\mathbf{T}_1 = \mathbf{T}_1^{u, \beta}(\omega)$ , and then for any  $k \geq 1$ , one has  $\mathbf{T}_{k+1} = \mathbf{T}_1(\theta^{\mathbf{T}_k} \omega)$ , so that the sequence  $(\mathbf{T}_{k+1} - \mathbf{T}_k)_{k \geq 0}$  is ergodic. Then, with Lemma 5.2, one gets that

$$\begin{aligned} \mathbb{F}(\beta, h) &= \lim_{N \rightarrow \infty} \frac{1}{\mathbf{T}_N} \log Z_{\mathbf{T}_N, h}^{\omega, \beta} \\ &\leq \lim_{N \rightarrow \infty} \frac{N}{\mathbf{T}_N} \frac{1}{N} \sum_{k=1}^N \log \left( \max_{y \in \mathbb{N}} \left\{ \sum_{t=1}^{\mathbf{T}_k - \mathbf{T}_{k-1}} \frac{K(y+t)}{\bar{K}(y)} Z_{[\mathbf{T}_{k-1}+t, \mathbf{T}_k], h}^{\omega, \beta} + \frac{\bar{K}(y + \mathbf{T}_k - \mathbf{T}_{k-1})}{\bar{K}(y)} \right\} \right) \\ &= \frac{1}{\mathbb{E}[\mathbf{T}_1]} \mathbb{E} \left[ \log \left( \max_{y \in \mathbb{N}} \left\{ \sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\bar{K}(y)} Z_{[t, \mathbf{T}_1], h}^{\omega, \beta} + \frac{\bar{K}(y + \mathbf{T}_1)}{\bar{K}(y)} \right\} \right) \right] \end{aligned} \quad (5.11)$$

where we used twice Birkhoff's Ergodic Theorem for the last equality.

We are therefore reduced to estimate the partition function on the first segment  $(0, \mathbf{T}_1]$ , more precisely all the  $Z_{[t, \mathbf{T}_1], h}^{\omega, \beta}$ . Note that one also gets a rougher bound, using the second part of Lemma 5.2:

$$\mathbb{F}(\beta, h) \leq \frac{1}{\mathbb{E}[\mathbf{T}_1]} \mathbb{E} \left[ \log \left( \left( \max_{t \in [0, \mathbf{T}_1]} Z_{[t, \mathbf{T}_1], h}^{\omega, \beta} \right) \vee 1 \right) \right]. \quad (5.12)$$

**Proof of Lemma 5.2** The proof is similar to what is done in [4, Lem. 5.2], and we sketch it briefly. The proof is done by induction. The case  $N = 1$  is trivial.

Then, one writes

$$\frac{Z_{\mathbf{T}_{N+1}, h}^{\omega, \beta}}{Z_{\mathbf{T}_N, h}^{\omega, \beta}} = \mathbf{E}_{\mathbf{T}_N, h}^{\omega, \beta} \left[ \exp \left( \sum_{j=\mathbf{T}_N+1}^{\mathbf{T}_{N+1}} (\beta\omega + h) \mathbf{1}_{\{j \in \tau\}} \right) \right]. \quad (5.13)$$

One decomposes according to the positions of the last renewal before  $\mathbf{T}_N$  (noted  $\mathbf{T}_N - y$ ) and the first one after  $\mathbf{T}_N$  (noted  $\mathbf{T}_N + t$ ) to get, as in [4, Eq. (4.31)]

$$\begin{aligned} &\mathbf{E}_{\mathbf{T}_N, h}^{\omega, \beta} \left[ \exp \left( \sum_{j=\mathbf{T}_N+1}^{\mathbf{T}_{N+1}} (\beta\omega + h) \mathbf{1}_{\{j \in \tau\}} \right) \right] \\ &\leq \max_{y \in [0, \mathbf{T}_N]} \left[ \sum_{t=1}^{\mathbf{T}_{N+1} - \mathbf{T}_N} \frac{K(y+t)}{\bar{K}(y)} Z_{[\mathbf{T}_N+t, \mathbf{T}_{N+1}], h}^{\omega, \beta} + \frac{\bar{K}(y + \mathbf{T}_{N+1} - \mathbf{T}_N)}{\bar{K}(y)} \right]. \end{aligned} \quad (5.14)$$

The second inequality comes easily from the first one, bounding uniformly  $Z_{[\mathbf{T}_{k-1}+t, \mathbf{T}_k], h}^{\omega, \beta}$  by  $\max_{x \in (\mathbf{T}_{k-1}, \mathbf{T}_k]} Z_{[x, T_i], h}^{\omega, \beta}$ .  $\square$

**5.2.2. Rough coarse-graining: proof of the bound (2.10).** We first use a very rough decomposition of our environment, taking  $\mathbf{T}_k = T_{2k}$  (recall definition (2.2)). Thanks to the coarse-graining lemma 5.2, and in view of (5.11), one is reduced to estimate the partition function on the segment  $(0, T_2]$ .

**Lemma 5.3.** *There exist some  $C, c_1 > 0$  and some constant  $\mathbf{c}_1 > 0$  such that for any  $\beta \in (0, 1)$  and  $u \in (0, c_1\beta)$ ,*

$$\begin{aligned} \max_{x \in (0, T_2]} Z_{[x, T_2], -\beta+u}^{\omega, \beta} &\leq 1 && \text{if } \xi_1 \leq \mathbf{c}_1\beta u^{-1}, \\ \max_{x \in (0, T_2]} Z_{[x, T_2], -\beta+u}^{\omega, \beta} &\leq e^{\xi_1 u} && \text{if } \xi_1 \geq \mathbf{c}_1\beta u^{-1}. \end{aligned} \quad (5.15)$$

Thanks to translation invariance, this tells that the contribution of a block  $(T_{2k}, T_{2k+2}]$  in the decompositions (5.9)-(5.10) is null if  $\xi_{2k+1} = T_{2k+1} - T_{2k} \leq \mathbf{c}_1\beta u^{-1}$ , and (possibly) non-zero otherwise. The bounds (5.15) are equivalent to Lemmas 4.3 (for  $\alpha > 1$ ) and 5.3 (for  $\alpha < 1$ ) in [4], but in the present case, we deal with the cases  $\alpha > 1$  and  $\alpha < 1$  at the same time (with however some loss in the case  $\alpha < 1$ ).

**Proof** On the block  $(0, T_2]$ , one has  $\omega \equiv +1$  for  $i \in (0, T_1]$ , and  $\omega_i = -1$  for  $i \in (T_1, T_2]$ . The second inequality is then trivial, using only that  $\delta_n \leq 1$  and that  $\beta\omega_i - \beta + u \leq 0$  for  $i \in (T_1, T_2]$ , provided  $u \leq 2\beta$ :

$$\sum_{n=x}^{T_2} (\beta\omega_n - \beta + u)\delta_n \leq \sum_{n=1}^{T_1} u = u\xi_1.$$

We therefore focus on the first inequality, in the case  $\alpha \in (0, 1)$  (the case  $\alpha > 1$  being treated in [4, Lem. 4.2], or equivalently with the following technique). If  $x \in (T_1, T_2)$ ,  $\omega_x = -1$ , and then one has for  $u \leq \beta$

$$Z_{[x, T_2], -\beta+u}^{\omega, \beta} = e^{-2\beta+u} \mathbf{E} \left[ \exp \left( (-2\beta + u) \sum_{n=1}^{T_2-x} \delta_n \right) \right] \leq e^{-\beta}. \quad (5.16)$$

If  $x \in (0, T_1]$ , since  $\omega_{T_1+1} = -1$ , one gets for  $u \leq \beta$

$$Z_{[x, T_2], -\beta+u}^{\omega, \beta} \leq e^u \mathbf{E} \left[ \exp \left( u \sum_{n=1}^{T_1+1-x} \delta_n \right) \exp(-\beta \mathbf{1}_{\{T_1+1-x \in \tau\}}) \right]. \quad (5.17)$$

Then, setting  $l = T_1 + 1 - x$ , we notice that

$$\mathbf{E} \left[ e^{u \sum_{n=1}^l \delta_n} e^{-\beta \mathbf{1}_{\{l \in \tau\}}} \right] \leq \mathbf{E} \left[ e^{u \sum_{n=1}^l \delta_n} \right] - c\beta \mathbf{E} \left[ e^{u \sum_{n=1}^l \delta_n} \mathbf{1}_{\{l \in \tau\}} \right] = Z_{l,u}^{\text{pur}} - c\beta Z_{l,u}^{\text{pur, pin}}, \quad (5.18)$$

where we used that  $\beta \mathbf{1}_{\{l \in \tau\}} \leq 1$  to expand  $e^{-\beta \mathbf{1}_{\{l \in \tau\}}}$ . We are therefore left with estimating the *pinned* and *free* partition function. When  $l \leq u^{-1}$ , we have easy bounds:

$$Z_{l,u}^{\text{pur}} = 1 + \sum_{k=1}^{+\infty} \frac{u^k}{k!} \mathbf{E} \left[ \left( \sum_{n=1}^l \delta_n \right)^k \right] \leq 1 + u \mathbf{E} \left[ \sum_{n=1}^l \delta_n \right] \sum_{k=1}^{+\infty} \frac{(ul)^{k-1}}{k!} \leq 1 + c'ul^\alpha, \quad (5.19)$$

and also

$$Z_{l,u}^{\text{pur, pin}} \geq \mathbf{P}(l \in \tau) \geq c''l^{\alpha-1}, \quad (5.20)$$

where we used twice that  $\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \text{cst. } n^{\alpha-1}$ , see [12]. Note that these bounds are actually sharp when  $l$  is smaller than the correlation length (which is  $F(u)^{-1}$ , cf. [18]).

All together, if  $\xi_1 \leq u^{-1}$ , one has that

$$Z_{[x, T_2], -\beta+u}^{\omega, \beta} \leq e^u \left( 1 + c'(\xi_1)^\alpha (u - c\beta(\xi_1)^{-1}) \right). \quad (5.21)$$

If  $\xi_1 \leq u^{-1} \times c\beta/2$ , one finally has

$$\max_{x \in (0, T_2]} Z_{[x, T_2], -\beta+u}^{\omega, \beta} \leq e^u (1 - \text{cst. } \beta(\xi_1)^{\alpha-1}) \leq \exp(u - \text{cst. } \beta(\xi_1)^{\alpha-1}), \quad (5.22)$$

which is smaller than 1 provided that  $u$  is small enough, since  $(\xi_1)^{\alpha-1} \geq cu^{1-\alpha} \gg u$ .  $\square$

In the end, Lemma 5.3 gives

$$\max_{x \in (0, T_2]} Z_{[x, T_2], -\beta+u}^{\omega, \beta} \leq \mathbf{1}_{\{\xi_1 < c_1\beta u^{-1}\}} + e^{u\xi_1} \mathbf{1}_{\{\xi_1 \geq c_1\beta u^{-1}\}}. \quad (5.23)$$

Then, the inequality (5.12) with  $\mathbf{T}_1 = T_2$  gives that

$$\mathbf{F}(\beta, -\beta + u) \leq \frac{1}{\mathbb{E}[T_2]} u \mathbb{E} [\xi_1 \mathbf{1}_{\{\xi_1 \geq c_1\beta u^{-1}\}}]. \quad (5.24)$$

**5.2.3. Refined coarse-graining: proof of bound (2.9).** We focus on the case  $\alpha < 1$  since the previous bound is sufficient for our purposes in the case  $\alpha > 1$ , but the following technique works also in the case  $\alpha > 1$ . The procedure that we apply here is actually the same as the one in [4, Sec. 5.1], but we recall it here (briefly), for the sake of completeness.

In view of the inequality (5.11), we only have to estimate the partition function on the first segment  $[0, \mathbf{T}_1]$ . We take  $\beta \in (0, 1)$ , and we choose

$$\mathbf{T}_1 := T_L, \quad (5.25)$$

$$\text{with } L = L(u) := \lfloor \mathbf{c}\beta^{1/\alpha} \mathbf{F}(u)^{-1} \rfloor,$$

where the constant  $\mathbf{c}$  has yet to be chosen (small), independently on  $\beta$ . We now give here the key lemma, that deals with the estimate of the partition function, and is extracted from [4, Lem. 5.3].

**Lemma 5.4.** *There exist constants  $c_1$  and  $\mathbf{c}$  (entering in the definition of  $L(u)$ ), such that for all  $\beta \in (0, 1)$  and  $u \in [0, c_1\beta)$ , we have the two following cases:*

*If  $\mathbf{T}_1 < 4\mathbb{E}[\mathbf{T}_1]$ , there is a constant  $C$  (independent of  $\beta \in (0, 1)$ ) such that*

$$\max_{y \geq 0} \left[ \sum_{t=1}^{\mathbf{T}_1} \frac{\mathbf{K}(y+t)}{\bar{\mathbf{K}}(y)} Z_{[t, \mathbf{T}_1], -\beta+u}^{\omega, \beta} + \frac{\bar{\mathbf{K}}(y + \mathbf{T}_1)}{\bar{\mathbf{K}}(y)} \right] \leq 1. \quad (5.26)$$

*If  $\mathbf{T}_1 \geq 4\mathbb{E}[\mathbf{T}_1]$ , then*

$$\max_{y \geq 0} \left[ \sum_{t=1}^{\mathbf{T}_1} \frac{\mathbf{K}(y+t)}{\bar{\mathbf{K}}(y)} Z_{[t, \mathbf{T}_1], -\beta+u}^{\omega, \beta} + \frac{\bar{\mathbf{K}}(y + \mathbf{T}_1)}{\bar{\mathbf{K}}(y)} \right] \leq e^{c\beta^{1-\nu^{\text{pur}} \mathbf{F}(u) \mathbf{T}_1}}, \quad (5.27)$$

*for some constant  $c$  (independent of  $\beta \in (0, 1)$ ).*

Inequality (5.12) then yields

$$\mathbf{F}(\beta, -\beta + u) \leq c\beta^{1-\nu^{\text{pur}} \mathbf{F}(u)} \mathbb{E} \left[ \frac{\mathbf{T}_1}{\mathbb{E}[\mathbf{T}_1]} \mathbf{1}_{\{\mathbf{T}_1 \geq 4\mathbb{E}[\mathbf{T}_1]\}} \right]. \quad (5.28)$$

which is exactly (2.9), once one recalled that  $\mathbf{T}_1 = T_{L(u)}$ , so that  $\mathbb{E}[\mathbf{T}_1]$  is of order  $L(u)$ .

**Proof** To prove (5.27), one just uses the trivial bound  $Z_{[t, \mathbf{T}_1], -\beta+u}^{\omega, \beta} \leq Z_{\mathbf{T}_1, u}^{\text{pur}}$ . Then, since  $\mathbf{T}_1 \geq L(u) \geq \mathbf{c}\beta^{1/\alpha} \mathbf{F}(u)^{-1}$ , one uses Lemma B.1 ( $n = \mathbf{T}_1$ ,  $h = u$ , and  $\epsilon = \mathbf{c}\beta^{1/\alpha}$  is small if  $\mathbf{c}$  is small) to get that  $Z_{\mathbf{T}_1, u}^{\text{pur}} \leq e^{c\beta^{(\alpha-1)/\alpha} \mathbf{F}(u) \mathbf{T}_1}$  for all  $u \in [0, c_1\beta)$  with suitable constants  $c$  and  $c_1$ .

We now focus on the proof of (5.26). We only give the steps of the proof for the sake of completeness, since the proof is identical to the one done in [4, Sec. 5.1]. Loosely



speaking, it relies on the same idea as the previous Section: if the block  $\mathbf{T}_1$  is smaller than  $4\mathbb{E}[\mathbf{T}_1]$ , it means that it contains  $-1$ 's that are close one to another, and the entropic cost of avoiding them is too high compared to the energetic gain one has on this segment.

We recall Lemma 4.5 in [4], that estimates the probability of avoiding a large region in the environment, whatever its shape is.

**Lemma 5.5.** *There exists some constant  $c > 0$  such that for any  $M > 0$ ,  $\varphi > 0$ , if one takes  $\mathcal{S}$  a subset of  $[1, M]$  of cardinality at least  $\varphi M$ , one has*

$$\mathbf{P}(\tau \cap \mathcal{S} \neq \emptyset) \geq c\varphi^{1+\alpha}. \quad (5.29)$$

From this Lemma, one is able to estimate the contribution of the different terms  $Z_{[x, \mathbf{T}_1], -\beta+u}^{\omega, \beta}$ , in the case  $\mathbf{T}_1 \leq 4\mathbb{E}[\mathbf{T}_1] \leq 4\mathbb{E}[T_2]L(u)$ .

If  $x > L/2$ , we use the trivial bound  $Z_{[x, \mathbf{T}_1], -\beta+u}^{\omega, \beta} \leq Z_{4\mathbb{E}[\mathbf{T}_1], u}^{\text{pur}}$ . Since  $n = 4\mathbb{E}[\mathbf{T}_1] \leq cL(u)$ , with  $L(u) \leq c\beta^{1/\alpha}\mathbf{F}(u)^{-1}$ , then Lemma B.1 gives that there is a constant  $c_1$  such that for all  $u \in [0, c_1\beta)$  one has  $Z_{4\mathbb{E}[\mathbf{T}_1], u}^{\text{pur}} \leq 1 + c\beta$ , where  $c = c\mathbf{c}^\alpha$  for some constant  $c$  ( $\mathbf{c}$  being the constant entering in the definition of  $L(u)$ ). In the end one gets

$$\max_{x \in (L/2, \mathbf{T}_1]} Z_{[x, \mathbf{T}_1], -\beta+u}^{\omega, \beta} \leq 1 + c\beta. \quad (5.30)$$

If  $x \leq L/2 \leq T_{L/2}$ , as  $\mathbf{T}_1 = T_L$ , there are at least  $L/4$   $-1$ 's in the segment  $[x, \mathbf{T}_1]$ , so that the proportion of  $-1$ 's is at least  $\frac{L}{4\mathbf{T}_1} \geq (16\mathbb{E}[T_2])^{-1}$  (since  $\mathbf{T}_1 \leq 4\mathbb{E}[T_2]L$ ). Take  $\varphi = (16\mathbb{E}[T_2])^{-1}$ , and  $\mathcal{S} := \{n \in [1, \mathbf{T}_1 - x] / \omega_{x+n} = -1\}$ , so that applying Lemma 5.5, one has

$$\begin{aligned} e^{-h} Z_{[x, \mathbf{T}_1], -\beta+u}^{\omega, \beta} &\leq \mathbf{E} \left[ e^{\sum_{n=1}^{\mathbf{T}_1-x} u \mathbf{1}_{n \in \tau}} \mathbf{1}_{\{\tau \cap \mathcal{S} = \emptyset\}} \right] + e^{-2\beta} \mathbf{E} \left[ e^{\sum_{n=1}^{\mathbf{T}_1-x} u \mathbf{1}_{n \in \tau}} \mathbf{1}_{\{\tau \cap \mathcal{S} \neq \emptyset\}} \right] \\ &\leq Z_{\mathbf{T}_1-x, u}^{\text{pur}} - (1 - e^{-2\beta}) \mathbf{P}(\tau \cap \mathcal{S} \neq \emptyset) \leq 1 + c\beta - c'\varphi^{1+\alpha}(1 - e^{-\beta}). \end{aligned} \quad (5.31)$$

If the constant  $c = c\mathbf{c}^\alpha$  is small enough (taking  $\mathbf{c}$  in the definition  $L(u)$  small), one has  $c\beta \leq \frac{c'}{2}\varphi^{1+\alpha}(1 - e^{-\beta})$  for all  $\beta \in (0, 1)$ . In the end one obtains, for all  $\beta \in (0, 1)$

$$\max_{x \in [0, L/2]} Z_{[x, \mathbf{T}_1], -\beta+u}^{\omega, \beta} \leq 1 - \frac{c'}{2}\varphi^{1+\alpha}(1 - e^{-\beta}) \leq 1 - 2C\beta, \quad (5.32)$$

where the constant  $C$  is independent of how small  $\mathbf{c}$  is.

Thanks to (5.30)-(5.32) (which are actually Lemma 4.4 in [4]), we have

$$\begin{aligned} \sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\bar{K}(y)} Z_{[t, \mathbf{T}_1], -\beta+u}^{\omega, \beta} &\leq (1 - 2C\beta) \sum_{t=1}^{L/2} \frac{K(y+t)}{\bar{K}(y)} + (1 + c\beta) \sum_{t=L/2}^{\mathbf{T}_1} \frac{K(y+t)}{\bar{K}(y)} \\ &\leq \sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\bar{K}(y)} - 2C\beta \sum_{t=1}^{L/2} \frac{K(y+t)}{\bar{K}(y)} + c\beta \sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\bar{K}(y)}. \end{aligned} \quad (5.33)$$

Then, using that  $\mathbf{T}_1 \leq 4\mathbb{E}[T_2]L$

$$\frac{\sum_{t=1}^{L/2} K(y+t)}{\sum_{t=1}^{\mathbf{T}_1} K(y+t)} \geq \frac{\sum_{t=1}^{L/2} K(y+t)}{\sum_{t=L/2}^{4\mathbb{E}[T_2]L} K(y+t)}$$

is bounded away from 0, uniformly in  $y$  and  $L$ . Therefore, if  $c$  is small enough (recall that the value of  $C$  is independent from that of  $c$ ), we have

$$\sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\overline{K}(y)} Z_{[t, \mathbf{T}_1], -\beta+u}^{\omega, \beta} \leq \sum_{t=1}^{\mathbf{T}_1} \frac{K(y+t)}{\overline{K}(y)}, \quad (5.34)$$

which gives (5.26).  $\square$

**5.3. On the way to getting a sharper upper bound.** In both of the previous cases, the coarse-graining procedure only singled out some favorable stretches, but did not keep track of the cost from avoiding the repulsive ones. Therefore, the bounds (2.9)-(5.12) on the free energy are really rough, and yet sufficient to our purpose. We present now a procedure to get better bounds on the free energy, but since the bounds obtained are hard to be estimated (but may be computed in some particular cases), we only give a sketch of the method. The idea is to use a refined multiscale coarse graining, dividing the system into larger and larger meta-blocks. We give a sketch of the proof for the first step of improvement, which follows the idea of Section 3.2 and 4.2 in [4], and we explain how one could iterate the procedure. We then give a heuristic justification of Conjecture 2.8.

**First step of improvement** We enlarge the scale of the coarse-graining procedure of Section 5.2.3: define  $\mathbf{T}_k^{(1)} := \mathcal{T}_k(A_1)$  where  $A_1 = A_1(u) := c_1 \beta u^{-1}$  is the threshold appearing in Lemma 5.3. One then divides the environment into meta-blocks  $(\mathbf{T}_k^{(1)}, \mathbf{T}_{k+1}^{(1)})$ . For each meta-block, there is a certain (random) number of segments  $(T_i, T_{i+1}]$  smaller than  $A_1$ , that, in view of Lemma 5.3, are repulsive (in the sense that the gain on such a segment is smaller than 1), and then there is one "favorable" ending  $A_1$ -block.

Then, one only has to estimate the partition function on the first meta-block, more precisely all the  $Z_{[x, \mathbf{T}_1^{(1)}], -\beta+u}^{\omega, \beta}$  (remember (5.11)). We now call upon Lemma 4.6 in [4]: it uses a coarse-graining argument to keep track that trajectories "avoid" repulsive regions, and jumps directly to the last  $A_1$ -block, the attractive one. In particular, denoting  $k_1 = k_1(u)$  the index of the first  $A_1$ -block, so that  $\mathbf{T}_1^{(1)} = T_{k_1}$ , one obtains that there exists a constant  $C_1 > 0$  such that

$$Z_{\mathbf{T}_1^{(1)}, -\beta+u}^{\omega, \beta} \leq e^{u \xi_{k_1}} T_{k_1}^{-C_1}. \quad (5.35)$$

Then, one has  $Z_{\mathbf{T}_1^{(1)}, -\beta+u}^{\omega, \beta} > 1$  only if  $u \xi_{k_1} \geq c_1 \log(1 + T_{k_1-1})$ , or more easily  $u \xi_{k_1} \geq c_1 \log k_1$ . One would therefore have

$$F(\beta, -\beta + u) \leq cu \mathbb{P}(\xi_1 \geq A_1(u)) \mathbb{E} \left[ \xi_{k_1} \mathbf{1}_{\{\xi_{k_1} \geq c_1 u^{-1} \log k_1\}} \right] \quad (5.36)$$

where we used that  $\mathbb{E}[\mathbf{T}_1^{(1)}] = \mathbb{E}[\mathcal{T}_1(A_1)] = c' \mathbb{P}(\xi_1 \geq A_1(u))^{-1}$ .

**Remark 5.6.** This presumably gives a sharper upper bound to the free energy than (2.9) (or (3.7)) since the condition  $\{\xi_{k_1} \geq c_1 u^{-1} \log k_1\}$  is restrictive. The difficulty is actually to estimate the influence of this condition in the case of a very general environment. If the sizes of the blocks were independent (which is the case in [4]), then the integer  $k_1$  is a geometric random variable of parameter  $\mathbb{P}(\xi_1 \geq A_1)$ , and one is able to estimate precisely the upper bound (5.36). This gives for example [4, Th. 2.1].

**Remark 5.7.** The method sketched here is valid for any  $\alpha > 0$ : it relies on Lemma 5.3 to identify repulsive and attractive blocks  $(T_i, T_{i+1}]$ . However, in the case  $\alpha < 1$ , one can identify repulsive and attractive segments  $(\mathbf{T}_i, \mathbf{T}_{i+1}]$ , thanks to Lemma 5.4 (with  $\mathbf{T}_i$  defined in (5.25)), and then obtain a refined bound.

**Iterating the procedure.** The above reasoning tells, among the segments  $(\mathbf{T}_k^{(1)}, \mathbf{T}_{k+1}^{(1)})$ , which are the favorable ones ( $\xi_{k_1} \geq c_1 u^{-1} \log k_1$ , as far as the first block is concerned) and which are the unfavorable ones ( $\xi_{k_1} < c_1 u^{-1} \log k_1$ ). One is then able to repeat the procedure presented above at larger scales:

- construct  $(\mathbf{T}_k^{(2)}, \mathbf{T}_{k+1}^{(2)})$ , composed of (many) blocks  $(\mathbf{T}_j^{(1)}, \mathbf{T}_{j+1}^{(1)})$  identified as *globally repulsive*, and then one *globally attractive* ending block;
- identify when the blocks  $(\mathbf{T}_k^{(2)}, \mathbf{T}_{k+1}^{(2)})$  are favorable or unfavorable (justifying that the main contribution comes from trajectories that jump over the first repulsive blocks  $(\mathbf{T}_j^{(1)}, \mathbf{T}_{j+1}^{(1)})$ , and aim directly at the last, favorable one, to obtain an analogue of (5.35));
- repeat the argument to enlarge the scale...

In the end, one obtains sharper and sharper bounds on the free energy, the difficulty being to give simple estimates of the bounds.

**Heuristic justification of the Conjecture 2.8.** One realizes that, after  $j$  steps of improvements, the main contribution in the first block  $(0, \mathbf{T}_1^{(j)})$  comes from trajectories aiming directly at its last *A-block*. Then, after  $j$  steps, if  $k_1^{(j)}$  is such that  $\mathbf{T}_1^{(j)} = T_{k_1^{(j)}}$ , the first block is considered attractive if  $u\xi_{k_1^{(j)}} \geq c \log T_{k_1^{(j)}}$ . This consideration meets the definition (3.3):  $A_{cu}^\omega$  is the first length  $\xi_k$  such that  $u\xi_k \geq c \log T_k$ , and the segment  $(0, \mathcal{T}_1(A_{cu}^\omega))$  is the first one to be favorable (with main contribution coming from trajectories aiming directly at the last *A-block*).

If  $A_{cu}^\omega < +\infty$ , there is some step  $j < +\infty$  such that the first block  $(0, \mathbf{T}_1^{(j)})$  is attractive: this justifies that the localization strategy sketched in Section 2.3 should be the right one. On the other hand, if  $A_{cu_0}^\omega = +\infty$ , then the first segment  $(0, \mathbf{T}_1^{(j)})$  remains unfavorable after any step  $j$  (with  $\mathbf{T}_1^{(j)}$  going to infinity as  $j$  goes to infinity). It would confirm that, in absence of infinite disorder (recall its different characterizations, in particular in terms of  $A_\delta^\omega$ , see Lemma 3.1), there should be some  $u_0$  sufficiently small such that  $\liminf_{n \rightarrow \infty} Z_{n, -\beta + u_0}^{\omega, \beta} \leq 1$ , so that  $F(\beta, -\beta + u_0) = 0$  and  $h_c(\beta) \geq -\beta + u_0$ .

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## APPENDIX A. ON THE SIGN OF GAUSSIAN SEQUENCES

Let  $W = \{W_n\}_{n \in \mathbb{N}}$  be a stationary Gaussian process, centered and with unitary variance, and with covariance matrix denoted by  $\Upsilon$ , the covariance function being  $\rho_k = \Upsilon_{i, i+k}$ . We also denote  $\Upsilon_l$  the covariance matrix of the vector  $(W_1, \dots, W_l)$ , which is just a restriction of  $\Upsilon$ . We recall Assumption (2.11): correlations are non-negative, and power-law decaying,

$$\rho_k \stackrel{k \rightarrow \infty}{\sim} ck^{-a}, \quad \text{for some } a > 0 \text{ and } c > 0. \quad (\text{A.1})$$

### **Proof** Proof of Proposition 3.2

We recall here a more general lower bound, dealing with the probability for a Gaussian vector to be componentwise larger than some given value.

**Lemma A.1** ([3], Lemma A.3). *Under assumption (2.11) with  $a < 1$ , there exist two constants  $c, C > 0$  such that for every  $l \in \mathbb{N}$ , one has*

$$\mathbb{P}(\forall i \in \{1, \dots, l\}, W_i \geq A) \geq c^{-1} \exp\left(-c(A \vee C\sqrt{\log l})^2 l^a\right). \quad (\text{A.2})$$

Taking  $A = 0$  in this Lemma gives (3.8).

To simplify notations, we prove the upper bound for the specific sequence  $i_k = k$ ,  $k \in \{1, \dots, n\}$ . The general proof follows the same reasoning. One first observes that for any subset  $\{k_1, \dots, k_m\} \subset \{1, \dots, n\}$ ,  $m \in \mathbb{N}$ , one has

$$\mathbb{P}(W_i \geq 0; \forall i \in \{1, \dots, n\}) \leq \mathbb{P}(W_{k_j} \geq 0; \forall j \in \{1, \dots, m\}) \quad (\text{A.3})$$

The idea is that, if the  $k_j$ 's are sufficiently far one from another, the Gaussian vector  $(W_{k_1}, \dots, W_{k_m})$  behaves like an independent one.

**Claim A.2.** • *If  $a < 1$ , then there exists some  $A > 0$  such that taking  $k_j := j \lfloor An^{1-a} \rfloor$  for  $j \in \{0, \dots, m := \lceil A^{-1}n^a \rceil\}$ , one has some constant  $c > 0$  such that for all  $n \in \mathbb{N}$*

$$\mathbb{P}(W_{k_j} \geq 0; \forall j \in \{1, \dots, m\}) \leq e^{-cm}. \quad (\text{A.4})$$

• *If  $a > 1$ , then there exists some integer  $A > 0$  such that taking  $k_j := jA$  for  $j \in \{0, \dots, m := \lceil A^{-1}n \rceil\}$ , one has some constant  $c > 0$  such that for all  $n \in \mathbb{N}$*

$$\mathbb{P}(W_{k_j} \geq 0; \forall j \in \{1, \dots, m\}) \leq e^{-cm}. \quad (\text{A.5})$$

This claim, together with (A.3), gives the conclusion. We now prove the claim.

Under  $\mathbb{P}$ , the vector  $(W_{k_1}, \dots, W_{k_m})$  is a Gaussian vector with covariance matrix  $\tilde{\Upsilon}_m$ , with  $\tilde{\Upsilon}_{ij} = \Upsilon_{k_i, k_j} = \rho_{|k_j - k_i|}$  for  $i, j \in \{1, \dots, m\}$ . We note  $\tilde{\mathbb{P}}$  the law of this  $m$ -dimensional vector. Then if  $\hat{\mathbb{P}}$  denotes the law of a  $m$ -dimensional independent standard Gaussian vector  $\mathcal{N}(0, \text{Id})$ , a change of measure procedure gives thanks to the Cauchy-Schwarz inequality

$$\tilde{\mathbb{P}}(W_j \geq 0; \forall j \in \{1, \dots, m\}) \leq \left(\frac{1}{2}\right)^{m/2} \hat{\mathbb{E}} \left[ \left( \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}} \right)^2 \right]^{1/2}. \quad (\text{A.6})$$

One has  $\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}(X) = (\det \tilde{\Upsilon}_m)^{-1/2} e^{-\frac{1}{2} \langle (\tilde{\Upsilon}_m^{-1} - I)X, X \rangle}$  from the definitions of  $\tilde{\mathbb{P}}$  and  $\hat{\mathbb{P}}$ , so that from a Gaussian computation, one gets

$$\hat{\mathbb{E}} \left[ \left( \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}} \right)^2 \right]^{1/2} = (\det \tilde{\Upsilon}_m)^{-1/2} (\det(2(\tilde{\Upsilon}_m)^{-1} - I))^{-1/4} = \det(I - V^2)^{-1/4} \quad (\text{A.7})$$

where we defined  $V := \tilde{\Upsilon}_m - I$ .

We now estimate  $\det(I - V^2)$ . Note that the maximal eigenvalue  $\tilde{\lambda}$  of  $\tilde{\Upsilon}_m$  verifies

$$\tilde{\lambda} \leq \max_{i \in \{1, \dots, n\}} \sum_{j=1}^n \tilde{\Upsilon}_{ij} \leq 1 + 2 \sum_{p=1}^m \rho_{k_p}. \quad (\text{A.8})$$

Then we use the definition of  $k_p$  and  $m$ , and the assumption (2.11) on  $(\rho_k)_{k \geq 0}$ . We get:

- if  $a < 1$  one has  $\tilde{\lambda} \leq 1 + cA^{-a}n^{-a(1-a)}m^{1-a} \leq 1 + cA^{-1}$ ,
- if  $a > 1$  one has  $\tilde{\lambda} \leq 1 + cA^{-a}$ .

In both cases one chooses  $A$  large enough so that  $\tilde{\lambda} \leq 3/2$ . Thus the eigenvalues of  $I - V^2$  are bounded from below by  $1 - (\tilde{\lambda} - 1)^2 \geq 3/4$ , so that in the end one has  $\det(I - V^2) \geq (3/4)^m$ . Combining (A.7) and (A.6) one gets

$$\bar{\mathbb{P}}(W_j \geq 0; \forall j \in \{1, \dots, m\}) \leq \left(\frac{1}{2}\right)^{m/2} \left(\frac{3}{4}\right)^{-m/4} \leq 3^{-m/4}. \quad (\text{A.9})$$

□

## APPENDIX B. ESTIMATE ON THE HOMOGENEOUS MODEL

The following Lemma tells that  $n \approx \mathbf{F}(h)^{-1}$  is the threshold at which the partition function starts growing exponentially.

**Lemma B.1.** *We take  $\alpha \neq 1$ . There exist  $\epsilon > 0$ , and constants  $c, c', c'' > 0$  not depending on  $\epsilon$ , such that if  $0 < h \leq c''\epsilon^{\alpha \wedge 1} \leq 1$ , then*

- *one has  $Z_{n,h}^{\text{pur}} \leq 1 + c(n\mathbf{F}(h))^{\alpha \wedge 1} \leq 1 + c\epsilon^{\alpha \wedge 1}$  for every  $n \leq \epsilon\mathbf{F}(h)^{-1}$ ;*
- *one has  $Z_{n,h}^{\text{pur}} \leq \exp(c'\epsilon^{\alpha \wedge 1 - 1}\mathbf{F}(h)n)$  for every  $n \geq \epsilon\mathbf{F}(h)^{-1}$ .*

**Proof** Because the case  $\alpha > 1$  is easier (essentially since  $\mathbf{F}(h)$  is proportional to  $h$ ), we restrict to the case  $\alpha < 1$ .

Using that  $\mathbf{P}(|\tau \cap [0, n]| \geq k) \leq (1 - \bar{\mathbf{K}}(n))^k$ , one writes

$$Z_{n,h}^{\text{pur}} = 1 + \sum_{k=1}^n (e^{kh} - e^{(k-1)h}) \mathbf{P}(|\tau \cap [0, n]| \geq k) \leq 1 + \sum_{k=1}^{n_0} h \left( e^h (1 - \bar{\mathbf{K}}(n)) \right)^k. \quad (\text{B.1})$$

Then, there is some constant  $cst. > 0$  such that  $\bar{\mathbf{K}}(n) \geq cst.n^{-\alpha}$ . Since  $h \in [0, 1]$ , we have that  $e^h(1 - \bar{\mathbf{K}}(n)) \leq 1 + 2h - cst.n^{-\alpha}$ . Since  $\mathbf{F}(h)$  is of order  $h^{1/\alpha}$ , we get that, uniformly for  $n \leq \epsilon\mathbf{F}(h)^{-1}$ , one has  $n^{-\alpha} \geq \epsilon^{-\alpha}h$ . Therefore, if  $\epsilon$  is small enough,  $e^h(1 - \bar{\mathbf{K}}(n)) \leq 1 - \frac{1}{2}cst.n^{-\alpha} < 1$ . From (B.1) one then gets

$$Z_{n_0,h}^{\text{pur}} \leq 1 + h \sum_{k=1}^n \left( 1 - \frac{1}{2}cst.n^{-\alpha} \right)^k \leq 1 + \frac{h}{\frac{1}{2}cst.n^{-\alpha}}, \quad (\text{B.2})$$

which gives  $Z_{n,h}^{\text{pur}} \leq 1 + c(n\mathbf{F}(h))^{\alpha \wedge 1}$ , since  $\mathbf{F}(h)$  is of order  $h^{1/\alpha}$ .

For the second point, one uses that for any two integers  $n_1, n_2$ , decomposing over the first return time after  $n_1$ , one gets that  $e^h Z_{n_1+n_2,h}^{\text{pur}} \leq e^h Z_{n_1,h}^{\text{pur}} e^h Z_{n_2,h}^{\text{pur}}$ . From this, one gets that for any  $p \in \mathbb{N}$

$$e^h Z_{pn_0,h}^{\text{pur}} \leq \left( e^h Z_{n_0,h}^{\text{pur}} \right)^p \leq e^{p \frac{c'}{2} \epsilon^\alpha} = e^{\frac{c'}{2} \epsilon^{\alpha-1} p n_0 \mathbf{F}(h)} \quad (\text{B.3})$$

where we used the previous bound on  $Z_{n_0,h}^{\text{pur}}$ , the fact that  $h \leq c''\epsilon^\alpha$ , and the definition of  $n_0 = \epsilon\mathbf{F}(h)^{-1}$ . The general bound for  $n \geq n_0$  relies on the monotonicity of  $Z_{n,h}^{\text{pur}}$  in  $n$ . □

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