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# Existence of guided waves due to a lineic perturbation of a 3D periodic medium

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## Abstract

In this note, we exhibit a three dimensional structure that permits to guide waves. This structure is obtained by a geometrical perturbation of a 3D periodic domain that consists of a three dimensional grating of equi-spaced thin pipes oriented along three orthogonal directions. Homogeneous Neumann boundary conditions are imposed on the boundary of the domain. The diameter of the section of the pipes, of order  $\varepsilon > 0$ , is supposed to be small. We prove that, for  $\varepsilon$  small enough, shrinking the section of one line of the grating by a factor of  $\sqrt{\mu}$  ( $0 < \mu < 1$ ) creates guided modes that propagate along the perturbed line. Our result relies on the asymptotic analysis (with respect to  $\varepsilon$ ) of the spectrum of the Laplace-Neumann operator in this structure. Indeed, as  $\varepsilon$  tends to 0, the domain tends to a periodic graph, and the spectrum of the associated limit operator can be computed explicitly.

**Keywords** : guided waves, periodic media, spectral theory.

**AMS codes** : 78M35, 35J05, 58C40

## 1 Statement of the problem

Let  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  be three Lipschitz bounded domains of  $\mathbb{R}^2$  of same area ( $|\omega_1| = |\omega_2| = |\omega_3|$ ) containing the origin  $(0,0)$ , let  $\varepsilon > 0$  be a parameter (that is going to be small), and let  $a_1$ ,  $a_2$  and  $a_3$  be three positive real numbers. We denote by  $(\mathbf{e}_i)_{i \in \{1,2,3\}}$ , the standard basis of  $\mathbb{R}^3$ . For any  $(k, \ell) \in \mathbb{Z}^2$ , we consider the three dimensional domain  $D_{k,\ell,3}^\varepsilon$  defined by

$$D_{k,\ell,3}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k)/\varepsilon, (x_2 - a_2 \ell)/\varepsilon) \in \omega_3\},$$

which is an unbounded cylinder of constant cross section  $\varepsilon\omega_3$ . It is infinite along the  $\mathbf{e}_3$  direction (invariant with respect to  $x_3$ ) and contains the point  $(a_1 k, a_2 \ell, 0)$ . Similarly, for any  $(k, \ell) \in \mathbb{Z}^2$ , we define the domains

$$D_{k,\ell,1}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_2 - a_2 k)/\varepsilon, (x_3 - a_3 \ell)/\varepsilon) \in \omega_1\},$$

$$D_{k,\ell,2}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k)/\varepsilon, (x_3 - a_3 \ell)/\varepsilon) \in \omega_2\},$$

and we consider the periodic domain  $\Omega_\varepsilon$  given by

$$\Omega_\varepsilon = \bigcup_{i \in \{1,2,3\}} \bigcup_{(k,\ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon. \quad (1)$$

The domain  $\Omega_\varepsilon$  is a three dimensional grating of equi-spaced parallel pipes (of constant cross section) oriented along the three orthogonal directions  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$ . It is  $a_j$ -periodic with respect to  $x_j$ ,  $j = 1, 2, 3$ . Moreover, the points  $(ka_1, \ell a_2, ma_3)$ ,  $(k, \ell, m) \in \mathbb{Z}^3$ , belong to  $\Omega_\varepsilon$ .

In order to create guided modes, we introduce a linear defect (see [1]-[5]-[2]) in the periodic structure by modifying the section size of one pipe of the grating (it is conjectured that guided modes cannot appear in the purely periodic structure, see [4] for the proof in the case of a symmetric medium). More precisely, we assume that the domain  $D_{0,0,3}^\varepsilon$  is replaced with the domain

$$D_{0,0,3}^{\varepsilon,\mu} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } (x_1/(\sqrt{\mu}\varepsilon), x_2/(\sqrt{\mu}\varepsilon)) \in \omega_3\},$$

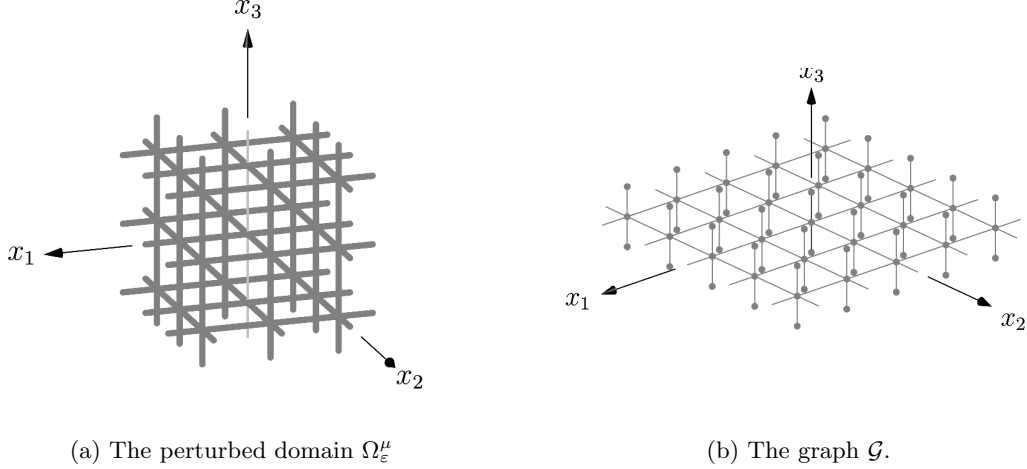


Figure 1: Illustration of the perturbed periodic domain  $\Omega_\varepsilon^\mu$  and the limit graph  $\mathcal{G}$ .

where  $\mu$  is a positive parameter. In other words, we enlarge ( $\mu > 1$ ) or shrink ( $0 < \mu < 1$ ) the section of one pipe of the domain by a factor  $\mu$  (see Fig 1a). The corresponding perturbed domain is denoted by  $\Omega_\varepsilon^\mu$ . Its precise definition is given by

$$\Omega_\varepsilon^\mu = \left( \bigcup_{i \in \{1,2\}} \bigcup_{(k,\ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon \right) \cup \left( \bigcup_{(k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} D_{k,\ell,3}^\varepsilon \right) \cup D_{0,0,3}^{\varepsilon,\mu}. \quad (2)$$

$\Omega_\varepsilon^\mu$  is still  $a_3$ -periodic with respect to  $x_3$ . However, the presence of the perturbed pipe  $D_{0,0,3}^{\varepsilon,\mu}$  breaks the periodicity with respect to  $x_1$  and  $x_2$ . We emphasize that the domain  $\Omega_\varepsilon^\mu$  (as well as  $\Omega_\varepsilon$ ) tends to a 3D periodic graph as  $\varepsilon$  tends to 0.

We look for guided modes, i.e. solutions to the wave equation  $\partial_t^2 u - \Delta u = 0$  in  $\Omega_\varepsilon^\mu$ , satisfying homogeneous Neumann boundary conditions on  $\partial\Omega_\varepsilon^\mu$  (see [7] for the investigation of the Dirichlet case), that propagate along the defect pipe  $D_{0,0,3}^{\varepsilon,\mu}$  (i.e. in the  $\mathbf{e}_3$  direction) but stay confined in the transversal directions. More precisely, denoting by  $B_\varepsilon^\mu$  the restriction of the domain  $\Omega_\varepsilon^\mu$  to the band  $|x_3| < a_3/2$ ,

$$B_\varepsilon^\mu = \{(x_1, x_2, x_3) \in \Omega_\varepsilon^\mu \text{ such that } |x_3| < a_3/2\}, \quad (3)$$

we search solutions of the form  $u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3)e^{i\omega t - \beta x_3}$ , where  $\beta$  is a real parameter and  $v(x_1, x_2, x_3) \in L_2(B_\varepsilon^\mu)$  is an  $a_3$ -periodic function in  $x_3$ . In fact, it is easily seen that the  $\beta$ -quasi-periodic function  $v(x_1, x_2, x_3)e^{-i\beta x_3}$  is an eigenfunction of the operator

$$A_\varepsilon^\mu(\beta) : D(A_\varepsilon^\mu(\beta)) \subset L_2(B_\varepsilon^\mu) \rightarrow L_2(B_\varepsilon^\mu), \quad A_\varepsilon^\mu(\beta) = -\Delta u \text{ in } B_\varepsilon^\mu, \quad (4)$$

with  $D(A_\varepsilon^\mu(\beta)) = \{u \in H_\Delta^1(B_\varepsilon^\mu), u|_{\Sigma^+} = e^{-i\beta} u|_{\Sigma^-}, \partial_{x_3} u|_{\Sigma^+} = e^{-i\beta} \partial_{x_3} u|_{\Sigma^-}, \partial_n u|_{\partial B_\varepsilon^\mu \setminus \Sigma^\pm} = 0\}$ , where

$$H_\Delta^1(B_\varepsilon^\mu) = \{u \in H^1(B_\varepsilon^\mu), \text{s.t. } \Delta u \in L_2(B_\varepsilon^\mu)\} \text{ and } \Sigma^\pm = \{(x_1, x_2, x_3) \in \partial B_\varepsilon^\mu, x_3 = \pm a_3/2\}.$$

To study the spectral properties of  $A_\varepsilon^\mu(\beta)$ , we investigate its (formal) limit  $\mathcal{A}^\mu(\beta)$  as  $\varepsilon$  tends to 0. The operator  $\mathcal{A}^\mu(\beta)$  is defined on the limit graph  $\mathcal{G}$  (see Fig. 1b) and its spectrum can be explicitly computed. In particular, its spectrum has infinitely many gaps (Lemma 2.1), i.e. open intervals  $(a, b) \subset \mathbb{R}$  such that the intersection of  $[a, b]$  with the spectrum is reduced to  $\{a, b\}$ . Moreover, for  $\mu < 1$ , there is at least one eigenvalue in each gap (Lemma 2.5). Since, in addition, for  $\varepsilon > 0$  sufficiently small, the spectrum of  $A_\varepsilon^\mu(\beta)$  is close to the spectrum of  $\mathcal{A}^\mu(\beta)$ , the existence of guided modes is guaranteed (Theorem 3.1).

## 2 The spectrum of the limit operator $\mathcal{A}^\mu(\beta)$

### 2.1 Definition of the limit operator $\mathcal{A}^\mu(\beta)$

The limit operator  $\mathcal{A}^\mu(\beta)$  is defined on the infinite periodic graph  $\mathcal{G} = \bigcap_{\varepsilon>0} B_\varepsilon^\mu$  obtained as the limit of  $B_\varepsilon^\mu$  as  $\varepsilon$  tends to 0:  $\mathcal{G}$  is made of the vertices  $\{v_{k,\ell} = (ka_1, \ell a_2, 0), v_{k,\ell}^\pm = (ka_1, \ell a_2, \pm a_3/2), (k, \ell) \in \mathbb{Z}^2\}$  connected by the edges

$$\{e_{k+1/2,\ell} = (v_{k,\ell}, v_{k+1,\ell}), e_{k,\ell+1/2} = (v_{k,\ell}, v_{k,\ell+1}), e_{k,\ell}^\pm = (v_{k,\ell}, v_{k,\ell}^\pm), (k, \ell) \in \mathbb{Z}^2\}.$$

It is  $a_1$ -periodic with respect to  $x_1$  and  $a_2$ -periodic with respect to  $x_2$  (see Fig. 1b).

For any function  $u$  defined on  $\mathcal{G}$ , we denote by  $\mathbf{u}_{k,\ell}$  (resp.  $\mathbf{u}_{k,\ell}^\pm$ ) its value at the vertex  $v_{k,\ell}$  (resp.  $v_{k,\ell}^\pm$ ). The restriction of  $u$  to the edge  $e_{k+1/2,\ell}$  (resp.  $e_{k,\ell+1/2}$  and  $e_{k,\ell}^\pm$ ) is denoted by  $u_{k+1/2,\ell}(x_1)$  (resp.  $u_{k,\ell+1/2}(x_2)$  and  $u_{k,\ell}^\pm(x_3)$ ).

The definition of  $\mathcal{A}^\mu(\beta)$  also requires the introduction of the function spaces  $L_2^\mu(\mathcal{G})$  and  $H^2(\mathcal{G})$  defined as

$$L_2^\mu(\mathcal{G}) = \left\{ u : \|u\|_{L_2^\mu(\mathcal{G})} < +\infty \right\}, \quad H^2(\mathcal{G}) = \left\{ u \in C(\mathcal{G}) : \|u\|_{H^2(\mathcal{G})} < +\infty \right\}, \quad (5)$$

where,

$$\|u\|_{L_2^\mu(\mathcal{G})}^2 = \sum_{(k,\ell) \in \mathbb{Z}^2} \left( w_{k,\ell}^\mu \sum_{\pm} \|u_{k,\ell}^\pm\|_{L_2(e_{k,\ell}^\pm)}^2 + \|u_{k+\frac{1}{2},\ell}\|_{L_2(e_{k+\frac{1}{2},\ell})}^2 + \|u_{k,\ell+\frac{1}{2}}\|_{L_2(e_{k,\ell+\frac{1}{2}})}^2 \right), \quad (6)$$

$$\|u\|_{H^2(\mathcal{G})}^2 = \sum_{(k,\ell) \in \mathbb{Z}^2} \left( \sum_{\pm} \|u_{k,\ell}^\pm\|_{H^2(e_{k,\ell}^\pm)}^2 + \|u_{k+\frac{1}{2},\ell}\|_{H^2(e_{k+\frac{1}{2},\ell})}^2 + \|u_{k,\ell+\frac{1}{2}}\|_{H^2(e_{k,\ell+\frac{1}{2}})}^2 \right), \quad (7)$$

and, for any  $(k, \ell) \in \mathbb{Z}^2$ ,  $w_{k,\ell}^\mu$  is the weight coefficient equal to  $\mu$  for  $k = \ell = 0$  and 1 otherwise.

The unbounded limit operator in  $L_2^\mu(\mathcal{G})$  has domain

$$D(\mathcal{A}^\mu(\beta)) = \left\{ u \in H^2(\mathcal{G}) : \forall (k, \ell) \in \mathbb{Z}^2, \quad \mathbf{u}_{k,\ell}^+ = e^{-i\beta} \mathbf{u}_{k,\ell}^-, \quad (u_{k,\ell}^+)'(a_3/2) = e^{-i\beta} (u_{k,\ell}^-)'(-a_3/2), \right. \\ \left. u'_{k+\frac{1}{2},\ell}(ka_1) - u'_{k-\frac{1}{2},\ell}(ka_1) + u'_{k,\ell+\frac{1}{2}}(\ell a_2) - u'_{k,\ell-\frac{1}{2}}(\ell a_2) + w_{k,\ell}^\mu \left( (u_{k,\ell}^+)'(0) - (u_{k,\ell}^-)'(0) \right) = 0 \right\}, \quad (8)$$

and is defined by

$$\forall u \in D(\mathcal{A}^\mu(\beta)), \quad \mathcal{A}^\mu(\beta)u = -u'' \quad \text{on any edge of the graph } \mathcal{G}. \quad (9)$$

The functions of  $D(\mathcal{A}^\mu(\beta))$  are continuous on  $\mathcal{G}$  and  $\beta$  quasi-periodic. Moreover, they satisfy the Kirchhoff conditions (8) that enforce the weighted sum of the outward derivatives of  $u$  to vanish at each vertex  $v_{k,\ell}$  ( $(k, \ell) \in \mathbb{Z}^2$ ). We point out that the perturbation, which results from a geometrical modification of the domain for the problem (4), is taken into account at the limit by means of the Kirchhoff condition (8) at the vertex  $v_{0,0}$  ( $w_{0,0}^\mu = \mu$ ). The formal derivation of the limit model can be found in [6]. It is easily verified that the operator  $\mathcal{A}^\mu(\beta)$  is self-adjoint (for the weighted scalar product associated with (6)), see also [3]. The objective of the following two sections is to study the spectrum of  $\mathcal{A}^\mu(\beta)$ .

### 2.2 Characterization and properties of the essential spectrum of $\mathcal{A}^\mu(\beta)$

By a compact perturbation argument, one can prove that  $\sigma_{ess}(\mathcal{A}^\mu(\beta)) = \sigma(\mathcal{A}(\beta))$ , where  $\mathcal{A}(\beta) = \mathcal{A}^1(\beta)$  is the purely periodic operator corresponding to  $\mathcal{A}^\mu(\beta)$  for  $\mu = 1$ . The computation of its spectrum relies on the Floquet-Bloch theory (see [9]). More precisely, we can prove that  $\lambda = \omega^2 \in \sigma(\mathcal{A}(\beta))$  if and only if either  $\omega = 0$  and  $\beta = 0$  or  $\omega \neq 0$  and there exists  $(k_1, k_2) \in [0, \pi]^2$  such that

$$\sin(\omega a_2) \sin(\omega a_3) (\cos(\omega a_1) - \cos k_1) + \sin(\omega a_3) \sin(\omega a_1) (\cos(\omega a_2) - \cos k_2) \\ + \sin(\omega a_1) \sin(\omega a_2) (\cos(\omega a_3) - \cos \beta) = 0. \quad (10)$$

Based on the previous characterization, we prove that the operator  $\mathcal{A}(\beta)$  has a countable infinity of gaps that can be separated into three categories (see [10] for the proof):

**Lemma 2.1** *The following properties hold :*

1-  $\sigma_1 \cup \sigma_2 \cup \sigma_3 \subset \sigma(\mathcal{A}(\beta))$ , where

$$\sigma_i = \left\{ (\pi n/a_i)^2, n \in \mathbb{Z} \right\} \text{ for } i \in \{1, 2\}, \text{ and } \sigma_3 = \left\{ ((\pm\beta + 2\pi n)/a_3)^2, n \in \mathbb{Z} \right\}.$$

2- For any  $\beta \in [0, \pi]$ , the operator  $\mathcal{A}(\beta)$  has infinitely many gaps whose ends tend to infinity.

3- Let  $\mathcal{W}(\beta) = \{\pi n/a_3, n \in \mathbb{N}^*\}$  if  $\beta \notin \{0, \pi\}$  and  $\mathcal{W}(\beta) = \{\beta/a_3 + (2n+1)\pi/a_3, n \in \mathbb{N}^*\}$  if  $\beta \in \{0, \pi\}$ ,

If an interval  $(\omega_b^2, \omega_t^2)$  is a spectral gap of  $\mathcal{A}(\beta)$ , then, one of the following possibilities holds:

- (i)  $\omega_b^2 \notin \sigma_1 \cup \sigma_2$ ,  $\omega_t^2 \notin \sigma_1 \cup \sigma_2$ , and there is a unique  $\omega_0 \in (\omega_b, \omega_t) \cap \mathcal{W}(\beta)$ .
- (ii)  $\omega_b^2 \in \sigma_1 \cup \sigma_2$ ,  $\omega_t^2 \notin \sigma_1 \cup \sigma_2$  and  $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \emptyset$ .
- (iii)  $\omega_b^2 \notin \sigma_1 \cup \sigma_2$ ,  $\omega_t^2 \in \sigma_1 \cup \sigma_2$  and  $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \emptyset$ .

### 2.3 Computation of the discrete spectrum

Let us now determine the discrete spectrum of  $\mathcal{A}^\mu(\beta)$ . If  $\lambda = \omega^2$  is an eigenvalue of  $\mathcal{A}^\mu(\beta)$ , then the corresponding eigenfunction  $u \in D(\mathcal{A}^\mu(\beta))$  satisfies the linear differential equation  $u'' + \omega^2 u = 0$  on each edge of the graph  $\mathcal{G}$ . Solving explicitly this equation (on each edge), taking into account the quasi-periodicity of  $u$  and the Kirchhoff conditions (8), we can replace the eigenvalue problem  $\mathcal{A}^\mu(\beta)u = \lambda u$  with a set of finite differences equations for  $(\mathbf{u}_{\ell,k})_{(\ell,k) \in \mathbb{Z}^2}$ :

**Lemma 2.2** *Assume that  $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$ .  $u \in D(\mathcal{A}^\mu(\beta))$  is an eigenfunction of  $\mathcal{A}^\mu(\beta)$  if and only if  $(\mathbf{u}_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$  belongs to  $\ell_2(\mathbb{Z}^2)$  and satisfies*

$$\begin{cases} \frac{\mathbf{u}_{k+1,\ell}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{k-1,\ell}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{k,\ell+1}}{\sin(\omega a_2)} + \frac{\mathbf{u}_{k,\ell-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{k,\ell} = 0, & \forall (k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ \frac{\mathbf{u}_{1,0}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{-1,0}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{0,1}}{\sin(\omega a_2)} + \frac{\mathbf{u}_{0,-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{0,0} = 2(\mu-1)\frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}\mathbf{u}_{0,0}, \end{cases} \quad (11)$$

where we have defined  $g_\beta(\omega) = \frac{1}{\tan(\omega a_1)} + \frac{1}{\tan(\omega a_2)} + \frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$ .

As well-known, finite difference schemes may be investigated using the discrete Fourier transform

$$\mathcal{F} : \mathbf{v} = (\mathbf{v}_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2} \mapsto \mathcal{F}(\mathbf{v}) = \widehat{\mathbf{v}}, \quad \widehat{\mathbf{v}}(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}} e^{i(k\xi + \ell\eta)} \mathbf{v}_{k,\ell}, \quad (\xi, \eta) \in [0, 2\pi]^2, \quad (12)$$

where  $\mathcal{F}$  is an isometry between  $\ell_2(\mathbb{Z}^2)$  and  $L_2((0, 2\pi)^2)$ . This, together with Lemma 2.2, provides the following characterization for the discrete spectrum of  $\mathcal{A}^\mu(\beta)$ :

**Lemma 2.3** *Assume that  $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$ .  $u \in D(\mathcal{A}^\mu(\beta))$  is an eigenfunction of  $\mathcal{A}^\mu(\beta)$  if and only if the discrete Fourier transform  $\widehat{\mathbf{u}}$  of  $(\mathbf{u}_{\ell,k})_{(\ell,k) \in \mathbb{Z}^2}$  belongs to  $L_2((0, 2\pi)^2)$  and satisfies*

$$(f(\xi, \eta, \omega) - \phi_\beta(\omega)) \widehat{\mathbf{u}}(\xi, \eta) = (\mu-1)\phi_\beta(\omega)\mathbf{u}_{0,0}. \quad (13)$$

where  $\phi_\beta(\omega) = \frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$  and  $f(\xi, \eta, \omega) = \frac{\cos\xi - \cos(\omega a_1)}{\sin(\omega a_1)} + \frac{\cos\eta - \cos(\omega a_2)}{\sin(\omega a_2)}$ .

Under the assumption of Lemma 2.3, (10) can be written as  $f(\xi, \eta, \omega) - \phi_\beta(\omega) = 0$ . It follows that  $\lambda = \omega^2$  does not belong to  $\sigma_{ess}(\mathcal{A}^\mu(\beta))$  if and only if, for any  $(\xi, \eta) \in [0, 2\pi]^2$ ,  $\phi_\beta(\omega) - f(\xi, \eta, \omega)$  does not vanish. As a consequence, as soon as  $\lambda = \omega^2 \notin \sigma_{ess}(\mathcal{A}^\mu(\beta))$ , the function  $(\xi, \eta) \mapsto \phi_\beta(\omega)/(\phi_\beta(\omega) - f(\xi, \eta, \omega))$  is continuous and bounded. Then, the inverse discrete Fourier transform can be applied to (13) to obtain

$$\mathbf{u}_{k,\ell} = \frac{(1-\mu)\mathbf{u}_{0,0}}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} e^{-i(k\xi + \ell\eta)} d\xi d\eta, \quad \forall (k, \ell) \in \mathbb{Z}^2.$$

Writing the previous relation for  $k = \ell = 0$  yields the following criterion of existence of an eigenvalue:

**Lemma 2.4** Assume that  $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$  and that  $\lambda = \omega^2 \notin \sigma_{\text{ess}}(\mathcal{A}^\mu(\beta))$ . Then,  $\lambda$  is an eigenvalue of  $\mathcal{A}^\mu(\beta)$  if and only if

$$\mu = 1 - F_\beta(\omega) \quad \text{where} \quad F_\beta(\omega) = \left( \frac{1}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} d\xi d\eta \right)^{-1}. \quad (14)$$

The study of the behavior of the function  $F_\beta$  leads to the existence of at least one eigenvalue in each gap of  $\mathcal{A}^\mu(\beta)$  as soon as  $\mu < 1$ , the minimal number of eigenvalues in each gap depending on the type of gaps (cf. Lemma. 2.1-3 for the classification):

**Lemma 2.5** For  $\mu > 1$ , the operator  $\mathcal{A}^\mu(\beta)$  has no eigenvalue. For  $0 < \mu < 1$ , let  $(\omega_b^2, \omega_t^2)$  be a spectral gap of the operator  $\mathcal{A}^\mu(\beta)$ :

- (a) If  $(\omega_b^2, \omega_t^2)$  is a gap of type (i), then  $\mathcal{A}^\mu(\beta)$  has at least two eigenvalues  $\lambda_1 = \omega_1^2$  and  $\lambda_2 = \omega_2^2$  that satisfy  $\omega_b < \omega_1 < \omega_0 < \omega_2 < \omega_t$  (see Lemma. 2.1-3 for the definition of  $\omega_0$ ).
- (b) If  $(\omega_b^2, \omega_t^2)$  is a gap of type (ii) or (iii), then  $\mathcal{A}^\mu(\beta)$  has at least one eigenvalue  $\lambda_1 = \omega_1^2$  such that  $\omega_b < \omega_1 < \omega_t$ .

The sketch of the proof of the previous lemma is the following, a complete proof being available in [10] (Theorem 5.2.1): First, one can verify that  $F_\beta(\omega) \geq 0$  in any gap, which, together with (14) proves that  $\mathcal{A}^\mu(\beta)$  has no eigenvalue for  $\mu > 1$ . Then, if  $(\omega_b^2, \omega_t^2)$  is a gap of type (i), one can show that

$$\lim_{\omega \rightarrow \omega_b^+} (1 - F_\beta(\omega)) = \lim_{\omega \rightarrow \omega_t^-} (1 - F_\beta(\omega)) = 1 \quad \text{and that} \quad \lim_{\omega \rightarrow \omega_0} (1 - F_\beta(\omega)) = 0.$$

By continuity of  $F_\beta$  inside the gap, (a) directly results from the intermediate value theorem and (14). If  $(\omega_b^2, \omega_t^2)$  is a gap of type (ii), the intermediate value theorem also permits us to conclude since

$$\lim_{\omega \rightarrow \omega_b} (1 - F_\beta(\omega)) \leq 0 \quad \text{and} \quad \lim_{\omega \rightarrow \omega_t^-} (1 - F_\beta(\omega)) = 1.$$

A similar argument works for a gap of type (iii).

### 3 Guided modes for the operator $A_\varepsilon^\mu(\beta)$ : an asymptotic result

Finally, thanks to the general result [8] (Theorem 2.13 convergence of the spectrum of  $A_\varepsilon^\mu(\beta)$  toward the spectrum of  $\mathcal{A}^\mu(\beta)$ ), we can prove the following result of existence of eigenvalue for the operator  $A_\varepsilon^\mu(\beta)$ :

**Theorem 3.1** Let  $\mu \in (0, 1)$ ,  $(\lambda_b, \lambda_t)$  be a spectral gap of the operator  $\mathcal{A}^\mu(\beta)$  and  $\lambda_0 \in (\lambda_b, \lambda_t)$  be a (simple) eigenvalue of this operator. Then, there exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$  the operator  $A_\varepsilon^\mu(\beta)$  has an eigenvalue  $\lambda_\varepsilon$  inside a spectral gap  $(\lambda_b^\varepsilon, \lambda_t^\varepsilon)$ . Moreover,  $\lambda_\varepsilon = \lambda_0 + O(\sqrt{\varepsilon})$ .

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