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Abstract

In this note, we exhibit a three dimensional structure that permits to guide waves. This structure is obtained by a geometrical perturbation of a 3D periodic domain that consists of a three dimensional grating of equi-spaced thin pipes oriented along three orthogonal directions. Homogeneous Neumann boundary conditions are imposed on the boundary of the domain. The diameter of the section of the pipes, of order $\varepsilon > 0$, is supposed to be small. We prove that, for ε small enough, shrinking the section of one line of the grating by a factor of $\sqrt{\mu}$ ($0 < \mu < 1$) creates guided modes that propagate along the perturbed line. Our result relies on the asymptotic analysis (with respect to ε) of the spectrum of the Laplace-Neumann operator in this structure. Indeed, as ε tends to 0, the domain tends to a periodic graph, and the spectrum of the associated limit operator can be computed explicitly.

Keywords : guided waves, periodic media, spectral theory.

AMS codes : 78M35, 35J05, 58C40

1 Statement of the problem

Let ω_1, ω_2 and ω_3 be three Lipschitz bounded domains of $\mathbb{R}^2$ of same area ($|\omega_1| = |\omega_2| = |\omega_3|$) containing the origin $(0, 0)$, let $\varepsilon > 0$ be a parameter (that is going to be small), and let a_1, a_2 and a_3 be three positive real numbers. We denote by $(\mathbf{e}_i)_{i\in\{1,2,3\}}$, the standard basis of $\mathbb{R}^3$. For any $(k, \ell) \in \mathbb{Z}^2$, we consider the three dimensional domain $D^\varepsilon_{k,\ell,3}$ defined by

$$D^\varepsilon_{k,\ell,3} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } \big((x_1 - a_1k)/\varepsilon, (x_2 - a_2\ell)/\varepsilon\big) \in \omega_3\right\},$$

which is an unbounded cylinder of constant cross section $\varepsilon\omega_3$. It is infinite along the $\mathbf{e}_3$ direction (invariant with respect to x_3) and contains the point $(a_1k, a_2\ell, 0)$. Similarly, for any $(k, \ell) \in \mathbb{Z}^2$, we define the domains

$$\begin{aligned} D^\varepsilon_{k,\ell,1} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } \big((x_2 - a_2k)/\varepsilon, (x_3 - a_3\ell)/\varepsilon\big) \in \omega_1\right\}, \\ D^\varepsilon_{k,\ell,2} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } \big((x_1 - a_1k)/\varepsilon, (x_3 - a_3\ell)/\varepsilon\big) \in \omega_2\right\}, \end{aligned}$$

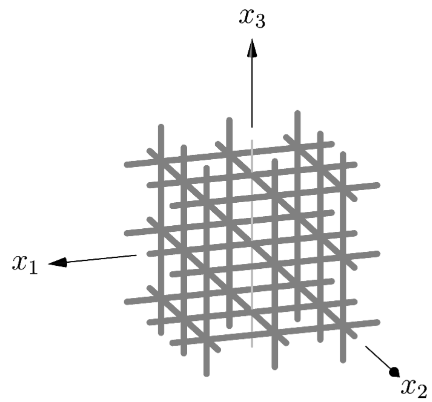
and we consider the periodic domain Ω_ε given by

$$\Omega_\varepsilon = \bigcup_{i\in\{1,2,3\}} \bigcup_{(k,\ell)\in\mathbb{Z}^2} D^\varepsilon_{k,\ell,i}. \tag{1}$$

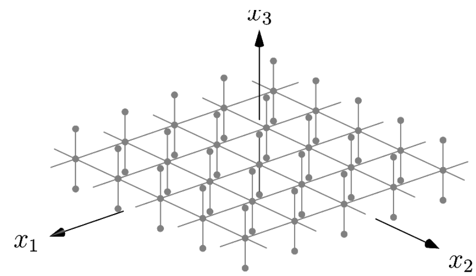
The domain Ω_ε is a three dimensional grating of equi-spaced parallel pipes (of constant cross section) oriented along the three orthogonal directions $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. It is a_j-periodic with respect to x_j, $j = 1, 2, 3$. Moreover, the points $(ka_1, \ell a_2, ma_3)$, $(k, \ell, m) \in \mathbb{Z}^3$, belong to Ω_ε.

In order to create guided modes, we introduce a linear defect (see [1]-[5]-[2]) in the periodic structure by modifying the section size of one pipe of the grating (it is conjectured that guided modes cannot appear in the purely periodic structure, see [4] for the proof in the case of a symmetric medium). More precisely, we assume that the domain $D^\varepsilon_{0,0,3}$ is replaced with the domain

$$D^{\varepsilon,\mu}_{0,0,3} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } \big(x_1/(\sqrt{\mu}\varepsilon), x_2/(\sqrt{\mu}\varepsilon)\big) \in \omega_3\right\},$$



(a) The perturbed domain Ω_ε^μ



(b) The graph $\mathcal{G}$.

Figure 1: Illustration of the perturbed periodic domain Ω_ε^μ and the limit graph $\mathcal{G}$.

where μ is a positive parameter. In other words, we enlarge ($\mu > 1$) or shrink ($0 < \mu < 1$) the section of one pipe of the domain by a factor μ (see Fig 1a). The corresponding perturbed domain is denoted by Ω_ε^μ. Its precise definition is given by

$$\Omega_\varepsilon^\mu = \left(\bigcup_{i\in\{1,2\}} \bigcup_{(k,\ell)\in\mathbb{Z}^2} D_{k,\ell,i}^\varepsilon \right) \bigcup \left(\bigcup_{(k,\ell)\in\mathbb{Z}^2\setminus\{(0,0)\}} D_{k,\ell,3}^\varepsilon \right) \bigcup D_{0,0,3}^{\varepsilon,\mu}. \tag{2}$$

Ω_ε^μ is still a_3-periodic with respect to x_3. However, the presence of the perturbed pipe $D_{0,0,0}^{\varepsilon,\mu}$ breaks the periodicity with respect to x_1 and x_2. We emphasize that the domain Ω_ε^μ (as well as Ω_ε) tends to a 3D periodic graph as ε tends to 0.

We look for guided modes, i.e. solutions to the wave equation $\partial_t^2 u - \Delta u = 0$ in Ω_ε^μ, satisfying homogeneous Neumann boundary conditions on $\partial\Omega_\varepsilon^\mu$ (see [7] for the investigation of the Dirichlet case), that propagate along the defect pipe $D_{0,0,0}^{\varepsilon,\mu}$ (i.e. in the $\mathbf{e}_3$ direction) but stay confined in the transversal directions. More precisely, denoting by B_ε^μ the restriction of the domain Ω_ε^μ to the band $|x_3| < a_3/2$,

$$B_\varepsilon^\mu = \{(x_1, x_2, x_3) \in \Omega_\varepsilon^\mu \text{ such that } |x_3| < a_3/2\}, \tag{3}$$

we search solutions of the form $u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3)e^{i\omega t - \beta x_3}$, where β is a real parameter and $v(x_1, x_2, x_3) \in L_2(B_\varepsilon^\mu)$ is an a_3-periodic function in x_3. In fact, it is easily seen that the β-quasiperiodic fonction $v(x_1, x_2, x_3)e^{-i\beta x_3}$ is an eigenfunction of the operator

$$A_\varepsilon^\mu(\beta) : D\left(A_\varepsilon^\mu(\beta)\right) \subset L_2\left(B_\varepsilon^\mu\right) \to L_2\left(B_\varepsilon^\mu\right), \quad A_\varepsilon^\mu(\beta) = -\Delta u \text{ in } B_\varepsilon^\mu, \tag{4}$$

with $D\left(A_\varepsilon^\mu(\beta)\right) = \left\{u \in H_\Delta^1\left(B_\varepsilon^\mu\right), u|_{\Sigma^+} = e^{-i\beta}\, u|_{\Sigma^-}, \partial_{x_3}u|_{\Sigma^+} = e^{-i\beta}\, \partial_{x_3}u|_{\Sigma^-},\ \partial_n u|_{\partial B_\varepsilon^\mu \setminus \Sigma^\pm} = 0\right\}$, where

$$H_\Delta^1(B_\varepsilon^\mu) = \left\{u \in H^1(B_\varepsilon^\mu), \text{s.t. } \Delta u \in L_2(B_\varepsilon^\mu)\right\} \text{ and } \Sigma^\pm = \{(x_1, x_2, x_3) \in \partial B_\varepsilon^\mu, x_3 = \pm a_3/2\}.$$

To study the spectral properties of $A_\varepsilon^\mu(\beta)$, we investigate its (formal) limit $\mathcal{A}^\mu(\beta)$ as ε tends to 0. The operator $\mathcal{A}^\mu(\beta)$ is defined on the limit graph $\mathcal{G}$ (see Fig. 1b) and its spectrum can be explicitly computed. In particular, its spectrum has infinitely many gaps (Lemma 2.1), i.e. open intervals $(a, b) \subset \mathbb{R}$ such that the intersection of $[a, b]$ with the spectrum is reduced to $\{a, b\}$. Moreover, for $\mu < 1$, there is at least one eigenvalue in each gap (Lemma 2.5). Since, in addition, for $\varepsilon > 0$ sufficiently small, the spectrum of $A_\varepsilon^\mu(\beta)$ is close to the spectrum of $\mathcal{A}^\mu(\beta)$, the existence of guided modes is guaranteed (Theorem 3.1).

2 The spectrum of the limit operator $\mathcal{A}^\mu(\beta)$

2.1 Definition of the limit operator $\mathcal{A}^\mu(\beta)$

The limit operator $\mathcal{A}^\mu(\beta)$ is defined on the infinite periodic graph $\mathcal{G} = \bigcap_{\varepsilon>0} B_\varepsilon^\mu$ obtained as the limit of B_ε^μ as ε tends to 0: $\mathcal{G}$ is made of the vertices $\{v_{k,\ell} = (ka_1, \ell a_2, 0), v_{k,\ell}^\pm = (ka_1, \ell a_2, \pm a_3/2), (k,\ell) \in \mathbb{Z}^2\}$ connected by the edges

$$\{e_{k+1/2,\ell} = (v_{k,\ell}, v_{k+1,\ell}),\ \ e_{k,\ell+1/2} = (v_{k,\ell}, v_{k,\ell+1}),\ \ e_{k,\ell}^\pm = (v_{k,\ell}, v_{k,\ell}^\pm),\ \ (k,\ell) \in \mathbb{Z}^2\}.$$

It is a_1-periodic with respect to x_1 and a_2-periodic with respect to x_2 (see Fig. 1b).

For any function u defined on $\mathcal{G}$, we denote by $\mathbf{u}_{k,\ell}$ (resp. $\mathbf{u}_{k,\ell}^\pm$) its value at the vertex $v_{k,\ell}$ (resp. $v_{k,\ell}^\pm$). The restriction of u to the edge $e_{k+1/2,\ell}$ (resp. $e_{k,\ell+1/2}$ and $e_{k,\ell}^\pm$) is denoted by $u_{k+1/2,\ell}(x_1)$ (resp. $u_{k,\ell+1/2}(x_2)$ and $u_{k,\ell}^\pm(x_3)$).

The definition of $\mathcal{A}^\mu(\beta)$ also requires the introduction of the function spaces $L_2^\mu(\mathcal{G})$ and $H^2(\mathcal{G})$ defined as

$$L_2^\mu(\mathcal{G}) = \Big\{u : \|u\|_{L_2^\mu(\mathcal{G})} < +\infty\Big\}, \quad H^2(\mathcal{G}) = \big\{u \in C(\mathcal{G}) : \|u\|_{H^2(\mathcal{G})} < +\infty\big\}, \tag{5}$$

where,

$$\|u\|^2_{L_2^\mu(\mathcal{G})} = \sum_{(k,\ell)\in\mathbb{Z}^2} \Big(w_{k\ell}^\mu \sum_{\pm} \|u_{k,\ell}^\pm\|^2_{L_2(e_{k,\ell}^\pm)} + \|u_{k+\frac12,\ell}\|^2_{L_2(e_{k+\frac12,\ell})} + \|u_{k,\ell+\frac12}\|^2_{L_2(e_{k,\ell+\frac12})} \Big), \tag{6}$$

$$\|u\|^2_{H^2(\mathcal{G})} = \sum_{(k,\ell)\in\mathbb{Z}^2} \Big(\sum_{\pm} \|u_{k,\ell}^\pm\|^2_{H^2(e_{k,\ell}^\pm)} + \|u_{k+\frac12,\ell}\|^2_{H^2(e_{k+\frac12,\ell})} + \|u_{k,\ell+\frac12}\|^2_{H^2(e_{k,\ell+\frac12})} \Big), \tag{7}$$

and, for any $(k,\ell) \in \mathbb{Z}^2$, $w_{k,\ell}^\mu$ is the weight coefficient equal to μ for $k = \ell = 0$ and 1 otherwise.

The unbounded limit operator in $L_2^\mu(\mathcal{G})$ has domain

$$\begin{aligned} D(\mathcal{A}^\mu(\beta)) = \Big\{ u \in H^2(\mathcal{G}) :\quad \forall (k,\ell)\in\mathbb{Z}^2, \quad \mathbf{u}_{k,\ell}^+ = e^{-i\beta}\mathbf{u}_{k,\ell}^-, \quad (u_{k,\ell}^+)'(a_3/2) = e^{-i\beta}(u_{k,\ell}^-)'(-a_3/2), \\ u'_{k+\frac12,\ell}(ka_1) - u'_{k-\frac12,\ell}(ka_1) + u'_{k,\ell+\frac12}(\ell a_2) - u'_{k,\ell-\frac12}(\ell a_2) + w_{k,\ell}^\mu \Big((u_{k,\ell}^+)'(0) - (u_{k,\ell}^-)'(0) \Big) = 0 \Big\}, \end{aligned} \tag{8}$$

and is defined by

$$\forall\, u \in D(\mathcal{A}^\mu(\beta)), \quad \mathcal{A}^\mu(\beta)u = -u'' \quad \text{on any edge of the graph}\, \mathcal{G}. \tag{9}$$

The functions of $D(\mathcal{A}^\mu(\beta))$ are continuous on $\mathcal{G}$ and β quasi-periodic. Moreover, they satisfy the Kirchhoff conditions (8) that enforce the weighted sum of the outward derivatives of u to vanish at each vertex $v_{k,\ell}$ ($(k,\ell) \in \mathbb{Z}^2$). We point out that the perturbation, which results from a geometrical modification of the domain for the problem (4), is taken into account at the limit by means of the Kirchhoff condition (8) at the vertex $v_{0,0}$ ($w_{0,0}^\mu = \mu$). The formal derivation of the limit model can be found in [6]. It is easily verified that the operator $\mathcal{A}^\mu(\beta)$ is self-adjoint (for the weighted scalar product associated with (6)), see also [3]. The objective of the following two sections is to study the spectrum of $\mathcal{A}^\mu(\beta)$.

2.2 Characterization and properties of the essential spectrum of $\mathcal{A}^\mu(\beta)$

By a compact perturbation argument, one can prove that $\sigma_{ess}(\mathcal{A}^\mu(\beta)) = \sigma(\mathcal{A}(\beta))$, where $\mathcal{A}(\beta) = \mathcal{A}^1(\beta)$ is the purely periodic operator corresponding to $\mathcal{A}^\mu(\beta)$ for $\mu = 1$. The computation of its spectrum relies on the Floquet-Bloch theory (see [9]). More precisely, we can prove that $\lambda = \omega^2 \in \sigma(\mathcal{A}(\beta))$ if and only if either $\omega = 0$ and $\beta = 0$ or $\omega \neq 0$ and there exists $(k_1, k_2) \in [0,\pi]^2$ such that

$$\begin{aligned} \sin(\omega a_2)\sin(\omega a_3)(\cos(\omega a_1) - \cos k_1) + \sin(\omega a_3)\sin(\omega a_1)(\cos(\omega a_2) - \cos k_2) \\ + \sin(\omega a_1)\sin(\omega a_2)(\cos(\omega a_3) - \cos\beta) = 0. \end{aligned} \tag{10}$$

Based on the previous characterization, we prove that the operator $\mathcal{A}(\beta)$ has a countable infinity of gaps that can be separated into three categories (see [10] for the proof):

Lemma 2.1 *The following properties hold :*

1- $\sigma_1 \cup \sigma_2 \cup \sigma_3 \subset \sigma(\mathcal{A}(\beta))$, *where*

$$\sigma_i = \left\{ (\pi n/a_i)^2 , n \in \mathbb{Z} \right\} \text{ for } i \in \{1,2\}, \text{ and } \sigma_3 = \left\{ ((\pm\beta + 2\pi n)/a_3)^2 , n \in \mathbb{Z} \right\}.$$

2- *For any* $\beta \in [0,\pi]$, *the operator* $\mathcal{A}(\beta)$ *has infinitely many gaps whose ends tend to infinity.*

3- *Let* $\mathcal{W}(\beta) = \{\pi n/a_3,\ n \in \mathbb{N}^*\}$ *if* $\beta \notin \{0,\pi\}$ *and* $\mathcal{W}(\beta) = \{\beta/a_3 + (2n+1)\pi/a_3,\ n \in \mathbb{N}^*\}$ *if* $\beta \in \{0,\pi\}$,.
If an interval (ω_b^2, ω_t^2) *is a spectral gap of* $\mathcal{A}(\beta)$, *then, one of the following possibilities holds:*

(i) $\omega_b^2 \notin \sigma_1 \cup \sigma_2$, $\omega_t^2 \notin \sigma_1 \cup \sigma_2$, *and there is a unique* $\omega_0 \in (\omega_b, \omega_t) \cap \mathcal{W}(\beta)$.
(ii) $\omega_b^2 \in \sigma_1 \cup \sigma_2$, $\omega_t^2 \notin \sigma_1 \cup \sigma_2$ *and* $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \varnothing$.
(iii) $\omega_b^2 \notin \sigma_1 \cup \sigma_2$, $\omega_t^2 \in \sigma_1 \cup \sigma_2$ *and* $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \varnothing$.

2.3 Computation of the discrete spectrum

Let us now determine the discrete spectrum of $\mathcal{A}^\mu(\beta)$. If $\lambda = \omega^2$ is an eigenvalue of $\mathcal{A}^\mu(\beta)$, then the corresponding eigenfunction $u \in D(\mathcal{A}^\mu(\beta))$ satisfies the linear differential equation $u'' + \omega^2 u = 0$ on each edge of the graph $\mathcal{G}$. Solving explicitly this equation (on each edge), taking into account the quasi-periodicity of u and the Kirchhoff conditions (8), we can replace the eigenvalue problem $\mathcal{A}^\mu(\beta)u = \lambda u$ with a set of finite differences equations for $(\mathbf{u}_{\ell,k})_{(\ell,k)\in\mathbb{Z}^2}$:

Lemma 2.2 *Assume that* $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$. $u \in D(\mathcal{A}^\mu(\beta))$ *is an eigenfunction of* $\mathcal{A}^\mu(\beta)$ *if and only if* $(\mathbf{u}_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}$ *belongs to* $\ell_2(\mathbb{Z}^2)$ *and satisfies*

$$\left\{ \begin{array}{l} \dfrac{\mathbf{u}_{k+1,\ell}}{\sin(\omega a_1)} + \dfrac{\mathbf{u}_{k-1,\ell}}{\sin(\omega a_1)} + \dfrac{\mathbf{u}_{k,\ell+1}}{\sin(\omega a_2)} + \dfrac{\mathbf{u}_{k,\ell-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{k,l} = 0, \qquad \forall (k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ \dfrac{\mathbf{u}_{1,0}}{\sin(\omega a_1)} + \dfrac{\mathbf{u}_{-1,0}}{\sin(\omega a_1)} + \dfrac{\mathbf{u}_{0,1}}{\sin(\omega a_2)} + \dfrac{\mathbf{u}_{0,-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{0,0} = 2(\mu-1)\dfrac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}\mathbf{u}_{0,0}, \end{array} \right. \tag{11}$$

where we have defined $g_\beta(\omega) = \dfrac{1}{\tan(\omega a_1)} + \dfrac{1}{\tan(\omega a_2)} + \dfrac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$.

As well-known, finite difference schemes may be investigated using the discrete Fourier transform

$$\mathcal{F} : \mathbf{v} = (\mathbf{v}_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2} \mapsto \mathcal{F}(\mathbf{v}) = \widehat{\mathbf{v}}, \quad \widehat{\mathbf{v}}(\xi,\eta) = \sum_{k,\ell\in\mathbb{Z}} e^{i(k\xi+\ell\eta)}\mathbf{v}_{k,\ell}, \qquad (\xi,\eta) \in [0,2\pi]^2, \tag{12}$$

where $\mathcal{F}$ is an isometry between $\ell_2(\mathbb{Z}^2)$ and $L_2((0,2\pi)^2)$. This, together with Lemma 2.2, provides the following characterization for the discrete spectrum of $\mathcal{A}^\mu(\beta)$:

Lemma 2.3 *Assume that* $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$. $u \in D(\mathcal{A}^\mu(\beta))$ *is an eigenfunction of* $\mathcal{A}^\mu(\beta)$ *if and only if the discrete Fourier transform* $\widehat{\mathbf{u}}$ *of* $(\mathbf{u}_{\ell,k})_{(\ell,k)\in\mathbb{Z}^2}$ *belongs to* $L_2((0,2\pi)^2)$ *and satisfies*

$$(f(\xi,\eta,\omega) - \phi_\beta(\omega))\ \widehat{\mathbf{u}}(\xi,\eta) = (\mu-1)\phi_\beta(\omega)\mathbf{u}_{0,0}. \tag{13}$$

where $\phi_\beta(\omega) = \dfrac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$ *and* $f(\xi,\eta,\omega) = \dfrac{\cos\xi - \cos(\omega a_1)}{\sin(\omega a_1)} + \dfrac{\cos\eta - \cos(\omega a_2)}{\sin(\omega a_2)}$.

Under the assumption of Lemma 2.3, (10) can be written as $f(\xi,\eta,\omega) - \phi_\beta(\omega) = 0$. It follows that $\lambda = \omega^2$ does not belong to $\sigma_{ess}(\mathcal{A}^\mu(\beta))$ if and only if, for any $(\xi,\eta) \in [0,2\pi]^2$, $\phi_\beta(\omega) - f(\xi,\eta,\omega)$ does not vanish. As a consequence, as soon as $\lambda = \omega^2 \notin \sigma_{ess}(\mathcal{A}^\mu(\beta))$, the function $(\xi,\eta) \mapsto \phi_\beta(\omega)/(\phi_\beta(\omega) - f(\xi,\eta,\omega))$ is continuous and bounded. Then, the inverse discrete Fourier transform can be applied to (13) to obtain

$$\mathbf{u}_{k,\ell} = \frac{(1-\mu)\mathbf{u}_{0,0}}{4\pi^2} \int\limits_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi,\eta,\omega)} e^{-i(k\xi+\ell\eta)} d\xi d\eta, \quad \forall (k,\ell) \in \mathbb{Z}^2.$$

Writing the previous relation for $k = \ell = 0$ yields the following criterion of existence of an eigenvalue:

Lemma 2.4 *Assume that* $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$ *and that* $\lambda = \omega^2 \notin \sigma_{ess}(\mathcal{A}^\mu(\beta))$. *Then,* λ *is an eigenvalue of* $\mathcal{A}^\mu(\beta)$ *if and only if*

$$\mu = 1 - F_\beta(\omega) \quad where \quad F_\beta(\omega) = \Big(\frac{1}{4\pi^2} \int\limits_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi,\eta,\omega)} d\xi d\eta \Big)^{-1}. \tag{14}$$

The study of the behavior of the function F_β leads to the existence of at least one eigenvalue in each gap of $\mathcal{A}^\mu(\beta)$ as soon as $\mu < 1$, the minimal number of eigenvalues in each gap depending on the type of gaps (cf. Lemma. 2.1-3 for the classification):

Lemma 2.5 *For* $\mu > 1$, *the operator* $\mathcal{A}^\mu(\beta)$ *has no eigenvalue. For* $0 < \mu < 1$, *let* (ω_b^2, ω_t^2) *be a spectral gap of the operator* $\mathcal{A}^\mu(\beta)$*:*

(a) If (ω_b^2, ω_t^2) *is a gap of type (i), then* $\mathcal{A}^\mu(\beta)$ *has at least two eigenvalues* $\lambda_1 = \omega_1^2$ *and* $\lambda_2 = \omega_2^2$ *that satisfy* $\omega_b < \omega_1 < \omega_0 < \omega_2 < \omega_t$ *(see Lemma. 2.1-3 for the definition of* ω_0*).*

(b) If (ω_b^2, ω_t^2) *is a gap of type (ii) or (iii), then* $\mathcal{A}^\mu(\beta)$ *has at least one eigenvalue* $\lambda_1 = \omega_1^2$ *such that* $\omega_b < \omega_1 < \omega_t$.

The sketch of the proof of the previous lemma is the following, a complete proof being available in [10] (Theorem 5.2.1): First, one can verify that $F_\beta(\omega) \geq 0$ in any gap, which, together with (14) proves that $\mathcal{A}^\mu(\beta)$ has no eigenvalue for $\mu > 1$. Then, if (ω_b^2, ω_t^2) is a gap of type (i), one can show that

$$\lim_{\omega\to\omega_b^+} (1 - F_\beta(\omega)) = \lim_{\omega\to\omega_t^-} (1 - F_\beta(\omega)) = 1 \text{ and that } \lim_{\omega\to\omega_0} (1 - F_\beta(\omega)) = 0.$$

By continuity of F_β inside the gap, (a) directly results from the intermediate value theorem and (14). If (ω_b^2, ω_t^2) is a gap of type (ii), the intermediate value theorem also permits us to conclude since

$$\lim_{\omega\to\omega_b} (1 - F_\beta(\omega)) \leq 0 \text{ and } \lim_{\omega\to\omega_t^-} (1 - F_\beta(\omega)) = 1.$$

A similar argument works for a gap of type (iii).

3 Guided modes for the operator $A_\varepsilon^\mu(\beta)$: an asymptotic result

Finally, thanks to the general result [8] (Theorem 2.13 convergence of the spectrum of $A_\varepsilon^\mu(\beta)$ toward the spectrum of $\mathcal{A}^\mu(\beta)$), we can prove the following result of existence of eigenvalue for the operator $A_\varepsilon^\mu(\beta)$:

Theorem 3.1 *Let* $\mu \in (0,1)$, (λ_b, λ_t) *be a spectral gap of the operator* $\mathcal{A}^\mu(\beta)$ *and* $\lambda_0 \in (\lambda_b, \lambda_t)$ *be a (simple) eigenvalue of this operator. Then, there exists* $\varepsilon_0 > 0$ *such that if* $\varepsilon < \varepsilon_0$ *the operator* $A_\varepsilon^\mu(\beta)$ *has an eigenvalue* λ_ε *inside a spectral gap* $(\lambda_b^\varepsilon, \lambda_t^\varepsilon)$. *Moreover,* $\lambda_\varepsilon = \lambda_0 + O(\sqrt{\varepsilon})$.

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