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Characterizing the maximum parameter of the total-variation denoising through the pseudo-inverse of the divergence

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Abstract—We focus on the maximum regularization parameter for anisotropic total-variation denoising. It corresponds to the minimum value of the regularization parameter above which the solution remains constant. While this value is well known for the Lasso, such a critical value has not been investigated in details for the total-variation. Though, it is of importance when tuning the regularization parameter as it allows fixing an upper-bound on the grid for which the optimal parameter is sought. We establish a closed form expression for the one-dimensional case, as well as an upper-bound for the two-dimensional case, that appears reasonably tight in practice. This problem is directly linked to the computation of the pseudo-inverse of the divergence, which can be quickly obtained by performing convolutions in the Fourier domain.

I. INTRODUCTION

We consider the reconstruction of a d -dimensional signal (in this study $d = 1$ or 2) from its noisy observation $y = x + w \in \mathbb{R}^n$ with $w \in \mathbb{R}^n$. Anisotropic TV regularization writes, for $\lambda > 0$, as [1]

$$x^* = \operatorname{argmin}_{x \in \mathbb{R}^n} \frac{1}{2} \|y - x\|_2^2 + \lambda \|\nabla x\|_1 \quad (1)$$

with $\nabla x \in \mathbb{R}^{dn}$ being the concatenation of the d components of the discrete periodical gradient vector field of x , and $\|\nabla x\|_1 = \sum_i |(\nabla x)_i|$ being a sparsity promoting term. The operator ∇ acts as a convolution which writes in the one dimensional case ($d = 1$)

$$\nabla = F^+ \operatorname{diag}(K_{\rightarrow})F \quad \text{and} \quad \operatorname{div} = F^+ \operatorname{diag}(K_{\leftarrow})F \quad (2)$$

where $\operatorname{div} = -\nabla^\top$ (where \top denotes the adjoint), $F : \mathbb{R}^n \mapsto \mathbb{C}^n$ is the Fourier transform, $F^+ = \operatorname{Re}[F^{-1}]$ is its pseudo-inverse and $K_{\rightarrow} \in \mathbb{C}^n$ and $K_{\leftarrow} \in \mathbb{C}^n$ are the Fourier transforms of the kernel functions performing forward and backward finite differences respectively. Similarly, we define in the two dimensional case ($d = 2$)

$$\nabla = \begin{pmatrix} F^+ & 0 \\ 0 & F^+ \end{pmatrix} \begin{pmatrix} \operatorname{diag}(K_{\downarrow}) \\ \operatorname{diag}(K_{\rightarrow}) \end{pmatrix} F \quad (3)$$

$$\text{and} \quad \operatorname{div} = F^+ \begin{pmatrix} \operatorname{diag}(K_{\uparrow}) & \operatorname{diag}(K_{\leftarrow}) \\ 0 & F \end{pmatrix} \quad (4)$$

where $K_{\rightarrow} \in \mathbb{C}^n$ and $K_{\leftarrow} \in \mathbb{C}^n$ (resp. $K_{\downarrow} \in \mathbb{C}^n$ and $K_{\uparrow} \in \mathbb{C}^n$) perform forward and backward finite difference in the horizontal (resp. vertical) direction.

II. GENERAL CASE

For the general case, the following proposition provides an expression of the maximum regularization parameter λ_{\max} as the solution of a convex but non-trivial optimization problem (direct consequence of the Karush-Khun-Tucker condition).

Proposition 1. Define for $y \in \mathbb{R}^n$,

$$\lambda_{\max} = \min_{\zeta \in \operatorname{Ker}[\operatorname{div}]} \|\operatorname{div}^+ y + \zeta\|_{\infty} \quad (5)$$

where div^+ is the Moore-Penrose pseudo-inverse of div and $\operatorname{Ker}[\operatorname{div}]$ its null space. Then, $x^* = \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top y$ if and only if $\lambda \geq \lambda_{\max}$.

III. ONE DIMENSIONAL CASE

In the 1d case, $\operatorname{Ker}[\operatorname{div}] = \operatorname{Span}[\mathbb{1}_n]$ and thus the optimization problem can be solved by computing div^+ in the Fourier domain, in $O(n \log n)$ operations, as shown in the next corollary.

Corollary 1. For $d = 1$, $\lambda_{\max} = \frac{1}{2} [\max(\operatorname{div}^+ y) - \min(\operatorname{div}^+ y)]$,

$$\text{where} \quad \operatorname{div}^+ = F^+ \operatorname{diag}(K_{\uparrow}^+)F \quad (6)$$

$$\text{and} \quad (K_{\uparrow}^+)_i = \begin{cases} \frac{(K_{\uparrow})_i^*}{|(K_{\uparrow})_i|^2} & \text{if } |(K_{\uparrow})_i|^2 > 0 \\ 0 & \text{otherwise} \end{cases},$$

and $*$ denotes the complex conjugate.

Note that the condition $|(K_{\uparrow})_i|^2 > 0$ is satisfied everywhere except for the zero frequency. In the non-periodical case, div is the incidence matrix of a tree whose pseudo-inverse can be obtained following [2].

IV. TWO DIMENSIONAL CASE

In the 2d case, $\operatorname{Ker}[\operatorname{div}]$ is the orthogonal of the vector space of signals satisfying Kirchhoff's voltage law on all cycles of the periodical grid. Its dimension is $n+1$. It follows that the optimization problem becomes much harder. Since our motivation is only to provide an approximation of λ_{\max} , we propose to compute an upper-bound in $O(n \log n)$ operations thanks to the following corollary.

Corollary 2. For $d = 2$, $\lambda_{\max} \leq \underbrace{\frac{1}{2} [\max(\operatorname{div}^+ y) - \min(\operatorname{div}^+ y)]}_{\lambda_{\text{bnd}}}$,

$$\text{where} \quad \operatorname{div}^+ = \begin{pmatrix} F^+ & 0 \\ 0 & F^+ \end{pmatrix} \begin{pmatrix} \operatorname{diag}(\tilde{K}_{\uparrow}^+) \\ \operatorname{diag}(\tilde{K}_{\leftarrow}^+) \end{pmatrix} F, \quad \text{and} \quad (7)$$

$$(\tilde{K}_{\uparrow}^+)_i = \begin{cases} \frac{(K_{\uparrow})_i^*}{|(K_{\uparrow})_i|^2 + |(K_{\leftarrow})_i|^2} & \text{if } |(K_{\uparrow})_i|^2 + |(K_{\leftarrow})_i|^2 > 0 \\ 0 & \text{otherwise} \end{cases},$$

$$(\tilde{K}_{\leftarrow}^+)_i = \begin{cases} \frac{(K_{\leftarrow})_i^*}{|(K_{\uparrow})_i|^2 + |(K_{\leftarrow})_i|^2} & \text{if } |(K_{\uparrow})_i|^2 + |(K_{\leftarrow})_i|^2 > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Note that the condition $|(K_{\uparrow})_i|^2 + |(K_{\leftarrow})_i|^2 > 0$ is again satisfied everywhere except for the zero frequency. Remark also that this result can be straightforwardly extended to the case where $d > 2$.

V. RESULTS AND DISCUSSION

Figure 1 and 2 provide illustrations of the computation of λ_{\max} and λ_{bnd} on a 1d signal and a 2d image respectively. The convolution kernel is a simple triangle wave in the 1d case but is more complex in the 2d case. The operator $\operatorname{div} \operatorname{div}^+$ is in fact the projector onto the space of zero-mean signals, i.e., $\operatorname{Im}[\operatorname{div}]$. Figure 3 illustrates the evolution of x^* with respect to λ (computed with the algorithm of [3]). Our upper-bound λ_{bnd} (computed in ~ 5 ms) appears to be reasonably tight (λ_{\max} computed in ~ 25 s with [3] on Problem (5)).

Future work will concern the generalization of these results to other ℓ_1 analysis regularization and to ill-posed inverse problems.

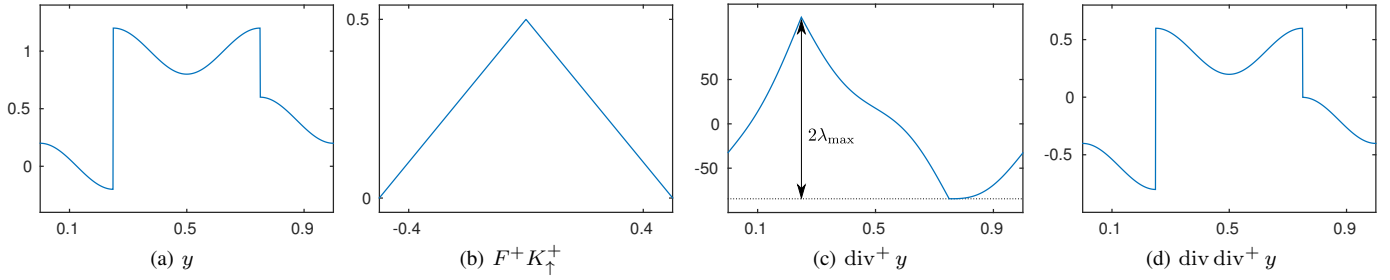


Fig. 1. (a) A 1d signal y . (b) The convolution kernel $F^+K_{\uparrow}^+$ that realizes the pseudo inversion of the divergence. (c) The signal $\text{div}^+ y$ on which we can read the value of λ_{\max} . (d) The signal $\text{div div}^+ y$ showing that one can reconstruct y from $\text{div}^+ y$ up to its mean component.

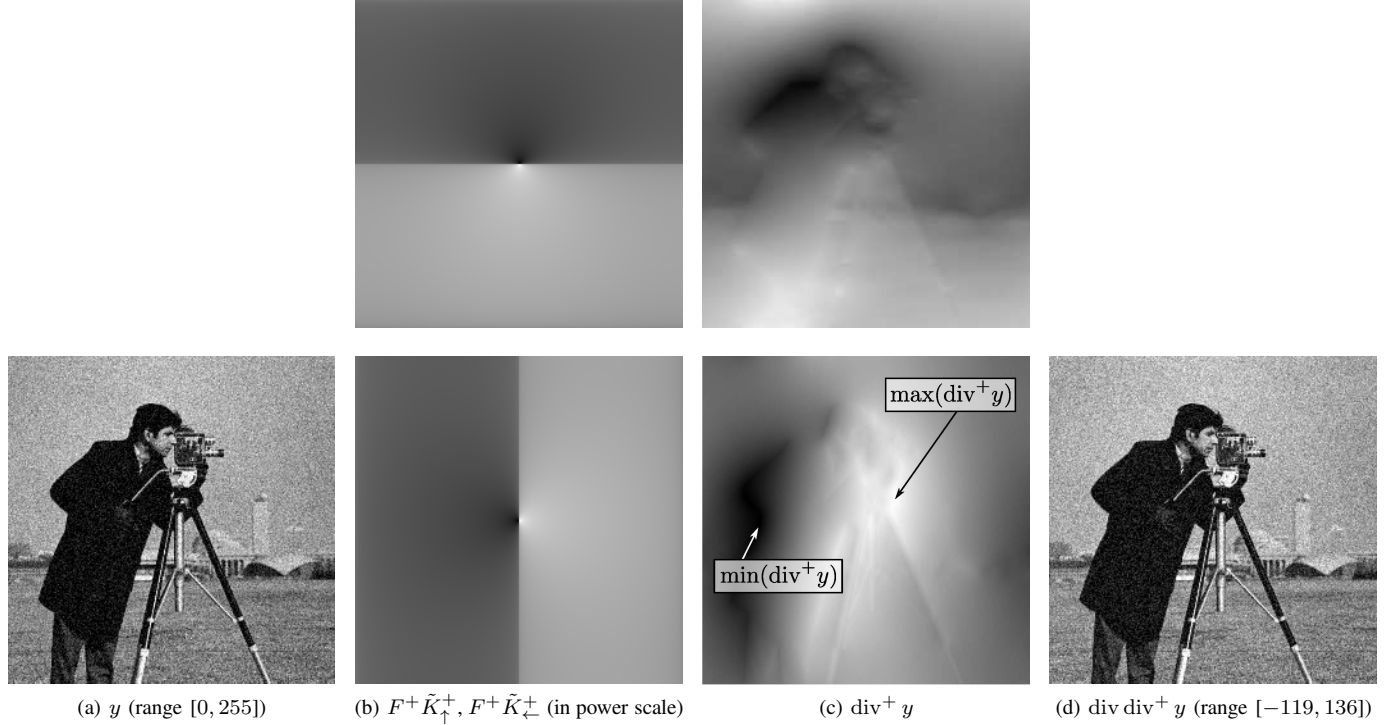


Fig. 2. (a) A 2d signal y . (b) The convolution kernels $F^+K_{\uparrow}^+$ and $F^+\tilde{K}_{\pm}^+$ that realizes the pseudo inversion of the divergence. (c) The absolute value of the two coordinates of the vector field $\text{div}^+ y$ on which we can read the upper-bound λ_{bnd} of λ_{\max} . (d) The image $\text{div div}^+ y$ showing again that one can reconstruct y from $\text{div}^+ y$ up to its mean component.

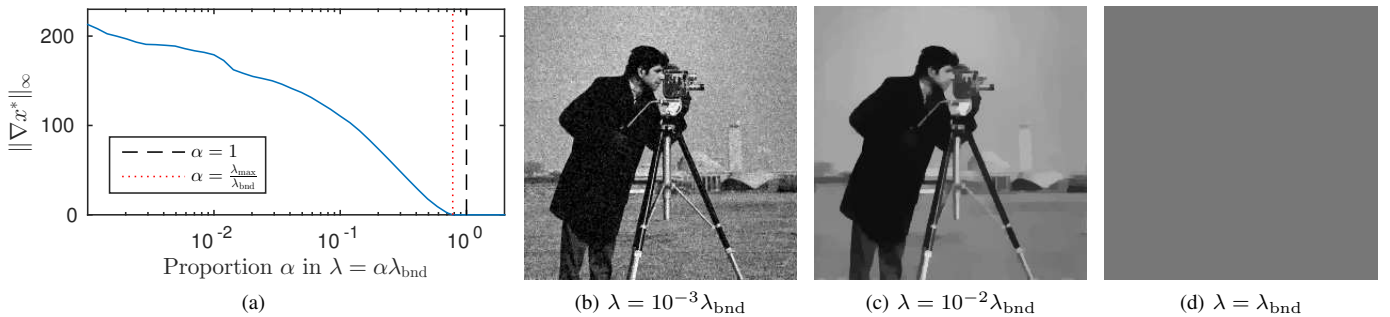


Fig. 3. (a) Evolution of $\|\nabla x^*\|_{\infty}$ as a function of λ . (b), (c), (d) Results x^* of the periodical anisotropic total-variation for three different values of λ .

REFERENCES

[1] L. I. Rudin, S. Osher, and E. Fatemi, "Nonlinear total variation based noise removal algorithms," *Physica D: Nonlinear Phenomena*, vol. 60, no. 1, pp. 259–268, 1992.

[2] R. Bapat, "Moore-penrose inverse of the incidence matrix of a tree," *Linear and Multilinear Algebra*, vol. 42, no. 2, pp. 159–167, 1997.

[3] A. Chambolle and T. Pock, "A first-order primal-dual algorithm for convex problems with applications to imaging," *J. Math. Imaging Vis.*, vol. 40, pp. 120–145, 2011.