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Calculus of the coarse Ricci curvatures κ and $\tilde{\kappa}$ on an example of degenerated diffusion

Laurent Veysseire

This work is about studying an example of a degenerated diffusion on which the Ollivier's generalization of coarse Ricci curvature is positive, thus allowing to get a Poincaré inequality.

In section 1, we give the formulas to compute the two different curvatures, κ and $\tilde{\kappa}$, introduced in [4]. The curvature κ is the classical Ollivier's coarse Ricci curvature, whereas the curvature $\tilde{\kappa}$ is a modified version of it, which only exists in special cases and is always smaller than κ . In the example presented in this paper, the curvature $\tilde{\kappa}$ is less useful than the curvature κ to get a good constant in the Poincaré inequality, because, due to the symmetries, the curvatures will be the same everywhere on the manifold, so the harmonic mean of $\tilde{\kappa}$ will always be smaller than the infimum of κ .

In section 2, the example of degenerated diffusion is introduced by defining its generator, along with a lot of notations which will be used in the further sections.

Section 3 describes the eigenvalues and eigenvectors of the generator, and in particular, this gives the true optimal Poincaré constant for the diffusion we are studying here. This section is mostly independent from the other sections and is quite long, so it can be skipped by the reader if he is more interested in the curvature computations than in the spectral analysis of the generator.

Section 4 is where the curvatures κ and $\tilde{\kappa}$ are actually computed thanks to the formulas given in section 1.

In the section 5 is a discussion about the regularity of the optimal couplings used to define the curvatures κ and $\tilde{\kappa}$ in this particular example of diffusion, and how it behaves differently than is the non-degenerate case.

The final section 6 shows that the more classical Bakry–Emery curvature approach fails in the example we study here, proving the utility of the Ollivier's coarse Ricci curvature.

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1 Formulas for κ and $\tilde{\kappa}$

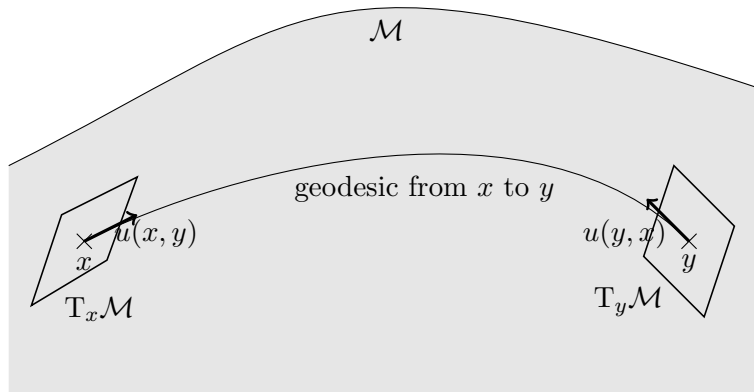
The coarse Ricci curvatures κ and $\tilde{\kappa}$ we are computing here are the ones introduced in [4] for diffusions. We first recall the formulas for computing them.

Let (\mathcal{M}, g) be a smooth Riemannian manifold, and let

$$L = \frac{1}{2}A^{ij}\nabla_i\nabla_j + F^i\nabla_i$$

be the generator of a diffusion on \mathcal{M} , with A a smooth symmetric nonnegative tensor field, F a smooth vector field, and ∇ the Levi-Civita connexion on \mathcal{M} .

We take x and y two different points on M , such that (x, y) is not on the cut-locus of \mathcal{M} . Then there exist $u(x, y) \in T_x\mathcal{M}$ and $u(y, x) \in T_y\mathcal{M}$ two unit norm vectors such that $y = \exp_x(d(x, y)u(x, y))$ and $x = \exp_y(d(x, y)u(y, x))$, and they are unique.



The distance admits the following Taylor expansion in the neighborhood of (x, y) :

$$\begin{aligned} d(\exp_x(\varepsilon v), \exp_y(\varepsilon w)) &= d(x, y) - \varepsilon[g(x)(u(x, y), v) + g(y)(u(y, x), w)] \\ &+ \frac{\varepsilon^2}{2d(x, y)}[q^{(1)}(x, y)(v, v) + q^{(2)}(x, y)(w, w) + 2q^{(12)}(x, y)(v, w)] + O(\varepsilon^3) \end{aligned}$$

where $q^{(1)}(x, y)$, $q^{(2)}(x, y)$ and $q^{(12)}(x, y)$ are quadratic forms, and can be computed by integrating the Jacobi field along the geodesic joining x to y .

Then the coarse Ricci curvature κ between x and y takes the following form:

$$\begin{aligned} \kappa(x, y) = & \frac{1}{d(x, y)} (g(x)(u(x, y), F(x)) + g(y)(u(y, x), F(y))) \\ & - \frac{1}{2} (\text{tr}(A(x)q^{(1)}(x, y)) + \text{tr}(A(y)q^{(2)}(x, y))) \\ & + \text{tr} \left(\sqrt{A(x)q^{(12)}(x, y)A(y)q^{(12)T}(x, y)} \right) \end{aligned}$$

The curvature $\tilde{\kappa}$ is only defined when the tensor field A satisfies the following condition:

$$(H) : \nabla_i A_{jk} + \nabla_j A_{ki} + \nabla_k A_{ij} = 0,$$

with $A_{ij} = g_{ik}A^{kl}g_{lj}$. In this case, we take

$$\rho(x, y) := (g(x)A(x)g(x))(u(x, y), u(x, y)) = (g(y)A(y)g(y))(u(y, x), u(y, x)).$$

If $\rho(x, y) \neq 0$ we define

$$\tilde{A}(x) = A(x) - \frac{A(x)g(x)u(x, y)u^T(x, y)g(x)A(x)}{\rho(x, y)}$$

and

$$\tilde{A}(y) = A(y) - \frac{A(y)g(y)u(y, x)u^T(y, x)g(y)A(y)}{\rho(x, y)}.$$

The tensor $\tilde{A}(x)$ corresponds to the orthogonal projection of $A(x)$ on $\text{Im}(A(x)) \cap (g(x)u(x, y))^\perp$, with respect to the scalar product induced by $A(x)$ on $\text{Im}(A(x))$, and the same holds by replacing $\tilde{A}(x)$ with $\tilde{A}(y)$, $A(x)$ with $A(y)$ and $g(x)u(x, y)$ with $g(y)u(y, x)$. Then the curvature $\tilde{\kappa}$ can be computed using the following formula:

$$\begin{aligned} \tilde{\kappa}(x, y) = & \frac{1}{d(x, y)} (g(x)(u(x, y), F(x)) + g(y)(u(y, x), F(y))) \\ & - \frac{1}{2} (\text{tr}(A(x)q^{(1)}(x, y)) + \text{tr}(A(y)q^{(2)}(x, y))) + \frac{q^{(12)}(A(x)g(x)u(x, y), A(y)g(y)u(y, x))}{\rho(x, y)} \\ & + \text{tr} \left(\sqrt{\tilde{A}(x)q^{(12)}(x, y)\tilde{A}(y)q^{(12)T}(x, y)} \right). \end{aligned}$$

(Sorry, I wrote a wrong formula in the first version of [4].)

In the case when $\rho(x, y) = 0$, the variance term of the distance is 0 for every coupling, so the maximal decay rate of the distance under the constraint that this term cancels (which was the definition of $\tilde{\kappa}$) is given by κ , so if $\rho(x, y) = 0$, we have

$$\tilde{\kappa}(x, y) = \kappa(x, y).$$

2 Our example of degenerated diffusion

In our example, the manifold \mathcal{M} will be the sphere of odd dimension S^{2n-1} , and the diffusion will be a Brownian motion restricted to the directions orthogonal to the fibers of the Hopf fibration on S^{2n-1} . Due to the symmetries, this diffusion is reversible with respect to the usual normalised volume measure. We will denote by σ_{2n-1} this measure.

So we will assume that $n \geq 2$, because the resulting diffusion in the case $n = 1$ is a bit too degenerated.

We will work on \mathbb{C}^n equipped with an hermitian product $\langle y, x \rangle$. Here, we take the convention where this product is semilinear with respect to y and linear with respect to x . We identify S^{2n-1} with the unit sphere of \mathbb{C}^n for the hermitian norm. The Hopf fiber of $x \in S^{2n-1}$ is $\{e^{i\theta}.x, \theta \in [0, 2\pi)\}$. The quotient space is naturally the complex projective space of dimension $n - 1$.

The hermitian scalar product on \mathbb{C}^n inducts an Euclidean one, which is its real part, and a symplectic form, which is its imaginary part. We denote by g_0 the symmetric tensor corresponding to the Euclidean norm, and by ξ the antisymmetric one which corresponds to the symplectic form. For any orthonormal \mathbb{C} -basis (x_1, \dots, x_n) , we have:

$$g_0 = \sum_{j=1}^n x_j^* \otimes x_j^* + (ix_j)^* \otimes (ix_j)^*$$

and

$$\xi = \sum_{j=1}^n x_j^* \otimes (ix_j)^* - (ix_j)^* \otimes x_j^*,$$

where $x^* = g_0.x$ is the \mathbb{R} -linear form on \mathbb{C}^n defined by $x^*(y) = \Re\langle x, y \rangle$.

Beware, the tensor products are here over \mathbb{R} , so for any two non-zero vectors x and y , the four tensors $x \otimes y$, $x \otimes iy$, $ix \otimes y$ and $ix \otimes iy$ form a linearly free system, even if $x = y$!

The metric we put on S^{2n-1} is the one induced by the Euclidean norm, and the tensors of the diffusion are:

$$A(x) = g_0^{-1} - x \otimes x - (ix) \otimes (ix) \text{ and } F(x) = 0.$$

In the particular case where $n = 2$, we can identify S^3 to the set of unit norm quaternions, and we may approximate our diffusion by Markov chains as follows. Let $\varepsilon > 0$, we discretize the time by taking intervals of length ε^2 . At each step N , we randomly choose $p_N = \pm 1$ and $q_N = \pm 1$, independently and taking the value $+1$ with probability $\frac{1}{2}$ and the value -1 with probability $\frac{1}{2}$, and we jump to $X_{(N+1)\varepsilon^2} = e^{\varepsilon(p_N j + q_N k)} X_{N\varepsilon^2}$, where the exponential function and the product are the ones in the set of quaternions. Our diffusion is the limit of this random walk when ε tends to 0. Then, with this interpretation, we see a simple coupling between pathes: we take one path Z_t of the diffusion starting at 1, and for any two starting points x and y , we set $X_t = Z_t.x$ and $Y_t = Z_t.y$. This is an isometric coupling between the pathes starting at x and those ones starting at y . The simple fact that such a coupling exists implies that A satisfies the (H) condition, and that $\tilde{\kappa}(x, y) \geq 0$ (and thus that $\kappa(x, y) \geq 0$, since $\kappa \geq \tilde{\kappa}$). This remark also holds more generally for diffusions unvariant by right multiplication (which means that $P^t f(y.x) = P^t f_x(y)$, with $f_x(z) = f(z.x)$) on finite dimensional Lie groups equipped with a metric which is unvariant by left multiplication (i.e. $d(x, y) = d(z.x, z.y)$).

For $n > 2$, we saw in Example 22 in [4] that the A which satisfy (H) on S^{2n-1} could be defined thanks to a four dimensional tensor in $T \in \mathbb{R}^{2n} \simeq \mathbb{C}^n$ having the symmetry of a Riemann tensor in the following way:

$$A(x)(g(x)u, g(x)u) = T(x, u, x, u)$$

for every $u \in T_x S^{2n-1}$, that is, for every u g_0 -orthogonal to x . Now let us consider the four dimensional tensor defined by:

$$T(x, y, z, t) = g_0(x, z)g_0(y, t) - g_0(x, t)g_0(y, z) - \frac{1}{3}(2\xi(x, y)\xi(z, t) + \xi(x, z)\xi(y, t) + \xi(x, t)\xi(z, y)).$$

It has the symmetry of a Riemann tensor, i.e.

$$\begin{aligned} T(x, y, z, t) &= -T(x, y, t, z) = T(z, t, x, y) \\ T(x, y, z, t) + T(x, z, t, y) + T(x, t, y, z) &= 0. \end{aligned}$$

Checking this equalities is easy and is left to the reader. Now, we have, for any u g_0 -orthogonal to x , $T(x, u, x, u) = g_0(u, u) - (\xi(x, u))^2$, and $A(x)(u, u) = g_0(u, u) - ((ix)^*.u)^2$, but we have

$$(ix)^*.u = \Re(\langle ix, u \rangle) = -\Im(\langle x, u \rangle) = -\xi(x, u)$$

so the tensor T defines the tensor field A in the way stated above, thus A satisfies (H).

3 Eigenvalues and eigenvectors of the generator

A positive lower bound of the curvature κ or $\tilde{\kappa}$ implies a positive lower bound for the spectral gap of the generator, as seen in [4].

In this section, we give the whole spectrum and a nice basis of eigenvectors of the generator of our diffusion. So we can get its spectral gap, and we will compare it later to the bounds obtained by computing κ and $\tilde{\kappa}$.

3.1 A simple description of the eigensubspaces

We first remark we can write our generator as a difference of two more simple operators:

$$L = \frac{1}{2}\Delta_S - \frac{1}{2}\Delta_H$$

where Δ_S is the classical Laplacian operator on the $2n - 1$ -dimensional sphere, and Δ_H is the Laplacian operator on the Hopf fiber (more explicitly, $\Delta_H f(x) = \frac{d^2}{d\theta^2} \Big|_{\theta=0} f(e^{i\theta}x)$). As we will see below, the operators Δ_S and Δ_H commute, the nontrivial intersections of an eigensubspace of Δ_S and an eigensubspace of Δ_H form an orthogonal decomposition of the space $L^2(\sigma_{2n-1})$ on subspaces of eigenvectors of L .

Let us remind the eigensubspaces and the eigenvalues of the classical Laplacian Δ on the sphere S^N , which can be found in [3], for example.

Definition 1 *Let $N \in \mathbb{N}$, $N \geq 1$. We denote by Π_d the space of restrictions to S^N of polynomials of degree d on $N + 1$ variables x_0, \dots, x_N . We also set $E_0 = \Pi_0$ and $E_d = \Pi_d \cap (\Pi_{d-1})^\perp$ for $d \geq 1$, where the orthogonal of Π_{d-1} is taken with respect to the scalar product of $L_2(\sigma_N)$.*

The following proposition gives us the eigenvalues and eigensubspaces of Δ .

Proposition 2 *The subspaces E_d are the eigensubspaces of Δ , and the corresponding eigenvalues are $-d(d + N - 1)$.*

Proof : Let f be a smooth function on \mathbb{R}^{N+1} , and $x \in S^N$. We want to compute $\Delta \tilde{f}(x)$, where \tilde{f} is the restriction of f to S^N . Let e_1, \dots, e_N be an

orthonormal basis of $T_x S^N$. Then we have:

$$\Delta \tilde{f}(x) = \sum_{i=1}^N \frac{d^2}{dt^2} \Big|_{t=0} f(\gamma_i(t))$$

where $\gamma_i(t) = \exp_x(t, e_i) = \cos(t)x + \sin(t)e_i$ is the geodesic starting at x with speed e_i . We have

$$\begin{aligned} \frac{d^2}{dt^2} f(\cos(t)x + \sin(t)e_i) &= -\cos(t)\nabla_x f(\cos(t)x + \sin(t)e_i) + \sin(t)\nabla_{e_i} f(\cos(t)x + \sin(t)e_i) \\ &\quad + \sin(t)^2 \nabla_{xx}^2 f(\cos(t)x + \sin(t)e_i) - 2\sin(t)\cos(t)\nabla_{xe_i}^2 f(\cos(t)x + \sin(t)e_i) \\ &\quad + \cos(t)^2 \nabla_{e_i e_i}^2 f(\cos(t)x + \sin(t)e_i), \end{aligned}$$

where ∇ and ∇^2 are the usual gradient and hessian in \mathbb{R}^{N+1} . So we get:

$$\Delta \tilde{f}(x) = -N\nabla_x f(x) + \sum_{i=1}^N \nabla_{e_i e_i}^2 f(x) = \Delta_0 f(x) - N\nabla_x f(x) - \nabla_{xx}^2 f(x),$$

where Δ_0 is the usual Laplacian on \mathbb{R}^{N+1} , and the last equality is due to the simple fact that $\Delta_0 f = \nabla_{xx}^2 f + \sum_{i=1}^N \nabla_{e_i e_i}^2 f$, because x, e_1, \dots, e_N is an orthonormal basis of \mathbb{R}^{N+1} .

Let us apply the preceding formula for f being a monomial of degree d : $f(x) = \prod_{i=0}^N x_i^{\alpha_i}$, with $\sum_{i=0}^N \alpha_i = d$. We have

$$\begin{aligned} \Delta \tilde{f}(x) &= \sum_{i=0}^N \left(\alpha_i(\alpha_i - 1)x_i^{\alpha_i-2} \prod_{j \in [0, N], j \neq i} x_j^{\alpha_j} \right) - N \sum_{i=0}^N x_i \left(\alpha_i x_i^{\alpha_i-1} \prod_{j \in [0, N], j \neq i} x_j^{\alpha_j} \right) \\ &\quad - \sum_{i=0}^N \sum_{j=0}^N x_i x_j \left(\mathbf{1}_{i=j} \alpha_i(\alpha_i - 1)x_i^{\alpha_i-2} \prod_{k \in [0, N], k \neq i} x_k^{\alpha_k} \right. \\ &\quad \left. + \mathbf{1}_{i \neq j} \alpha_i \alpha_j x_i^{\alpha_i-1} x_j^{\alpha_j-1} \prod_{k \in [0, N], k \neq i, k \neq j} x_k^{\alpha_k} \right) \\ &= g(x) + f(x) \left(-N \sum_{i=0}^N \alpha_i - \sum_{i=0}^N \sum_{j=0}^N (\alpha_i \alpha_j - \mathbf{1}_{i=j} \alpha_i) \right) \\ &= g(x) - d(d + N - 1)f(x) \end{aligned}$$

where $g = \Delta_0 f$ is an homogenous polynomial of degree $d - 2$ (or $g = 0$ if $d < 2$).

By linearity, we get that Π_d is stabilized by Δ , and more precisely, for any $f \in \Pi_d$, we have $\Delta f + d(d+N-1)f \in \Pi_{d-1}$. Let $f \in E_d$, we set $g = \Delta f + d(d+N-1)f \in \Pi_{d-1}$. We have $\int_{S^N} \bar{g} \Delta f d\sigma_N = \int_{S^N} \bar{g} (-d(d+N-1)f + g) d\sigma_N = \int_{S^N} |g|^2 d\sigma_N$ because $f \in E_d$, so f is orthogonal to g . But Δ is a symmetric operator of $L_2(\sigma_N)$, so we have $\int_{S^N} \bar{g} \Delta f d\sigma_N = \int_{S^N} f \Delta(\bar{g}) d\sigma_N = 0$ because $\Delta g \in \Pi_{d-1}$, so it is orthogonal to f . Thus $\int_{S^N} |g|^2 d\sigma_N = 0$, hence $g = 0$ and $\Delta f = -d(d+N-1)f$, that is to say, provided f is not 0, it is an eigenvector of Δ with the eigenvalue $-d(d+N-1)$.

The linear span of the E_d is the set of the restrictions of polynomials to S^N , which is dense in the space of continuous functions on S^N , and thus in $L_2(\sigma_N)$, so we have found all the eigenvalues. \square

One can also compute the dimensions of the spaces E_d to get the multiplicity of the eigenvalues. As $\Pi_{d-1} \subset \Pi_d$, we have $\dim(E_d) = \dim(\Pi_d) - \dim(\Pi_{d-1})$. Two distinct polynomials can have the same restriction on S^N , in which case their difference is a polynomial that cancels on S^N , so it is a multiple of $P_0 = \left(\sum_{i=0}^N x_i^2\right) - 1$. So the kernel of the linear map which associates to a polynomial of degree d or less its restriction to S^N is exactly the set of the products of P_0 and a polynomial of degree less or equal to $d-2$.

The dimension of the space of polynomials of degree d or less in $N+1$ variables is the binomial coefficient $\binom{N+d+1}{N+1}$. So the dimension of Π_d is $\binom{N+d+1}{N+1} - \binom{N+d-1}{N+1} = \binom{N+d}{N} + \binom{N+d-1}{N}$. And thus the dimension of E_d is $\binom{N+d}{N} - \binom{N+d-2}{N} = \binom{N+d-1}{N-1} + \binom{N+d-2}{N-1}$.

Classical Fourier analysis gives us the eigenvalues and eigenvectors of Δ_H .

Definition 3 Let $k \in \mathbb{Z}$, we set F_k the set of L^2 functions on S^{2n-1} satisfying $f(e^{i\theta}x) = e^{ki\theta}f(x)$ for all $x \in S^{2n-1}$ and $\theta \in [0, 2\pi)$.

Proposition 4 Any non 0 element of F_k is an eigenvector of Δ_H for the eigenvalue $-k^2$. The eigensubspaces of Δ_H are F_0 , and the $F_k \oplus F_{-k}$ for $k \geq -1$.

Proof : The fact that the vectors of F_k are eigenvectors of Δ_H is trivial. Let $f \in L^2(\sigma_{2n-1})$, then, by Fubini, the restriction of f on almost every Hopf fibers (with respect to the quotient measure of σ_{2n-1} on the set of Hopf fibers PC^n) is L^2 . One can use the classical Fourier decomposition on each Hopf fiber for which the restriction of f is L^2 , so using Fubini again, we have $f = \sum_{k \in \mathbb{Z}} f_k$, where $f_k(z) = \int_0^{2\pi} e^{-ik\theta} f(e^{i\theta}z) d\theta$, which is well defined for almost all z , and $f_k \in F_k$. \square

Polynomials on the real and imaginary parts x_j and y_j of the complex coordinates z_j ($j \in [1, n]$) can be written as polynomials on the z_j 's and their complex conjugates \bar{z}_j 's.

Remark 5 For any monomial $\prod_{j=1}^n z_j^{\alpha_j} \bar{z}_j^{\beta_j}$, its restriction to S^{2n-1} belongs to F_k , with $k = \sum_{j=1}^n \alpha_j - \beta_j$.

Definition 6 Let $M_{d,k}$ be the set of monomials $\prod_{j=1}^n z_j^{\alpha_j} \bar{z}_j^{\beta_j}$ of degree $d = \sum_{j=1}^n \alpha_j + \beta_j$ whose restriction to S^{2n-1} belongs to F_k .

We note that $M_{d,k}$ is empty if d and k do not have the same parity or if $|k| > d$. More precisely, we have

$$\#M_{d,k} = \binom{\frac{d+k}{2} + n - 1}{n - 1} \binom{\frac{d-k}{2} + n - 1}{n - 1} \mathbb{1}_{d+k \in 2\mathbb{Z}, |k| \leq d},$$

since the α_j 's and β_j 's are nonnegative integers satisfying $\sum_{j=1}^n \alpha_j = \frac{d+k}{2}$ and $\sum_{j=1}^n \beta_j = \frac{d-k}{2}$.

Definition 7 We denote by $\Pi_{d,k}$ the vector space of the restrictions to S^{2n-1} of the polynomials which are linear combinations of the monomials in $M_{d,k}$.

The dimension of $\Pi_{d,k}$ is $\#M_{d,k}$, because the restriction to S^{2n-1} of homogeneous polynomials of degree d is injective. We remark that $\Pi_{d,k} \subset \Pi_{d+2,k}$, since when one multiplies a monomial of $M_{d,k}$ by the polynomial $\prod_{j=1}^n z_j \bar{z}_j$, whose restriction to S^{2n-1} is 1, every monomial of the product belongs to $M_{d+2,k}$.

Definition 8 We set $E_{d,k} = \Pi_{d,k} \cap \Pi_{d-2,k}^\perp$, where $\Pi_{d-2,k} = \{0\}$ if $d < 2$.

The dimension of $E_{d,k}$ is $\#M_{d,k} - \#M_{d-2,k}$, which is non-zero if $n > 1$, $k + d \in 2\mathbb{Z}$ and $|k| \leq d$ or $n = 1$ and $|k| = d$.

Lemma 9 Let E_d be as in Definition 1, with $N = 2n - 1$. Then, we have $E_{d,k} = E_d \cap F_k$.

Proof : We first prove $E_{d,k} \subset E_d \cap F_k$. The inclusion $E_{d,k} \subset F_k$ is trivial, let us prove that $E_{d,k} \subset E_d$. Let $f \in E_{d,k}$, and $g \in \Pi_{d-1}$. We trivially have $f \in \Pi_d$, it only remains to check $\int_{S^{2n-1}} \bar{f}g = 0$. The function g is the restriction of some polynomial $P(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$ of degree smaller than

$d - 1$. The restriction of each monomial of P belongs to some $F_{k'}$, with $|k'| \leq d - 1$. So we can write $P = \sum_{k'=-d+1}^{d-1} P_{k'}$, where each monomial of $P_{k'}$ belongs to some $M_{d',k'}$, with $d' \leq d - 1$. The restriction $g_{k'}$ of $P_{k'}$ to S^{2n-1} belongs to $F_{k'}$ (actually, $g_{k'}$ is the orthogonal projection of g on $F_{k'}$). Thus f is orthogonal to all the $g_{k'}$ with $k' \neq k$ (remember $f \in F_k$). So it suffices to show that f is orthogonal to g_k . If $d + k \notin 2\mathbb{Z}$, then $f = 0$, and this is trivial. If $d + k \in 2\mathbb{Z}$, every monomial of P_k lies in some $M_{d',k}$, with $d' \leq d - 1$ and $d' + k \in 2\mathbb{Z}$. Thus $g_k \in \sum_{j=1}^{\lfloor \frac{d}{2} \rfloor} \Pi_{d-2j,k} = \Pi_{d-2,k}$, so f is orthogonal to g_k , since $f \in E_{d,k}$. So we have just proved the inclusion $E_{d,k} \subset E_d \cap F_k$.

It remains to prove $E_d \cap F_k \subset E_{d,k}$. Let $f \in E_d \cap F_k$. Since $f \in E_d$, there exists a polynomial P of degree at most d such that f is the restriction of P to S^{2n-1} . One can use the same decomposition as above $P = \sum_{k'=-d}^d P_{k'}$. As $f \in F_k$, the restriction of $P_{k'}$ to S^{2n-1} must be 0 for all $k' \neq k$, so we can assume $P = P_k$. If P has no term of degree d , then $f \in \Pi_{d-1}$, but since $f \in E_d$, f must be orthogonal to f , and then $f = 0$, so $f \in E_{d,k}$. If $P = P_k$ has at least one term of degree d , then $k + d \in 2\mathbb{Z}$, and every monomial of P_k lies in some $M_{d',k}$ with $d' + k \in 2\mathbb{Z}$ and $d' \leq d$, thus $f \in \Pi_{d,k}$. Let $g \in \Pi_{d-2,k} \subset \Pi_{d-2} \subset \Pi_{d-1}$. Since $f \in E_d$, f is orthogonal to g , so $f \in \Pi_{d,k} \cap \Pi_{d-2,k}^\perp = E_{d,k}$. Hence $E_d \cap F_k \subset E_{d,k}$.

So the lemma is proved. \square

Now we are able to prove the following theorem, which gives the eigenvalues and eigensubspaces of the operator L :

Theorem 10 *For $n \geq 2$, the eigenvalues of L are non-positive integers, and more precisely those ones which can be written*

$$\lambda_{d,k} := -\frac{d(d+2n-2) - k^2}{2},$$

where d is a non-negative integer and k an integer such that $k + d \in 2\mathbb{Z}$ and $|k| \leq d$. For a such eigenvalue λ , the corresponding eigensubspace is $\bigoplus_{(d,k)|\lambda_{d,k}=\lambda} E_{d,k}$, which has finite dimension.

Proof : According to Lemma 9, any non-0 vector of $E_{d,k}$ belongs to E_d , and then is an eigenvector of Δ_S for the eigenvalue $-d(d+2n-2)$ and also belongs to F_k , and then is an eigenvector of Δ_H for the eigenvalue $-k^2$. Thus such a vector is an eigenvector of L for the eigenvalue $\lambda_{d,k}$. As we remarked just below the definition 8, if $n \geq 2$, such a vector does exist as soon as d and k have the same parity and $|k| \leq d$. The eigenvalue $\lambda_{d,k}$ is an integer because $d(d+2n-2) - k^2$ is an even integer if k and d are both odd integers or both even integers.

Doing as in the proof of Lemma 9, we can write each polynomial of degree less than d as a sum of polynomials of degree less than d whose restriction to S^{2n-1} belongs to F_k , so $\Pi_d = \sum_{d' \leq d, |k| \leq d'} \Pi_{d',k}$. Furthermore, we trivially have, by induction $\Pi_{d,k} = \bigoplus_{0 \leq p \leq \frac{d-|k|}{2}} E_{d-2p,k}$, so the linear span of the $E_{d,k}$ is nothing but the set of the restrictions of polynomials to S^{2n-1} , which is dense in L^2 , so Δ_H and Δ_S commute, and any eigenvalue of L is a $\lambda_{d,k}$.

It remains to check that the $\lambda_{d,k}$ are non-positive, and that for a given λ , the equation $\lambda = \lambda_{d,k}$ has a finite set of solutions among the set of pairs of integers (d, k) satisfying $k + d \in 2\mathbb{Z}$ and $|k| \leq d$. For a fixed d , the maximal possible $\lambda_{d,k}$ is obtained when $k = d$ or $-d$, so $\lambda_{d,k} \leq \lambda_{d,d} = -(n-1)d \leq 0$, so the $\lambda_{d,k}$ are non-positive. If we have $\lambda = \lambda_{d,k}$, we get $\lambda \leq -(n-1)d$, so $d \leq \frac{\lambda}{n-1}$, thus we have a finite number of possible d , and for each one a finite number of possible k . \square

Corollary 11 *The spectral gap of L is $n - 1$.*

Proof : If $n = 1$, $L = 0$, so it is trivial. If $n \geq 2$, we have $\lambda_{0,0} = 0$ and $E_{0,0}$ is the set of constant functions, and so has dimension 1. We have $\lambda_{1,1} = \lambda_{1,-1} = -(n-1)$, and for any $d \geq 2$, we have for any k such that $k + d \in 2\mathbb{Z}$ and $|k| \leq d$ $\lambda_{d,k} \leq -d(n-1) \leq -2(n-1) < \lambda_{1,1}$, so $\lambda_{1,1}$ is the greatest non-zero eigenvalue. \square

3.2 A nice orthonormal eigenvector basis

In this subsection, we give a nice orthonormal basis of $E_{d,k}$. To check that a given set of vectors of $E_{d,k}$ is an orthonormal basis, it suffices to show they are orthogonal with each other, have norm 1, and that the cardinal of the set equals the (finite) dimension of $E_{d,k}$.

We introduce some systems of coordinates to carry out the computations. We set

$$r_j := |z_j|^2 \text{ and } \theta_j := \arg(z_j) \text{ for } j \in \llbracket 1, n \rrbracket.$$

The angular coordinate θ_j is not uniquely defined if $z_j = 0$, but this only happens on a set of measure 0 with respect to σ_{2n-1} . The coordinates r_j are nonnegative and satisfy the relation

$$\sum_{j=1}^n r_j = 1.$$

So we parametrize S^{2n-1} with the $2n-1$ coordinates $\theta_1, \dots, \theta_n, r_1, \dots, r_{n-1}$, where the θ_j 's belong to $[0, 2\pi)$ and the $n-1$ -uple (r_1, \dots, r_{n-1}) belongs to

the domain D_n of \mathbb{R}^{2n-1} defined by $r_j \geq 0$ and $\sum_{j=1}^{n-1} r_j \leq 1$. This domain D_n is the closed simplex whose vertices are 0 and the $n-1$ vectors of the canonical basis of \mathbb{R}^{n-1} .

If we denote by φ the parametrization:

$$\varphi : \begin{cases} [0, 2\pi)^n \times D_n & \mapsto & S^{2n-1} \\ (\theta_1, \dots, \theta_n, r_1, \dots, r_{n-1}) & \rightarrow & (\sqrt{r_1}e^{i\theta_1}, \dots, \sqrt{r_{n-1}}e^{i\theta_{n-1}}, \sqrt{1 - \sum_{j=1}^{n-1} r_j}e^{i\theta_n}) \end{cases}$$

we have the following.

Proposition 12 *The pulled back measure $\varphi_{\#}(\sigma_{2n-1})$ of σ_{2n-1} by φ is the normalized Lebesgue measure on $[0, 2\pi)^n \times D_n$.*

Proof : We compute the matrix G of the pulled back metric, since the density of the pulled back Riemannian volume measure with respect to the Lebesgue measure is $\sqrt{\det(G)}$. We have for $1 \leq j \leq n$

$$\frac{\partial \varphi}{\partial \theta_j} = (0, \dots, 0, i\sqrt{r_j}e^{i\theta_j}, 0, \dots, 0)$$

with $r_n = 1 - \sum_{j=1}^{n-1} r_j$, and for $1 \leq j \leq n-1$

$$\frac{\partial \varphi}{\partial r_j} = (0, \dots, 0, \frac{e^{i\theta_j}}{2\sqrt{r_j}}, 0, \dots, 0, -\frac{e^{i\theta_n}}{2\sqrt{r_n}}).$$

We get then

$$G(\theta_1, \dots, \theta_n, r_1, \dots, r_{n-1}) = \begin{pmatrix} \text{Diag}(r_1, \dots, r_n) & 0 \\ 0 & \text{Diag}(\frac{1}{r_1}, \dots, \frac{1}{r_{n-1}}) + \frac{1}{r_n} \mathbf{1}_{n-1} \end{pmatrix},$$

where $\text{Diag}(a_1, \dots, a_k)$ is the diagonal matrix whose diagonal coefficients are a_1, \dots, a_k , and $\mathbf{1}_k$ is the $k \times k$ square matrix whose coefficients all equal to 1.

The determinant of G is thus

$$\det(G) = r_1 \dots r_n \begin{vmatrix} \frac{1}{r_1} + \frac{1}{r_n} & \frac{1}{r_n} & \dots & \frac{1}{r_n} \\ \frac{1}{r_n} & \frac{1}{r_2} + \frac{1}{r_n} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{r_n} \\ \frac{1}{r_n} & \dots & \frac{1}{r_n} & \frac{1}{r_{n-1}} + \frac{1}{r_n} \end{vmatrix} = r_1 \dots r_n \begin{vmatrix} \frac{1}{r_1} + \frac{1}{r_n} & \frac{1}{r_n} & \dots & \frac{1}{r_n} \\ -\frac{1}{r_1} & & & \\ \vdots & & & \\ -\frac{1}{r_1} & & & \end{vmatrix} \text{Diag}(\frac{1}{r_2}, \dots, \frac{1}{r_{n-1}}),$$

where we have subtracted the first line of the matrix to the other ones to get the last equality.

Now we use the formula $\det(M) = \sum_{\sigma \in \mathfrak{S}_{n-1}} \epsilon(\sigma) \prod_{j=1}^{n-1} m_{j, \sigma(j)}$ for the determinant of a $(n-1) \times (n-1)$ square matrix. The last matrix has a lot of zeros, and to get a non-zero term, the permutation σ must satisfy $\sigma(j) = 1$ or j for every $2 \leq j \leq n-1$, which is only possible if σ is the identity or a transposition $(1, j)$. Thus we get:

$$\det(G) = r_1 \dots r_n \left(\left(\frac{1}{r_1} + \frac{1}{r_n} \right) \prod_{j=2}^{n-1} \frac{1}{r_j} - \sum_{j=2}^{n-1} \left(- \prod_{k \neq j} \frac{1}{r_k} \right) \right) = r_n + r_1 + \sum_{j=2}^{n-1} r_j = 1.$$

Hence the pulled back Riemannian volume measure on S^{2n-1} by φ is the Lebesgue measure on $[0, 2\pi)^n \times D_n$. We can remark that the Lebesgue measure of $[0, 2\pi)^n \times D_n$ is $\frac{(2\pi)^n}{(n-1)!}$, which is indeed the Riemannian volume of S^{2n-1} .

We only have to renormalize the measures to get the proposition. \square

The domain D_n can be parametrized on a simple way. For $1 \leq j \leq n-1$, we set $X_j := \frac{\sum_{k=j+1}^n r_k}{\sum_{k=j}^n r_k} \in [0, 1]$. The X_j 's are all well defined as soon as r_{n-1} and r_n are not both 0, which occurs on a subset of Lebesgue measure 0. We have $r_j = (1 - X_j) \prod_{k=1}^{j-1} X_k$ for $1 \leq j \leq n-1$. The application

$$\psi : \begin{cases} [0, 1]^{n-1} & \mapsto & D_n \\ (X_1, \dots, X_{n-1}) & \rightarrow & (1 - X_1, X_1(1 - X_2), \dots, X_1 \dots X_{n-2}(1 - X_{n-1})) \end{cases}$$

is a diffeomorphism from $(0, 1)^{n-1}$ to the interior of D_n . The matrix of its differential is lower triangular, and its diagonal coefficients are $-1, -X_1, -X_1 X_2, \dots, -X_1 \dots X_{n-2}$. So its determinant is $(-1)^n \prod_{j=1}^{n-1} X_j^{n-1-j}$, and thus we have:

$$d\psi_{\sharp}(\mathbb{1}_{D_n} \text{Leb}^{n-1}) = \prod_{j=1}^{n-1} X_j^{n-1-j} \mathbb{1}_{[0,1]}(X_j) dX_j. \quad (1)$$

To give the expression of our eigenvectors, we define the following polynomials:

Definition 13 *Let a and b be two real numbers greater than -1 . For any $k \in \mathbb{N}$, we set $P_{a,b,d}$ the polynomial of degree d obtained by performing the Gram-Schmidt orthogonalisation algorithm on the canonical basis of polynomials for the following scalar product*

$$\langle f, g \rangle_{a,b} = \int_0^1 \bar{f}(x) g(x) x^a (1-x)^b dx.$$

Remark 14 The polynomials $P_{a,b,d}$ have the following expressions:

$$\begin{aligned}
P_{a,b,d}(x) &= (-1)^d \sqrt{\frac{(a+b+2d+1)\Gamma(a+b+d+1)}{d!\Gamma(a+d+1)\Gamma(b+d+1)}} \frac{1}{x^a(1-x)^b} \frac{d^d}{dx^d} (x^{a+d}(1-x)^{b+d}) \\
&= (-1)^d \sqrt{\frac{(a+b+2d+1)\Gamma(a+b+d+1)}{d!\Gamma(a+d+1)\Gamma(b+d+1)}} \sum_{k=0}^d \frac{(-1)^k d!\Gamma(a+d+1)\Gamma(b+d+1)}{k!(d-k)!\Gamma(a+k+1)\Gamma(b+d-k+1)} x^k (1-x)^{d-k} \\
&= (-1)^d \sqrt{\frac{(a+b+2d+1)\Gamma(a+b+d+1)}{d!\Gamma(a+d+1)\Gamma(b+d+1)}} \sum_{k=0}^d \frac{(-1)^k d!\Gamma(a+d+1)\Gamma(a+b+d+k+1)}{k!(d-k)!\Gamma(a+k+1)\Gamma(a+b+d+1)} x^k \\
&= (-1)^d P_{b,a,d}(1-x),
\end{aligned}$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is the Euler's gamma function.

Proof: Let a and b be two real numbers greater than -1 . We first show that the functions $Q_{a,b,d}(x) := \frac{1}{x^a(1-x)^b} \frac{d^d}{dx^d} (x^{a+d}(1-x)^{b+d})$ are polynomials of degree d and are orthogonal to each other for $\langle \cdot, \cdot \rangle_{a,b}$. After that, we will compute $\|Q_{a,b,d}\|_{a,b}^2$ to renormalize $Q_{a,b,d}$.

We have

$$\begin{aligned}
Q_{a,b,d}(x) &= \frac{1}{x^a(1-x)^b} \frac{d^d}{dx^d} (x^{a+d}(1-x)^{b+d}) \\
&= \sum_{k=0}^d \binom{d}{k} \frac{1}{x^a} \frac{d^{d-k}}{dx^{d-k}} (x^{a+d}) \frac{1}{(1-x)^b} \frac{d^k}{dx^k} ((1-x)^{b+d}) \\
&= \sum_{k=0}^d (-1)^k \binom{d}{k} \frac{\Gamma(a+d+1)}{\Gamma(a+k+1)} x^k \frac{\Gamma(b+d+1)}{\Gamma(b+d-k+1)} (1-x)^{d-k}, \quad (2)
\end{aligned}$$

so $Q_{a,b,d}$ is a polynomial of degree at most d .

Let $0 \leq d_1 \leq d_2$. We have:

$$\begin{aligned}
\langle Q_{a,b,d_1}, Q_{a,b,d_2} \rangle_{a,b} &= \int_0^1 x^a (1-x)^b Q_{a,b,d_1}(x) Q_{a,b,d_2}(x) dx \\
&= \int_0^1 Q_{a,b,d_1}(x) \frac{d^{d_2}}{dx^{d_2}} (x^{a+d_2} (1-x)^{b+d_2}) dx \\
&= \sum_{k=1}^{d_2} \left[(-1)^{k-1} Q_{a,b,d_1}^{(k-1)}(x) \frac{d^{d_2-k}}{dx^{d_2-k}} (x^{a+d_2} (1-x)^{b+d_2}) \right]_{x=0}^1 \\
&\quad + (-1)^{d_2} \int_0^1 Q_{a,b,d_1}^{(d_2)}(x) x^{a+d_2} (1-x)^{b+d_2} dx \quad (n-2 \text{ integrations by parts}) \\
&= \sum_{k=1}^{d_2} \left[(-1)^{k-1} Q_{a,b,d_1}^{(k-1)}(x) x^{a+k} (1-x)^{b+k} Q_{a+k,b+k,d_2-k}(x) \right]_{x=0}^1 \\
&\quad + (-1)^{d_2} \int_0^1 Q_{a,b,d_1}^{(d_2)}(x) x^{a+d_2} (1-x)^{b+d_2} dx \\
&= (-1)^{d_2} \int_0^1 Q_{a,b,d_1}^{(d_2)}(x) x^{a+d_2} (1-x)^{b+d_2} dx, \tag{3}
\end{aligned}$$

where the last equality occurs because when $k \geq 1$, $a+k$ and $b+k$ are positive, so each term of the sum is 0. If $d_1 < d_2$, one gets 0 because Q_{a,b,d_1} is a polynomial of degree at most d_1 , so differentiating it d_2 times, one gets 0. So Q_{a,b,d_1} and Q_{a,b,d_2} are orthogonal for $\langle \cdot, \cdot \rangle_{a,b}$, as announced.

Using equation 3 with $d_1 = d_2 = d$, one has

$$\|Q_{a,b,d}\|_{a,b}^2 = (-1)^d \frac{d! \text{lc}(Q_{a,b,d}) \Gamma(a+d+1) \Gamma(b+d+1)}{\Gamma(a+b+2d+2)},$$

where $\text{lc}(Q_{a,b,d})$ is the leading coefficient of the polynomial $Q_{a,b,d}$.

According to equation 2, we have

$$\begin{aligned}
Q_{a,b,d}(x) &= \sum_{k=0}^d (-1)^k \binom{d}{k} \frac{\Gamma(a+d+1)}{\Gamma(a+k+1)} x^k \frac{\Gamma(b+d+1)}{\Gamma(b+d-k+1)} (1-x)^{d-k} \\
&= \sum_{k=0}^d \sum_{l=0}^{d-k} (-1)^{k+l} \binom{d}{k} \binom{d-k}{l} \frac{\Gamma(a+d+1)\Gamma(b+d+1)}{\Gamma(a+k+1)\Gamma(b+d-k+1)} x^{k+l} \\
&= \sum_{k'=0}^d (-x)^{k'} \binom{d}{k'} \frac{\Gamma(a+d+1)\Gamma(b+d+1)}{\Gamma(a+b+d+1)} \sum_{k=0}^{k'} \binom{k'}{k} \frac{\Gamma(a+b+d+1)}{\Gamma(a+k+1)\Gamma(b+d-k+1)} \\
&\quad (\text{change of variables } k' = k+l) \\
&= \sum_{k'=0}^d (-x)^{k'} \binom{d}{k'} \frac{\Gamma(a+d+1)\Gamma(b+d+1)}{\Gamma(a+b+d+1)} \frac{\Gamma(a+b+d+k'+1)}{\Gamma(b+d+1)\Gamma(a+k'+1)},
\end{aligned} \tag{4}$$

where the last equality occurs because of the identity

$$\sum_{k=0}^{k'} \binom{k'}{k} \frac{\Gamma(z_1+1)}{\Gamma(z_2-k+1)\Gamma(z_1-z_2+k+1)} = \frac{\Gamma(z_1+k'+1)}{\Gamma(z_2+1)\Gamma(z_1-z_2+k'+1)},$$

which holds for any $k' \in \mathbb{N}$, and for any z_1 and z_2 in \mathbb{C} such that z_1 is not a negative integer.

This identity can be easily shown by induction on k' , using the fact that $\binom{k'}{k} = \binom{k'-1}{k-1} + \binom{k'-1}{k}$, but it also has a combinatoric interpretation when z_1 is a nonnegative integer and z_2 is an integer not less than k' . In this case, the identity can be written as

$$\sum_{k=0}^{k'} \binom{k'}{k} \binom{z_1}{z_2-k} = \binom{z_1+k'}{z_2}.$$

The right hand term is the number of ways to choose z_2 elements among a set of z_1+k' elements, and the generic term of the sum of the left hand term is the number of ways to choose z_2 elements among a set of z_1+k' elements with the constraint that k of them belong to a fixed subset of cardinal k' .

So according to equation 4, we have $\text{lc}(Q_{a,b,d}) = (-1)^d \frac{\Gamma(a+b+2d+1)}{\Gamma(a+b+d+1)}$. Thus we have

$$\|Q_{a,b,d}\|_{a,b}^2 = \frac{d!\Gamma(a+b+1)\Gamma(b+d+1)}{(a+b+2d+1)\Gamma(a+b+d+1)}.$$

Hence $P_{a,b,d} = \pm \sqrt{\frac{(a+b+2d+1)\Gamma(a+b+d+1)}{d!\Gamma(a+d+1)\Gamma(b+d+1)}} Q_{a,b,d}$. Because of its definition, the leading coefficient of $P_{a,b,d}$ is positive, so finally we have

$$P_{a,b,d} = (-1)^d \sqrt{\frac{(a+b+2d+1)\Gamma(a+b+d+1)}{d!\Gamma(a+d+1)\Gamma(b+d+1)}} Q_{a,b,d}.$$

Replacing $Q_{a,b,d}$ with its definition in this last equality gives us the first equality of Remark 14. For the second one, one uses equation 2, and for the third one, one uses equation 4. The last equality of Remark 14 is a straightforward consequence of the first one or the second one. \square

Now our eigenvectors will be defined on the following way:

Definition 15 *Let $\alpha_1, \dots, \alpha_n$ be n integers, and d_1, \dots, d_{n-1} be $n-1$ non-negative integers. We set*

$$F_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}(\theta_1, \dots, \theta_n, X_1, \dots, X_{n-1}) = \frac{1}{\sqrt{(n-1)!}} \prod_{j=1}^n \left[((1-X_j) \prod_{k=1}^{j-1} X_k)^{\frac{|\alpha_j|}{2}} e^{i\alpha_j \theta_j} \right] \prod_{j=1}^{n-1} \left[P_{a_j, |\alpha_j|, d_j}(X_j) \prod_{k=1}^{j-1} X_k^{d_j} \right],$$

where we take $X_n = 0$ in the last term of the first product, and for $1 \leq j \leq n-1$,

$$a_j = n-1-j + |\alpha_n| + \sum_{k=j+1}^{n-1} (|\alpha_k| + 2d_k).$$

The function $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$ will be defined on almost all S^{2n-1} by

$$f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}(z_1, \dots, z_n) = F_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}(\theta_1, \dots, \theta_n, X_1, \dots, X_{n-1}),$$

with $\theta_j = \arg(z_j)$ and $X_j = \frac{\sum_{k=j+1}^n |z_k|^2}{\sum_{k=j}^n |z_k|^2}$.

Then we have

Theorem 16 *The functions $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$, with $\sum_{j=1}^n \alpha_j = k$ and $\sum_{j=1}^n |\alpha_j| + 2 \sum_{j=1}^{n-1} d_j = d$, form an orthonormal basis of $E_{d,k}$.*

Proof : The fact that $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$ belongs to $F_{\alpha_1 + \dots + \alpha_n}$ is rather trivial, since when one multiplies z by $e^{i\theta}$, one just adds θ (modulo 2π) to each θ_j , and so the value of the function is multiplied by $e^{i(\alpha_1 + \dots + \alpha_n)\theta}$.

To prove that $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$ is the restriction to S^{2n-1} of a polynomial function, it suffices to check that the j -th term of the first product in the definition of $F_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$ is nothing but $z_j^{\alpha_j}$ if α_j is nonnegative, and $\bar{z}_j^{-\alpha_j}$ else, and that the j -th term of the second product is an homogenous

polynomial of degree d_j of the $|z_l|^2$'s (and thus an homogenous polynomial of degree $2d_j$ of the z_l 's and \bar{z}_l 's). More precisely, according to Remark 14, it is

$$(-1)^{d_j} \sqrt{\frac{(a_j+|\alpha_j|+2d_j+1)(a_j+|\alpha_j|+d_j)!}{d_j!(a_j+d_j)!}} \sum_{h=0}^{d_j} \frac{(-1)^h d_j!(a_j+d_j)! (|\alpha_j|+d_j)!}{h!(d_j-h)!(a_j+h)! (|\alpha_j|+d_j-h)!} (|z_{j+1}|^2 + \dots + |z_n|^2)^h (|z_j|^2)^{d_j-h}.$$

Thus the product of these guys is an homogenous polynomial of degree $\sum_{j=1}^n |\alpha_j| + 2 \sum_{j=1}^{n-1} d_j$.

Let us check the orthonormality of these functions. According to Proposition 12 and equation 1, we have:

$$\begin{aligned} \int_{S^{2n-1}} \overline{f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}} f_{\alpha'_1, \dots, \alpha'_n, d'_1, \dots, d'_{n-1}} d\sigma_{2n-1} &= (n-1)! \int \overline{F_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}(\theta_1, \dots, \theta_n, X_1, \dots, X_{n-1})} \\ &\quad F_{\alpha'_1, \dots, \alpha'_n, d'_1, \dots, d'_{n-1}}(\theta_1, \dots, \theta_n, X_1, \dots, X_{n-1}) \\ &\quad X_1^{n-2} \dots X_{n-2} dX_1 \dots dX_{n-1} \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} \\ &= \prod_{j=1}^n \int_0^{2\pi} e^{i(\alpha'_j - \alpha_j)\theta_j} \frac{d\theta_j}{2\pi} \\ \prod_{j=1}^{n-1} \int_0^1 (1-X_j) \frac{|\alpha_j|+|\alpha'_j|}{2} X_j^{n-1-j+\sum_{k=j+1}^n |\alpha_k|+|\alpha'_k|} X_j^{n-1-j+\sum_{k=j+1}^n d_k+d'_k} P_{\alpha_j, |\alpha_j|, d_j}(X_j) P_{\alpha'_j, |\alpha'_j|, d'_j}(X_j) dX_j. \end{aligned}$$

If the $\alpha_j \neq \alpha'_j$ for some j , the j -th integral of the first product is 0. Now assume that $\alpha_j = \alpha'_j$ for all $1 \leq j \leq n$. If for some j , we have $d_j \neq d'_j$, let's assume that j is the largest index satisfying this inequality. Then the j -th integral of the second product is

$$\int_0^1 (1-X_j)^{|\alpha_j|} X_j^{a_j} P_{\alpha_j, |\alpha_j|, d_j}(X_j) P_{\alpha_j, |\alpha_j|, d'_j}(X_j),$$

and this integral is 0 because of the orthogonality of $P_{\alpha_j, |\alpha_j|, d_j}$ and $P_{\alpha_j, |\alpha_j|, d'_j}$ for the scalar product $\langle \cdot, \cdot \rangle_{\alpha_j, |\alpha_j|}$. Finally, if $\alpha_j = \alpha'_j$ and $d_j = d'_j$, all the integrals are 1, so our functions have norm 1.

It remains to prove that $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$ is orthogonal to every polynomial of degree less than $d = \sum_{j=1}^n |\alpha_j| + 2 \sum_{j=1}^{n-1} d_j$, and that there are $\dim(E_{d,k})$ $(2n-1)$ -uples $(\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1})$ satisfying $\sum_{j=1}^n |\alpha_j| + 2 \sum_{j=1}^{n-1} d_j = d$ and $\sum_{j=1}^{n-1} \alpha_j = k$.

For the first fact, we use induction on d . Assume the theorem is proved for $d' < d$. Then the set of the $f_{\alpha'_1, \dots, \alpha'_n, d'_1, \dots, d'_{n-1}}$ such that $\sum_{j=1}^n |\alpha'_j| + 2 \sum_{j=1}^{n-1} d'_j < d$ is an orthonormal basis of Π_{d-1} , and each of this functions is orthogonal to $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}$. So $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}} \in E_d$, and thus from Lemma 9, we get $f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}} \in E_{d,k}$.

For the second fact, we associate to each $(2n-1)$ -uple $(\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1})$ that satisfies $\sum_{j=1}^n |\alpha_j| + 2 \sum_{j=1}^{n-1} d_j = d$ and $\sum_{j=1}^{n-1} \alpha_j = k$ the monomial

$\prod_{j=1}^n z_j^{\alpha'_j} \bar{z}_j^{\beta'_j}$, where for each $1 \leq j \leq n$ we take $\alpha'_j = \alpha_j + d_j$ and $\beta'_j = d_j$ if $\alpha_j \geq 0$, and $\alpha'_j = d_j$ and $\beta'_j = -\alpha_j + d_j$ if $\alpha_j < 0$, with $d_n = 0$. This is a bijection between our set of $(2n-1)$ -uples and the monomials in $M_{d,k}$ which satisfy $\alpha'_n = 0$ or $\beta'_n = 0$. The monomials in $M_{d,k}$ which satisfy $\alpha'_n > 0$ and $\beta'_n > 0$ are just $z_n \bar{z}_n$ times a monomial of $M_{d-2,k}$. So the cardinal of our set of $(2n-1)$ -uples is $\#M_{d,k} - \#M_{d-2,k}$, which is precisely the dimension of $E_{d,k}$. So the theorem is proved. \square

Remark 17 *If one wants a real valued eigenvectors basis, we can replace the pairs of conjugated complex eigenvectors $(f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}, f_{-\alpha_1, \dots, -\alpha_n, d_1, \dots, d_{n-1}})$ by the pairs $(\sqrt{2}\Re(f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}), \sqrt{2}\Im(f_{\alpha_1, \dots, \alpha_n, d_1, \dots, d_{n-1}}))$ when the α_j 's are not all 0.*

4 Calculus of κ and $\tilde{\kappa}$

Since our diffusion is degenerated, we can't use the formulas given in [4] for the close-points limit curvatures $\kappa(x, u)$ and $\tilde{\kappa}(x, u)$. We will compute $\kappa(x, y)$ and $\tilde{\kappa}(x, y)$ and take the limit when $d(x, y) \rightarrow 0$.

Now we assume that x and y are fixed on S^{2n-1} , with $y \neq \pm x$. Then there exists an unit norm vector $z \in \mathbb{C}^n$, \mathbb{C} -orthogonal to x , such that $u(x, y) = \cos(\alpha)ix + \sin(\alpha)z$, with α the angle between ix and $u(x, y)$ (z is unique if $\alpha \notin \{0, \pi\}$).

We choose e_3, e_4, \dots, e_n an orthonormal \mathbb{C} -basis of the \mathbb{C} -orthogonal of $\text{Vect}_{\mathbb{C}}(x, z)$. The orthonormal basis of $T_x S^{2n-1}$ we choose for our computations will be

$$\mathcal{B}(x) = (ix, z, iz, e_3, ie_3, e_4, ie_4, \dots, e_n, ie_n).$$

If we note $d = d(x, y) \in (0, \pi)$, we have $y = \cos(d)x + \sin(d)u(x, y)$ and $u(y, x) = -\cos(d)u(x, y) + \sin(d)x$. Then we get $u(y, x) = -\cos(\alpha)iy - \sin(\alpha)z'$, with

$$z' = \cos(d)z - \sin(d)\cos(\alpha)iz - \sin(d)\sin(\alpha)x.$$

z' is \mathbb{C} -orthogonal to y , its norm is 1. Then $\text{Vect}_{\mathbb{C}}(y, z') = \text{Vect}_{\mathbb{C}}(x, z)$, because the first one is clearly included in the second one and they have both dimension 2. So the following basis of $T_y S^{2n-1}$ is orthonormal:

$$\mathcal{B}(y) = (iy, z', iz', e_3, ie_3, \dots, e_n, ie_n).$$

The matrix of $A(x)$ in the basis $\mathcal{B}(x)$ is the same that the matrix of $A(y)$ in $\mathcal{B}(y)$. They both are

$$A = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ & 0 & & & I_{2n-4} \end{pmatrix}.$$

It remains to compute the matrices of $q^{(1)}$, $q^{(2)}$ and $q^{(12)}$. For $v \in \mathbb{T}_x S^{2n-1}$ and $w \in \mathbb{T}_y S^{2n-1}$, we have

$$\exp_x(\varepsilon v) = \left(1 - \frac{\varepsilon^2 \|v\|^2}{2}\right) x + \varepsilon v + O(\varepsilon^3) \text{ and } \exp_y(\varepsilon w) = \left(1 - \frac{\varepsilon^2 \|w\|^2}{2}\right) y + \varepsilon w + O(\varepsilon^3).$$

Then, we have

$$\begin{aligned} d(\exp_x(\varepsilon v), \exp_y(\varepsilon w)) &= \arccos \left(g_0 \left(\left(1 - \frac{\varepsilon^2 \|v\|^2}{2}\right) x + \varepsilon v, \left(1 - \frac{\varepsilon^2 \|w\|^2}{2}\right) y + \varepsilon w \right) + O(\varepsilon^3) \right) \\ &= \arccos(\cos(d) + \varepsilon(g_0(x, w) + g_0(v, y)) + \frac{\varepsilon^2}{2}(2g_0(v, w) - (\|v\|^2 + \|w\|^2)\cos(d)) + O(\varepsilon^3)) \\ &= d - \varepsilon \frac{g_0(x, w) + g_0(y, v)}{\sin(d)} + \frac{\varepsilon^2}{2} \left(\frac{(\|v\|^2 + \|w\|^2)\cos(d) - 2g_0(v, w)}{\sin(d)} - \frac{\cos(d)(g_0(x, w) + g_0(y, v))^2}{\sin^3(d)} \right) + O(\varepsilon^3). \end{aligned}$$

As we have $x = \cos(d)y + \sin(d)u(y, x)$ and $y = \cos(d)x + \sin(d)u(x, y)$, the linear term becomes

$$\begin{aligned} \frac{g_0(x, w) + g_0(y, v)}{\sin(d)} &= \frac{\cos(d)g_0(y, w) + \sin(d)g_0(u(y, x), w) + \cos(d)g_0(x, v) + \sin(d)g_0(u(x, y), v)}{\sin(d)} \\ &= g_0(u(y, x), w) + g_0(u(x, y), v) \end{aligned}$$

since $v \in \mathbb{T}_x S^{2n-1}$ and $w \in \mathbb{T}_y S^{2n-1}$. That is well what was expected in the Taylor expansion of the distance.

We identify the second order terms after simplification, and get:

$$\begin{aligned} q^{(1)}(v, v) &= \frac{\cos(d)}{d \sin(d)} (\|v\|^2 - g_0(v, u(x, y)))^2, \quad q^{(2)}(w, w) = \frac{\cos(d)}{d \sin(d)} (\|w\|^2 - (g_0(w, u(y, x))))^2 \\ \text{and } q^{(12)}(v, w) &= \frac{1}{d \sin(d)} (-g_0(v, w) - \cos(d)g_0(v, u(x, y))g_0(w, u(y, x))). \end{aligned}$$

To get the matrices of $q^{(1)}$, $q^{(2)}$ and $q^{(12)}$ in the bases $\mathcal{B}(x)$ and $\mathcal{B}(y)$, we only have to apply these quadratic forms to the vectors of the bases and so we get the coefficients.

The matrix of $q^{(1)}$ in the basis $\mathcal{B}(x)$ is the same that the matrix of $q^{(2)}$ in the basis $\mathcal{B}(y)$ and equals to $\frac{\cos(d)}{d \sin(d)} Q$, where

$$Q = \begin{pmatrix} \sin^2(\alpha) & -\sin(\alpha) \cos(\alpha) & 0 & & \\ -\sin(\alpha) \cos(\alpha) & \cos^2(\alpha) & 0 & 0 & \\ 0 & 0 & 1 & & \\ & 0 & & & I_{2n-4} \end{pmatrix}.$$

The matrix of $q^{(12)}$ in the bases $\mathcal{B}(x)$ and $\mathcal{B}(y)$ is $\frac{1}{d \sin(d)} Q_{12}$, with

$$Q_{12} = \begin{pmatrix} -\cos(d) \sin^2(\alpha) & \cos(d) \sin(\alpha) \cos(\alpha) & \sin(d) \sin(\alpha) & & \\ \cos(d) \sin(\alpha) \cos(\alpha) & -\cos(d) \cos^2(\alpha) & -\sin(d) \cos(\alpha) & 0 & \\ -\sin(d) \sin(\alpha) & \sin(d) \cos(\alpha) & -\cos(d) & & \\ & 0 & & & -I_{2n-4} \end{pmatrix}.$$

To compute $\tilde{\kappa}$, we also need the expressions of $\tilde{A}(x)$ and $\tilde{A}(y)$. Here, we have $A(x)g(x)u(x, y) = \sin(\alpha)z$, $A(y)g(y)u(y, x) = -\sin(\alpha)z'$ and thus $\rho(x, y) = \sin^2(\alpha)$. So we need to compute $\tilde{\kappa}$ only if $\alpha \notin \{0, \pi\}$, in which case, the matrices of $\tilde{A}(x)$ and $\tilde{A}(y)$ in the bases $\mathcal{B}(x)$ and $\mathcal{B}(y)$ both equal to the very nice

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & & \\ & 0 & & & I_{2n-4} \end{pmatrix}.$$

Then, according to our formulas, we have

$$\kappa(x, y) = 0 - \frac{\cos(d)}{2d \sin(d)} (\text{tr}(AQ) + \text{tr}(AQ)) + \frac{1}{d \sin(d)} \text{tr} \left(\sqrt{AQ_{12}AQ_{12}^T} \right)$$

for any α and

$$\tilde{\kappa}(x, y) = 0 - \frac{\cos(d)}{2d \sin(d)} (\text{tr}(AQ) + \text{tr}(AQ)) - \frac{\sin^2(\alpha)q^{(12)}(z, z')}{\sin^2(\alpha)} + \frac{1}{d \sin(d)} \text{tr} \left(\sqrt{\tilde{A}Q_{12}\tilde{A}Q_{12}^T} \right)$$

in the case when $\sin(\alpha) \neq 0$, i.e in the case when x and y are not on the same Hopf fiber. We have $\text{tr}(AQ) = \cos^2(\alpha) + 2n - 3$, and $q^{(12)}(z, z') = -\frac{\cos(d) \cos^2(\alpha)}{d \sin(d)}$.

The eigenvalues of the matrix $AQ_{12}AQ_{12}^T$ are 0, r_1 , r_2 and 1 with multiplicity $2n - 4$, with

$$r_1 = \frac{\cos^2(d) \sin^4(\alpha) + 2 \cos^2(\alpha) - |\cos(d)| \sin^2(\alpha) \sqrt{\cos^2(d) \sin^4(\alpha) + 4 \cos^2(\alpha)}}{2}$$

and

$$r_2 = \frac{\cos^2(d) \sin^4(\alpha) + 2 \cos^2(\alpha) + |\cos(d)| \sin^2(\alpha) \sqrt{\cos^2(d) \sin^4(\alpha) + 4 \cos^2(\alpha)}}{2}.$$

Using a simple trick, we can simplify $\sqrt{r_1} + \sqrt{r_2}$: we have $(\sqrt{r_1} + \sqrt{r_2})^2 = r_1 + r_2 + 2\sqrt{r_1 r_2}$. Thus, we get

$$\begin{aligned} \sqrt{r_1} + \sqrt{r_2} &= \sqrt{r_1 + r_2 + 2\sqrt{r_1 r_2}} = \sqrt{\cos^2(d) \sin^4(\alpha) + 2 \cos^2(\alpha) + 2\sqrt{\cos^4(\alpha)}} \\ &= \sqrt{\cos^2(d) \sin^4(\alpha) + 4 \cos^2(\alpha)}. \end{aligned}$$

The eigenvalues of $\tilde{A}Q_{12}\tilde{A}Q_{12}^T$ are 0 with multiplicity 2, $\cos^2(d)$ and 1 with multiplicity $2n - 4$.

So finally we get the following expressions of κ and $\tilde{\kappa}$:

$$\begin{aligned} \kappa(x, y) &= \frac{\sqrt{\cos^2(d) \sin^4(\alpha) + 4 \cos^2(\alpha)} + 2n - 4 - \cos(d)(\cos^2(\alpha) + 2n - 3)}{d \sin(d)} \\ \tilde{\kappa}(x, y) &= \frac{(2n - 4)(1 - \cos(d)) + |\cos(d)| - \cos(d)}{d \sin(d)}. \end{aligned}$$

To get $\kappa(x, u)$ with $u \in T_x S^{2n-1}$, we just make d tend to 0. The denominator in the formulas is equivalent to d^2 , so we have to compute the second order Taylor expansion of the numerator. For $\tilde{\kappa}$, we get

$$\tilde{\kappa}(x, u) = n - 2$$

for every u not colinear to ix . For κ , we have:

$$\begin{aligned} \sqrt{\cos^2(d) \sin^4(\alpha) + 4 \cos^2(\alpha)} &= \sqrt{(1 - d^2)(1 - \cos^2(\alpha))^2 + 4 \cos^2(\alpha) + O(d^4)} \\ &= \sqrt{(1 + \cos^2(\alpha))^2 - d^2(1 - \cos^2(\alpha))^2 + O(d^4)} \\ &= 1 + \cos^2(\alpha) - d^2 \frac{(1 - \cos^2(\alpha))^2}{2(1 + \cos^2(\alpha))} + O(d^4). \end{aligned}$$

So we get

$$\kappa(x, u) = \frac{2n - 3}{2} + \frac{3 \cos^2(\alpha) - 1}{2(1 + \cos^2(\alpha))}.$$

We see that $\kappa(x, u)$ is not the restriction to the unit sphere of a quadratic form, but it is however the quotient of two quadratic forms. The maximal value of κ is $n - 1$, when $\alpha = 0$ or π , which is the half (remind the $\frac{1}{2}$ factor

in L) of the Ricci curvature of S^{2n-1} . The minimal value of κ is $n - 2$, when $\alpha = \frac{\pi}{2}$, so exactly the value of $\tilde{\kappa}$. Remark that $\tilde{\kappa}$ is not a continuous function of u , because it jumps from the value $n - 1$ when u is colinear to ix to the value $n - 2$ everywhere else. This is due to the fact that our optimal coupling defining $\tilde{\kappa}$ is not continuous.

5 unicity and regularity of the optimal couplings

We first recall what happens in the non degenerate case.

We saw in [4] that the unicity of the covariance of the optimal coupling is equivalent to $\text{rk}(A(x)q^{(12)}A(y)q^{(12)T}) = \min(\text{rk}(A(x)), \text{rk}(A(y)))$, with $\text{rk}(M)$ the rank of the matrix M . In this case, the optimal coupling between the tangent spaces is unique and linear.

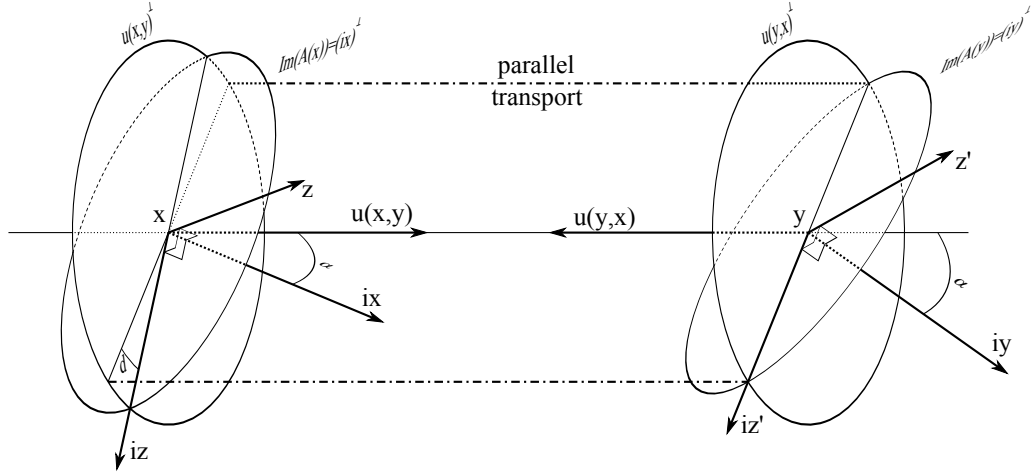
In the case of non degenerated diffusions, the optimal coupling defining κ was not unique. Outside the cut-locus of the manifold, the set of optimal covariances was a segment $[C^-, C^+]$. An optimal coupling for κ realized a kind of parallel transport between the $A(x)$ -orthogonal projection of v on the $A(x)$ -orthogonal of u on one side, and the $A(y)$ -orthogonal projection of w on the $A(y)$ -orthogonal of u on the other side. The coupling between the remaining u components of v and w could be freely chosen.

In the case of a non degenerated diffusion which satisfied (H) , the optimal coupling under the constraint that the variance terms vanishes was unique and linear (for x and y close enough again). It sent $A(x)g(x)u$ on $A(y)g(y)u$ and realized a kind of parallel transport between the g orthogonals of u in $T_x\mathcal{M}$ and $T_y\mathcal{M}$ (not the true one, because the gaussian measures are not exactly the same ones). This linear coupling varied smoothly in a neighborhood of each point such that $\text{rk}(q^{(12)}) = \dim(\mathcal{M}) - 1$.

In our degenerated diffusion, we have $r_1 r_2 = \cos^4(\alpha)$, so $\text{rk}(A(x)q^{(12)}A(y)q^{(12)T}) = 2n - 2$ if $\alpha \neq \frac{\pi}{2}$, in which case the optimal coupling in the sense of κ is unique and linear. If $\alpha = \frac{\pi}{2}$, then $r_1 = 0$ and the optimal coupling is not unique, and if furthermore $d = \frac{\pi}{2}$, then $r_2 = 0$ too and the set of optimal couplings is even greater.

With the constraint to cancel the variance term, the optimal coupling in the sense of $\tilde{\kappa}$ is unique if and only if $\text{rk}(\tilde{A}Q_{12}\tilde{A}Q_{12}^T) = 2n - 3$, which is the case if and only if $d \neq \frac{\pi}{2}$.

The following figure illustrates how do the Hopf fibers and thus the hyperplanes of the diffusion evolve along a geodesic:



There are $2n - 4$ more dimensions not represented on this figure, which correspond to the e_j 's and ie_j 's ($j = 3, \dots, n$), but these vectors are orthogonal to those ones represented, and are left invariant by the parallel transport along the geodesic joining x to y . While doing this parallel transport, we see the Hopf fiber, which is directed by ix , revolving around the space of codimension 2 generated by $u(x, y)$ and the e_j 's and the ie_j 's, so the hyperplane $\text{Im}(A(x))$ in which the diffusion moves revolves on the same way.

The couplings corresponding to the extremal covariances are the isometries between $\text{Im}(A(x))$ and $\text{Im}(A(y))$. Such a coupling will be optimal if he maximizes the mean of the scalar product of the orthogonal projections of v and w on $u(x, y)^\perp$ and $u(y, x)^\perp$, these both spaces being identified via the parallel transport, with v and w being standard gaussian vectors in $A(x)$ and $A(y)$.

In the case $\alpha = \frac{\pi}{2}$, one optimal coupling consists in making a rotation of angle d or $\pi - d$ (depending on whether or not $d < \frac{\pi}{2}$), but the fact that the functional to minimize does not depend on the u component allows to take the coupling we can obtain from this last one by a reflection with respect to u^\perp . If $\alpha = d = \frac{\pi}{2}$, then any coupling preserving the e_j 's and the ie_j 's is optimal.

For $\alpha \neq 0, \pi$, the supplementary constraint that the covariance term of the distance cancels imposes that the coupling sends z to z' . It remains to make the coupling between the remaining u^\perp components of v and w . The optimal coupling in the sense of $\tilde{\kappa}$ leaves the e_j 's and ie_j 's invariant, sends z to z' , as previously said, and sends iz to iz' if $d \leq \frac{\pi}{2}$ (in which case it is a rotation), or to $-iz'$ if $d \geq \frac{\pi}{2}$ (in which case it is a reflection).

When $\alpha \rightarrow 0$ or π , with a fixed $d \neq 0, \frac{\pi}{2}, \pi$, the optimal coupling in

the sense of $\tilde{\kappa}$ tends to a rotation of angle d or a reflection (in fact, that is not totally true, because the 2-plane of the rotation, or the hyperplane of the reflection depends on the direction in which y approaches the Hopf fiber except in the case when $n = 2$ and $d < \frac{\pi}{2}$), which is not the optimal coupling for $\alpha = 0$ or π , this latter one being the parallel transport. This explains the discontinuity of $\tilde{\kappa}$.

6 The Bakry–Émery curvature of our diffusion

In this section, we prove that our diffusion does not satisfy any curvature-dimension inequality $CD(\rho, N)$ in the sense of Bakry–Émery (these inequalities were introduced in [1] and [2]).

In the non degenerated case, one usually take a metric adapted to the generator of the diffusion. Here, it would imply that the metric takes the value $+\infty$ in the directions which are not orthogonal to the Hopf fibers. The distance we would obtain will be similar to the Carnot–Carathéodory distance on the Heisenberg group. So we prefer to keep the usual metric on S^{2n-1} to do our computations.

We first recall the definition of the Bakry–Émery curvature of a diffusion process on a manifold. Let $L = \frac{1}{2}A^{ij}\nabla_i\nabla_j + F^i\nabla_i$ be the generator of a diffusion on a Riemannian manifold. The carré du champ operator is defined as

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fL(g) - gL(f)) = \frac{1}{2}A^{ij}\nabla_i f \nabla_j g.$$

The iterated carré du champ opérateur is defined as

$$\Gamma_2(f, g) = \frac{1}{2}(L(\Gamma(f, g)) - \Gamma(f, L(g)) - \Gamma(g, L(f))).$$

The general formula of Γ_2 is a bit long and usually people do not write it, but here we need it, so it is given below

$$\begin{aligned} \Gamma_2(f, f) = & \left[\frac{1}{8}A^{kl}\nabla_k\nabla_l A^{ij} + \frac{1}{4}A^{kl}A^{im}R_{kml}{}^j + \frac{1}{4}F^k\nabla_k A^{ij} - \frac{1}{2}A^{ik}\nabla_k F^j \right] \nabla_i f \nabla_j f \\ & + \left[\frac{1}{2}A^{jl}\nabla_l A^{ik} - \frac{1}{4}A^{il}\nabla_l A^{jk} \right] \nabla_i f \nabla_j \nabla_k f \\ & + \frac{1}{4}A^{ik}A^{jl}\nabla_i\nabla_j f \nabla_k \nabla_l f. \end{aligned}$$

The diffusion is said to satisfy a curvature-dimension inequality $CD(\rho, N)$, in the sense of Bakry–Émery, if for any regular enough function f , we have:

$$\Gamma_2(f, f) \geq \rho\Gamma(f, f) + \frac{1}{N}(Lf)^2.$$

So the optimal ρ for a given N is the infimum of $\frac{(\Gamma_2(f, f) - \frac{1}{N}(Lf)^2)(x)}{\Gamma(f, f)(x)}$ (or $-\infty$ if there exists a point x and a function f such that $\Gamma(f, f) = 0$ and $\Gamma_2(f, f) < 0$).

Thanks to the expressions of Γ and Γ_2 , we see that $(\Gamma_2(f, f) - \frac{1}{N}(Lf)^2)(x) = G_2(\nabla f(x), \nabla^2 f(x))$ and $\Gamma(f, f)(x) = G_1(\nabla f(x))$, where G_1 and G_2 are quadratic functions, and G_1 is nonnegative. At a given point x , for any $u^* \in \mathbb{T}_x^* \mathcal{M}$ and any symmetric $m \in (\mathbb{T}_x^* \mathcal{M})^{\otimes 2}$, there exists a regular function f such that $\nabla f = u^*$ and $\nabla^2 f = m$.

It seems then natural to try to minimize $(\Gamma_2(f, f) - \frac{1}{N}(Lf)^2)(x)$ with a fixed ∇f . So for any $u^* \in \mathbb{T}_x^* \mathcal{M}$ such that $G_1(u^*) \neq 0$, we set

$$\rho_N(x, u^*) = \frac{\inf_{\substack{m \in (\mathbb{T}_x^* \mathcal{M})^{\otimes 2} \\ m \text{ symmetric}}} G_2(u^*, m)}{G_1(u^*)}.$$

We have

$$G_2(u^*, m) = G_2^{(1)}(u^*, u^*) + 2G_2^{(12)}(u^*, m) + G_2^{(2)}(m, m),$$

with $G_2^{(1)}$, $G_2^{(12)}$ and $G_2^{(2)}$ three quadratic forms, and we can assume that $G_2^{(1)}$ and $G_2^{(2)}$ are symmetric. We have to minimize this quantity over m with a fixed u^* .

If $G_2^{(2)}$ is not nonnegative, then the infimum is $-\infty$, by taking $m = \lambda m_0$, with m_0 such that $G_2^{(2)}(m_0, m_0) < 0$ and making λ tend to $+\infty$. In the classical case where $A^{ij} = g^{ij}$, it was the case if and only if $\frac{1}{N} > \frac{1}{n}$, with n the dimension of the manifold.

If $G_2^{(2)}$ is nonnegative, then the infimum is $G_2^{(1)}(u^*, u^*) - G_2^{(12)}(u^*, m_{u^*})$ if there exists a symmetric $m_{u^*} \in (\mathbb{T}_x^* \mathcal{M})^{\otimes 2}$ such that $G_2^{(12)}(u^*, \cdot) = G_2^{(2)}(m_{u^*}, \cdot)$, because $G_2(u^*, m) = G_2(u^*, -m_{u^*}) + G_2^{(2)}(m + m_{u^*}, m + m_{u^*})$. In the case when $G_2^{(2)}$ is invertible, the minimum can be written as

$$G_2^{(1)}(u^*, u^*) - G_2^{(2)-1}(G_2^{(12)}(u^*, \cdot), G_2^{(12)}(u^*, \cdot))$$

so it is a quadratic function of u^* .

If no such m_{u^*} exist, the infimum is $-\infty$ because the non existence of m_{u^*} is equivalent to the existence of a m'_{u^*} such that $G_2^{(2)}(m'_{u^*}, m'_{u^*}) = 0$

and $G_2^{(12)}(u^*, m'_{u^*}) \neq 0$, and then we can take $m = \lambda m'_{u^*}$, and make λ tend to $+\infty$ or $-\infty$.

Now let us see what we get for our degenerated diffusion. We have $A(x) = g_0^{-1} - x \otimes x - (ix) \otimes (ix) = g(x)^{-1} - (ix) \otimes (ix)$. Then we have

$$G_2^{(2)}(m, m') = \frac{1}{4} [\text{tr}(mg^{-1}(x)m'g^{-1}(x)) - 2g^{-1}(x)(m(ix), m'(ix)) + m(ix, ix)m'(ix, ix) - \frac{1}{N}(\text{tr}(mg^{-1}(x)) - m(ix, ix))(\text{tr}(m'g^{-1}(x)) - m'(ix, ix))].$$

If M and M' are the matrices of m and m' in the basis $(ix, e_2, ie_2, \dots, e_n, ie_n)$ of $\mathbb{T}_x S^{2n-1}$, with the e_j chosen such that (x, e_2, \dots, e_n) is a \mathbb{C} -orthonormal basis, then we have

$$G_2^{(2)}(m, m') = \frac{1}{4} (\langle \bar{M}, \bar{M}' \rangle_{HS} - \frac{1}{N} \text{tr}(\bar{M}) \text{tr}(\bar{M}')),$$

with \bar{M} and \bar{M}' obtained from M and M' by removing the first line and the first column. So $G_2^{(2)}$ is nonnegative if and only if $\frac{1}{N} \leq \frac{1}{2n-2}$.

To compute $G_2^{(12)}$, we have to compute ∇A , and thus $\nabla(ix)$, since we have

$$\nabla_i A^{jk} = \nabla_i \left((g^{-1})^{jk} - (ix)^j (ix)^k \right) = -\nabla_i (ix)^j (ix)^k - (ix)^j \nabla_i (ix)^k.$$

As previously, we take $u \in \mathbb{T}_x \mathcal{M}$ an unit norm vector and $y = \exp_x(du)$. Then $\nabla_u(ix) = \frac{d}{dd} \tau_{yx}(iy)|_{d=0}$, where τ_{yx} is the parallel transport along the geodesic joining y to x . This parallel transport corresponds to a rotation in the 2-plane generated by x and u . We have:

$$\begin{aligned} \tau_{yx}(iy) &= iy + g_0(iy, x)((\cos(d) - 1)x - \sin(d)u) + g_0(iy, u)(\sin(d)x + (\cos(d) - 1)u) \\ &= \cos(d)(ix) + \sin(d)(iu) - \sin(d)g_0(u, ix)((\cos(d) - 1)x - \sin(d)u) + \cos(d)g_0(u, ix)(\sin(d)x + (\cos(d) - 1)u) \\ &= \cos(d)(ix) + \sin(d)(iu) + g_0(u, ix)(\sin(d)x + (1 - \cos(d))u) \end{aligned}$$

So we get

$$\nabla_u(ix) = iu + g_0(u, ix)x = iu - g_0(iu, x)x,$$

wich is the orthogonal projection of iu on $\mathbb{T}_x S^{2n-1}$. Let us denote by $J(x)$ the endomorphism of $\mathbb{T}_x S^{2n-1}$ which sends u to $iu + g_0(u, ix)x$. Then $g(x)J(x)$ is antisymmetric, $J(x)J(x) = -I + (ix) \otimes (ix)^*$ and $J(x)(ix) = 0$. We have

$$\nabla_i (ix)^j = J^j_i.$$

Thus we get

$$\begin{aligned}
G_2^{(12)}(u^*, m) &= \left[\frac{1}{2} A^{jl} \nabla_l A^{ik} - \frac{1}{4} A^{il} \nabla_l A^{jk} \right] u_i^* m_{jk} \\
&= \left[\frac{1}{2} (g^{jl} - (ix)^j (ix)^l) (-J^i l (ix)^k - J^k_l (ix)^i) - \frac{1}{4} (g^{il} - (ix)^i (ix)^l) (-J^j_l (ix)^k - J^k_l (ix)^j) \right] u_i^* m_{jk} \\
&= \left[-\frac{1}{2} (J^i_l g^{jl} (ix)^k + J^k_l g^{lj} (ix)^i) + \frac{1}{4} (J^j_l g^{il} (ix)^k + J^k_l g^{il} (ix)^j) \right] u_i^* m_{jk} \\
&= m(J(x)(g^{-1}(x)u^*), ix),
\end{aligned}$$

where we used the antisymmetry of $J(x)g^{-1}(x)$ and the symmetry of m_{jk} to cancel the second term and to remark that the three other ones were equal (up to the prefactor $\frac{1}{2}$ or $\frac{1}{4}$).

Here we are in the case where m_{u^*} does not exist since for any u^* not colinear to $(ix)^*$ (in which case $G_1(u^*, u^*) = 0$), we can take

$$m'_{u^*} = \frac{1}{2} ((J(x)^T u^*) \otimes (ix)^* + (ix)^* \otimes (J(x)^T u^*)).$$

Thus $\rho_N(x, u^*) = -\infty$, and we have no Bakry–Émery curvature-dimension inequality.

So in our example, it is useful to consider a metric not adapted to the diffusion and to use the curvatures κ and $\tilde{\kappa}$ (to get Poincaré inequalities when $n \geq 3$, for example).

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