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SEMI-FREDHOLMNESS OF THE DISCRETE GAUSS-BONNET OPERATOR

HÈLA AYADI

ABSTRACT. In the context of an infinite locally finite weighted graph, we give a necessary and sufficient condition for semi-Fredholmness of the Gauss-Bonnet operator. This result is a discrete version of the theorem of Gilles Carron in the continuous case [5]. In addition, using a criterion of Anghel [2], we give a sufficient condition to have an operator of Gauss-Bonnet with closed range. Finally, this work can be considered as an extension of the work of Colette Anné and Nabila Torki-Hamza [3].

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1. INTRODUCTION

Dirac type operators have become of central importance in many branches of mathematics such as PDE's, differential geometry and topology (see [4], [7], [12]..), since the introduction in 1928 by the physicist Paul Dirac of a first-order linear differential operator whose square is the Laplacian operator. In particular, this paper focuses on the conditions to have semi-Fredholmness of the discrete Gauss-Bonnet operator needed to approach the Hodge decomposition theorem [3]. In fact, we present a discrete version of the work of G. Carron [5], which defines a new concept "non-parabolicity at infinity" to have the Gauss-Bonnet operator with closed range. Indeed, G. Carron's condition is quite weaker than the one given by Anghel [2]. Moreover, we provide a new sufficient condition to obtain a Gauss-Bonnet operator semi-Fredholm. Finally, we give two explicit examples one example verifying the property of non-parabolicity at infinity, and the other not.

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2. PRELIMINARIES

2.1. Definitions and notations.

- A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where \mathcal{V} is a set at most countable whose elements are called vertices and \mathcal{E} is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.
- If the graph G has a finite set of vertices, it's called a finite graph. Otherwise, G is called infinite graph.
- We assume that \mathcal{E} is symmetric without loops:

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

- Choosing an orientation of G consists of defining a partition of \mathcal{E} : $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

- For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \quad e^+ = v_2 \text{ and } -e = (v_2, v_1).$$

- The graph G is connected if, any two vertices x, y in \mathcal{V} can be joined by a path of edges γ_{xy} , that means,

$$\gamma_{xy} = \{e_k\}_{k=1, \dots, n} \text{ with } e_1^- = x, \quad e_n^+ = y \text{ and if } n \geq 2, \quad \forall j; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

- The degree (or valence) of a vertex x is the number of edges emanating from x . We denote

$$\text{deg}(x) := \#\{e \in \mathcal{E}; e^- = x\}.$$

- If $\text{deg}(x) < \infty, \forall x \in \mathcal{V}$, we say that G is a locally finite graph.

2.2. **The weighted graph.** The weighted graph (G, c, r) is given by the graph $G = (\mathcal{V}, \mathcal{E})$, a weight on the vertices $c : \mathcal{V} \rightarrow]0, \infty[$ and a weight on the edges $r : \mathcal{E} \rightarrow]0, \infty[$ such that $r(-e) = r(e)$.

Examples: - An infinite electrical network is a weighted graph (G, c, r) where the weights of the edges called resistances r ; their reciprocals are called conductances. And the weights of the vertices given by $c(x) = \sum_{y \in \mathcal{V}} \frac{1}{r(x, y)} < \infty, \forall x \in \mathcal{V}$.

-The graph G called a simple graph where the weights of the edges and the vertices equals 1.

All the graphs we shall consider on the sequel will be weighted, connected and locally finite.

2.3. **The notion of subgraph.** A subgraph of a graph G is a graph $G_K := (K, \mathcal{E}_K)$ such that $K \subset \mathcal{V}$ and $\mathcal{E}_K := \{e \in \mathcal{E}; e^-, e^+ \in K\}$.

For such a subgraph we define:

- the vertex boundary :

$$\partial K := \{x \in \mathcal{V} \setminus K; \exists y \in K, (x, y) \in \mathcal{E}\},$$

- the edge boundary:

$$\partial \mathcal{E}_K := \{e \in \mathcal{E}; e^- \in K \text{ and } e^+ \notin K \text{ or } e^+ \in K \text{ and } e^- \notin K\}.$$

2.4. Functional spaces. We denote the set of real functions on \mathcal{V} by:

$$\mathcal{C}(\mathcal{V}) = \{f : \mathcal{V} \rightarrow \mathbb{R}\}$$

and the set of functions of finite support by $\mathcal{C}_0(\mathcal{V})$.

Moreover, we denote the set of real skewsymmetric functions on \mathcal{E} by:

$$\mathcal{C}^a(\mathcal{E}) = \{\varphi : \mathcal{E} \rightarrow \mathbb{R} ; \varphi(-e) = -\varphi(e)\}$$

and the set of functions of finite support by $\mathcal{C}_0^a(\mathcal{E})$.

We define on the weighted graph (G, c, r) the following function spaces endowed of the scalar products.

a):

$$l^2(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V}); \sum_{x \in \mathcal{V}} c(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g \rangle_{\mathcal{V}} = \sum_{x \in \mathcal{V}} c(x) f(x) g(x)$$

and the norm

$$\|f\|_{l^2(\mathcal{V})} = \sqrt{\langle f, f \rangle_{\mathcal{V}}}.$$

b):

$$l^2(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^a(\mathcal{E}); \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \varphi(e) \psi(e)$$

and the norm

$$\|\varphi\|_{l^2(\mathcal{E})} = \sqrt{\langle \varphi, \varphi \rangle_{\mathcal{E}}}.$$

As a consequence, we define the direct sums of $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ by:

$$l^2(G) := l^2(\mathcal{V}) \oplus l^2(\mathcal{E}) = \{(f, \varphi), f \in l^2(\mathcal{V}) \text{ and } \varphi \in l^2(\mathcal{E})\},$$

with the norm

$$\|(f, \varphi)\|_{l^2(G)}^2 := \|f\|_{l^2(\mathcal{V})}^2 + \|\varphi\|_{l^2(\mathcal{E})}^2.$$

2.5. Operators and properties. The difference operator: it is the operator

$$d : \mathcal{C}_0(\mathcal{V}) \longrightarrow \mathcal{C}_0^a(\mathcal{E}),$$

given by

$$d(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator: it is δ the formal adjoint of d . Thus it satisfies

$$(2.1) \quad \langle df, \varphi \rangle_{\mathcal{E}} = \langle f, \delta\varphi \rangle_{\mathcal{V}}$$

for all $f \in \mathcal{C}_0(\mathcal{V})$ and for all $\varphi \in \mathcal{C}_0^a(\mathcal{E})$.

As consequence, we have the following formula characterizing δ :

Lemma 2.1. *The coboundary operator δ is characterized by the formula*

$$\delta\varphi(x) = \frac{1}{c(x)} \sum_{e, e^+=x} r(e)\varphi(e),$$

for all $\varphi \in \mathcal{C}_0^a(\mathcal{E})$.

Proof

For $f \in \mathcal{C}_0(\mathcal{V})$ and $\varphi \in \mathcal{C}_0^a(\mathcal{E})$, using (2.1), we get

$$\begin{aligned} \langle df, \varphi \rangle_{\mathcal{E}} &= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) df(e) \varphi(e) \\ &= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) (f(e^+) - f(e^-)) \varphi(e) \\ &= \frac{1}{2} \sum_{x \in \mathcal{V}} f(x) \left(\sum_{e, e^+=x} r(e) \varphi(e) - \sum_{e, e^-=x} r(e) \varphi(e) \right). \end{aligned}$$

But, $r(-e) = r(e)$ and $\sum_{e, e^+=x} r(e) \varphi(e) = - \sum_{e, e^-=x} r(e) \varphi(e)$.

So we have,

$$\begin{aligned} \langle df, \varphi \rangle_{\mathcal{E}} &= \sum_{x \in \mathcal{V}} c(x) f(x) \left(\frac{1}{c(x)} \sum_{e, e^+=x} r(e) \varphi(e) \right) \\ &= \langle f, \delta\varphi \rangle_{\mathcal{V}}. \end{aligned}$$

□

We introduce now a very important result inspired by [11].

Lemma 2.2. *Let x and x_0 in \mathcal{V} , then there exists a positive constant C_{xx_0} such that*

$$(2.2) \quad |f(x)| \leq C_{xx_0} \left(|f(x_0)| + \|df\|_{l^2(\mathcal{E})} \right),$$

for all $f \in \mathcal{C}_0(\mathcal{V})$.

Proof

As G is connected, then we can find a path γ_{xx_0} joining x to x_0 , i.e.

$$\gamma_{xx_0} = \{e_k\}_{k=1, \dots, n} \text{ with } e_1^- = x, e_n^+ = x_0 \text{ and if } n \geq 2, \forall j; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

Then, using the triangle inequality, we have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f(e_1^+) + f(e_1^+) - f(e_2^+) + \dots + f(e_{n-1}^+) - f(x_0)| \\ &\leq |df(e_1)| + |df(e_2)| + \dots + |df(e_n)| \\ &\leq \sum_{e \in \gamma_{xx_0}} \frac{1}{\sqrt{r(e)}} \sqrt{r(e)} |df(e)|. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |f(x) - f(x_0)| &\leq \left(\sum_{e \in \gamma_{xx_0}} \frac{1}{r(e)} \right)^{\frac{1}{2}} \left(\sum_{e \in \gamma_{xx_0}} r(e) (df(e))^2 \right)^{\frac{1}{2}} \\ &\leq S_{xx_0} \left(\sum_{e \in \mathcal{E}} r(e) (df(e))^2 \right)^{\frac{1}{2}} \\ &\leq S_{xx_0} \|df\|_{l^2(\mathcal{E})}, \end{aligned}$$

$$\text{with } S_{xx_0} = \left(\sum_{e \in \gamma_{xx_0}} \frac{1}{r(e)} \right)^{\frac{1}{2}}.$$

Thus, we deduce that

$$\begin{aligned} |f(x)| &\leq |f(x) - f(x_0)| + |f(x_0)| \\ &\leq S_{xx_0} \|df\|_{l^2(\mathcal{E})} + |f(x_0)| \\ &\leq C_{xx_0} \left(\|df\|_{l^2(\mathcal{E})} + |f(x_0)| \right), \end{aligned}$$

with $C_{xx_0} = \max(S_{xx_0}, 1)$. □

Before giving another important result, for $f \in \mathcal{C}_0(\mathcal{V})$, we define the mean value \bar{f} of f by

$$\bar{f}(e) = \frac{f(e^+) + f(e^-)}{2}$$

for all $e \in \mathcal{E}$.

And we have from [10] the following derivation property:

Lemma 2.3. *For $f, g \in \mathcal{C}_0(\mathcal{V})$ and $\varphi \in \mathcal{C}_0^a(\mathcal{E})$, it follows*

$$(2.3) \quad d(fg)(e) = f(e^+)dg(e) + g(e^-)d(f)(e).$$

$$(2.4) \quad \delta(\bar{f}\varphi)(x) = f(x)\delta\varphi(x) - \frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(f)(e)\varphi(e).$$

Proof

For $f, g \in \mathcal{C}_0(\mathcal{V})$ and $e \in \mathcal{E}$,

$$\begin{aligned} d(fg)(e) &= (fg)(e^+) - (fg)(e^-) \\ &= f(e^+) (g(e^+) - g(e^-)) + g(e^-) (f(e^+) - f(e^-)) \\ &= f(e^+)d(g)(e) + g(e^-)d(f)(e). \end{aligned}$$

On the other hand, for $\varphi \in \mathcal{C}_0^a(\mathcal{E})$ applying the characterization of δ from Lemma (2.1) to the function $\bar{f}\varphi \in \mathcal{C}_0^a(\mathcal{E})$, we have

$$\begin{aligned}
\delta(\bar{f}\varphi)(x) &= \frac{1}{c(x)} \sum_{e, e^+=x} r(e)(\bar{f}\varphi)(e) \\
&= \frac{1}{c(x)} \sum_{e, e^+=x} r(e) \left(\frac{f(e^+) + f(e^-)}{2} \right) \varphi(e) \\
&= \frac{1}{c(x)} \sum_{e, e^+=x} r(e)f(e^+)\varphi(e) + \frac{1}{c(x)} \sum_{e, e^+=x} r(e) \left(\frac{f(e^-) - f(e^+)}{2} \right) \varphi(e) \\
&= f(x) \frac{1}{c(x)} \sum_{e, e^+=x} r(e)\varphi(e) + \frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(f)(-e)\varphi(e) \\
&= f(x)\delta(\varphi)(x) - \frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(f)(e)\varphi(e).
\end{aligned}$$

□

The Gauss-Bonnet operator: it is the endomorphism

$$D = d + \delta : \mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E}) \longrightarrow \mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$$

with,

$$D(f, \varphi) = \delta\varphi + df, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E}).$$

And it is a symmetric operator.

3. NON-PARABOLICITY AT INFINITY

Now we introduce the discrete result of Carron [5]:

Definition 3.1. We say that D is non-parabolic at infinity if there is a finite subgraph G_K of G such that for all finite subset U of $G \setminus G_K$, there exists a positive constant $C = C(U)$ such that holds the following inequality

$$C \|(f, \varphi)\|_{l^2(U)} \leq \|D(f, \varphi)\|_{l^2(G \setminus G_K)}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V} \setminus K) \times \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Remark 3.1. We call a finite subset U of G a couple $U := (\mathcal{V}_U, \mathcal{E}_U)$ such that \mathcal{V}_U is a finite subset of \mathcal{V} and \mathcal{E}_U is a finite subset of \mathcal{E} . And, we denote

$$\|(f, \varphi)\|_{l^2(U)}^2 = \|f\|_{l^2(\mathcal{V}_U)}^2 + \|\varphi\|_{l^2(\mathcal{E}_U)}^2.$$

Definition 3.2. $G_{\tilde{K}}$ is a **neighborhood** of G_K if $G_{\tilde{K}} := (\tilde{K}, \mathcal{E}_{\tilde{K}})$ is a finite subgraph of G such that

$$\left\{ \begin{array}{l}
i) K \subset \tilde{K} \text{ finite,} \\
ii) \mathcal{E}_K \sqcup \partial\mathcal{E}_K \subset \mathcal{E}_{\tilde{K}}, \\
iii) e = (x, y) \in \mathcal{E}_{\tilde{K}} \Rightarrow x, y \in \tilde{K}.
\end{array} \right.$$

Since we can define **the smallest neighborhood** of G_K by $G_{\tilde{K}_0}$, where $G_{\tilde{K}_0}$ is a finite subgraph of G contains G_K and its boundary.

Remark 3.2. In [9], $G_{\tilde{K}_0}$ is called a combinatorial neighborhood of G_K .

Lemma 3.1. *If D is non-parabolic at infinity then, for every finite subset U of G there exists a positive constant $C' = C'(U)$ such that*

$$(3.5) \quad C' \|(f, \varphi)\|_{l^2(U)} \leq \|D(f, \varphi)\|_{l^2(G)} + \|(f, \varphi)\|_{l^2(G_{\tilde{K}})}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E}),$$

where $G_{\tilde{K}}$ is a neighborhood of G_K .

Proof

Since U is a finite subset of G it can be reduced to a point or an edge.

Let x any vertex of G , we start by proving

$$C' |f(x)| \leq \|df\|_{l^2(\mathcal{E})} + \|f\|_{l^2(\tilde{K})}, \quad \forall f \in \mathcal{C}_0(\mathcal{V}).$$

$G_{\tilde{K}}$ is a finite subgraph of G , so according to Lemma 2.2, we obtain

$$(3.6) \quad f^2(x) \leq C_1 \left(\|f\|_{l^2(\tilde{K})}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right),$$

where C_1 is a positive constant which depends on x and \tilde{K} . Indeed: let $x \in \mathcal{V}$ and $x_0 \in \tilde{K}$, using Lemma 2.2, we obtain

$$(3.7) \quad f^2(x) \leq C_{xx_0} \left(f^2(x_0) + \|df\|_{l^2(\mathcal{E})}^2 \right).$$

Multiplying (3.7) by $c(x_0) > 0$, we get

$$\begin{aligned} c(x_0)f^2(x) &\leq C_{xx_0} \left(c(x_0)f^2(x_0) + c(x_0) \|df\|_{l^2(\mathcal{E})}^2 \right) \\ &\leq C_{xx_0} \left(\|f\|_{l^2(\tilde{K})}^2 + c(x_0) \|df\|_{l^2(\mathcal{E})}^2 \right) \\ &\leq C'_{xx_0} \left(\|f\|_{l^2(\tilde{K})}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right), \end{aligned}$$

where $C'_{xx_0} = \max(C_{xx_0}, c(x_0)C_{xx_0})$.

Then, we have

$$f^2(x) \leq \frac{C'_{xx_0}}{c(x_0)} \left(\|f\|_{l^2(\tilde{K})}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right).$$

Finally, we obtain

$$f^2(x) \leq C_1 \left(\|f\|_{l^2(\tilde{K})}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right)$$

where $C_1 = \frac{C'_{xx_0}}{c(x_0)}$.

On the other hand, we want to show the following inequality, for any edge $e \in \mathcal{E}$

$$C'' |\varphi(e)| \leq \|\delta\varphi\|_{l^2(\mathcal{V})} + \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}, \quad \forall \varphi \in \mathcal{C}_0(\mathcal{E}).$$

For $e \in \mathcal{E}_K \subset \mathcal{E}_{\tilde{K}}$ finite, we have

$$\varphi^2(e) \leq \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2 \leq \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2 + \|\delta\varphi\|_{l^2(\mathcal{V})}^2.$$

And if $e \in \mathcal{E} \setminus \mathcal{E}_K$, we consider the indicator function of K^c , denoted by χ

$$(3.8) \quad \chi(x) = \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{otherwise.} \end{cases}$$

which gives

$$d\chi(e) = \begin{cases} 0 & \text{if } e \in \mathcal{E}_K, \\ \pm 1 & \text{if } e \in \partial\mathcal{E}_K, \\ 0 & \text{otherwise.} \end{cases} \quad \& \quad \bar{\chi}(e) = \begin{cases} 0 & \text{if } e \in \mathcal{E}_K, \\ \frac{1}{2} & \text{if } e \in \partial\mathcal{E}_K, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\varphi \in \mathcal{C}_0^a(\mathcal{E})$, we have then $\bar{\chi}\varphi$ with finite support in $\mathcal{E} \setminus \mathcal{E}_K$. Thus, applying the definition of the non-parabolicity at infinity of D to the function $(0, \bar{\chi}\varphi)$, we obtain

$$\|\bar{\chi}\varphi\|_{l^2(U)}^2 \leq C \|\delta(\bar{\chi}\varphi)\|_{l^2(\mathcal{V})}^2,$$

where $C = \frac{1}{c(U)}$.

Since we have $e \in \mathcal{E} \setminus \mathcal{E}_K$, this implies that

$$(3.9) \quad \varphi^2(e) \leq C \|\delta(\bar{\chi}\varphi)\|_{l^2(\mathcal{V})}^2.$$

The derivation property of Lemma (2.3), gives

$$\delta(\bar{\chi}\varphi)(x) = \chi(x)\delta\varphi(x) - \frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(\chi)(e)\varphi(e).$$

And by the inequality $(a - b)^2 \leq 2(a^2 + b^2)$, we obtain

$$\begin{aligned} \|\delta(\bar{\chi}\varphi)\|_{l^2(\mathcal{V})}^2 &= \sum_{x \in \mathcal{V}} c(x) (\delta(\bar{\chi}\varphi))^2 \\ &\leq 2 \left[\underbrace{\sum_{x \in \mathcal{V}} c(x) (\chi(x)\delta\varphi(x))^2}_I + \underbrace{\sum_{x \in \mathcal{V}} c(x) \left(\frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(\chi)(e)\varphi(e) \right)^2}_J \right]. \end{aligned}$$

So, for the first term we have

$$(3.10) \quad I = \sum_{x \in \mathcal{V} \setminus K} c(x) (\delta\varphi(x))^2 \leq \|\delta\varphi\|_{l^2(\mathcal{V})}^2$$

and for the second one, we get

$$(3.11) \quad J = \underbrace{\sum_{x \in K} \frac{1}{2c(x)} \left(\sum_{e, e^+=x} r(e)d(\chi)(e)\varphi(e) \right)^2}_{J_1} + \underbrace{\sum_{x \in \mathcal{V} \setminus K} \frac{1}{2c(x)} \left(\sum_{e, e^+=x} r(e)d(\chi)(e)\varphi(e) \right)^2}_{J_2}.$$

Using that $\text{supp}(d\chi) = \partial\mathcal{E}_K \subset \mathcal{E}_{\tilde{K}}$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
J_1 &= \sum_{x \in K} \frac{1}{2c(x)} \left(\sum_{\substack{e, e^+ = x \\ e \in \text{supp}(d\chi)}} r(e)\varphi(e) \right)^2 \\
&= C_K \left(\sum_{e \in \text{supp}(d\chi)} r(e)\varphi(e) \right)^2 \\
&\leq C_K \left(\sum_{e \in \text{supp}(d\chi)} r(e) \right) \left(\sum_{e \in \text{supp}(d\chi)} r(e)\varphi^2(e) \right) \\
&\leq C_K C'_K \sum_{e \in \mathcal{E}_{\tilde{K}}} r(e)\varphi^2(e) \\
&= C_2 \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2,
\end{aligned}$$

where $C_K = \max_{x \in K} \frac{1}{2c(x)}$, $C'_K = \#\mathcal{E}_{\tilde{K}} \max_{e \in \mathcal{E}_{\tilde{K}}} r(e)$ and $C_2 = C_K C'_K$.

And for J_2 , we have $e = (e^-, e^+) \in \text{supp}(d\chi) = \partial\mathcal{E}_K$, so if $e^- \in K$, $e^+ \in \partial K$.

$$\begin{aligned}
J_2 &= \sum_{x \in \partial K} \frac{1}{2c(x)} \left(\sum_{\substack{e, e^+ = x \\ e \in \text{supp}(d\chi)}} r(e)\varphi(e) \right)^2 \\
&= C''_K \left(\sum_{e \in \text{supp}(d\chi)} r(e)\varphi(e) \right)^2 \\
&\leq C''_K \left(\sum_{e \in \text{supp}(d\chi)} r(e) \right) \left(\sum_{e \in \text{supp}(d\chi)} r(e)\varphi^2(e) \right) \\
&\leq C''_K C'_K \sum_{e \in \mathcal{E}_{\tilde{K}}} r(e)\varphi^2(e) \\
&= C'_2 \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2,
\end{aligned}$$

where $C''_K = \max_{x \in \partial K} \frac{1}{2c(x)}$ and $C'_2 = C''_K C'_K$.

Thus, (3.11) becomes

$$(3.12) \quad J \leq C''_2 \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2,$$

where $C''_2 = \max(C_2, C'_2)$.

So by (3.10) and (3.12), we get

$$(3.13) \quad \|\delta(\bar{\chi}\varphi)\|_{l^2(\mathcal{V})}^2 \leq \max(2, 2C''_2) \left(\|\delta\varphi\|_{l^2(\mathcal{V})}^2 + \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2 \right).$$

Finally, (3.9) and (3.13) give

$$\varphi^2(e) \leq \tilde{C} \left(\|\delta\varphi\|_{l^2(\mathcal{V})}^2 + \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})}^2 \right)$$

where $\tilde{C} = \frac{2\max(1, C_2'')}{C}$.

□

Proposition 3.1. *If D is non-parabolic at infinity, then we can construct a Hilbert space W such that :*

- (1) $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ is dense in W .
- (2) The injection of $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ to $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ extends by continuity to W .
- (3) $D : W \longrightarrow l^2(G)$ is a bounded operator.

Remark 3.3. *In 1) and 2) we use the topology of ponctual convergence on $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$, it means, the sequence (f_n, φ_n) converges ponctually to (f, φ) on $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ if $f_n(x)$ converges to $f(x)$, $\forall x \in \mathcal{V}$ and $\varphi_n(e)$ converges to $\varphi(e)$, $\forall e \in \mathcal{E}$.*

Remark 3.4. *In Carron's paper [5], the injection of the space of functions with compact support to l_{loc}^2 extends by continuity to W . But, in our case we didn't need to introduce the space l_{loc}^2 because in discrete case this notion is trivial.*

Proof

Let us denote by W the closure of $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ for the norm

$$N_{\tilde{K}}(f, \varphi) = \left(\|(f, \varphi)\|_{l^2(G_{\tilde{K}})}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 \right)^{\frac{1}{2}},$$

where $G_{\tilde{K}}$ is a neighborhood of G_K (see Definition (3.2)).

Aim i): $N_{\tilde{K}}$ is a norm on W , we just look at the nullity, we have

$$\begin{aligned} N_{\tilde{K}}(f, \varphi) = 0 &\Leftrightarrow \|(f, \varphi)\|_{l^2(G_{\tilde{K}})} = 0 \text{ and } \|D(f, \varphi)\|_{l^2(G)} = 0 \\ &\Leftrightarrow \|f\|_{l^2(\tilde{K})} = 0, \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})} = 0, \|df\|_{l^2(\mathcal{E})} = 0 \text{ and } \|\delta\varphi\|_{l^2(\mathcal{V})} = 0. \end{aligned}$$

For any $x \in \mathcal{V}$ and as $\#\tilde{K} < \infty$, from Lemma (3.1), we get

$$(3.14) \quad f^2(x) \leq C_1 \left(\|f\|_{l^2(\tilde{K})}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right).$$

But, $\|f\|_{l^2(\tilde{K})} = 0$ and $\|df\|_{l^2(\mathcal{E})} = 0$. So it follows immediately that $f = 0$ on \mathcal{V} .

It remains to show that if $\|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})} = 0$ and $\|\delta\varphi\|_{l^2(\mathcal{V})} = 0$ then $\varphi = 0$. We suppose that $\varphi \neq 0$.

φ is a finite support function in $\mathcal{E} \setminus \mathcal{E}_{\tilde{K}}$ and therefore, by Lemma (3.1) where U equals to the support of φ , there exists a positive constant C such that

$$C \|\varphi\|_{l^2(\mathcal{E}_U)} \leq \|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})} + \|\delta\varphi\|_{l^2(\mathcal{V})}.$$

But, $\|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}})} = \|\delta\varphi\|_{l^2(\mathcal{V})} = 0$, since we get $\varphi = 0$ on \mathcal{E}_U , which is impossible.

Aim ii) Show that the space W is independent of the choice of $G_{\tilde{K}}$.

Let $G_{\tilde{K}_1}$ be another neighborhood of G_K such that $K \subset \tilde{K}_0 \subset \tilde{K}_1$.

So, we have

$$N_{\tilde{K}_0}(f, \varphi) \leq N_{\tilde{K}_1}(f, \varphi).$$

Moreover, to show the existence of a constant $C > 0$ such that $N_{\tilde{K}_1}(f, \varphi) \leq CN_{\tilde{K}_0}(f, \varphi)$, it suffices to show the existence of a constant $C > 0$ such that $\|(f, \varphi)\|_{l^2(\tilde{K}_1 \setminus \tilde{K}_0)}^2 \leq CN_{\tilde{K}_0}^2(f, \varphi)$. Indeed, we have

$$\begin{aligned} N_{\tilde{K}_1}^2(f, \varphi) &= \|(f, \varphi)\|_{l^2(\tilde{K}_1)}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 \\ &= \|(f, \varphi)\|_{l^2(\tilde{K}_1 \setminus \tilde{K}_0)}^2 + \|(f, \varphi)\|_{l^2(\tilde{K}_0)}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 \\ &= \|(f, \varphi)\|_{l^2(\tilde{K}_1 \setminus \tilde{K}_0)}^2 + N_{\tilde{K}_0}^2(f, \varphi). \end{aligned}$$

Using lemma (3.1) and as we have $\sharp(\tilde{K}_1 \setminus \tilde{K}_0) < \infty$, we get

$$\|f\|_{l^2(\tilde{K}_1 \setminus \tilde{K}_0)}^2 \leq C \left(\|f\|_{l^2(\tilde{K}_0)}^2 + \|df\|_{l^2(\mathcal{E})}^2 \right),$$

where $C = C(\tilde{K}_1 \setminus \tilde{K}_0, \tilde{K}_0)$.

And

$$\|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}_1} \setminus \mathcal{E}_{\tilde{K}_0})}^2 \leq C \left(\|\varphi\|_{l^2(\mathcal{E}_{\tilde{K}_0})}^2 + \|\delta\varphi\|_{l^2(\mathcal{V})}^2 \right).$$

where $C = C(\tilde{K}_1 \setminus \tilde{K}_0, \tilde{K}_0)$.

So, we obtain

$$\|(f, \varphi)\|_{l^2(G_{\tilde{K}_1} \setminus G_{\tilde{K}_0})}^2 \leq CN_{\tilde{K}_0}^2(f, \varphi).$$

Thus, we have shown that the construction of a norm on W is independent of the choice of the neighborhood associated to the subgraph G_K . We set:

$$\|(f, \varphi)\|_W := \left(\|(f, \varphi)\|_{l^2(G_{\tilde{K}_0})}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 \right)^{\frac{1}{2}},$$

for $(f, \varphi) \in \mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$.

Aim iii): By Lemma (3.1), we have the injection of $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ to $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ extends by continuity to W .

Aim iv): we have

$$\|D(f, \varphi)\|_{l^2(G)}^2 \leq \|(f, \varphi)\|_{l^2(G_{\tilde{K}})}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 = \|(f, \varphi)\|_W^2.$$

Consequently, $D : W \longrightarrow l^2(G)$ is a bounded operator. □

4. SEMI-FREDHOLMNESS OF THE DISCRETE GAUSS-BONNET OPERATOR

Definition 4.1. *An operator is semi-Fredholm if its range is closed and its kernel is finite dimensional.*

Now we come to our main result:

Theorem. Let W be a Hilbert space satisfying:

- (1) $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ is dense in W .
- (2) The injection of $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ to $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ extends by continuity to W .
- (3) $D : W \longrightarrow l^2(G)$ is a bounded operator.

Then, the following conditions are equivalent:

i) $D : W \rightarrow l^2(G)$ is semi-Fredholm.

ii) There exists a finite subgraph G_K of G and a positive constant $C = C_K$ such that

$$(4.15) \quad C \|(f, \varphi)\|_W \leq \|D(f, \varphi)\|_{l^2(G)}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V} \setminus K) \times \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Proof

We take the same arguments used by Carron [5]. We start by showing the direct implication, we assume that the conclusion is false. Then, we can find an increasing sequence of finite subgraph $\{G_{K_n}\}_n$ such that $G = \bigcup_n G_{K_n}$ and a sequence $\{\sigma_n\}_n$ with finite support in $\mathcal{V} \setminus K_n$ satisfying the following conditions, for all $n \geq 1$

$$\begin{cases} \sigma_n = (f_n, \varphi_n) \in \mathcal{C}_0(\mathcal{V} \setminus K_n) \times \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_{K_n}), \\ \|\sigma_n\|_W = 1, \\ \|D\sigma_n\|_{l^2(G)} \leq \frac{1}{n}. \end{cases}$$

On the other hand, it was assumed that $D : W \rightarrow l^2(G)$ is semi-Fredholm. Therefore, by [13] there exists a bounded operator $P : l^2(G) \rightarrow W$ such that

$$(4.16) \quad P \circ D = \text{Id}_W - H,$$

where H is the orthogonal projection onto the kernel of D , it is an operator with finite rank.

Then, we obtain

$$\begin{aligned} \|\sigma_n\|_W &\leq \|(P \circ D)\sigma_n\|_W + \|H\sigma_n\|_W \\ &\leq \|P\| \|D\sigma_n\|_{l^2(G)} + \|H\sigma_n\|_W \\ &\leq \left(\frac{\|P\|}{n} + \|H\sigma_n\|_W \right). \end{aligned}$$

If

$$\lim_{n \rightarrow \infty} \|H\sigma_n\|_W = 0 \implies \lim_{n \rightarrow \infty} \|\sigma_n\|_W = 0,$$

which contradicts the assumption $\|\sigma_n\|_W = 1$.

So, our aim is to prove that $\{H\sigma_n\}_n$ converges to 0 in W . Indeed, we set

$$(4.17) \quad \sigma_n = \sigma_n^1 + \sigma_n^2$$

with $\sigma_n^1 (= H\sigma_n) \in \ker D$ and $\sigma_n^2 \in (\ker D)^\perp$.

Such as

$$\begin{cases} (P \circ D)\sigma_n = \sigma_n^2, \\ \|P \circ D\sigma_n\|_W \leq \|P\| \|D\sigma_n\|_{l^2(G)} \xrightarrow{n \rightarrow \infty} 0. \end{cases}$$

Then, for the norm of W

$$(4.18) \quad \lim_{n \rightarrow \infty} \sigma_n^2 = 0.$$

Moreover, $\{\sigma_n^1\}_n$ is a bounded sequence of $\ker D$ which is of finite dimension. So we can extract a subsequence converging to σ in W , which we denote $\{\sigma_{\varphi(n)}^1\}_n$.

Using (4.17) and (4.18), $\{\sigma_{\varphi(n)}\}_n$ converges in W to σ (as a sum of two converging sequences) and as a consequence $\|\sigma\|_W = 1$.

Let us prove that $\sigma = 0$ where $\sigma = \lim \sigma_{\varphi(n)} = \lim \sigma_{\varphi(n)}^1$.

We suppose that $\sigma \neq 0$. As W is injected continuously in $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$, there exists $x \in V$ such that $\{\sigma_{\varphi(n)}(x)\}_n$ converges to $\sigma(x) \neq 0$. But, by construction the sequence $\{\sigma_{\varphi(n)}\}_n$ converges punctually to 0 (the sequence $\{\sigma_{\varphi(n)}\}_n$ has a finite support outside of G_{K_n}). Hence, we conclude that $\sigma(x) = 0$ which is absurd.

It remains to prove $ii) \Rightarrow i)$.

First step: We construct a bounded operator $Q : l^2(G) \rightarrow W$ such that $Q \circ D - \text{Id}_W$ is a compact operator, this will show that $D : W \rightarrow l^2(G)$ has a finite kernel and a closed range [13].

Let D_1 be the restriction of D on $G \setminus G_K$, so $D_1 : W(G \setminus G_K) \rightarrow l^2(G)$ is bounded, where $W(G \setminus G_K) = \{\sigma = (f, \varphi) \in W; \sigma = 0 \text{ on } G_K\}$. Moreover, by assumption we have

$$C \|(f, \varphi)\|_W \leq \|D(f, \varphi)\|_{l^2(G)}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V} \setminus K) \times \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Then, D_1 is injective with closed range, which allows the existence of a left inverse P_1 such that

$$P_1 \circ D_1 = \text{Id}.$$

On the other hand, we denote

$$D_2 : l^2(\tilde{K}_1) \rightarrow l^2(G)$$

where \tilde{K}_1 is a neighborhood (see Definition (3.2)) of \tilde{K}_0 , such that \tilde{K}_0 is the smallest neighborhood of K .

Since $l^2(\tilde{K}_1)$ is a vector space of finite dimension, then D_2 is continuous with closed range. We denote P_2 "the parametrix" which is a continuous operator satisfying

$$P_2 \circ D_2 = \text{Id} - H_2,$$

where H_2 is the orthogonal projection onto the kernel of D_2 .

We consider now the indicator function χ as in (3.8) by replacing K by \tilde{K}_0 , which gives $d\chi, \bar{\chi}, 1 - \chi$ and $1 - \bar{\chi}$ where

$$(1 - \chi)(x) = \begin{cases} 1 & \text{if } x \in \tilde{K}_0 \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } (1 - \bar{\chi})(e) = \begin{cases} 1 & \text{if } e \in \mathcal{E}_{\tilde{K}_0}, \\ \frac{1}{2} & \text{if } e \in \partial\mathcal{E}_{\tilde{K}_0}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define the operator χ . depending on the domain by:

If $\chi. : C_0(\mathcal{V}) \rightarrow C_0(\mathcal{V})$ so we have $\chi.f = \chi f$, for all $f \in C_0(\mathcal{V})$.

If $\chi. : C_0^a(\mathcal{E}) \longrightarrow C_0^a(\mathcal{E})$ we get $\chi.\varphi = \bar{\chi}\varphi$, for all $\varphi \in C_0^a(\mathcal{E})$.

If $\chi. : C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E}) \longrightarrow C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ hence we obtain $\chi.(f, \varphi) = (\chi f, \bar{\chi}\varphi)$,

for all $(f, \varphi) \in C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$.

We set

$$Q\sigma := P_2(1 - \chi)\sigma + P_1\chi\sigma,$$

where $\sigma = (f, \varphi)$.

Second step: Let us check that the operator $Q \circ D - \text{Id}$ is compact.

We denote the following bracket for any two operators A and B :

$$[A, B] = AB - BA.$$

Then, we obtain

$$\begin{aligned} Q \circ D &= P_2(1 - \chi)D + P_1\chi D \\ &= P_2D(1 - \chi) + P_2[1 - \chi, D] + P_1D\chi + P_1[\chi, D] \\ &= P_2D_2(1 - \chi) + P_2[1 - \chi, D] + P_1D_1\chi + P_1[\chi, D] \\ &= (\text{Id} - H_2)(1 - \chi) + P_2[1 - \chi, D] + \text{Id}(\chi) + P_1[\chi, D] \\ &= \text{Id} - H_2(1 - \chi) + P_2[1 - \chi, D] + P_1[\chi, D]. \end{aligned}$$

We just calculate $P_2[\chi, D]$. We have

$$[\chi, D] = [\chi, d] + [\chi, \delta].$$

For the first bracket, we obtain

$$\begin{aligned} [\chi, d]f(e) &= \bar{\chi}(e)d(f)(e) - d(\chi f)(e) \\ &= \frac{1}{2} [\chi(e^+) + \chi(e^-)] d(f)(e) - \chi(e^+)d(f)(e) - f(e^-)d\chi(e) \\ &= -\frac{1}{2}d\chi(e)d(f)(e) - f(e^-)d\chi(e). \end{aligned}$$

And for the second one, we get

$$\begin{aligned} [\chi, \delta]\varphi(x) &= \chi(x)\delta(\varphi)(x) - \delta(\bar{\chi}\varphi)(x) \\ &= \chi(x)\delta(\varphi)(x) - \chi(x)\delta(\varphi)(x) + \frac{1}{2} \sum_{e, e^+=x} d(\chi)(e)\varphi(e) \\ &= \frac{1}{2} \sum_{e, e^+=x} d(\chi)(e)\varphi(e). \end{aligned}$$

But, the support of $d(\chi)$ is included in $\partial\mathcal{E}_{\tilde{K}_0} \subset \tilde{K}_1$ which is finite. Then, $[\chi, D]$ has a finite range so it is a compact operator.

Finally, $Q \circ D = \text{Id} + H$ where H is a compact operator .

□

Remark 4.1. In the Theorem , we obtain D Fredholm if it is an essential-selfadjoint operator [5].

Remark 4.2. *There is a second method inspired from [2] to show $ii) \Rightarrow i)$ of the Theorem . This can be demonstrated with the aid of the following **claim**: "If $\sigma_n = (f_n, \varphi_n) \in \mathcal{C}_0(\mathcal{V}) \times \mathcal{C}_0^a(\mathcal{E})$ is W -bounded and $(D\sigma_n)_n$ is convergent in $l^2(G)$, then $(\sigma_n)_n$ has a W -convergent subsequence".*

We have the following result:

Proposition 4.1. *Let W be a Hilbert space satisfying:*

- (1) $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ is dense in W .
- (2) The injection of $\mathcal{C}_0(\mathcal{V}) \oplus \mathcal{C}_0^a(\mathcal{E})$ to $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ extends by continuity to W .
- (3) $D : W \rightarrow l^2(G)$ is a bounded operator.

Then if there exists a finite subgraph G_K of G and a positive constant $C = C_K$ such that

$$(4.19) \quad C \|(f, \varphi)\|_W \leq \|D(f, \varphi)\|_{l^2(G)}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V} \setminus K) \times \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_K),$$

so necessarily, the operator $D : W \rightarrow l^2(G)$ is semi-Fredholm.

Proof

We start by proving the following **claim**: if $\sigma_n = (f_n, \varphi_n) \in \mathcal{C}_0(\mathcal{V}) \times \mathcal{C}_0^a(\mathcal{E})$ is W -bounded and $(D\sigma_n)_n$ is convergent in $l^2(G)$, then $(\sigma_n)_n$ has a W -convergent subsequence.

Let $G_{\tilde{K}}$ be a neighborhood of the subgraph G_K (see Definition 3.2), then $(\sigma_n \upharpoonright_{\tilde{K}})_n$ is a bounded sequence in a vector space with finite dimension. Hence, it admits a convergent subsequence.

In $G \setminus G_{\tilde{K}}$, we consider the indicator function χ as in (3.8) by replacing K by \tilde{K} . Then, we obtain a function $\chi\sigma_n$ with finite support in $G \setminus G_K$ and we can apply the inequality (4.19) to $\chi\sigma_n$, in particular to $(\chi f_n, 0)$ and $(0, \bar{\chi}\varphi_n)$. First, we obtain

$$\|\chi f_n\|_W \leq C \|d(\chi f_n)\|_{l^2(\mathcal{E})}.$$

But, from the equality (2.3) of Lemma (2.3), we get

$$d(\chi f_n)(e) = \chi(e^+)d(f_n)(e) + f_n(e^-)d(\chi)(e).$$

We have $(d(f_n))_n$ is a convergent sequence and $\text{supp}(d\chi) \subset \mathcal{E}_{\tilde{K}}$ is finite, thus, $f_n(x) \upharpoonright_{\tilde{K}}$ admits a convergent subsequence.

Then we may conclude that χf_n admits a W -convergent subsequence, i.e., $(f_n \upharpoonright_{V \setminus \tilde{K}})_n$ admits a W -convergent subsequence.

Second, we have

$$\|\bar{\chi}\varphi_n\|_W \leq C \|\delta(\bar{\chi}\varphi_n)\|_{l^2(\mathcal{V})}.$$

Since the equality (2.4) of Lemma (2.3) gives

$$\delta(\bar{\chi}\varphi_n)(x) = \chi(x)\delta(\varphi_n)(x) - \frac{1}{2c(x)} \sum_{e, e^+=x} r(e)d(\chi)(e)\varphi_n(e), \quad \forall x \in \mathcal{V}.$$

Furthermore by assumptions the sequence $(\delta(\varphi_n))_n$ is convergent and $\text{supp}(d\chi) \subset \mathcal{E}_{\tilde{K}}$ is finite, hence, $(\varphi_n \upharpoonright_{\mathcal{E}_{\tilde{K}}})_n$ admits a convergent subsequence. As a result, we deduce that the sequence $(\bar{\chi}\varphi_n)_n$ admits a W -convergent subsequence. So, the sequence $(\varphi_n \upharpoonright_{\mathcal{E} \setminus \mathcal{E}_{\tilde{K}}})_n$ admits a W -convergent subsequence.

Now we can show that our operator D is semi-Fredholm.

- (1) We start by proving that $\ker D$ is finite dimensional, which is equivalent to show that $\{\sigma \in \ker D; \|\sigma\|_W = 1\}$ is compact.

Let $(\sigma_n)_n \subset \ker D$ be such that $\|\sigma_n\|_W = 1$ and $D\sigma_n = 0$. Then, by the claim, $(\sigma_n)_n$ admits a convergent subsequence. So the result occurs.

- (2) Let us show that $\text{Im}D$ is closed.

Let $(y_n)_n$ be a sequence of $\text{Im}D$ such that $(y_n)_n$ converges to y in $l^2(G)$. Is that y in $\text{Im}D$?

Since $(y_n)_n \subset \text{Im}D$, then there exist $(\sigma_n)_n \subset \ker D^\perp$ and $\sigma_n \neq 0 \forall n$, such that $y_n = D\sigma_n$. $(\sigma_n)_n$ must be bounded. If not, by extraction we can construct $s_n = \frac{\sigma_n}{\|\sigma_n\|_W}$, such that

$$\left\{ \begin{array}{l} (s_n)_n \subset \ker D^\perp \\ \|s_n\|_W = 1 \\ Ds_n \rightarrow 0. \end{array} \right.$$

Using the claim, we can conclude that $(s_n)_n$ admits a convergent subsequence with limit denoted s such that

$$\left\{ \begin{array}{l} s \in \ker D^\perp \\ \|s\|_W = 1 \\ Ds = 0. \end{array} \right.$$

Then, $s \in \ker D \cap \ker D^\perp = \{0\}$. So $s = 0$, which is absurd.

Hence the sequence $(\sigma_n)_n$ is bounded and since $(D\sigma_n)_n$ converges to y , using the claim, the sequence $(\sigma_n)_n$ admits a convergent subsequence and let σ be this limit. But, the operator D is bounded. Then, $D\sigma_n$ converges to $D\sigma$ and by uniqueness of the limit $y = D\sigma$.

□

Corollary 4.1. *D is non-parabolic at infinity if and only if there exists a finite subgraph G_K of G such that if we complete $C_0(\mathcal{V}) \times C_0^a(\mathcal{E})$ by the norm*

$$\|(f, \varphi)\|_W = \left(\|(f, \varphi)\|_{l^2(\tilde{K})}^2 + \|D(f, \varphi)\|_{l^2(G)}^2 \right)^{\frac{1}{2}},$$

in order to obtain W satisfying

- (1) $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ is dense in W .
- (2) The injection of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ to $\mathcal{C}(\mathcal{V}) \oplus \mathcal{C}^a(\mathcal{E})$ extends by continuity to W .
- (3) $D : W \rightarrow l^2(G)$ is semi-Fredholm.

5. EXAMPLES

5.1. A star-like graph.

Definition 5.1. *The disjoint union of two graphs $G_\alpha = (\mathcal{V}_\alpha, \mathcal{E}_\alpha)$ and $G_\beta = (\mathcal{V}_\beta, \mathcal{E}_\beta)$ is the disjoint union of their vertex and edge with no edge joining \mathcal{V}_α and \mathcal{V}_β .*

According to [6], we have the following definition:

Definition 5.2. An infinite graph $G = (\mathcal{V}, \mathcal{E})$ is called *star-like*, if there exists a finite subgraph G_K of G so that $G \setminus G_K$ is the union of a finite number of disjoint copies G_α of the graph \mathbb{N} .

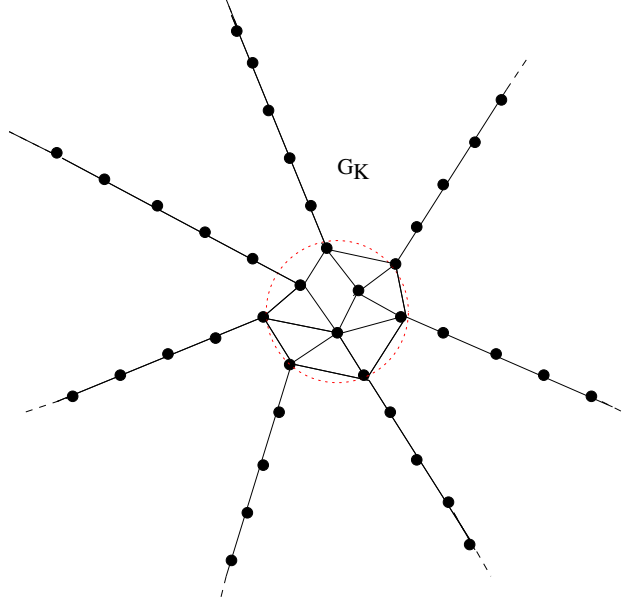


FIGURE 1

Proposition 5.1. In the case where $c = r = 1$, D is non-parabolic at infinity in the star-like graph.

Proof

By the definition of the star-like graph, there exists a finite subgraph G_K of G so that $G \setminus G_K = \bigsqcup_{\alpha \in J} G_\alpha$. Let U be a finite subset of $G \setminus G_K$ then, there exists $\alpha \in J$ such that $U \subset G_\alpha$. We look for a positive constant $C = C(U)$ such that

$$(5.20) \quad C \|(f, \varphi)\|_{l^2(U)} \leq \|D(f, \varphi)\|_{l^2(G)}, \quad \forall (f, \varphi) \in \mathcal{C}_0(\mathcal{V} \setminus K) \oplus \mathcal{C}_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Let $f \in \mathcal{C}_0(\mathcal{V} \setminus K)$ such that U is included in the support of f .

For $U = \{a\}$, we have

$$\|f\|_{l^2(U)}^2 = f^2(a).$$

For $o \in K$ and as G is connected we can find a path γ_{oa} joining o to a . Suppose that this path is of length n such that $x_0 = a$ and $x_n = o$, using the Jensen's inequality and $f(x_n) = 0$, we obtain

$$\begin{aligned} f^2(a) &= (f(x_0) - f(x_1) + f(x_1) - f(x_2) + f(x_2) - \dots - f(x_{n-1}) + f(x_{n-1}) - f(x_n) + f(x_n))^2 \\ &\leq n \left((f(a) - f(x_1))^2 + (f(x_1) - f(x_2))^2 + \dots + (f(x_{n-1}) - f(x_n))^2 \right), \end{aligned}$$

which implies

$$(5.21) \quad f^2(a) \leq n \|df\|_{l^2(\mathcal{V})}^2.$$

Remark 5.1. n depends only on U and K .

Similarly, for $\varphi \in C_0^a(\mathcal{E} \setminus \mathcal{E}_K)$, we obtain

$$\|\varphi\|_{l^2(\mathcal{E}_U)}^2 \leq C_U \|\delta\varphi\|_{l^2(\mathcal{V})}^2.$$

Moreover, for $U = \{a_1, \dots, a_n\}$, we prove the inequality (5.20).

By the inequality (5.21), for all $i \in \{1, \dots, n\}$, we get

$$f^2(a_i) \leq n_i \|\mathrm{d}f\|_{l^2(\mathcal{V})}^2$$

where n_i is the number of edge of the shortest path between a_i and any vertex of K .

For thus, we have

$$\sum_{i=1}^n f^2(a_i) \leq \sum_{i=1}^n n_i \|\mathrm{d}f\|_{l^2(\mathcal{V})}^2.$$

Hence

$$\|f\|_{l^2(U)}^2 \leq C_U \|\mathrm{d}f\|_{l^2(\mathcal{V})}^2.$$

And similarly, we show that

$$\|\varphi\|_{l^2(\mathcal{E}_U)}^2 \leq C_U \|\delta\varphi\|_{l^2(\mathcal{V})}^2.$$

□

5.2. The triadic tree.

Definition 5.3. A tree is a connected graph containing no cycles. The **triadic tree** is the tree such that all the vertices have degree 3.

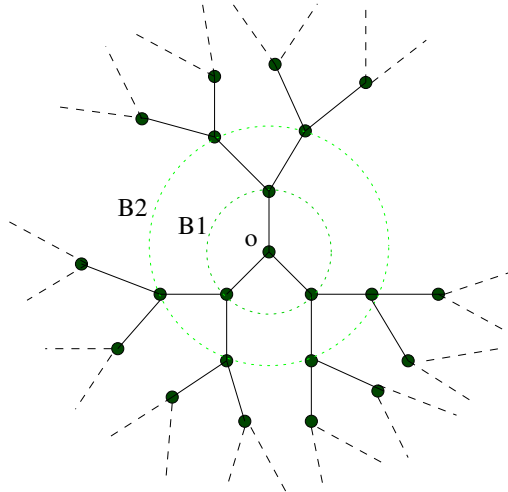


FIGURE 2

Proposition 5.2. In the triadic graph the condition of "non-parabolicity at infinity" is not verified.

Proof

We fix a vertex o , see the figure 2, we can find an increasing sequence of finite subgraph $\{G_n\}_n$ such that $G_n = \{x \in \mathcal{V}; d(o, x) \leq n\}$ and $G = \bigcup_n G_n$. The contradiction of non-parabolicity at infinity property could be: for all n there exists U outside of G_n and a 1-form φ_n with finite support outside of G_n such that $\delta\varphi_n = 0$ and $\|\varphi_n\|_{l^2(U)} \neq 0$. Such φ_n exist. Indeed one can construct a skewsymmetric function φ_n supported on the outward tree of every vertex $x_n \in G_n$ with $\delta\varphi_n = 0$ in the following way: let e_0 and b_0 be the two outward edges of x_n (the third one rely x_n to x_{n-1}) and denote $e_m^k, m \geq 1, 1 \leq k \leq 2^m$, resp. $b_m^k, m \geq 1, 1 \leq k \leq 2^m$, the outward edges emanating from e_0 , resp. b_0 , of generation m . We define φ_n to be 0 excepted on these edges where $\varphi_n(e_m^k) = \frac{1}{2^m}$ and $\varphi_n(b_m^k) = -\frac{1}{2^m}$ (the edge are oriented outward). So, we deduce that δ does not satisfy the property of non-parabolicity at infinity. □

Remark 5.2. We can generalize this example for the tree with degree $d \geq 3$, we can use the same argument with $\varphi_n = \pm(\frac{1}{d-1})^m$.

Remark 5.3. a) The importance of non-parabolicity at infinity appears with the operator δ . In fact, this property for the operator d is always true on any connected graph.

b) In probability [8] and potential theory [15] there exists an interesting notion of non-parabolic for the graph which is equivalent ([1] Theorem 2.1) to the following statement: there exists $x \in \mathcal{V}$ and $C > 0$ such that

$$f^2(x) \leq C \|df\|_{l^2(\mathcal{E})}^2, \forall f \in \mathcal{C}_0(G).$$

This notion is different from the non-parabolicity at infinity. Indeed, the graph \mathbb{Z} and \mathbb{Z}^2 are parabolic, but $\mathbb{Z}^n, n \geq 3$ is non-parabolic. On the other side, we have δ is non-parabolic at infinity in \mathbb{Z} but in $\mathbb{Z}^n, n \geq 2$, δ does not verify this property (since it has cycles supported outside any finite subgraph).

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