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# Subdivisions of oriented cycles in digraphs with large chromatic number

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## EXTENDED ABSTRACT

The *chromatic number*  $\chi(D)$  of a digraph  $D$  is the chromatic number of its underlying graph. The *chromatic number* of a class of digraphs  $\mathcal{D}$ , denoted by  $\chi(\mathcal{D})$ , is the smallest  $k$  such that  $\chi(D) \leq k$  for all  $D \in \mathcal{D}$ , or  $+\infty$  if no such  $k$  exists. If  $\chi(\mathcal{D}) \neq +\infty$ , we say that  $\mathcal{D}$  has *bounded chromatic number*.

We are interested in the following question : which are the digraph classes  $\mathcal{D}$  such that every digraph with sufficiently large chromatic number contains an element of  $\mathcal{D}$  ? Let us denote by  $\text{Forb}(H)$  (resp.  $\text{Forb}(\mathcal{H})$ ) the class of digraphs that do not contain  $H$  (resp. any element of  $\mathcal{H}$ ) as a subdigraph. The above question can be restated as follows :

**Problem 1.** *Which are the classes of digraphs  $\mathcal{D}$  such that  $\chi(\text{Forb}(\mathcal{D})) < +\infty$  ?*

An *oriented graph* is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said *directed* if all vertices have in-degree and out-degree at most 1.

Observe that if  $D$  is an orientation of a graph  $G$  and  $\text{Forb}(D)$  has bounded chromatic number, then  $\text{Forb}(G)$  has also bounded chromatic number. A classical result by Erdős implies that  $G$  must be a tree. Burr proved that every  $(k-1)^2$ -chromatic digraph contains every oriented tree of order  $k$  and conjectured Burr [3] that it could be further improved to  $(2k-2)$ -chromatic digraphs.

For special oriented trees  $T$ , better bounds on the chromatic number of  $\text{Forb}(T)$  are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that  $\chi(\text{Forb}(P^+(k))) = k$ , where  $P^+(k)$  is the directed path of length  $k$  (a *directed path* is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path  $P$  on  $n$  vertices with two blocks (*blocks* are maximal directed subpaths) has been determined by Addario-Berry et al. [1] :

**Theorem 2** (Addario-Berry et al. [1]). *Let  $P$  be an oriented path with two blocks on  $n \geq 4$  vertices, then  $\chi(\text{Forb}(P)) = n - 1$ .*

In this paper, we are interested in the chromatic number of  $\text{Forb}(\mathcal{H})$  when  $\mathcal{H}$  is an infinite family of oriented cycles. Let us denote by  $\text{S-Forb}(D)$  (resp.  $\text{S-Forb}(\mathcal{D})$ ) the class of digraphs that contain no subdivision of  $D$  (resp. any element of  $\mathcal{D}$ ) as a subdigraph. We are particularly interested in the chromatic number of  $\text{S-Forb}(\mathcal{C})$ , where  $\mathcal{C}$  is a family of oriented cycles.

Let us denote by  $\vec{C}_k$  the directed cycle of length  $k$ . For all  $k$ ,  $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$  because transitive tournaments have no directed cycle. Let us denote by  $C(k, \ell)$  the oriented cycle with two blocks, one of length  $k$  and the other of length  $\ell$ . Observe that the oriented cycles with two blocks are the subdivisions of  $C(1, 1)$ . As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore  $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$ . We first generalise this result to every oriented cycle.

**Theorem 3.** *For any oriented cycle  $C$ ,  $\chi(\text{S-Forb}(C)) = +\infty$ .*

In fact, we show the following stronger theorem.

**Theorem 4.** *For any positive integers  $b, c$ , there exists an acyclic digraph  $D_c$  with  $\chi(D_c) \geq c$  in which all oriented cycles have more than  $b$  blocks.*

We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number.

**Theorem 5.** [5, Theorem 1'] For  $k, g, c \in \mathbb{N}$ , there exists a  $k$ -uniform hypergraph with girth larger than  $g$  and weak chromatic number larger than  $c$ .

We assume  $g$  is being fixed, the following construction allow us to find  $D_{c+1}$  from  $D_c$ . Let  $p$  be the number of proper  $c$ -colourings of  $D_c$ , and let those colourings be denoted by  $col_c^1, \dots, col_c^p$ . By Theorem 5 there exists a  $c \times p$ -uniform hypergraph  $\mathcal{H}$  with weak chromatic number  $> p$  and girth  $> g/2$ . Let  $X = \{x_1, \dots, x_n\}$  be the ground set of  $\mathcal{H}$ .

We construct  $D_{c+1}$  from  $n$  disjoint copies  $D_c^1, \dots, D_c^n$  of  $D_c$  as follows. For each hyperedge  $S \in \mathcal{H}$ , we do the following :

- We partition  $S$  into  $p$  sets  $S_1, \dots, S_p$  of cardinality  $c$ .
- For each set  $S_i = \{x_{k_1}, \dots, x_{k_c}\}$ , we choose vertices  $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$  such that  $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$ , and add a new vertex  $w_{S,i}$  with  $v_{k_1}, \dots, v_{k_c}$  as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by  $\mathcal{S}$ .

**Theorem 6** (Bondy [2]).  $\chi(\text{S-Forb}(\vec{C}_k) \cap \mathcal{S}) = k - 1$ .

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

**Problem 7.** Let  $k$  and  $\ell$  be two positive integers then  $\chi(\text{S-Forb}(C(k, \ell)) \cap \mathcal{S}) < k + \ell$ .

We give evidence for this problem by showing the following weaker statement.

**Theorem 8.** Let  $k$  and  $\ell$  be two positive integers such that  $k \geq \max\{\ell, 3\}$ , and let  $D$  be a digraph in  $\text{S-Forb}(C(k, \ell)) \cap \mathcal{S}$ . Then,  $\chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$ .

We need the following lemma.

The union of two digraphs  $D_1$  and  $D_2$  is the digraph  $D_1 \cup D_2$  with vertex set  $V(D_1) \cup V(D_2)$  and arc set  $A(D_1) \cup A(D_2)$ .

**Lemma 9.** Let  $D_1$  and  $D_2$  be two digraphs.  $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$ .

A consequence of the previous lemma is that, if we partition the arc set of  $D$  into set  $A_1 \dots A_k$ , then bounding the chromatic number of all digraphs induced by the  $A_i$  implies that  $D$  has bounded chromatic number.

*Proof.* Let  $D$  be a strong digraph without any copy of  $C(k, \ell)$ , we exhibit a colouring of  $D$  using a bounded number of colours. The proof heavily relies on the technique of *levelling*. Let  $u$  be a vertex of  $D$ . The *level* of a vertex  $x$ , noted  $\text{lvl}(x)$  is the length of the shortest dipath from  $u$  to  $x$ .  $L(i)$  is the set of vertices at level  $i$ .

Since  $D$  is strongly connected, it has an out-generator  $u$ . Let  $T$  be a BFS-tree with root  $u$ . We define the following sets of arcs.

$$\begin{aligned} A_0 &= \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\ A_1 &= \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3\}; \\ A' &= \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}. \end{aligned}$$

Since  $k + \ell - 3 > 0$  and there is no arc  $xy$  with  $\text{lvl}(y) > \text{lvl}(x) + 1$ ,  $(A_0, A_1, A')$  is a partition of  $A(D)$ . Observe moreover that  $A(T) \subseteq A_1$ . We further partition  $A'$  into two sets  $A_2$  and  $A_3$ , where  $A_2 = \{xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T\}$  and  $A_3 = A' \setminus A_2$ . Then  $(A_0, A_1, A_2, A_3)$  is a partition of  $A(D)$ . Let  $D_j = (V(D), A_j)$  for all  $j \in \{0, 1, 2, 3\}$ .

**Claim 10.**  $\chi(D_0) \leq k + \ell - 2$ .

*Proof.* Observe that  $D_0$  is the disjoint union of the  $D[L_i]$  where  $L_i = \{v \mid \text{dist}_D(u, v) = i\}$ . Therefore it suffices to prove that  $\chi(D[L_i]) \leq k + \ell - 2$  for all non-negative integer  $i$ .

$L_0 = \{u\}$  so the result holds trivially for  $i = 0$ .

Assume now  $i \geq 1$ . Suppose for a contradiction  $\chi(D[L_i]) \geq k + \ell - 1$ . Since  $k \geq 3$ , by Theorem 2,  $D[L_i]$  contains a copy  $Q$  of  $P^+(k-1, \ell-1)$ , the path on two blocks of length  $k-1$  and  $\ell-1$  with one vertex of indegree 2. Let  $v_1$  and  $v_2$  be the initial and terminal vertices of  $Q$ , and let  $x$  be the least common ancestor of  $v_1$  and  $v_2$ . By definition, for  $j \in \{1, 2\}$ , there exists a dipath  $P_j$  from  $x$  to  $v_j$  in  $T$ . By definition of least common ancestor,  $V(P_1) \cap V(P_2) = \{x\}$ ,  $V(P_j) \cap L_i = \{v_j\}$ ,  $j = 1, 2$ , and both  $P_1$  and  $P_2$  have length at least 1. Consequently,  $P_1 \cup P_2 \cup Q$  is a subdivision of  $C(k, \ell)$ , a contradiction.  $\square$

**Claim 11.**  $\chi(D_1) \leq k + \ell - 3$ .

*Proof.* Let  $\phi_1$  be the colouring of  $D_1$  defined by  $\phi_1(x) = \text{lvl}(x) \pmod{k + \ell - 3}$ . By definition of  $D_1$ , this is clearly a proper colouring of  $D_1$ .  $\square$

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

**Claim 12.**  $\chi(D_2) \leq 2\ell + 2$ .

**Claim 13.**  $\chi(D_3) \leq k + \ell + 1$ .

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result.  $\square$

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for  $\hat{C}_4$  the cycle with 4 blocks.

**Theorem 14.** *Let  $D$  be a digraph in  $\text{S-Forb}(\hat{C}_4)$ . If  $D$  admits an out-generator, then  $\chi(D) \leq 24$ .*

*Proof.* The general idea is the same as in the proof of Theorem 8.

Suppose that  $D$  admits an out-generator  $u$  and let  $T$  be an BFS-tree with root  $u$ . We partition  $A(D)$  into three sets according to the levels of  $u$ .

$$\begin{aligned} A_0 &= \{(x, y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\ A_1 &= \{(x, y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\}; \\ A_2 &= \{(x, y) \in A(D) \mid \text{lvl}(y) \leq \text{lvl}(x) - 2\}. \end{aligned}$$

For  $i = 0, 1, 2$ , let  $D_i = (V(D), A_i)$ .

**Claim 15.**  $\chi(D_0) \leq 3$ .

*Proof.* Suppose for a contradiction that  $\chi(D) \geq 4$ . By Theorem 2, it contains a  $P^-(1, 1)$   $(y_1, y, y_2)$ , that is  $(y, y_1)$  and  $(y, y_2)$  are in  $A(D_0)$ . Let  $x$  be the least common ancestor of  $y_1$  and  $y_2$  in  $T$ . The union of  $T[x, y_1]$ ,  $(y, y_1)$ ,  $(y, y_2)$ , and  $T[x, y_2]$  is a subdivision of  $\hat{C}_4$ , a contradiction.  $\square$

**Claim 16.**  $\chi(D_1) \leq 2$ .

*Proof.* Since the arc are between consecutive levels, then the colouring  $\phi_1$  defined by  $\phi_1(x) = \text{lvl}(x) \pmod{2}$  is a proper 2-colouring of  $D_1$ .  $\square$

Let  $y \in V_i$  we denote by  $N'(y)$  the out-degree of  $y$  in  $\bigcup_{0 \leq j \leq i-1} V_j$ . Let  $D' = (V, A')$  with  $A' = \bigcup_{x \in V} \{(x, y), y \in N'(x)\}$  and  $D_x = (V, A_x)$  where  $A_x$  is the set of arc inside the level and from  $V_i$  to  $V_{i+1}$  for all  $i$ . Note that  $A = A' \cup A_x$  and

**Claim 17.**  $\chi(D_2) \leq 4$ .

*Proof.* We refer to [4] for the proof of this statement. □

Claims 15, 16, 17, and Lemma 9 implies  $\chi(D) \leq 24$ . □

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