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# $H_-/H_\infty$ fault detection observer design based on generalized output for polytopic LPV system

Meng Zhou<sup>1,2</sup>, Mickael Rodrigues<sup>2</sup>, Yi Shen<sup>1</sup>, Didier Theilliol<sup>3</sup>

[1] School of Astronautics, Harbin Institute of Technology, Harbin 150001, P. R. China

[2] Univ Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France

[3] Centre de Recherche en Automatique de Nancy (CRAN), Université de Lorraine, CNRS UMR 7039, F-54506 Vandoeuvre-les-Nancy, France

E-mail: [didier.theilliol@univ-lorraine.fr](mailto:didier.theilliol@univ-lorraine.fr)

**Abstract.** This paper proposes an  $H_-/H_\infty$  fault detection observer design method by using generalized output for a class of polytopic linear parameter-varying (LPV) system. First, with the aid of the relative degree of output, a new output vector is generated by gathering the original and its time derivative. The actuator fault is introduced into the measurement equation of the new system. An  $H_-/H_\infty$  observer is designed for the new LPV polytopic system to guarantee the robustness against disturbances and to improve the fault sensitivity, simultaneously. The existence conditions of the  $H_-/H_\infty$  observer are given and solved by a set of linear matrix inequalities (LMIs). Finally simulation results are given to illustrate the effectiveness of the proposed method.

## 1. Introduction

Most nonlinear control systems can be approximated into a linear parameter varying (LPV) systems, besides, it is convenient to extend the methods and theories of linear systems into LPV system models. Control theories for LPV systems have received considerable attention [1]. In recent decades, with the increasing demand of safety and reliability for modern complex control system, fault detection for LPV systems has become more and more important and fruitful literatures on this topic can be found, see, e.g. [2-4] and the references therein.

A robust fault detection observer should guarantee the generated residual significantly robust to unknown disturbances and sensitive to faults, simultaneously. In order to achieve a proper trade-off between the sensitivity to faults and robustness to disturbances, mixed  $H_-/H_\infty$  observer method has received many researchers' attention, in which the  $H_\infty$  norm is regarded as the criterion to evaluate the influence of the disturbances and the  $H_-$  index is used to measure the sensitivity of the faults. Mixed  $H_-/H_\infty$  fault detection observer is firstly proposed in [5], in which the  $H_-$  norm is defined as the smallest nonzero singular value of the transfer function matrix at the particular frequency  $\omega = 0$ . In [6, 7], the  $H_-$  index is defined as the minimum singular value of the transfer function matrix. Besides, the  $H_-$  index definition is also suitable for a finite frequency domain situation. Based on this  $H_-$  index definition, many literatures focus on  $H_-/H_\infty$  observer design for LTI system can be found in [8], [9], [10] and so on. Recently, based on the  $H_-$  index definition in full-frequency domain and finite-frequency domain,  $H_-/H_\infty$

fault detection observe design method has been extended into nonlinear systems [3], [11], [12], [13], [14], [17], just to name a few.

It should be mentioned that mixed  $H_-/H_\infty$  observer design method is not feasible for strictly proper systems in full-frequency domain, as the  $H_-$  index over  $[0, \infty)$  is always zeros[18]. Recently, in [19], a new actuator fault diagnosis approach is proposed with the relative degree of the output respect to fault. It is proved that with the aid of the relative degree of output respect to fault, the actuator fault vector can be considered into the output equation. Hence mixed  $H_-/H_\infty$  observer is feasible for the new system. However, in [19], the authors only address the fault diagnosis approach considering the relative degree of output into linear system. Besides, the  $H_-$  index do not be taken into consideration, which motives the work of this paper.

Based on the idea of [19], the main contribution of this paper lies in the following aspects. First, an  $H_-/H_\infty$  fault detection observer method is designed to a class of polytopic LPV system for actuator fault detection. Thanks to the relative degree of output respect to fault, it is feasible to consider  $H_-$  index for system with only actuator faults in full-frequency domain. Second, the  $H_-$  index is considered in observer design to improve the fault sensitivity. Besides, both the system unknown disturbance and measurement noise are considered in this paper.

The rest of this paper is organised as follows. The preliminaries are given in Section II. The problem statements is described in Section III. In Section IV, a new system is first constructed with generalized output, then a mixed  $H_-/H_\infty$  observer is designed for the new polytopic LPV system. At last, the residual evaluation and decision making logic are proposed. Simulation results are given in Section V to demonstrate the proposed approach. Finally, conclusions are given in Section VI.

**Notation 1.** Throughout the paper,  $P > 0$  and  $Q < 0$  represent that  $P$  is positive-definite and  $Q$  is negative-definite, respectively. The star symbol ( $\star$ ) in a symmetric matrix denotes the transposed block in the symmetric position.  $\bar{\sigma}(X)$  denotes the maximum singular value of  $X$  and  $\underline{\sigma}(X)$  is the minimum singular value of  $X$ . If  $\mathcal{L}_2$  is a set of square integrable functions, then the  $\mathcal{L}_2$  norm of  $x(t) \in \mathcal{L}_2$  is defined as  $\|x(t)\|_2 = (\int_0^\infty x^T(t)x(t)dt)^{1/2}$ .

## 2. Preliminaries

The following definitions and assumption are given and used throughout the paper.

**Definition 1.** [13] For a stable linear parameter-varying system described as

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) = C(\rho(t))x(t) + D(\rho(t))u(t) \end{cases} \quad (1)$$

$H_\infty$  norm for LPV system (1) is defined as

$$\|G(\rho(t))\|_\infty = \sup_{\forall \rho, \omega} \bar{\sigma}[G(j\omega)] \quad (2)$$

and  $H_-$  index is defined as

$$\|G(\rho(t))\|_- = \inf_{\forall \rho, \omega} \underline{\sigma}[G(j\omega)] \quad (3)$$

where  $G(\rho(t))$  represents the transfer function from input  $u(t)$  to output  $y(t)$ .

**Definition 2.** (Relative Degree [21]). Considering a linear control system as

$$\begin{cases} \dot{x}(t) = Ax(t) + Ef(t) \\ y(t) = Cx(t) \end{cases} \quad (4)$$

where  $x \in \mathbb{R}^{n_x}$ ,  $f \in \mathbb{R}^{n_f}$  and  $y \in \mathbb{R}$  are the state vector, actuator fault signal and measurement output, respectively. The relative degree of the output  $y(t)$  respect to fault  $f(t)$  is  $\lambda_f$  satisfying

$$\begin{cases} CA^{i-1}E = 0, \forall i = 1, \dots, (\lambda_f - 1) \\ CA^{\lambda_f-1}E \neq 0 \end{cases} \quad (5)$$

The relative degree of output  $y(t)$  respect to fault  $f(t)$  is  $\lambda_f$  means that the  $\lambda_f$ th time-derivative of output  $y^{\lambda_f}(t)$  depends on fault explicitly, while all lower order time-derivatives of output do not depend on the fault explicitly, such as

$$y^{\lambda_f}(t) = CA^{\lambda_f}x(t) + \underbrace{CA^{(\lambda_f-1)}E}_{\neq 0}f(t) \quad (6)$$

**Assumption 1.** In this paper, it is assumed that the relative degree of output  $y(t)$  respect to actuator fault  $f(t)$  is 1, such that  $CE(\rho(t)) \neq 0$ .

### 3. Problem formulation

Considering the following LPV systems subject to actuator faults and unknown disturbances as

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + E(\rho(t))f(t) + D_x(\rho(t))w(t) \\ y(t) = Cx(t) + D_yv(t) \end{cases} \quad (7)$$

where  $x(t) \in \mathbb{R}^{n_x}$ ,  $u(t) \in \mathbb{R}^{n_u}$ ,  $y(t) \in \mathbb{R}^{n_y}$  are the state vector, input vector and measurement output vector, respectively.  $f(t) \in \mathbb{R}^{n_f}$  is the actuator fault which needs to be detected.  $w(t) \in \mathbb{R}^{n_w}$  is the system unknown disturbance and  $v(t) \in \mathbb{R}^{n_v}$  is the measurement noise which are assumed bounded.  $A(\rho(t)) \in \mathbb{R}^{n_x \times n_x}$ ,  $B(\rho(t)) \in \mathbb{R}^{n_x \times n_u}$ ,  $E(\rho(t)) \in \mathbb{R}^{n_x \times n_f}$ ,  $D_x(\rho(t)) \in \mathbb{R}^{n_x \times n_w}$ ,  $C \in \mathbb{R}^{n_y \times n_x}$ ,  $D_y \in \mathbb{R}^{n_y \times n_v}$  are system matrices.  $\rho(t) = [\rho_1(t), \dots, \rho_s(t)]$  is the scheduling vector assumed measurable online,  $s$  is the number of the scheduling vector. It is assumed that the scheduling vectors are measurable. This paper mainly focuses on the actuator fault, i.e.  $B(\rho(t)) = E(\rho(t))$ , so each of the output  $y_i(t)$ ,  $i = (1, 2, \dots, n_y)$  has the same relative degree respect to the input  $u(t)$  and fault  $f(t)$ . Without loss of generality, it is assumed the system  $(A(\rho(t)), B(\rho(t)))$  is controllable,  $(A(\rho(t)), C)$  is observable, and the rank condition is satisfied as  $\text{rank}(CB(\rho(t))) = \text{rank}(B(\rho(t)))$ .

In this paper, the LPV system (7) is assumed that the time-varying parameter vector  $\rho(t)$  varies within a polytope  $\Theta$  of vertices  $\{\rho_1, \dots, \rho_s\}$  with  $\rho_i \in [\underline{\rho}_i, \bar{\rho}_i]$ ,  $i = (1, 2, \dots, s)$ . The system matrices range in a polytope of matrices defined as the convex hull, each polytope vertex corresponds to a particular value of scheduling variable [23]. Then the polytopic LPV system (7) is expressed as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^N h_i(\rho(t))(A_i x(t) + B_i u(t) + B_i f(t) + D_{xi} w(t)) \\ y(t) = Cx(t) + D_y v(t) \end{cases} \quad (8)$$

where  $A(\rho(t)) = \sum_{i=1}^N h_i(\rho(t))A_i$ ,  $B(\rho(t)) = \sum_{i=1}^N h_i(\rho(t))B_i$ ,  $D_x(\rho(t)) = \sum_{i=1}^N h_i(\rho(t))D_{xi}$ .  $N$  is the number of weighting functions.  $A_i$ ,  $B_i$ ,  $D_{xi}$  are time-invariant matrices of the  $i$ th model. The polytopic system is scheduled through the weighting functions that lie in a convex set and satisfy the following properties

$$\sum_{i=1}^N h_i(\rho(t)) = 1, \quad 0 \leq h_i(\rho(t)) \leq 1, \quad i = (1, 2, \dots, N) \quad (9)$$

Classically, a mixed  $H_-/H_\infty$  observer is generated as

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^N h_i(\rho(t)) [A_i \hat{x}(t) + B_i u(t) + L_i (y(t) - \hat{y}(t))] \\ \hat{y}(t) = C \hat{x}(t) \\ r(t) = M [y(t) - \hat{y}(t)] \end{cases} \quad (10)$$

where  $\hat{x}(t) \in \mathbb{R}^{n_x}$  is the state estimation of observer (10),  $L_i \in \mathbb{R}^{n_x \times n_y}$ ,  $i = (1, 2, \dots, N)$  and  $M \in \mathbb{R}^{n_y \times n_y}$  are the gain matrices of residual generator based on an observer (10) that have to be designed.

Defining the state estimation error vector as

$$e(t) = x(t) - \hat{x}(t) \quad (11)$$

Subtracting (8) from (10), the estimation error dynamic system is obtained as

$$\dot{e}(t) = \sum_{i=1}^N h_i(\rho(t)) [(A_i - L_i C) e(t) + B_i f(t) + (\bar{D}_{xi} - L_i \bar{D}_y) d(t)] \quad (12)$$

$$r(t) = M C e(t) + M \bar{D}_y d(t) \quad (13)$$

where

$$d(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \bar{D}_{xi} = [D_{xi} \quad 0], \bar{D}_y = [0 \quad D_y] \quad (14)$$

According to [5], observer (10) is called an  $H_-/H_\infty$  observer if the following conditions hold

- (i) The state estimation error system (12) is asymptotically stable;
- (ii) The residual  $r(t)$  (13) is robust to the unknown disturbance  $d(t)$ , i.e.

$$\|r(t)\|_2 < \gamma \|d(t)\|_2 \quad (15)$$

- (iii) The residual  $r(t)$  (13) is sensitive to the actuator fault  $f(t)$ , i.e.

$$\|r(t)\|_2 > \beta \|f(t)\|_2 \quad (16)$$

Obviously, the goal (i) and (ii) can be satisfied easily based on the  $H_\infty$  theory. While considering the fault sensitivity condition (16), the observer matrices should satisfy the following LMI

$$\begin{bmatrix} C^T M^T M C - P(A_i - L_i C) - (A_i - L_i C)^T P & -P B_i \\ \star & -\beta^2 I \end{bmatrix} > 0 \quad (17)$$

where  $P$  is a symmetrical positive definite matrix.

Unfortunately, due to term  $-\beta^2 I < 0$ , there is no feasible solution for LMI (17). As a result, the  $H_-/H_\infty$  observer cannot be obtained for system in (8) in full frequency domain. It is known that if a system consists a sensor fault, such that  $y(t) = Cx(t) + Df(t) + D_y v(t)$ , then the generated residual signal will be influenced by the sensor fault, the  $H_-$  condition will become

$$\begin{bmatrix} C^T M^T M C - P(A_i - L_i C) - (A_i - L_i C)^T P & C^T M^T M D - P B_i \\ \star & D^T M^T M D - \beta^2 I \end{bmatrix} > 0 \quad (18)$$

There may be a feasible solution for (18) with the matrix  $D$ . Therefore, the mixed  $H_-/H_\infty$  observer is theoretically valid for a system with faults in the measurement equations. Actually, the actuator fault also influence the outputs. If the actuator fault can be transformed into the measurement output equation, the  $H_-/H_\infty$  observer is theoretically feasible. Fortunately, relative degree of output is a useful method to bring the actuator fault to the measurement output. In this paper, we mainly focus on  $H_-/H_\infty$  fault detection observer design based on the relative degree of output.

## 4. Main Results

### 4.1. $H_-/H_\infty$ observer design based on generalized output

According to [19], for multiple output systems, each output  $y_i(t)$ ,  $i = (1, \dots, n_y)$  may have different relative degree respect to fault  $f(t)$ . The first step is to generate a new output component  $\tilde{y}_i(t)$  for each output  $y_i(t)$ . Then gather all the new component  $\tilde{y}_i(t)$  as a whole to become the new output  $\tilde{y}(t)$ . However in this paper, under assumption 1, the relative degree of the each output respect to fault is all assumed to be 1, such that the output  $y(t)$  can be treat as a whole and the time derivative  $\dot{y}(t)$  is expressed as

$$\dot{y}(t) = \sum_{i=1}^N h_i(\rho(t)) [CA_i x(t) + CB_i(u(t) + f(t)) + CD_{xi}w(t)] + D_y \dot{v}(t) \quad (19)$$

A new generalized output  $\tilde{y}(t)$  is generated by gathering the original output  $y(t)$  and the time derivative  $\dot{y}(t)$

$$\tilde{y}(t) = \sum_{i=1}^N h_i(\rho(t)) (\tilde{C}_i x(t) + R_i f(t) + \bar{D}_{yi} d(t)) \quad (20)$$

where

$$\begin{aligned} \tilde{y}(t) &= \begin{bmatrix} y(t) \\ \dot{y}(t) - \sum_{i=1}^N h_i(\rho(t)) CB_i u(t) \end{bmatrix}, \tilde{C}_i = \begin{bmatrix} C \\ CA_i \end{bmatrix}, R_i = \begin{bmatrix} 0 \\ CB_i \end{bmatrix}, \\ \bar{D}_{yi} &= \begin{bmatrix} 0 & D_y & 0 \\ CD_{xi} & 0 & D_y \end{bmatrix} \text{ and } d(t) = \begin{bmatrix} w(t) \\ v(t) \\ \dot{v}(t) \end{bmatrix} \end{aligned} \quad (21)$$

By letting,

$$\bar{D}_{xi} = [D_{xi} \quad 0 \quad 0] \quad (22)$$

a new LPV system can be expressed as

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^N h_i(\rho(t)) [A_i \hat{x}(t) + B_i(u(t) + f(t)) + \bar{D}_{xi} d(t)] \\ \hat{\tilde{y}}(t) = \sum_{i=1}^N h_i(\rho(t)) [\tilde{C}_i \hat{x}(t) + R_i f(t) + \bar{D}_{yi} d(t)] \end{cases} \quad (23)$$

For system (23), an  $H_-/H_\infty$  observer with LPV structure is designed as

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^N h_i(\rho(t)) [A_i \hat{x}(t) + B_i u(t) + L_i (\tilde{y}(t) - \hat{\tilde{y}}(t))] \\ \hat{\tilde{y}}(t) = \sum_{i=1}^N h_i(\rho(t)) [\tilde{C}_i \hat{x}(t)] \\ r(t) = M [\tilde{y}(t) - \hat{\tilde{y}}(t)] \end{cases} \quad (24)$$

Define the state estimation error vector as

$$e(t) = x(t) - \hat{x}(t) \quad (25)$$

Subtracting (23) from (24), the estimation error dynamic system is obtained as

$$\dot{e}(t) = \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t))h_j(\rho(t))[(A_i - L_i\tilde{C}_j)e(t) + (B_i - L_iR_j)f(t) + (\bar{D}_{xi} - L_i\bar{D}_{yj})d(t)] \quad (26)$$

$$r(t) = \sum_{i=1}^N h_i(\rho(t))(M\tilde{C}_i e(t) + M\bar{D}_{yi}d(t) + MR_i f(t)) \quad (27)$$

The main objective of this paper is to design the matrices  $L_i$ ,  $i = (1, 2, \dots, N)$  and  $M$  of the  $H_-/H_\infty$  observer (24) to guarantee the residual  $r(t)$  robust to disturbance  $d(t)$  and sensitive to fault  $f(t)$ , simultaneously, such that

- (i) The estimation error system (26) is asymptotically stable;
- (ii) The residual  $r(t)$  (27) is robust to the disturbance  $d(t)$ , i.e.

$$\|r(t)\|_2 < \gamma \|d(t)\|_2 \quad (28)$$

- (iii) The residual  $r(t)$  (27) is sensitive to the actuator fault  $f(t)$ , i.e.

$$\|r(t)\|_2 > \beta \|f(t)\|_2 \quad (29)$$

The following Theorem is proposed to synthesize the matrices  $L_i$  and  $M$  of observer in (24).

**Theorem 1.** For given positive scalars  $\gamma$  and  $\beta$  and  $\theta_1$ , the observer (24) is called an  $H_-/H_\infty$  observer, if there exist a symmetrical positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$ , matrices  $W_i \in \mathbb{R}^{n_x \times n_y}$  and symmetrical non-negative matrix  $U \in \mathbb{R}^{n_y \times n_y}$  to satisfy the following LMIs hold

$$\begin{aligned} \Psi_{ii} &< 0, i = 1, \dots, N \\ \Psi_{ij} + \Psi_{ji} &< 0, 1 \leq i < j \leq N \end{aligned} \quad (30)$$

$$\begin{aligned} \Pi_{ii} &< 0, i = 1, \dots, N \\ \Pi_{ij} + \Pi_{ji} &< 0, 1 \leq i < j \leq N \end{aligned} \quad (31)$$

. where

$$\Psi_{ij} = \begin{bmatrix} \psi_{11} & \theta_1 P \bar{D}_{xi} - \theta_1 W_i \bar{D}_{yj} + \tilde{C}_i^T U \bar{D}_{yj} \\ \star & \bar{D}_{yi}^T U \bar{D}_{yj} - \gamma^2 I \end{bmatrix} \quad (32)$$

$$\psi_{11} = \theta_1 P A_i - \theta_1 W_i \tilde{C}_j + \theta_1 A_i^T P - \theta_1 \tilde{C}_j^T W_i^T + \tilde{C}_i^T U \tilde{C}_j \quad (33)$$

$$\Pi_{ij} = \begin{bmatrix} \pi_{11} & P B_i - W_i R_j - \tilde{C}_i^T U R_j \\ \star & -R_i^T U R_j + \beta^2 I \end{bmatrix} \quad (34)$$

$$\pi_{11} = P A_i - W_i \tilde{C}_j + A_i^T P - \tilde{C}_j^T W_i^T - \tilde{C}_i^T U \tilde{C}_j \quad (35)$$

$$W_i = P L_i, U = M^T M \quad (36)$$

**Proof:** (i) *Disturbance attenuation condition*

The robustness of the residual signal  $r(t)$  (27) against to disturbance  $d(t)$  is first considered. Making  $f(t) = 0$  in (26) and (27), then it becomes

$$\begin{cases} \dot{e}(t) = \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t))h_j(\rho(t))[(A_i - L_i\tilde{C}_j)e(t) + (\bar{D}_{xi} - L_i\bar{D}_{yj})d(t)] \\ r(t) = \sum_{i=1}^N h_i(\rho(t))[M\tilde{C}_i e(t) + M\bar{D}_{yi}d(t)] \end{cases} \quad (37)$$

Choosing a Lyapunov function as

$$V_1(t) = e(t)^T P_1 e(t) > 0 \quad (38)$$

where  $P_1 \in \mathbb{R}^{n_x \times n_x}$  is a symmetrical positive definite matrix.

Then the time derivative of  $V_1(t)$  is

$$\begin{aligned} \dot{V}_1(t) = & \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t)) h_j(\rho(t)) \{ e^T(t) [P_1(A_i - L_i \tilde{C}_j) + (A_i - L_i \tilde{C}_j)^T P_1] e(t) \\ & + 2e^T(t) [P_1(\bar{D}_{xi} - L_i \bar{D}_{yj})] d(t) \} \end{aligned} \quad (39)$$

To satisfy the  $H_\infty$  index  $\|r(t)\|_2 < \gamma \|d(t)\|_2$ , a criterion is defined as

$$J_\infty = \int_0^\infty [r^T(t)r(t) - \gamma^2 d^T(t)d(t)] dt \quad (40)$$

Then

$$J_\infty = \int_0^\infty [r^T(t)r(t) - \gamma^2 d^T(t)d(t) + \dot{V}_1(t)] dt - V_1(t) \quad (41)$$

Due to the fact that  $V_1(t) > 0$ , if

$$r^T(t)r(t) - \gamma^2 d^T(t)d(t) + \dot{V}_1(t) < 0 \quad (42)$$

then it is satisfied  $J_\infty < 0$ . Substitute (27) into (42), it becomes

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t)) h_j(\rho(t)) \{ [e^T(t) (P_1(A_i - L_i \tilde{C}_j) + (A_i - L_i \tilde{C}_j)^T P_1) e(t)] \\ & + 2e^T(t) [P_1(\bar{D}_{xi} - L_i \bar{D}_{yj})] d(t) + \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t)) h_j(\rho(t)) [e^T(t) \tilde{C}_i^T M^T M \tilde{C}_j e(t) \\ & + 2e^T(t) \tilde{C}_i^T M^T M \bar{D}_{yj} d(t) + d^T(t) \bar{D}_{yi}^T M^T M \bar{D}_{yj} d(t)] - \gamma^2 d^T(t)d(t) \} \\ = & \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t)) h_j(\rho(t)) \xi^T(t) \Omega_{ij} \xi(t) < 0 \end{aligned} \quad (43)$$

where

$$\xi(t) = \begin{bmatrix} e(t) \\ d(t) \end{bmatrix}, \quad \Omega_{ij} = \begin{bmatrix} \omega_{11} & P_1(\bar{D}_{xi} - L_i \bar{D}_{yj}) + \tilde{C}_i^T M^T M \bar{D}_{yj} \\ \star & \bar{D}_{yi}^T M^T M \bar{D}_{yj} - \gamma^2 I \end{bmatrix} \quad (44)$$

$$\omega_{11} = P_1(A_i - L_i \tilde{C}_j) + (A_i - L_i \tilde{C}_j)^T P_1 + \tilde{C}_i^T M^T M \tilde{C}_j \quad (45)$$

To make  $J_\infty < 0$ , it is required

$$\Omega_{ij} < 0, \quad i, j = (1, 2, \dots, N) \quad (46)$$

Note that (43) can be re-written as

$$\sum_{i=1}^N h_i(\rho(t))^2 \xi^T(t) \Omega_{ii} \xi(t) + \sum_{i=1}^N \sum_{i < j}^N h_i(\rho(t)) h_j(\rho(t)) \xi^T(t) (\Omega_{ij} + \Omega_{ji}) \xi(t) < 0 \quad (47)$$



Then we have the following conditions

$$\begin{cases} \Omega_{ii} < 0, & i = 1, \dots, N \\ \Omega_{ij} + \Omega_{ji} < 0, & 1 \leq i < j \leq N \end{cases} \quad (48)$$

According to [22], the conditions in (48) is more relaxed than that in (46).

(ii) *Fault sensitivity condition*

Next the fault sensitivity of the residual  $r(t)$  in (27) is considered, letting  $d(t) = 0$  in (26) and (27), then

$$\begin{cases} \dot{e}(t) = \sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t))h_j(\rho(t))[(A_i - L_i\tilde{C}_j)e(t) + (B_i - L_iR_j)f(t)] \\ r(t) = \sum_{i=1}^N h_i(\rho(t))(M\tilde{C}_i e(t) + MR_i f(t)) \end{cases} \quad (49)$$

The proof for fault sensitivity condition is similar to disturbance attenuation condition. A Lyapunov function is chosen as

$$V_2(t) = e^T(t)P_2e(t) > 0 \quad (50)$$

To maintain the  $H_-$  index  $\|r(t)\|_2 > \beta\|f(t)\|_2$ , defining a criterion as

$$J_- = \int_0^\infty [r^T(t)r(t) - \beta^2 f^T(t)f(t)]dt \quad (51)$$

which is equivalent to

$$J_- = \int_0^\infty [r^T(t)r(t) - \beta^2 f^T(t)f(t) - \dot{V}_2(t)]dt + V_2(t) \quad (52)$$

As  $V_2(t) > 0$ , if

$$r^T(t)r(t) - \beta^2 f^T(t)f(t) - \dot{V}_2(t) > 0 \quad (53)$$

Then the  $H_-$  index  $J_- > 0$  is satisfied. Combining (27), then (53) can be written such that

$$\sum_{i=1}^N \sum_{j=1}^N h_i(\rho(t))h_j(\rho(t))\eta^T(t)\Upsilon_{ij}\eta(t) > 0 \quad (54)$$

where

$$\begin{aligned} \eta(t) &= \begin{bmatrix} e(t) \\ f(t) \end{bmatrix}, \quad \Upsilon_{ij} = \begin{bmatrix} v_{11} & -P_2(E_i - L_iR_j) + \tilde{C}_i^T M^T MR_j \\ \star & -R_i^T M^T MR_j + \beta^2 I \end{bmatrix} \\ v_{11} &= -P_2(A_i - L_i\tilde{C}_j) - (A_i - L_i\tilde{C}_j)^T P_2 + \tilde{C}_i^T M^T M\tilde{C}_j \end{aligned} \quad (55)$$

For similar reasons explained in [22], the inequality (54) can be replaced by

$$\sum_{i=1}^N h_i(\rho(t))^2 \eta^T(t)\Upsilon_{ii}\eta(t) + \sum_{i=1}^N \sum_{i < j}^N h_i(\rho(t))h_j(\rho(t))\eta^T(t)(\Upsilon_{ij} + \Upsilon_{ji})\eta(t) > 0 \quad (56)$$

As a result,

$$\begin{cases} \Upsilon_{ii} > 0, & i = 1, \dots, N \\ \Upsilon_{ij} + \Upsilon_{ji} > 0, & 1 \leq i < j \leq N \end{cases} \quad (57)$$

Noted that there is coupling between (48) and (57), in order to facilitate the observer solving process, letting

$$P_1 = \theta_1 P, \quad P_2 = P, \quad W_i = PL_i, \quad U = M^T M \quad (58)$$

Then (48) and (57) is equivalent to (30) and (31), respectively. And the observer gain matrices  $L_i$  and  $M$  can be obtained by  $L_i = P^{-1}W_i$ ,  $M = \sqrt{U}$ .

#### 4.2. Residual evaluation and decision making

After the  $H_-/H_\infty$  observer is designed, a residual evaluation function and fault decision making scheme are given. Due to the presence of exogenous disturbances, the residual signals are different from zero even under fault-free operation. Since the proposed  $H_-/H_\infty$  observer guarantees that  $\|r(t)\|_2 < \gamma\|d(t)\|_2$ , the  $\mathcal{L}_2$  norm of the residual signal  $\|r(t)\|_2$  can be used as the residual evaluation function. However, since it is usually unrealistic to calculate the evaluation function over the whole time [20], in this paper, the Root Mean Square (RMS) function is chosen as the evaluation function

$$J = \|r(t)\|_{\text{RMS}} = \sqrt{\frac{1}{\Delta t} \int_{t-\Delta t}^t r^T(\tau)r(\tau)d\tau} \quad (59)$$

where  $J$  measures the average energy of the residual over a time interval  $(t - \Delta t, t)$ .  $\Delta t$  is a large enough time window which is usually determined depends on the applications [11].

Once the evaluation function is selected, a threshold  $J_{th}$  is needed. The threshold should represent the maximum influence of the unknown disturbances  $d(t)$  respect to the residual  $r(t)$  under a fault-free situation, therefore the threshold can be set as

$$J_{th} = \sup_{d(t), f(t)=0} \|r(t)\|_{\text{RMS}} \quad (60)$$

Then comparing the residual evaluation function  $J$  with the predefined threshold  $J_{th}$ , the decision logic is formulated as

$$\begin{cases} J > J_{th} \Rightarrow \text{alarm} \\ J \leq J_{th} \Rightarrow \text{no alarm} \end{cases} \quad (61)$$

### 5. Simulation Results

Numerical simulations are proposed to demonstrate the effectiveness of the proposed  $H_-/H_\infty$  observer design method. Considering a Vertical Takeoff and Landing (VTOL) aircraft model in the vertical plane described as an LPV form from [23]

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))(u(t) + f(t)) + D_x w(t) \\ y(t) = Cx(t) + D_y v(t) \end{cases} \quad (62)$$

where the state vectors  $x(t) = [V_h \ V_v \ q \ \theta]^T$  are horizontal velocity, vertical velocity, pitch rate and pitch angle, respectively, and the control input  $u(t) = [u_c \ u_l]^T$  are collective pitch control and longitudinal cyclic pitch control, respectively.  $w(t)$  represents the model uncertainties and unknown disturbances,  $v(t)$  denotes the measurement noise. The system matrices with varying parameters is expressed as

$$A(\rho(t)) = \begin{bmatrix} -9.9477 & -0.7476 & 0.2632 & 5.0337 \\ 52.1659 & 2.7452 & 5.5532 & -24.4221 \\ 26.0922 & 2.6361 + \rho_1(t) & -4.1975 & -19.2774 + \rho_2(t) \\ 0 & 0 & 1 & 0 \end{bmatrix}, B(\rho(t)) = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 + \rho_2(t) & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$$

The output and the unknown input distribution matrices are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, D_x = \begin{bmatrix} 0.001 \\ 0 \\ 0 \\ 0 \end{bmatrix}, D_y = \begin{bmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0.001 \\ 0 & 0.001 & 0 & 0 \end{bmatrix}$$

Obviously the relative degree of each output respect to actuator faults is 1, based on (21), the new output and its corresponding matrices can be obtained. In this paper, it is assumed that  $\rho(t) = [\rho_1(t) \ \rho_2(t)]^T$  are the scheduling vectors with  $\rho_1(t) \in [-0.5, 0.5]$  and  $\rho_2(t) \in [-2, 2]$ . The scheduling variables  $\rho_1(t)$  and  $\rho_2(t)$  are shown as figure 1 and figure 2. Then four local models are derived and the weighting functions  $h_i(\rho(t))$ ,  $i = (1, \dots, N)$ ,  $N = 4$  are assumed as

$$\begin{aligned} h_1(\rho(t)) &= \frac{\bar{\rho}_1 - \rho_1(t)}{\bar{\rho}_1 - \underline{\rho}_1} \cdot \frac{\bar{\rho}_2 - \rho_2(t)}{\bar{\rho}_2 - \underline{\rho}_2}, & h_2(\rho(t)) &= \frac{\bar{\rho}_1 - \rho_1(t)}{\bar{\rho}_1 - \underline{\rho}_1} \cdot \frac{\rho_2(t) - \underline{\rho}_2}{\bar{\rho}_2 - \underline{\rho}_2} \\ h_3(\rho(t)) &= \frac{\rho_1(t) - \underline{\rho}_1}{\bar{\rho}_1 - \underline{\rho}_1} \cdot \frac{\bar{\rho}_2 - \rho_2(t)}{\bar{\rho}_2 - \underline{\rho}_2}, & h_4(\rho(t)) &= \frac{\rho_1(t) - \underline{\rho}_1}{\bar{\rho}_1 - \underline{\rho}_1} \cdot \frac{\rho_2(t) - \underline{\rho}_2}{\bar{\rho}_2 - \underline{\rho}_2} \end{aligned} \quad (63)$$

The local weighting function  $h_i(\rho(t))$  of each local model is depicted in figure 3. According to the separation principle of the controller and the observer design, the controller has been synthesized with a state feedback structure so as to maintain the closed-loop system stable.

Generally speaking, the disturbance reduction performance  $\gamma$  and fault sensitivity performance  $\beta$  are need to be made a trade-off. The principle of determining these parameters in this paper is described as follows. First, the robustness performance index  $\gamma$  is fixed based on experience, then finding the largest value of the fault sensitive index  $\beta$  by tuning the parameter  $\theta_1$  to satisfy the LMIs of Theorem 1. In this paper, the robustness index is chosen as  $\gamma = 0.08$ , and the largest values of the sensitivity performance is obtained as  $\beta = 2.83$  with the parameter  $\theta_1 = 30$  by solving the LMIs in Theorem 1, and the gain matrices of the proposed  $H_-/H_\infty$  observer for the new system are obtained as

$$\begin{aligned} L_1 &= \begin{bmatrix} 0.8715 & 0.7565 & 0.9320 & 0.7549 & -0.1939 & 0.2457 & -0.9359 & 0.2418 \\ 0.0641 & 2.0349 & 0.1363 & 1.8852 & -0.0315 & 0.6062 & -0.3151 & 0.4274 \\ -0.0516 & -0.4609 & -0.1530 & 0.1742 & -0.1402 & -0.0592 & 0.3670 & 0.1549 \\ -0.0278 & 0.0194 & -0.0130 & 0.0390 & 0.0275 & -0.0080 & 0.0166 & -0.0045 \end{bmatrix} \\ L_2 &= \begin{bmatrix} 0.7548 & 1.1492 & 1.0942 & 1.1406 & 0.5589 & -0.5021 & -0.0639 & 0.5281 \\ -0.0484 & 2.1406 & 0.1995 & 1.9285 & 0.4204 & 0.3959 & -0.0583 & 0.5371 \\ -0.0166 & -0.4570 & -0.2079 & 0.2097 & -0.1869 & 0.0686 & 0.2528 & 0.1135 \\ -0.0069 & 0.0039 & 0.0019 & 0.0462 & -0.0413 & 0.0187 & -0.0336 & -0.0130 \end{bmatrix} \\ L_3 &= \begin{bmatrix} 0.9134 & 0.7841 & 0.9556 & 0.7921 & -0.5989 & 0.2851 & -1.0435 & 0.1972 \\ 0.1258 & 2.0508 & 0.2880 & 1.9941 & -0.1817 & 0.5794 & -0.4080 & 0.4595 \\ -0.0677 & -0.4371 & -0.1755 & 0.1941 & -0.0765 & -0.0581 & 0.3888 & 0.1552 \\ -0.0282 & 0.0196 & -0.0110 & 0.0428 & 0.0489 & -0.0107 & 0.0204 & -0.0013 \end{bmatrix} \\ L_4 &= \begin{bmatrix} 1.0496 & 0.7056 & 0.7324 & 0.9308 & -1.9622 & -0.1131 & -0.5318 & 0.0875 \\ 0.1454 & 2.0137 & 0.2207 & 2.0076 & -0.5766 & 0.4963 & -0.3034 & 0.4136 \\ -0.0861 & -0.4153 & -0.1269 & 0.1993 & 0.0462 & -0.0027 & 0.3099 & 0.1804 \\ -0.0295 & 0.0249 & 0.0010 & 0.0405 & 0.1112 & 0.0021 & 0.0023 & 0.0054 \end{bmatrix} \\ M &= \begin{bmatrix} 71.2831 & 7.1480 & 5.7026 & -6.5965 & 3.8840 & -2.2201 & -1.6248 & 1.8577 \\ 7.1480 & 61.0252 & -5.0354 & -5.9652 & 0.7846 & 2.2000 & 1.0432 & -1.7922 \\ 5.7026 & -5.0354 & 37.6864 & 3.3263 & 0.7685 & -4.8937 & -3.8885 & 4.6469 \\ -6.5965 & -5.9652 & 3.3263 & 59.1665 & -0.2383 & -1.8249 & -0.0181 & 1.4833 \\ 3.8840 & 0.7846 & 0.7685 & -0.2383 & 27.9935 & -2.0691 & 3.6357 & 2.3351 \\ -2.2201 & 2.2000 & -4.8937 & -1.8249 & -2.0691 & 10.7606 & 8.0019 & -10.3884 \\ -1.6248 & 1.0432 & -3.8885 & -0.0181 & 3.6357 & 8.0019 & 12.2915 & -7.8525 \\ 1.8577 & -1.7922 & 4.6469 & 1.4833 & 2.3351 & -10.3884 & -7.8525 & 10.6633 \end{bmatrix} \end{aligned}$$

In the simulations, initial states of the systems is chosen as  $[0 \ 0.01 \ 0 \ 0]^T$ , the unknown inputs are set as random signals bounded by  $[-1, 1]$ . The time window of the the residual evaluation function is chosen as  $\Delta t = 10s$  and according to (60), the threshold is calculated as  $J_{th} = 0.15$ . In

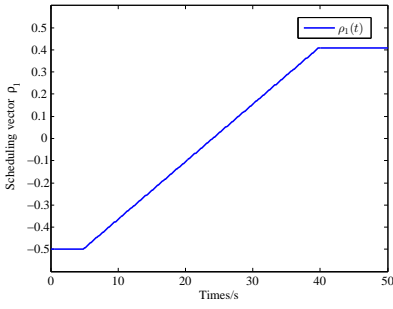


Figure 1.  $\rho_1(t)$

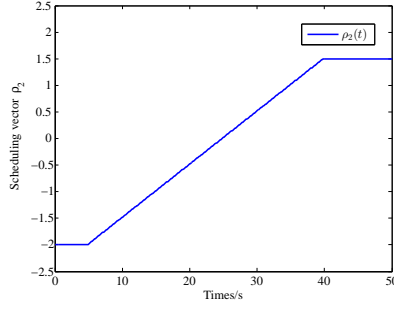


Figure 2.  $\rho_2(t)$

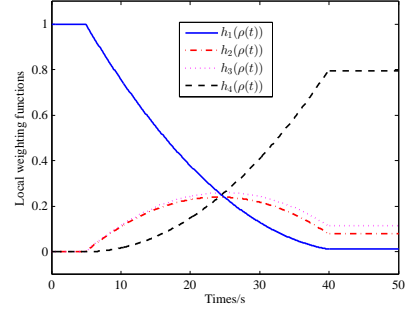


Figure 3.  $h_i(\rho)$

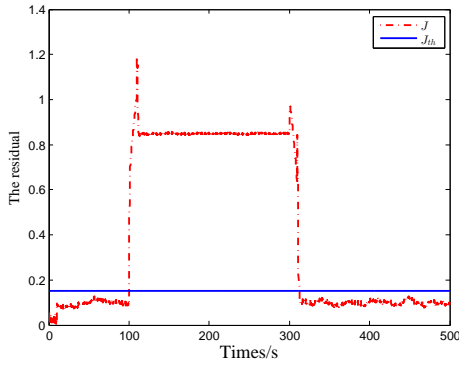


Figure 4. Residual of fault (64)

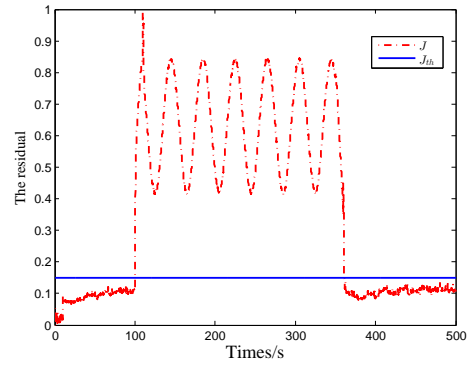


Figure 5. Residual of fault (65)

order to demonstrate the effectiveness of the proposed method, two fault scenarios are assumed to be detected, i.e. abrupt fault and time-varying fault. First, an abrupt fault is formulated as

$$f(t) = \begin{cases} [0 \ 0]^T & 0s < t \leq 100s \\ [0.1 \ 0]^T & 100s < t \leq 300s \\ [0 \ 0]^T & 300s < t \leq 500s \end{cases} \quad (64)$$

In the second scenario, it is assumed that a time-varying fault occurs with the formulation as

$$f(t) = \begin{cases} [0 \ 0]^T & 0s < t \leq 100s \\ [0 \ 0.05 + 0.02 \sin(0.1\pi(t - 50))]^T & 100s < t \leq 350s \\ [0 \ 0]^T & 350s < t \leq 500s \end{cases} \quad (65)$$

The simulation results are depicted as in figures 4–5. Figure 4 shows the residual generated by the proposed method with abrupt fault (64). From the picture it can be seen that the residual exceeds the predefined threshold between  $100s < t \leq 300s$ , the fault (64) can be detected by the proposed  $H_-/H_\infty$  observer successfully. Figure 5 depicts the residual respect to the time-varying fault (65), which shows that the time-varying fault (65) is also detected successfully.

## 6. Conclusions

In this paper, an  $H_-/H_\infty$  fault detection observer is investigated for a class of polytopic LPV systems with the aid of relative degree of output. This paper proposes a new method to consider the  $H_-$  index for systems only consider actuator faults in full-frequency domain, which may has some conservatism compared with finite-frequency domain method. In the future, we will

combine the proposed method with finite-frequency domain method. Besides, note that the proposed observer in this paper is depended on numerical derivation of the outputs, which is difficult to be available in practice, and it may enlarge the effect of the noise. As a result, in our future work, we will consider  $H_-/H_\infty$  fault detection observer design methods without using the numerical derivation.

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