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Smoothed extended finite element method for continua

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Résumé — The extended finite element method (XFEM) introduced by Belytschko [1] allows discontinuities to be arbitrarily present in the domain, i.e, discontinuities can be independent of the mesh. However this complicates the numerical integration of the stiffness matrix, because it requires sub-division of elements cut by discontinuity, which in 3D may be computationally costly. In this paper, we address this issue by proposing a new method named as the Smoothed eXtended Finite Element Method (SmXFEM) [4]. This method combines the strain smoothing technique proposed by Liu *et al.*, [6] for finite element method, coined as Smoothed Finite Element Method (SFEM) with the XFEM. The SmXFEM shares the properties of both the SFEM [9] and XFEM. The SmXFEM alleviates the need to compute the derivative of shape functions and numerical integration of singular functions, commonly encountered in linear elastic fracture mechanics. This is achieved by transforming the surface integration to line integration along the boundary of an element. The effectiveness of the method is illustrated by a few benchmark examples taken from linear elastic fracture mechanics. The results obtained with SmXFEM compare well with those of the standard XFEM.

Mots clés — strain smoothing, extended finite element method, boundary integration, isoparametric mapping

1 Introduction

The eXtended Finite Element Method (XFEM) is a numerical method to model arbitrary discontinuities in continuous bodies that does not require the mesh to conform to the discontinuities [2, 3]. Recently smoothed finite element method (SFEM) has been proposed by Liu *et al.* [6] by means of combining the strain smoothing technique with the conventional FEM technology and further extended by Nguyen *et al.* [9]. In this method, strain smoothing operation is performed on an entire or part of an element to replace the conventional strains obtained using displacement-compatible equations in FEM. Due to the presence of strain projection process, only shape function is involved in the calculation of field gradients and no isoparametric transformation is required. Hence, the element is allowed to be of arbitrary shape and thus the field domain can be discretised in a more flexible way. Interested readers can refer to work of Nguyen *et al.* [9] for detailed description about the SFEM. The SFEM is based on strain smoothing stabilization, originally introduced by Chen [5] for meshfree methods. Refer to [8] for a recent review on meshfree methods. The SFEM was subsequently extended to plate and shell analyses [10, 7].

We present a technique to combine the strain smoothing technique to the XFEM, to get the Smoothed eXtended Finite Element Method (SmXFEM) [4]. Section 2 briefly discusses the basics behind the XFEM and section 3 gives an overview of the proposed method and numerical results are presented in section 4.

2 Extended Finite Element Method

The main idea in XFEM is to extend the approximation basis by set of enrichment functions, that are chosen based on the local behavior of the problem. For the case of linear elastic fracture mechanics, two such functions are used : Heaviside jump function to capture the jump across the crack face and asymptotic branch functions that span the two-dimensional asymptotic crack tip fields. The enriched approximation for fracture mechanics problems takes the form [1, 2, 3] :

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in \mathcal{N}^{\text{fem}}} N_I(\mathbf{x}) \mathbf{q}_I + \sum_{J \in \mathcal{N}^c} N_J(\mathbf{x}) H(\mathbf{x}) \mathbf{a}_J + \sum_{K \in \mathcal{N}^f} N_K(\mathbf{x}) \sum_{\alpha=1}^4 B_\alpha(\mathbf{x}) \mathbf{b}_K^\alpha \quad (1)$$

where \mathbf{a}_J and \mathbf{b}_K are nodal degrees of freedom corresponding to the Heaviside function H and the near-tip functions, $\{B_\alpha\}_{1 \leq \alpha \leq 4}$. Nodes in set \mathcal{N}^c are such that their support is split by the crack and nodes in set \mathcal{N}^f belong to the elements that contain a crack tip. These nodes are enriched with the Heaviside and near-tip (branch functions) fields, respectively.

This modification of the displacement field approximation doesn't introduce a new form of the discretised finite element equilibrium equation, but leads to an enlarged problem to solve :

$$\begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa} \end{bmatrix} \begin{Bmatrix} u \\ a \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_a \end{Bmatrix} \quad (2)$$

In the XFEM, the discontinuities are independent of the mesh, i.e., the discontinuities can lie anywhere within or on the element. And the numerical integration of the above equation involves sub-division of elements, over which the integration can be done. Since it is not the scope of the present work, it is suffice to say that the numerical integration in XFEM is cumbersome. For more detailed information, interested readers can refer to the XFEM literature.

3 Smoothed eXtended Finite Element Method

The SmXFEM is a novel technique that combines the strain smoothing technique proposed by Chen for meshless [5] with the XFEM to model discontinuities without having to remesh locally. SmXFEM uses a similar approximation for the displacement field as the XFEM Equation (1).

The element stiffness matrix writes as the sum of the contributions from each of the nc subcells of the element e

$$\tilde{\mathbf{K}}^e = \sum_{C=1}^{nc} \int_{\Omega_C} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C d\Omega \quad (3)$$

where $C \in \{1, 2, \dots, nc\}$ is the number of the subcell Ω_C .

All entries in matrix $\tilde{\mathbf{B}}_C$ in Equation (3) are constants over each subcell Ω_C –each of these entries are line integrals calculated along the boundaries of the subcells. From the displacement approximation Equation (1), the strain field is deduced by differentiation of Equation (1)

$$\begin{aligned} \boldsymbol{\varepsilon}^h(\mathbf{x}) &= \sum_{I \in \mathcal{N}^{\text{fem}}} B_{I\text{fem}}(\mathbf{x}) \mathbf{q}_I + \sum_{J \in \mathcal{N}^{\text{enr}}} B_{J\text{enr}}(\mathbf{x}) \mathbf{a}_J = [\mathbf{B}_{\text{fem}} \parallel \mathbf{B}_{\text{enr}}][\mathbf{q}] \\ \boldsymbol{\varepsilon}^h(\mathbf{x}) &= [\mathbf{B}_{\text{xfem}}][\mathbf{q}] \end{aligned} \quad (4)$$

The \mathbf{B}_{xfem} matrix in Equation (4) includes two terms \mathbf{B}_{fem} and \mathbf{B}_{enr} corresponding to the standard nodes (FEM) and enriched nodes (ERN).

The smoothed enriched stiffness matrix for subcell C , $\tilde{\mathbf{K}}_{\text{xfem}}^C$ is computed by

$$\tilde{\mathbf{K}}_{\text{xfem}}^C = \int_{\Omega_C} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C d\Omega = \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C A_C \quad (5)$$

where $\tilde{\mathbf{B}}_C \equiv \tilde{\mathbf{B}}_{\text{xfem}}$, A_C is the area of the subcell. And the smoothed enriched element stiffness matrix $\tilde{\mathbf{K}}_{\text{xfem}}^e$ is the sum of the $\tilde{\mathbf{K}}_{\text{xfem}}^C$, for all subcells, C

$$\tilde{\mathbf{K}}^e = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C \int_{\Omega_C} d\Omega = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C A_C \quad (6)$$

where nc is the number of smoothing subcells of the element.

4 Numerical Examples

The effectiveness of the proposed method is illustrated by taking two benchmark problems from linear elastic fracture mechanics. We first consider a plate with an edge crack and then examine the case of plate with shear traction.

4.1 Edge crack under tension

A plate of dimension 1×2 is loaded by a tension $\sigma = 1$ psi over the top edge. The displacements along y -axis is fixed at the bottom right corner and plate is clamped at the bottom left corner. The geometry, loading, boundary conditions and domain discretisation are shown in Figure 1. The reference mode I SIF is given by

$$K_I = F\left(\frac{a}{b}\right) \sigma \sqrt{\pi a} \quad (7)$$

where a is the crack length, b is the plate width and $F\left(\frac{a}{b}\right)$ is an empirical function given as (For $\left(\frac{a}{b}\right) \leq 0.6$)

$$F\left(\frac{a}{b}\right) = 1.12 - 0.231\left(\frac{a}{b}\right) + 10.55\left(\frac{a}{b}\right)^2 - 21.72\left(\frac{a}{b}\right)^3 + 30.39\left(\frac{a}{b}\right)^4 \quad (8)$$

Tableau 1 – Normalized stress intensity factor K_I for edge crack under tension for different crack sizes

Number of nodes	Crack Size				
	0.1	0.2	0.3	0.4	0.5
288	1.0721	1.0027	1.0058	0.9554	0.8885
1152	0.9715	0.9567	0.9119	1.0478	0.9603
3444	1.0554	1.0302	1.0194	1.0087	0.9864
4608	0.9762	1.0061	1.0466	0.9658	0.9904
5000	0.9622	0.9710	0.9781	0.9886	0.9916

Table 1 shows the normalized stress intensity factor for various discretizations and crack sizes. Note that even for the coarsest meshes, the error is around a few percent and decreases with mesh refinement although the convergence is not absolutely monotonic.

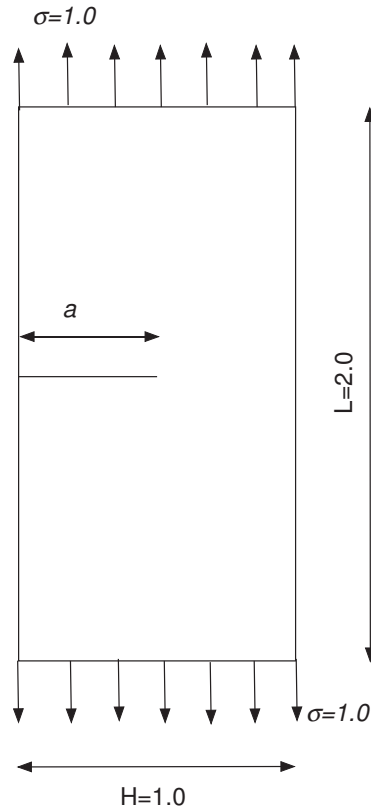


Figure 1 – Plate with edge crack under tension

4.2 Edge crack under shear stress

As a last example, in order to demonstrate the capabilities of proposed method for a general case with mixed-mode loading, we consider an edge cracked plate subjected to shear load. The plate is clamped at the bottom edge and loaded by a shear traction $\tau = 1\text{psi}$ over the top edge. The material parameters are Young's modulus, $E = 3 \times 10^7$ and Poisson's ratio $\nu = 0.25$. The geometry, loading and boundary conditions are shown in Figure 2. The reference mixed mode stress intensity factors are :

$$K_I = 34.0 \text{ psi}\sqrt{\text{in}} \quad (9)$$

$$K_{II} = 4.55 \text{ psi}\sqrt{\text{in}} \quad (10)$$

Tableau 2 – Normalized stress intensity factors for edge crack under shear for different mesh sizes, Exact $K_I = 34.0$ and $K_{II} = 4.55$

Number of nodes	K_I	K_{II}
288	0.8921	0.9302
1152	0.9634	0.9630
3444	0.9894	0.9924
4608	0.9929	0.9968
5000	0.9941	0.9980

Table 2 gives the SIFs for various mesh refinement. It is evident that with mesh refinement, the computed SIFs converge to the reference mixed mode SIFs.

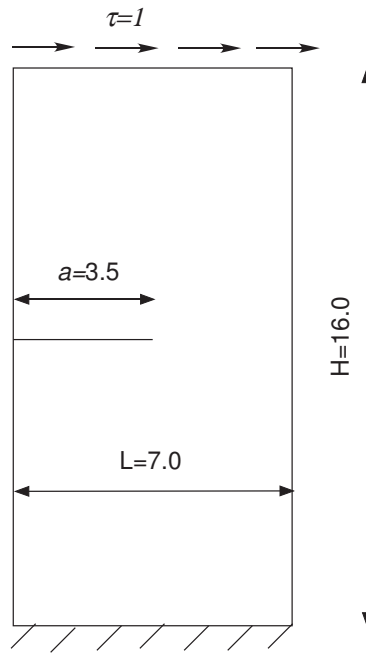


Figure 2 – Plate with edge crack under shear

5 Conclusion

In this paper, we presented a method to couple the strain smoothing technique with the XFEM to obtain the SmXFEM and to solve linear elastic fracture mechanics problems. The main advantage of this method is that it avoids the need to integrate the singular functions present in the XFEM stiffness matrix in linear elastic fracture mechanics problems. With the help of a few numerical analysis, we have shown the effectiveness of the proposed method. And since isoparametric mapping is eliminated, the elements can take any arbitrary shape.

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