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► **To cite this version:**

Farrell Brumley, Simon Marshall. LOWER BOUNDS FOR MAASS FORMS ON SEMISIMPLE GROUPS. 2016. hal-01408447

**HAL Id: hal-01408447**

**<https://hal.science/hal-01408447v1>**

Preprint submitted on 4 Dec 2016

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# LOWER BOUNDS FOR MAASS FORMS ON SEMISIMPLE GROUPS

FARRELL BRUMLEY AND SIMON MARSHALL

ABSTRACT. Let  $G$  be an anisotropic semisimple group over a totally real number field  $F$ . Suppose that  $G$  is compact at all but one infinite place  $v_0$ . In addition, suppose that  $G_{v_0}$  is  $\mathbb{R}$ -almost simple, not split, and has a Cartan involution defined over  $F$ . If  $Y$  is a congruence arithmetic manifold of non-positive curvature associated to  $G$ , we prove that there exists a sequence of Laplace eigenfunctions on  $Y$  whose sup norms grow like a power of the eigenvalue.

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## 1. INTRODUCTION

Let  $M$  be a compact Riemannian manifold of dimension  $n$  and with Laplace operator  $\Delta$ . Let  $\{\psi_i\}$  be an orthonormal basis of Laplace eigenfunctions for  $L^2(M)$ , which satisfy  $\|\psi_i\|_2 = 1$  and  $(\Delta + \lambda_i^2)\psi_i = 0$ . We assume that  $\{\psi_i\}$  are ordered by eigenvalue, so that  $0 = \lambda_1 \leq \lambda_2 \leq \dots$ . It is an important question in harmonic analysis to determine the asymptotic size of  $\psi_i$ , i.e. the growth rate of  $\|\psi_i\|_\infty$  in terms of  $\lambda_i$ . The basic upper bound for  $\|\psi_i\|_\infty$  was proved by Avacumović [1] and Levitan [19], and is

$$(1) \quad \|\psi_i\|_\infty \ll \lambda_i^{(n-1)/2}.$$

Moreover, this bound is sharp on the round  $n$ -sphere.

Our interest in this paper is in lower bounds on sup norms. One way of viewing the existence of large eigenfunctions on  $S^n$  is via the link between the asymptotics of  $\psi_i$  and the geodesic flow on  $M$  provided by the microlocal lift. Roughly speaking, this implies that the eigenfunctions on  $M$  should exhibit the same degree of chaotic behaviour as the geodesic flow. On  $S^n$ , for instance, the geodesic flow is totally integrable, and this is reflected in the fact

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Supported by ANR grant 14-CE25.

Supported by NSF grants DMS-1509331 and DMS-1501230.

that one can both write down an explicit basis of eigenfunctions, and find eigenfunctions with large peaks. Conversely, if  $M$  is negatively curved then its geodesic flow is highly chaotic, and one expects this to be reflected in the asymptotics of  $\psi_i$ . One example of this is the Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak [24], which predicts that the measures  $|\psi_i|^2 dV$  tend weakly to  $dV$  for these  $M$ .

In light of this, one would like to know to what extent the upper bounds (1) can be improved when  $M$  is negatively curved. When  $n = 2$ , Iwaniec and Sarnak [13] conjecture the strong bound

$$(2) \quad \|\psi_i\|_\infty \ll_\epsilon \lambda_i^\epsilon.$$

This is often referred to as a Lindelöf type bound, as it implies the classical Lindelöf conjecture on the Riemann zeta function in the case of the modular surface. Their conjecture is consistent with the Random Wave model and is supported by numerical computations as well as a power improvement over (1) established in [13]. Moreover it is generally believed that *generic* sequences of  $L^2$ -normalised eigenfunctions on a negatively curved manifold satisfy a Lindelöf type bound. Any sequence violating (2) will then be called *exceptional*.

Unlike the setting of the QUE conjecture, compact manifolds  $M$  of negative curvature which support exceptional sequences in the above sense do in fact exist. Rudnick and Sarnak were the first to investigate this question, proving in [24] the existence of an arithmetic hyperbolic 3-manifold  $Y$  and a sequence of  $L^2$ -normalised eigenfunctions  $\phi_i$  on  $Y$  for which  $\|\phi_i\|_\infty \gg_\epsilon \lambda_i^{1/2-\epsilon}$ . This result was generalised to  $n$ -dimensional hyperbolic manifolds for  $n \geq 5$  by Donnelly [4], and an alternative proof, valid in a wide class of arithmetic hyperbolic 3-manifolds, was given by Milićević [22]. Finally, in the spirit of the above example involving the Riemann zeta function, Lapid and Offen in [18] discovered a series of higher rank arithmetic manifolds supporting exceptional sequences through the link with automorphic  $L$ -functions (conditionally on standard conjectures on the size of automorphic  $L$ -functions at the edge of the critical strip). A synthesis of these developments, as well as a general conjecture restricting the possible limiting exponents for exceptional sequences, can be found in the influential letter [29].

In this paper, we extend these results to a wide range of compact congruence manifolds.

**Theorem 1.1.** *Let  $F$  be a totally real number field, and let  $v_0$  be a real place of  $F$ . Let  $G/F$  be a connected anisotropic semisimple  $F$ -group. We make the following additional assumptions on  $G$ .*

- (1)  $G_{v_0}$  is noncompact, not split, and  $\mathbb{R}$ -almost simple.
- (2)  $G_v$  is compact for all real  $v \neq v_0$ .
- (3) There is an involution  $\theta$  of  $G$  defined over  $F$  that induces a Cartan involution of  $G_{v_0}$ .

*Let  $Y$  be a congruence manifold associated to  $G$  as in Section 3.8. Then there exists  $\delta > 0$  and a sequence of linearly independent Laplacian eigenfunctions  $\psi_i$  on  $Y$  that satisfy*

$$\|\psi_i\|_2 = 1, \quad (\Delta + \lambda_i^2)\psi_i = 0, \quad \text{and} \quad \|\psi_i\|_\infty \gg \lambda_i^\delta.$$

Theorem 1.1 goes some distance toward answering the basic question of determining the precise conditions under which one should expect a Lindelöf type bound on a compact congruence negatively curved manifold. The three numbered conditions on the group  $G$  are a particularly convenient way of asking that a large enough compact subgroup of  $G_\infty$  admits a rational structure, which is a key ingredient in our proof. Although the condition that  $G_{v_0}$

is not split should be necessary, one should be able to relax the other conditions somewhat. For example, throughout most of the paper, the condition that  $G_{v_0}$  is  $\mathbb{R}$ -almost simple could be weakened to  $G$  being  $F$ -almost simple. The stronger form of this condition is only used in Lemma 6.1, to simplify the application of a theorem of Blomer and Pohl [2, Theorem 2] and Matz-Templier [21, Proposition 7.2].

Note that Theorem 1.1 includes the examples of Rudnick-Sarnak, Donnelly, and Milićević, although without explicit exponents. It is unable to reproduce the examples of Lapid and Offen due to the compactness requirement, but it can produce compact quotients of  $SL(n, \mathbb{C})/SU(n)$  with an exceptional sequence of eigenfunctions.

We address the question of whether one can find many groups satisfying the rationality hypothesis of Theorem 1.1 in Section 9. One consequence of the results proved there is that, for any  $G/\mathbb{R}$  that is connected,  $\mathbb{R}$ -almost simple, not compact, and not split, Theorem 1.1 produces a manifold  $Y$  of the form  $\Gamma \backslash G/K$  with an exceptional sequence of eigenfunctions.

Finally, while our approach was largely inspired by that of Milićević, we have made an effort to emphasize (in Section 2) the common features it shares with the techniques of Rudnick-Sarnak and Lapid-Offen.

**1.1. A hybrid result in the level-eigenvalue aspect.** We in fact prove a stronger result than that described in Theorem 1.1, establishing a lower bound in the level and eigenvalue aspects simultaneously. We present this separately, as it requires more care to state; indeed, any notion of non-trivial lower bound must overcome the lower bound one may prove when the eigenspaces have large dimension. More precisely, if  $M$  is a compact Riemannian manifold and  $V$  is the space of  $\psi \in L^2(M)$  with a given Laplace eigenvalue, one may show that there is  $\psi \in V$  satisfying  $\|\psi\|_\infty \geq \sqrt{\dim V} \|\psi\|_2$ .

If we consider a tower of congruence covers  $Y_N$  of  $Y$ , then the Laplace eigenspaces will have growing dimension because of multiplicities in the corresponding representations at places dividing  $N$ . Computationally, one observes that this (and its stronger form involving Arthur packets) is the only source of dimension growth. Although we believe that the dimensions of the joint eigenspaces we consider should be small (partly as a result of our choice of “large” congruence subgroup), we do not know how to prove this in general. As a result, we shall be satisfied if we can beat the bound  $\sqrt{\dim V}$ , where  $V$  is now a space of Hecke-Maass forms with the same Laplace and Hecke eigenvalues. This motivates the following definitions.

Let  $G$  be as in Theorem 1.1. Let  $H$  be the identity component of the group of fixed points of  $\theta$ . We let  $D$  be a positive integer such that  $G$  and  $H$  are unramified at places away from  $D$  and  $\infty$ ; see Section 3 for a precise definition. Let  $K$  and  $K_H$  be compact open subgroups of  $G(\mathbb{A}_f)$  and  $H(\mathbb{A}_f)$  that are hyperspecial away from  $D$ . If  $N$  is a positive integer prime to  $D$ , we let  $K(N)$  be the corresponding principal congruence subgroup of  $K$ , and define  $Y_N = G(F) \backslash G(\mathbb{A}) / K(N) K_H K_\infty$ . We give each  $Y_N$  the probability volume measure.

Let  $A \subset G_\infty$  be a maximal  $\mathbb{R}$ -split torus with real Lie algebra  $\mathfrak{a}$  and Weyl group  $W$ . We let  $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes \mathbb{C}$ . Let  $G_\infty^0$  be the connected component of  $G_\infty$  in the real topology. Any unramified irreducible unitary representation of  $G_\infty^0$  gives rise to an element  $\xi \in \mathfrak{a}_\mathbb{C}^*/W$  via the Harish-Chandra isomorphism, where we have normalised so that the tempered spectrum corresponds to  $\mathfrak{a}^*/W$ . We let  $\|\cdot\|$  be the norm on  $\mathfrak{a}$  and  $\mathfrak{a}^*$  coming from the Killing form, and if  $\mu, \lambda \in \mathfrak{a}^*/W$  we will sometimes abuse notation and write  $\|\mu - \lambda\|$  to mean the minimum of this norm over representatives for the orbits.

By a Hecke-Maass form we mean a joint eigenfunction  $\psi \in L^2(Y_N)$  for the Hecke algebra (away from  $N$  and  $D$ ) and the ring of invariant differential operators  $\mathcal{D}$  on  $Y_N$ . We may view the associated eigenvalues as elements in the unramified unitary dual of  $G_v$  at finite places  $v$  (via the Satake isomorphism), while at infinity they determine an element  $\xi \in \mathfrak{a}_{\mathbb{C}}^*/W$ . We define a spectral datum  $c$  for  $(G, N)$  to be a choice of element  $\xi(c) \in \mathfrak{a}^*/W$  and an element  $\pi_v(c)$  in the unramified unitary dual of  $G_v$  for all  $v \nmid ND\infty$ . Given a spectral datum  $c$  for  $(G, N)$ , we define  $V(N, c)$  to be the space of Hecke-Maass forms on  $Y_N$  whose  $\mathcal{D}$ -eigenvalues are given by  $\xi(c)$  (the spectral parameter) and whose Hecke eigenvalues at  $v \nmid ND\infty$  are given by  $\pi_v(c)$ . In particular, we require that these Maass forms are tempered at infinity.

**Theorem 1.2.** *With the notation and hypotheses of Theorem 1.1, there is  $\delta > 0$  and  $Q > 1$  with the following property. For any positive integer  $N$  with  $(N, D) = 1$  and spectral parameter  $\xi \in \mathfrak{a}^*$  such that  $N(1 + \|\xi\|)$  is sufficiently large, there is a spectral datum  $c$  for  $(G, N)$  with  $\|\xi(c) - \xi\| \leq Q$  and a Hecke-Maass form  $\psi \in V(N, c)$  such that*

$$\|\psi\|_{\infty} \gg N^{\delta} (1 + \|\xi\|)^{\delta} \sqrt{\dim V(N, c)} \|\psi\|_2.$$

Note that a Hecke-Maass form as in Theorem 1.1 has Laplacian eigenvalue of size roughly  $(1 + \|\xi\|)^2$ .

The only previous results giving lower bounds in the level aspect are for  $GL_2$  over a number field, due to Saha [25] and Templier [32]. They use the fact that local Whittaker functions of highly ramified  $p$ -adic representations are large high in the cusp, and in particular rely on the noncompactness of the manifold.

**1.2. An example: complex hyperbolic manifolds.** We now give an example of a family of manifolds to which our theorem can be applied, and which to our knowledge does not already appear in the literature. Let  $F$  be a totally real number field, and let  $E$  be a CM extension of  $F$ . Let the rings of integers of these fields be  $\mathcal{O}_F$  and  $\mathcal{O}_E$  respectively. Let  $v_0$  be a distinguished real place of  $F$ , and let  $w_0$  be the place of  $E$  over  $v_0$ . Let  $V$  be a vector space of dimension  $n + 1$  over  $E$  with a Hermitian form  $\langle \cdot, \cdot \rangle$  with respect to  $E/F$ . Assume that  $\langle \cdot, \cdot \rangle$  has signature  $(n, 1)$  at  $w_0$  and is definite at all other infinite places of  $E$ . Let  $G$  be the  $F$ -algebraic group  $SU(V, \langle \cdot, \cdot \rangle)$ , so that  $G_{v_0} \simeq SU(n, 1)$ .

Let  $L \subset V$  be an  $\mathcal{O}_E$  lattice on which the form  $\langle \cdot, \cdot \rangle$  is integral. Let  $L^*$  be the dual lattice  $L^* = \{x \in V : \langle x, y \rangle \in \mathcal{O}_E \text{ for all } y \in L\}$ . Let  $\Gamma$  be the group of isometries of  $V$  that have determinant 1, preserve  $L$ , and act trivially on  $L^*/L$ . If  $F \neq \mathbb{Q}$ , completion at  $w_0$  allows us to consider  $\Gamma$  as a discrete, cocompact subgroup of  $SU(n, 1)$ , which will be torsion free if  $L$  is chosen sufficiently small.

One may associate a complex hyperbolic manifold to  $\Gamma$  in the following way. Let  $D$  denote the space of lines in  $V_{w_0}$  on which the Hermitian form is negative definite.  $SU(n, 1)$  acts on  $D$ , and  $D$  carries a natural  $SU(n, 1)$ -invariant metric under which it becomes a model for complex hyperbolic  $n$ -space. The quotient  $Y = \Gamma \backslash D$  is then a compact complex hyperbolic  $n$ -manifold, and is an example of a congruence manifold associated to  $G$  as in Theorem 1.1.

If  $n \geq 2$ ,  $G$  satisfies conditions (1) and (2) of Theorem 1.1. We now show that (3) is satisfied. Let  $W \subset V$  be a codimension 1 subspace defined over  $E$  such that the Hermitian form is positive definite on  $W_{w_0}$ . Let  $\theta$  be the isometry of reflection in  $W$ . Then  $g \mapsto \theta g \theta^{-1}$  gives an  $F$ -involution of  $G$  that is a Cartan involution on  $G_{v_0}$ , as required. Theorem 1.1 then implies that there is a sequence of Laplace eigenfunctions  $\{\psi_i\}$  on  $Y$  satisfying  $\|\psi_i\|_{\infty} \gg \lambda_i^{\delta} \|\psi_i\|_2$ .

**1.3. The method of proof.** There are several methods that may be used to prove power growth of eigenfunctions on arithmetic manifolds. The original proof of Rudnick and Sarnak uses a distinction principle. This means that, for certain period integrals, if an automorphic form  $\phi$  has a nonzero period then  $\phi$  is exceptional in some sense, which can mean being a transfer from a smaller group, or being nontempered. We illustrate this in a special case, taken from Rudnick and Sarnak's proof.

Let  $Q$  be the quadratic form  $Q(x) = x_1^2 + x_2^2 + x_3^2 - 7x_4^2$ . If we let  $V = \{x \in \mathbb{R}^4 : Q(x) = -1\}$ , then  $V$  is a two-sheeted hyperboloid and the upper sheet is a model for  $\mathbb{H}^3$ . If we let  $\Gamma$  be the intersection of  $O(Q, \mathbb{Z})$  with the identity component of  $O(Q, \mathbb{R})$ , then  $Y = \Gamma \backslash \mathbb{H}^3$  is a compact hyperbolic 3-manifold. The distinction result that Rudnick and Sarnak prove is that if  $\psi \in L^2(Y)$  is orthogonal to all theta lifts of cusp forms of weight 1 on  $\Gamma_1(28)$ , then  $\psi((2, 1, 1, 1)) = 0$ . The result then follows from the local Weyl law and a counting argument. Indeed, the local Weyl law says that the average size of  $|\psi((2, 1, 1, 1))|^2$  must be 1. However, the number of eigenfunctions on  $X$  with eigenvalue  $\lambda \leq R$  is roughly  $R^3$ , while the number of theta lifts in this range is roughly  $R^2$ . Because the number of nonvanishing eigenfunctions is small, their values must be large to make up the right average.

The generalisation of this principle, namely that an automorphic form on  $SO(n, 1)$  that is orthogonal to theta lifts from  $SL_2$  must have vanishing  $SO(n)$  periods, was used by Donnelly [4]. It is likely that this could be used to prove Theorem 1.1 on other groups of the form  $SO(m, n)$ ,  $U(m, n)$ , or  $Sp(m, n)$ . Another distinction principle that one could apply is due to Jacquet [14] (and later refined by Feigon, Lapid, and Offen in [6, 18]), which states that a form on  $GL(n, \mathbb{C})$  with a nonvanishing  $U(n)$  period must come from quadratic base change. See [15] for a general discussion of these ideas.

The proof of power growth for  $\mathbb{H}^3$  by Milićević [22] does not use distinction, and instead compares an amplified trace and pre-trace formula. Our proof works by extending this method to general groups. The main work is in proving asymptotics for the trace formula. While writing this paper, Erez Lapid pointed out to us that there was another approach to proving Theorems 1.1 and 1.2 based on a theorem of Sakellaridis on the unramified  $C^\infty$  spectrum of symmetric varieties. We have included a discussion of this in Section 2. We have also included an explanation of why the condition of  $G$  being nonsplit at  $v_0$  is natural, and motivated our choice of test functions in the trace formula, based on a related conjecture of Sakellaridis and Venkatesh on the  $L^2$  spectrum.

**Acknowledgements.** We would like to thank Tobias Finis, Erez Lapid, Zeev Rudnick, Yiannis Sakellaridis, Sug Woo Shin, Nicolas Templier, and Akshay Venkatesh for helpful conversations during the elaboration of this paper. The first author benefited from many enlightening discussions with Nicolas Bergeron and Djordje Milićević.

## 2. RELATIONS WITH THE SPECTRA OF SYMMETRIC VARIETIES

In this section, we describe how a conjecture of Sakellaridis and Venkatesh on the  $L^2$  spectrum of symmetric varieties relates to the method we have used in this paper, in particular our choices of test functions in the trace formula. We also describe an alternative approach to proving Theorems 1.1 and 1.2 based on a theorem of Sakellaridis on the unramified  $C^\infty$  spectrum.

**2.1. Symmetric varieties.** Let  $F$  be a field of characteristic 0. A symmetric variety over  $F$  is a variety  $X = G/H$  where  $G$  is a reductive  $F$ -group,  $\theta$  is an involution of  $G$  over  $F$ , and  $H$  is an open  $F$ -subgroup of the fixed point group  $G^\theta$ . We say that a torus  $A \subset G$  is  $(F, \theta)$ -split if it is  $F$ -split and  $\theta$  acts on it by  $-1$ . All maximal  $(F, \theta)$ -split tori are conjugate in  $G$ , and we define the  $\theta$ -split rank of  $G$  to be their common dimension. We say that  $G$  is  $\theta$ -split if its  $\theta$ -split rank is equal to its absolute rank, that is if  $G$  contains a  $(F, \theta)$ -split maximal torus. We say that  $G$  is  $\theta$ -quasi-split if  $G$  contains a Borel subgroup  $B$  over  $F$  such that  $B$  and  $\theta(B)$  are opposed. If  $A$  is a maximal  $(F, \theta)$ -split torus in  $G$ ,  $G$  is  $\theta$ -quasi-split if and only if  $Z_G(A)$  is a torus.

**2.2. Plancherel measures.** We now let  $F$  be a  $p$ -adic field. We will denote  $G(F)$  by  $G$  etc. in this section. We assume that  $G$  is split (and therefore unramified) over  $F$ , and let  $K$  be a hyperspecial maximal compact subgroup. We let  $\mathcal{H}$  be the spherical Hecke algebra with respect to  $K$ . We let  $\widehat{G}$  be the unitary dual of  $G$ , and let  $\widehat{G}^{\text{sph}}$  be the spherical unitary dual with respect to  $K$ . We recall the existence of a Plancherel measure  $\mu_X$  associated with the separable Hilbert space  $L^2(X)$ , viewed as a  $G$ -representation; roughly speaking, this is a measure on  $\widehat{G}$  satisfying

$$L^2(X) = \int_{\widehat{G}} M(\pi) \otimes \pi d\mu_X(\pi),$$

where  $M(\pi)$  is some multiplicity space. Notice that  $\mu_X$  is only defined up to absolutely continuous equivalence (we shall only be concerned with its support).

We let  $\Pi_H : L^1(G) \rightarrow L^1(X)$  be given by integration over  $H$ . If we let  $v^0 = \Pi_H(\mathbf{1}_K)$ , there is a second measure, the spherical Plancherel measure  $\mu_X^{\text{sph}}$ , which satisfies

$$\langle \omega \cdot v^0, v^0 \rangle_{L^2(X)} = \int_{\widehat{G}^{\text{sph}}} \widehat{\omega}(\nu) d\mu_X^{\text{sph}}(\nu)$$

for all  $\omega \in \mathcal{H}$ . In particular, the support of  $\mu_X^{\text{sph}}$  is contained in the support of  $\mu_X$ . Note that, here and in Lemma 2.1,  $\omega \cdot v^0$  denotes the action of  $\omega$  on  $v^0$ , given by

$$\omega \cdot v^0 = \int_G \omega(g)(g \cdot v^0) dg.$$

The next lemma shows that  $\mu_X^{\text{sph}}$  determines the period of  $\omega$  along  $H$ .

**Lemma 2.1.** *The measure  $\mu_X^{\text{sph}}$  satisfies*

$$\Pi_H \omega(1) = \int_{\widehat{G}^{\text{sph}}} \widehat{\omega}(\nu) d\mu_X^{\text{sph}}(\nu)$$

for any  $\omega \in \mathcal{H}(G)$ .

*Proof.* First note that for any  $\omega_1, \omega_2 \in \mathcal{H}(G)$ , we have

$$(3) \quad \langle \Pi_H \omega_1, \Pi_H \omega_2 \rangle_{L^2(X)} = \int_G \int_H \omega_1(g) \omega_2(gh) dg dh,$$

by unfolding the integral in  $\Pi_H \omega_1$ . We then have

$$\begin{aligned} \Pi_H \omega(1) &= \int_H \omega(h) dh = \int_H \int_K (\omega \cdot \mathbf{1}_K)(kh) dh dk \\ &= \int_G \int_H (\omega \cdot \mathbf{1}_K)(gh) \mathbf{1}_K(g) dh dg = \langle \Pi_H(\omega \cdot \mathbf{1}_K), v^0 \rangle_{L^2(X)}, \end{aligned}$$

where we have used (3). But  $\Pi_H(\omega \cdot \mathbf{1}_K) = \omega \cdot v^0$ , by the  $G$ -equivariance of the map  $\Pi_H$ .  $\square$

**2.3. The conjecture of Sakellaridis and Venkatesh.** Let  $A_X^0$  be a maximal  $(F, \theta)$ -split torus in  $G$ . By [12, Lemma 4.5], our assumption that  $G$  is split implies that  $A_X^0$  is also a maximal  $\theta$ -split torus in  $G \times \overline{F}$ . This implies that  $G$  is  $\theta$ -(quasi)-split if and only if  $G \times \overline{F}$  is.

Let  $\check{G}$  be the complex dual group of  $G$ . In [28, Section 2.2], Sakellaridis and Venkatesh define a dual group  $\check{G}_X$ , which is a reductive complex algebraic group associated to  $X$ , and a homomorphism  $\iota : \check{G}_X \times SL(2, \mathbb{C}) \rightarrow \check{G}$  whose restriction to  $\check{G}_X$  has finite kernel. (Note that this requires imposing certain conditions on  $X$ , which we shall ignore as this section is purely expository.) In our special case where  $X$  is symmetric, and again under technical assumptions that we shall ignore,  $\iota(\check{G}_X)$  is equal to the group  $\check{H}$  constructed by Nadler in [23]. (Note that Nadler works over  $\mathbb{C}$  rather than a  $p$ -adic field, but passing to  $\overline{F}$  as in the remark above allows us to compare the two constructions.) We recall the following facts about  $\check{G}_X$  and  $\iota$ .

- The rank of  $\check{G}_X$  is equal to the  $\theta$ -split rank of  $G$ .
- $\iota(\check{G}_X) = \check{G}$  if and only if  $G$  is  $\theta$ -split.
- $\iota$  is trivial on the  $SL(2, \mathbb{C})$  factor if and only if  $G$  is  $\theta$ -quasi-split.

The first claim is stated in [23, Section 1.1] and proved in Proposition 10.6.1 there, and the second may be shown by examining the construction in [23, Section 10]. The third follows from the condition that  $\iota$  be a distinguished morphism in the sense of [28, Section 2.2]. Indeed, by the comment before Theorem 2.2.3 there, if we define  $L$  to be the Levi  $Z_G(A_X^0)$ , then  $\iota$  is trivial on  $SL(2, \mathbb{C})$  if and only if  $\rho_L$  is trivial, i.e.  $L$  is a torus. Sakellaridis and Venkatesh conjecture that the support of  $\mu_X$  may be described in terms of the tempered dual of  $\check{G}_X$  and the map  $\iota$ . They define an  $X$ -distinguished Arthur parameter to be a commutative diagram

$$\begin{array}{ccc} & \check{G}_X \times SL_2 & \\ \phi \otimes \text{Id} \nearrow & & \searrow \iota \\ \mathcal{L}_F \times SL_2 & \longrightarrow & \check{G} \end{array}$$

where  $\mathcal{L}_F$  is the local Langlands group of  $F$ , and  $\phi$  is a tempered Langlands parameter for  $\check{G}_X$ . This naturally gives rise to an Arthur parameter for  $\check{G}$ . We shall say that an Arthur parameter for  $\check{G}$  is  $X$ -distinguished if it arises from such a diagram, and likewise for an  $X$ -distinguished Arthur packet.

**Conjecture 1** (Sakellaridis-Venkatesh). *The support of  $\mu_X$  is contained in the Fell closure of the union of the  $X$ -distinguished Arthur packets for  $\check{G}$ .*

Note that this conjecture has been proved for  $\mu_X^{\text{sph}}$  in [27] under certain combinatorial assumptions. Let us now discuss what Conjecture 1 implies for  $\mu_X^{\text{sph}}$  under the assumptions



that  $G$  is  $\theta$ -split,  $\theta$ -quasi-split, or neither. We say that  $\mu_X^{\text{sph}}$  is *strongly tempered* if  $\mu_X^{\text{sph}} \leq C\mu_G^{\text{sph}}$  for some  $C > 0$ , where  $\mu_G^{\text{sph}}$  is the spherical Plancherel measure on  $G$ .

- If  $G$  is  $\theta$ -split, then  $\iota(\check{G}_X) = \check{G}$  and  $\iota$  is trivial on  $SL(2, \mathbb{C})$ . Conjecture 1 then implies that  $\mu_X$  is supported on the tempered dual of  $G$ . In fact, it may be shown in this case that  $\mu_X^{\text{sph}}$  is strongly tempered.
- If  $G$  is  $\theta$ -quasi-split but not  $\theta$ -split,  $\iota$  is still trivial on the  $SL(2, \mathbb{C})$  factor. This implies that  $\mu_X$  is still tempered. However, because  $\text{rank}(\check{G}_X) < \text{rank}(\check{G})$ , if we identify the tempered spherical dual of  $G$  with a quotient of a compact torus by the Weyl group, the support of  $\mu_X^{\text{sph}}$  will be contained in a union of lower dimensional tori. In particular,  $\mu_X^{\text{sph}}$  will not be strongly tempered.
- If  $G$  is not  $\theta$ -quasi-split, then all  $X$ -distinguished Arthur parameters have nontrivial  $SL(2, \mathbb{C})$  factor. It follows that if  $\psi$  is  $X$ -distinguished with packet  $\Pi_\psi$ , and  $\pi \in \Pi_\psi$  is spherical, then  $\pi$  must be non-tempered. From Conjecture 1 we deduce that the same is true for any  $\pi$  in the support of  $\mu_X^{\text{sph}}$ .

**2.4. The relation to this paper.** As mentioned in §1.3, we prove Theorems 1.1 and 1.2 by comparing a trace formula for  $G$  with a trace formula for  $G$  relative to  $H$ . After proving asymptotics for both, the problem reduces to finding  $\omega$  in the global Hecke algebra with the property that  $\Pi_H(\omega\omega^*)(1)/(\omega\omega^*)(1)$  is large. This may in turn be reduced to a local problem, namely that of finding  $k \in \mathcal{H}_v$  with  $k(1) = 0$ ,  $\|k\|_2 = 1$ , and  $\Pi_H k(1) \gg 1$ , for  $v$  in a set of places having positive density. Let  $v$  be a finite place at which  $G$  is split and all data are unramified. We let  $\mu_{G,v}^{\text{sph}}$  and  $\mu_{X,v}^{\text{sph}}$  denote the spherical Plancherel measures of  $G_v$  and  $X_v = G_v/H_v$ . If we rephrase our conditions on  $k$  in terms of the Satake transform using Lemma 2.1, they are equivalent to

$$(4) \quad \int \widehat{k}(\nu) d\mu_{G,v}^{\text{sph}}(\nu) = 0, \quad \int |\widehat{k}(\nu)|^2 d\mu_{G,v}^{\text{sph}}(\nu) = 1, \quad \int \widehat{k}(\nu) d\mu_{X,v}^{\text{sph}}(\nu) \gg 1.$$

We also note that  $G_{v_0}$  is (quasi-)split over  $\mathbb{R}$  if and only if  $G_{v_0} \times \mathbb{C}$  is  $\theta$ -(quasi-)split, and by the comment at the start of Section 2.3 this is equivalent to  $G_v$  being  $\theta$ -(quasi-)split. We obtain the following consequences of Conjecture 1 for the existence of the required function  $k$ .

- If  $G_{v_0}$  is not quasi-split,  $\mu_{X,v}^{\text{sph}}$  has non-tempered support. The exponential growth of  $\widehat{k}(\nu)$  away from  $\widehat{G}_{v,\text{temp}}^{\text{sph}}$  should make it easy (barring unforeseen cancellation) to obtain (4).
- If  $G_{v_0}$  is split, the oscillation of  $\widehat{k}$  and the strong temperedness of  $\mu_{X,v}^{\text{sph}}$  should prevent one from satisfying (4).
- If  $G_{v_0}$  is quasi-split but not split, the existence of  $k$  satisfying (4) depends on how singular  $\mu_{X,v}^{\text{sph}}$  is. By the remarks below, it seems that  $\mu_{X,v}^{\text{sph}}$  is still singular enough to allow (4) to be satisfied.

In practice, we take a much easier approach to constructing  $k$ . In [20], it is shown that  $G_{v_0}$  being non-split is equivalent to a certain inequality on the roots of  $G$  and  $H$ . We refer to this by saying that  $H$  is large in  $G$ ; see Definition 3.1. (Note that the proof of this involves studying the spectra of symmetric varieties.) It is then straightforward to show that if  $H$  is large in  $G$  then a function  $k$  of the required type exists; see Section 6.

**2.5. A result of Sakellaridis on the unramified  $C^\infty$  spectrum.** We now describe a  $C^\infty$  version of the above ideas that we expect would provide an alternative approach to Theorems 1.1 and 1.2 in the case when  $G_{v_0}$  is not quasi-split. We begin by stating a local result, due to Sakellaridis. As in Section 2.2 we let  $F$  be a  $p$ -adic field, and assume that  $G$  is split over  $F$ .

We again let  $A_X^0$  be a maximal  $(F, \theta)$ -split torus of  $G$ , and let  $A$  be a maximal split torus containing  $A_X^0$ . It is  $\theta$ -stable. We let  $A_X$  be the quotient  $A/A^\theta$ , which is also a quotient of  $A_X^0$ . Let  $\check{A}_X$  and  $\check{A}$  be the dual tori, so that we have a map  $\check{A}_X \rightarrow \check{A}$  with finite kernel. In fact,  $\check{A}_X$  is a maximal torus of  $\check{G}_X$ , and the map  $\check{A}_X \rightarrow \check{A}$  extends to the map  $\iota : \check{G}_X \rightarrow \check{G}$  of Section 2.3. Let  $W$  be the Weyl group of  $A$  and  $\check{A}$ . Then irreducible admissible unramified representations of  $G$  are in bijection with  $\check{A}/W$ , via the map taking  $\pi$  to its Satake parameter. Let  $B$  be a Borel subgroup of  $G$  containing  $A$ , and let  $\delta$  denote the modular character of  $A$  with respect to  $B$ . One may consider the positive square root  $\delta^{1/2}$  as an element of  $\check{A}$ . The following theorem is an immediate consequence of [26, Theorem 1.2.1]; note that the torus denoted  $A_X^*$  there is equal to  $\iota(\check{A}_X)$ .

**Theorem 2.2.** *If an irreducible admissible unramified representation  $\pi$  of  $G$  occurs as a subrepresentation of  $C^\infty(X)$ , then the Satake parameter of  $\pi$  lies in the image of  $\delta^{-1/2}\iota(\check{A}_X)$  in  $\check{A}/W$ .*

It is known that  $\delta^{1/2} \in \iota(\check{A}_X)$  if and only if  $G$  is  $\theta$ -quasi-split. Combined with the above theorem, this gives the following.

**Corollary 2.3.** *If  $G$  is not  $\theta$ -quasi-split, any irreducible admissible unramified representation  $\pi$  of  $G$  that occurs as a subrepresentation of  $C^\infty(X)$  must be non-tempered.*

**2.6. Period integrals.** We now describe how one might use Corollary 2.3 to prove asymptotic lower bounds for periods. The argument is in the same style as that of Rudnick and Sarnak described in Section 1.3.

Let  $G/\mathbb{Q}$  be a semisimple group. Assume that  $G(\mathbb{R})$  has a Cartan involution  $\theta$  defined over  $\mathbb{Q}$ . If we let  $H = G^\theta$ , then  $H(\mathbb{R})$  is a maximal compact subgroup of  $G(\mathbb{R})$ . Let  $K_f = \prod K_p$  be a compact open subgroup of  $G(\mathbb{A}_f)$ , and let  $K_\infty = H(\mathbb{R})$ . Define  $Y = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K$ . The image of  $H(\mathbb{A})$  in  $Y$  is a finite number of points, and we shall assume for simplicity that it is a single point  $p$ . Let  $\mathcal{P}_H : C^\infty([G]) \rightarrow \mathbb{C}$  be the period map  $f \mapsto \int_{[H]} f dh$ . (Here  $[G]$  denotes the usual adelic quotient  $G(F) \backslash G(\mathbb{A})$ .) When restricted to  $C^\infty(Y)$ ,  $\mathcal{P}_H$  is just evaluation at  $p$ .

Suppose that  $G$  is not quasi-split over  $\mathbb{R}$ . If  $G$  is split at a finite place  $v$ , then passing between places as before and applying Corollary 2.3 gives that any unramified representation  $\pi_v$  occurring in  $C^\infty(G_v/H_v)$  must be non-tempered.

Let  $\pi$  be a cuspidal automorphic representation of  $G$ , and let  $\phi \in \pi$  be invariant under  $K$ . If  $\mathcal{P}_H(\phi) \neq 0$ , this implies that each factor  $\pi_v$  admits a non-zero smooth linear functional invariant under  $H_v$ . This is equivalent to the existence of an embedding  $\pi_v \rightarrow C^\infty(G_v/H_v)$ , so that if  $v$  is finite,  $G$  splits at  $v$ , and all data are unramified, then  $\pi_v$  is non-tempered. The strategy would then be to use the trace formula to show that the number of such  $\phi$  is a power smaller than the total number of  $\phi$ . Combining this with the local Weyl law would then produce asymptotic growth. Note that this approach does not deal with the case when  $G$  is quasi-split but not split over  $\mathbb{R}$ , unlike Theorems 1.1 and 1.2.

**2.7. Higher dimensional periods.** One advantage of the method described in Section 2.6 is that it applies equally well to periods along positive dimensional submanifolds of  $Y$  arising from rational subgroups. Note that this would produce a result of the form “there are certain eigenfunctions whose periods are larger than the average by a power of the eigenvalue”, while determining the size of the average period is a separate problem. It should be pointed out that the average size of a positive dimensional period should be a negative power of the eigenvalue, so even if one could improve over this one would not necessarily obtain power growth of sup norms as a result.

In comparison, the relative trace formula approach we use is more difficult in the positive dimensional case, because the analysis of the error terms becomes much more complicated. In the case of a point, one needs to bound the value of a spherical function  $\varphi_\lambda$  away from its center of symmetry. In the positive dimensional case, one needs to bound the value of an oscillatory integral whose kernel is constructed from  $\varphi_\lambda$ , and which is taken over two copies of the submanifold in question. Moreover, the bound obtained must be uniform as the submanifolds move.

### 3. NOTATION

**3.1. Algebraic groups.** Let  $\mathcal{O}$  denote the ring of integers of  $F$ . Let  $\mathbb{A}$  and  $\mathbb{A}_f$  be the adèles and finite adèles of  $F$ .

Let  $G$  and  $\theta$  be as in Theorems 1.1 and 1.2. Let  $H$  again denote the identity component of the group of fixed points of  $\theta$ . We let  $T \subset G$  and  $T_H \subset H$  be maximal tori defined over  $F$  with  $T_H \subset T$ . We fix an  $F$ -embedding  $\rho : G \rightarrow SL_d$ . We choose an integer  $D > 1$  such that  $G$ ,  $H$ ,  $T$ , and  $T_H$  extend to smooth group schemes over  $\mathcal{O}[1/D]$ , all of whose fibers are connected reductive. We assume that all the inclusions we have defined over  $F$  extend to inclusions of smooth closed subgroups over  $\mathcal{O}[1/D]$ . Let  $Z$  be the center of  $G$ .

Let  $X^*(T)$  and  $X_*(T)$  denote the group of characters and cocharacters of  $T \times_F \overline{F}$ . Let  $\Delta$  be the set of roots of  $T$  in  $G$ , and let  $\Delta^+$  be a choice of positive roots. Let  $W$  be the Weyl group of  $(G, T)$  over  $\overline{F}$ . We define

$$X_*^+(T) = \{\mu \in X_*(T) : \langle \mu, \alpha \rangle \geq 0, \alpha \in \Delta^+\}.$$

Similarly, we may define  $\Delta_H, \Delta_H^+, W_H$ , and  $X_*^+(T_H)$ . Letting  $\rho$  and  $\rho_H$  denote, as usual, the half-sum of positive roots for  $G$  and  $H$ , we introduce the norms

$$\begin{aligned} \|\mu\|^* &= \max_{w \in W} \langle w\mu, \rho \rangle \\ \|\mu\|_H^* &= \max_{w \in W_H} \langle w\mu, \rho_H \rangle \end{aligned}$$

on  $X_*(T)$  and  $X_*(T_H)$  respectively. (Note that these are in fact norms; the condition that  $\|\mu\|^* = \|-\mu\|^*$  follows from the fact that  $\rho$  and  $-\rho$  lie in the same Weyl orbit.)

**Definition 3.1.** We say that  $G$  is  $H$ -large if there exists a nonzero  $\mu \in X_*(T_H)$  such that

$$(5) \quad 2\|\mu\|_H^* \geq \|\mu\|^*.$$

It follows from [20, Prop 7.2] that a group  $G$  satisfying the hypotheses of Theorem 1.1 is  $H$ -large. Indeed, by that proposition it is equivalent to know that  $G \times_F \overline{F}$  is not  $\theta$ -split. However, this is equivalent to  $G_{v_0} \times \mathbb{C}$  not being  $\theta$ -split, and this is equivalent to  $G_{v_0}$  not being split over  $\mathbb{R}$ .

**3.2. Local fields.** If  $v$  is a place of  $F$ , we denote the completion by  $F_v$ . If  $v$  is finite, we denote the ring of integers, uniformiser, and cardinality of the residue field by  $\mathcal{O}_v$ ,  $\pi_v$ , and  $q_v$  respectively. If  $v \nmid D\infty$ , we have the following consequences of our assumptions on  $D$  and  $\rho$  above.

- We have  $G(\mathcal{O}_v) = \rho^{-1}(\rho(G_v) \cap SL_d(\mathcal{O}_v))$  and  $H(\mathcal{O}_v) = \rho^{-1}(\rho(H_v) \cap SL_d(\mathcal{O}_v))$ , so that  $G(\mathcal{O}_v) \cap H_v = H(\mathcal{O}_v)$
- $G(\mathcal{O}_v)$  and  $H(\mathcal{O}_v)$  are hyperspecial maximal compact subgroups of  $G_v$  and  $H_v$  respectively.
- If  $T$  (and hence  $T_H$ ) split at  $v$ , the subgroups  $G(\mathcal{O}_v)$  and  $H(\mathcal{O}_v)$  correspond to points in the Bruhat-Tits buildings of  $G_v$  and  $H_v$  that lie in the apartments of  $T$  and  $T_H$  respectively.

We let  $\mathcal{P}$  be the set of finite places of  $F$  that do not divide  $D$  and at which  $T$  splits. If  $v \in \mathcal{P}$ , our assumptions imply that  $G_v$  has a Cartan decomposition

$$G_v = \coprod_{\mu \in X_*^+(T)} G(\mathcal{O}_v)\mu(\pi_v)G(\mathcal{O}_v)$$

with respect to  $G(\mathcal{O}_v)$  and  $T$ .

**3.3. Metrics.** For any place  $v$  of  $F$  and  $g \in G(F_v)$  let  $\|g\|_v$  denote the maximum of the  $v$ -adic norms of the matrix entries of  $\rho(g)$ . For  $g \in G(\mathbb{A}_f)$ , let  $\|g\|_f = \prod_{v \nmid \infty} \|g_v\|_v$ . Fix a left-invariant Riemannian metric on  $G(F_{v_0})$ . Let  $d(\cdot, \cdot)$  be the associated distance function. We define  $d(x, y) = \infty$  when  $x$  and  $y$  are in different connected components of  $G(F_{v_0})$  with the topology of a real manifold.

**3.4. Compact subgroups.** We choose a compact subgroup  $K = \prod_v K_v$  of  $G(\mathbb{A})$  such that

- $K_{v_0} = H_{v_0}$ ,
- $K_v = G_v$  for all other real places,
- $\rho(K_v) \subset SL_d(\mathcal{O}_v)$  for all finite  $v$ ,
- $K_v = G(\mathcal{O}_v)$  for finite places  $v \nmid D$ , and
- $K_f = \prod_{v \nmid \infty} K_v$  is open in  $G(\mathbb{A}_f)$ .

We shall suppose that  $K_v$  for  $v|D$  is sufficiently small to ensure that the finite group  $Z(F) \cap K_f$  is reduced to  $\{e\}$ .

The following lemma implies that the compact subgroup  $K_{v_0} = H_{v_0}$  is connected in the real topology; it will then follow, by passing to the Lie algebra, that  $K_{v_0}$  is a maximal compact connected subgroup of  $G_{v_0}$ .

**Lemma 3.2.** *Let  $K/\mathbb{R}$  be a Zariski-connected reductive algebraic group such that  $K(\mathbb{R})$  is compact. Then  $K(\mathbb{R})$  is connected in the real topology.*

*Proof.* Because  $K$  is a connected algebraic group, it is irreducible, and [30, Ch. VII §2.2] implies that  $K(\mathbb{C})$  is a connected Lie group. Now  $K(\mathbb{R})$  is a compact subgroup of  $K(\mathbb{C})$ ; it is therefore contained in some maximal compact subgroup  $K'$ , which must be connected. However,  $K'$  and  $K(\mathbb{R})$  must have the same Lie algebra, so that  $K' = K(\mathbb{R})$ . □

**3.5. Measure normalizations.** For any place  $v$  of  $F$ , let  $\mu_{G,v}^{\text{can}}$  be the canonical measure on  $G(F_v)$  as defined by Gross in [9, Section 11]; we recall this construction in Section 5.2. Then for all finite places  $v \nmid D$  one has  $\mu_{G,v}^{\text{can}}(K_v) = 1$ . We may then form the product measure  $\mu_G^{\text{can}} = \prod_v \mu_{G,v}^{\text{can}}$  on  $G(\mathbb{A})$ . All convolutions (local and global) on  $G$  will be defined with respect to these measures. If  $f \in C_c^\infty(G(\mathbb{A}))$ , we define the operator  $\pi(f)$  on  $L^2(G(F)\backslash G(\mathbb{A}))$  by

$$[\pi(f)\phi](x) = \int_{G(\mathbb{A})} \phi(xg)f(g)d\mu_G^{\text{can}}(g).$$

If  $f \in C_c^\infty(G(\mathbb{A}))$ , we define  $f^*$  by  $f^*(g) = \overline{f}(g^{-1})$ , so that  $\pi(f)$  and  $\pi(f^*)$  are adjoints.

The choice of canonical measure for  $G$  is imposed by the use of the Arthur-Selberg trace formula in Section 5; indeed one wants a uniform way of normalizing measures on the collection of connected reductive groups appearing as centralizers. We can afford to be more casual with measure normalizations for  $H$ , in light of our treatment of the geometric side of the relative trace formula in Section 4. For finite places  $v$  we choose Haar measures  $dh_v$  on  $H(F_v)$  so that  $H(F_v) \cap K_v$  is assigned measure 1. Because  $H_v$  is compact for archimedean  $v$ , at these places we choose Haar measures  $dh_v$  so that  $H_v$  has volume 1. We set  $dh = \otimes_v dh_v$ .

**3.6. Hecke algebras.** If  $S$  is any set of places prime to  $D_\infty$ , let  $\mathcal{H}_S$  be the convolution algebra of functions on  $G(F_S)$  that are compactly supported and bi-invariant under  $K_S$ . We identify  $\mathcal{H}_S$  with a subalgebra of  $C_c^\infty(G(\mathbb{A}_f))$  in the natural way. We define  $\mathcal{H}_f$  to be  $\mathcal{H}_S$  with  $S$  the set of all places prime to  $D_\infty$ . If  $G_\infty^0$  denotes the connected component of the identity in  $G_\infty$  in the real topology, we define  $\mathcal{H}_\infty$  to be the subspace of  $C_c^\infty(G_\infty^0)$  consisting of functions that are bi-invariant under  $K_\infty$ , and define  $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$ . We will sometimes denote  $\mathcal{H}_v = \mathcal{H}(G_v)$ .

**3.7. Lie algebras.** Let  $\mathfrak{g}$  be the real Lie algebra of  $G(F_\infty)$ , and let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition associated to  $K_\infty$ . Let  $\mathfrak{a} \subset \mathfrak{p}$  be a Cartan subalgebra. We let  $\Delta_\mathbb{R}$  be the roots of  $\mathfrak{a}$  in  $\mathfrak{g}$ , and let  $\Delta_\mathbb{R}^+$  be a choice of positive roots. We let  $W_\mathbb{R}$  be the Weyl group of  $\Delta_\mathbb{R}$ . For  $\alpha \in \Delta_\mathbb{R}$ , we let  $m(\alpha)$  denote the dimension of the corresponding root space. We denote the Killing form on  $\mathfrak{g}$  and  $\mathfrak{g}^*$  by  $\langle \cdot, \cdot \rangle$ , and let  $\| \cdot \|$  be the associated norm on  $\mathfrak{a}$  and  $\mathfrak{a}^*$ . For  $\xi \in \mathfrak{a}^*$ , we define

$$\beta(\xi) = \prod_{\alpha \in \Delta_\mathbb{R}^+} (1 + |\langle \alpha, \xi \rangle|)^{m(\alpha)}.$$

**3.8. Congruence subgroups and adelic quotients.** Fix an integer  $N \geq 1$  prime to  $D$ . For  $v \nmid D_\infty$  we write  $K_v(N)$  for the level  $N$  principal congruence subgroup, given by  $K_v(N) = G_v \cap SL_d(\mathcal{O}_v)(N)$ , and likewise for  $H$ , where  $SL_d(\mathcal{O}_v)(N)$  denotes the principal congruence subgroup of  $SL_d(\mathcal{O}_v)$ .

Then, recalling from §3.4 the choice of compact open subgroup  $K_v$  for  $v \mid D$ , we put

$$K(N) = \prod_{v \nmid D_\infty} K_v(N) \prod_{v \mid D} K_v \quad \text{and} \quad K^S(N) = \prod_{\substack{v \nmid D_\infty \\ v \notin S}} K_v(N) \prod_{\substack{v \mid D \\ v \notin S}} K_v.$$

For every finite  $v$  we define  $K_{H,v} = H_v \cap K_v$ . Letting

$$K_H = \prod_{v \nmid \infty} K_{H,v} \quad \text{and} \quad K_H^S = \prod_{v \notin S \cup \infty} K_{H,v},$$

then the congruence manifold of interest to us is

$$Y_N = G(F)\backslash G(\mathbb{A})/K(N)K_H K_\infty.$$

Let  $\text{Vol}_N$  be the volume assigned by  $\prod_{v \nmid \infty} \mu_{G,v}^{\text{can}}$  to the compact open subgroup  $K(N)K_H$ . For convenience, in the following proposition we use the standard notation

$$[G] = G(F)\backslash G(\mathbb{A}) \quad \text{and} \quad [H] = H(F)\backslash H(\mathbb{A})$$

for the automorphic spaces associated with  $G$  and  $H$ . We shall identify  $L^2(Y_N)$  with the functions in  $L^2([G])$  fixed under  $K(N)K_H K_\infty$ .

**3.9. Hecke-Maass forms.** Let  $\mathcal{D}$  be the algebra of differential operators on  $G_\infty^0/K_\infty$  that are invariant under the left action of  $G_\infty^0$ . Note that if we define  $K_{v_0}^+$  to be the group of fixed points of  $\theta$  on  $G_{v_0}$ , and let  $K_\infty^+$  be the group obtained by replacing  $K_{v_0}$  with  $K_{v_0}^+$  in  $K_\infty$ , then  $K_\infty^+$  is a maximal compact subgroup of  $G_\infty$ , we have  $G_\infty/K_\infty^+ \simeq G_\infty^0/K_\infty$ , and the elements of  $\mathcal{D}$  are also invariant under the larger group  $G_\infty$ . It follows that  $\mathcal{D}$  descends to an algebra of operators on  $Y_N$  in a natural way.

We define a Hecke-Maass form to be a function on  $Y_N$  that is an eigenfunction of the ring  $\mathcal{D}$  on  $Y_N$  and the Hecke algebra  $\mathcal{H}_f$  (and hence of  $\mathcal{H}$ ). If  $\psi$  is a Hecke-Maass form and  $\omega \in \mathcal{H}$ , we define  $\widehat{\omega}(\psi)$  by the equation  $\pi(\omega)\psi = \widehat{\omega}(\psi)\psi$ . We define the spectral parameter of  $\psi$  to be the unique  $\xi \in \mathfrak{a}_\mathbb{C}^*/W_\mathbb{R}$  such that  $\psi$  has the same eigenvalues under the action of  $\mathcal{D}$  as the associated spherical function  $\varphi_\xi$ . The Laplace eigenvalue of  $\psi$  is given by  $(\Delta + C_1(G) + \langle \xi, \xi \rangle)\psi = 0$  for some  $C_1(G) \in \mathbb{R}$ .

#### 4. THE AMPLIFIED RELATIVE TRACE FORMULA

For  $\psi \in C^\infty(G(F)\backslash G(\mathbb{A}))$  we consider the  $H$ -automorphic period

$$\mathcal{P}_H(\psi) = \int_{H(F)\backslash H(\mathbb{A})} \psi(h)dh.$$

In this section, we examine the average size of these periods over an orthonormal basis  $\{\psi_i\}$  of Hecke-Maass forms for  $L^2(Y_N)$ .

**4.1. Statement and reduction to off-diagonal estimates.** Our first task is to define a certain class of test functions to insert into the relative trace formula.

Let  $S$  be a finite set of finite places  $v$  of  $F$  such that  $v \nmid D$ . Let  $N$  be a positive integer prime to  $S$  and  $D$  and let  $\xi \in \mathfrak{a}^*$ . We shall consider test functions of the form  $\phi = \mathbf{1}_N^S \otimes k_S \otimes k_\xi$ , for  $k_S \in \mathcal{H}_S$  and  $k_\xi \in \mathcal{H}_\infty$ , where we have put

$$(6) \quad \mathbf{1}_N^S = \mathbf{1}_{K^S(N)K_H^S}.$$

We assume that

- (P<sub>S</sub>) : there is  $R > 1$  such that  $\|g\|_S \leq R$  for all  $g \in \text{supp}(k_S)$ ,
- (P<sub>∞</sub>) :  $k_\xi$  is supported in  $\{g \in G(F_\infty) : d(g, H_{v_0}) < 1\}$ , and satisfies

$$k_\xi(g) \ll \beta(\xi)(1 + \|\xi\|d(g, H_{v_0}))^{-1/2}.$$

The resulting formula will be expressed in terms of the averaging map

$$\Pi_H : L^1(G(F_S)) \longrightarrow L^1(G(F_S)/H(F_S))$$

given by

$$\Pi_H k_S(g) = \int_{H(F_S)} k_S(gh) dh.$$

**Proposition 4.1.** *There is  $A > 0$  such that the following holds. Let  $S$ ,  $N$ , and  $\xi$  be as above. Let  $k_S \in \mathcal{H}_S$  satisfy  $(\mathbf{P}_S)$  and  $k_\xi \in \mathcal{H}_\infty$  satisfy  $(\mathbf{P}_\infty)$ . Then*

$$\begin{aligned} \text{Vol}_N \sum_{i \geq 0} \widehat{k_S \otimes k_\xi}(\psi_i) |\mathcal{P}_H(\psi_i)|^2 \\ = \text{vol}([H]) \Pi_H k_S(1) k_\xi(1) + O\left(\#(\text{supp } k_S / K_S) \beta(\xi) (1 + \|\xi\|)^{-1/4} N^{-1/4} R^A \|k_S\|_\infty\right), \end{aligned}$$

where  $\{\psi_i\}$  runs over an orthonormal basis of Hecke-Maass forms for  $L^2(Y_N)$ .

*Proof.* For  $\phi = \mathbf{1}_N^S \otimes k_S \otimes k_\xi$  we let

$$K(x, y) = \sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y) \quad \text{and} \quad K_H(x, y) = \sum_{\gamma \in H(F)} \phi(x^{-1}\gamma y).$$

Integrating the spectral expansion

$$K(x, y) = \text{Vol}_N \sum_{i \geq 0} \widehat{k_S \otimes k_\xi}(\psi_i) \psi_i(x) \overline{\psi_i(y)}$$

over  $[H] \times [H]$ , we obtain

$$\int_{[H] \times [H]} K(x, y) dx dy = \text{Vol}_N \sum_{i \geq 0} \widehat{k_S \otimes k_\xi}(\psi_i) |\mathcal{P}_H(\psi_i)|^2.$$

On the other hand, by unfolding we have

$$\int_{[H] \times [H]} K_H(x, y) dx dy = \int_{H(\mathbb{A})} k(x) dx = \text{vol}([H]) \Pi_H k_S(1) k_\xi(1).$$

We have used the fact that  $K^S(N) K_H^S \cap H(\mathbb{A}^S) = K_H^S$  and the volume of this is  $\text{vol } K_H^S = 1$ .

For the remaining terms, first observe that

$$\#\{G(F) \cap \text{supp}(k)\} \ll \#(\text{supp } k_S / K_S),$$

uniformly in  $N$ . Indeed, suppose  $g, g' \in G(F) \cap \text{supp}(k)$  lie in the same coset in  $\text{supp } k_S / K_S$ . Then  $g_\infty^{-1} g'_\infty$  lies in a fixed compact set and  $g_f^{-1} g'_f \in K_f$ , so there are finitely many possibilities for  $g^{-1} g'$ . Therefore the map  $G(F) \cap \text{supp}(k) \rightarrow \text{supp } k_S / K_S$  has  $O(1)$  fibers. Using this, we simply estimate

$$\int_{[H] \times [H]} \sum_{\gamma \in G(F) - H(F)} k(x^{-1}\gamma y) dx dy$$

with the pointwise bounds of Corollary 4.3. □

**4.2. Bounding the off-diagonal contributions.** In this section we establish Corollary 4.3, which was used in the proof of Proposition 4.1 above. It is based on the following Diophantine lemma which shows, roughly speaking, that any  $\gamma \in G(F) - H(F)$  cannot be too close to  $H_v$  for various  $v$ . We shall use the notation introduced in §3.3.

**Lemma 4.2.** *There are  $A, C > 0$  such that the following properties hold for any  $\gamma \in G(F) - H(F)$ :*

- (1)  $d(\gamma, H_{v_0}) \geq C \|\gamma\|_f^{-A}$ ;
- (2) if  $(N, D) = 1$  is such that  $N > C \|\gamma\|_f^A$ , and  $d(\gamma, H_{v_0}) < 1$ , then there is a place  $v|N$  such that  $\gamma_v \notin K(N)_v K_{H,v}$ .

*Proof.* We consider  $H$  and  $G$  as subvarieties of  $F^{d^2}$  via the embedding  $\rho$ . Let  $p_1, \dots, p_k \in \mathcal{O}[1/D, x_1, \dots, x_{d^2}]$  be a set of defining polynomials for  $H$  that are integral over  $\mathcal{O}[1/D]$ . Now if  $\gamma \in G(F) - H(F)$  then  $p_i(\gamma) \neq 0$  for some  $i$ .

There are  $A, C_1 > 0$  such that for all  $\gamma \in G(F)$  one has  $\prod_{v|\infty} |p_i(\gamma)|_v \leq C_1 \|\gamma\|_f^A$ . For  $\gamma \in G(F) - H(F)$  the product formula applied to  $p_i(\gamma) \in F^\times$  implies that the archimedean norms satisfy  $\prod_{v|\infty} |p_i(\gamma)|_v \geq \|\gamma\|_f^{-A}/C_1$ . Because  $G_v$  is compact for all  $v|\infty$  other than  $v_0$ ,  $|p_i(\gamma)|_v$  is bounded for all such  $v$ . But then we have  $|p_i(\gamma)|_{v_0} \geq C \|\gamma\|_f^{-A}$  for some  $C > 0$ , and so  $d(\gamma, H_{v_0})$  satisfies the same bound. This establishes (1).

As above there are  $A_1, C_1 > 0$  such that  $\prod_{v|\infty N} |p_i(\gamma)|_v \leq C_1 \|\gamma\|_f^{A_1}$  for all  $\gamma \in G(F)$ . Moreover, we have  $|p_i(\gamma)|_v \ll 1$  for  $v|\infty$  by our assumption  $d(\gamma, H_{v_0}) < 1$ . Now suppose  $\gamma_v \in K(N)_v K_{H,v}$  for all  $v|N$ . Then  $p_i$  descends to a map  $K_v/K(N)_v \rightarrow \mathcal{O}_v/N\mathcal{O}_v$  which is trivial on  $K_{H,v}$ , and so  $|p_i(\gamma)|_v \leq |N|_v$ . It follows that  $\prod_{v|N} |p_i(\gamma)|_v \leq N^{-|F:\mathbb{Q}|}$ . If  $N > C \|\gamma\|_f^A$  for suitable  $A, C > 0$ , we obtain a contradiction by again applying the product formula to  $p_i(\gamma) \in F^\times$ . This establishes (2).  $\square$

**Corollary 4.3.** *There is  $A > 0$  such that the following holds. Let  $S, N$ , and  $\xi$  be as in §4.1. Let  $k_S \in \mathcal{H}_S$  satisfy  $(P_S)$  and  $k_\xi \in \mathcal{H}_\infty$  satisfy  $(P_\infty)$ . Put  $\phi = \mathbf{1}_N^S \otimes k_S \otimes k_\xi$ . Then for all  $\gamma \in G(F) - H(F)$  and all  $x, y \in H(\mathbb{A})$ , we have*

$$\phi(x^{-1}\gamma y) \ll \beta(\xi)(1 + \|\xi\|)^{-1/4} N^{-1/4} R^A \|k_S\|_\infty.$$

*Proof.* Let  $\Omega_H \subset H(\mathbb{A})$  be a compact set containing a fundamental domain for  $[H]$ . We assume that  $\Omega_H = \Omega_{H,D\infty} \times \prod_{v|D\infty} K_{H,v}$  after possibly enlarging  $D$ . Because  $G(F) - H(F)$  is bi-invariant under  $H(F)$ , we may assume that  $x, y \in \Omega_H$ .

We may also clearly assume that  $k(x^{-1}\gamma y) \neq 0$ . It then follows from Property  $(P_S)$  that  $\|x^{-1}\gamma y\|_S \leq R$ ; in fact we have  $\|x^{-1}\gamma y\|_f \leq R$ , using the condition  $x^{-1}\gamma y \in K^S(N)K_H^S \subset K^S$ . When combined with  $x, y \in \Omega_H$  this gives  $\|\gamma\|_f \ll R$ . We may now apply part (1) of Lemma 4.2, to find that  $d(\gamma, H_{v_0}) \gg R^{-A}$ . All together, since  $x, y \in H_{v_0}$ , we deduce that  $d(x^{-1}\gamma y, H_{v_0}) \gg R^{-A}$ . Similarly, from  $k_\xi(x^{-1}\gamma y) \neq 0$  and Property  $(P_\infty)$  it follows that  $d(x^{-1}\gamma y, H_{v_0}) < 1$ , and hence  $d(\gamma, H_{v_0}) \ll 1$ .

Suppose that  $1 + \|\xi\| \geq N$ . We have  $(k_S \otimes k_\xi)(x^{-1}\gamma y) \leq \|k_S\|_\infty k_\xi(x^{-1}\gamma y)$ . We then combine  $(P_\infty)$  with  $d(x^{-1}\gamma y, H_{v_0}) \gg R^{-A}$  to get

$$\begin{aligned} k_\xi(x^{-1}\gamma y) &\ll \beta(\xi)(1 + \|\xi\| d(x^{-1}\gamma y, H_{v_0}))^{-1/2} \\ &\ll \beta(\xi)(1 + \|\xi\| C R^{-A})^{-1/2} \\ &\ll \beta(\xi)(1 + \|\xi\|)^{-1/2} R^{A/2} \\ &\ll \beta(\xi)(1 + \|\xi\|)^{-1/4} N^{-1/4} R^{A/2}, \end{aligned}$$

which completes the proof in this case.

Now suppose that  $1 + \|\xi\| < N$ . Because  $\|\gamma\|_f \ll R$ , part (2) of Lemma 4.2 implies that there are  $A, C > 0$  such that if  $N > C R^A$ , then there is a place  $v|N$  for which  $\gamma_v \notin K(N)_v K_{H,v}$ . Because  $x, y \in \Omega_H$ , we have  $x, y \in K_{H,v}$ , and so  $x^{-1}\gamma y \notin K(N)_v K_{H,v}$ . It



follows that if  $N > CR^A$ , then  $\phi(x^{-1}\gamma y) = 0$ . We may rephrase this as saying that

$$\phi(x^{-1}\gamma y) \leq \|k_S k_\xi\|_\infty N^{-1} CR^A \ll \|k_S\|_\infty \beta(\xi) N^{-1} CR^A,$$

and the bound  $N^{-1} \leq N^{-1/2}(1 + \|\xi\|)^{-1/2}$  completes the proof.  $\square$

## 5. THE AMPLIFIED TRACE FORMULA

In this section we establish a trace formula asymptotic with uniform error term. Our proof relies crucially on recent work of Shin-Templier [31] on bounding centralizer volumes and  $p$ -adic orbital integrals as well as work of Finis-Lapid [7] bounding intersection volumes of conjugacy classes with congruence subgroups. We must supply our own bounds on archimedean orbital integrals; these are proven in Sections 7 and 8. Note that in this section, we can and will relax our condition that  $G_{v_0}$  is  $\mathbb{R}$ -almost simple to the condition that  $G$  is  $F$ -almost simple.

**5.1. Statement of main theorem.** To control for the degree of Hecke operators in our estimates, it will be convenient to work with the truncated Hecke algebras  $\mathcal{H}_T^{\leq \kappa}$  defined in [31]. Recall the Weyl-invariant norm  $\|\cdot\|^*$  on  $X_*(T)$ . Let  $v$  be a finite place not dividing  $D$ . Because  $G_v$  is unramified, we may let  $A_v$  be a maximal  $F_v$ -split torus in  $G_v$  such that  $K_v$  corresponds to a point in the apartment of  $A_v$ . We may conjugate  $A_v$  inside  $T \times_F \overline{F}_v$  over  $\overline{F}_v$ , and obtain a norm  $\|\cdot\|_v$  on  $X_*(A_v)$  that is independent of our choice of conjugation by the Weyl invariance of  $\|\cdot\|^*$ . If  $\mu \in X_*(A_v)$  we define  $\tau(v, \mu) \in \mathcal{H}_v$  to be the function supported on  $G(\mathcal{O}_v)\mu(\pi_v)G(\mathcal{O}_v)$  and taking the value  $q_v^{-\|\mu\|_v}$  there. We then define

$$\mathcal{H}_v^{\leq \kappa} = \text{span}_{\mathbb{C}} \{ \tau(v, \mu) : \mu \in X_*^+(A_v), \|\mu\|_v \leq \kappa \},$$

and if  $T$  is any finite set of places not dividing  $D$ ,  $\mathcal{H}_T^{\leq \kappa} = \otimes_{v \in T} \mathcal{H}_v^{\leq \kappa}$ .

If  $U$  is any finite set of finite places, we define  $q_U = \prod_{v \in U} q_v$ .

**Theorem 5.1.** *There are constants  $A, B, \delta, \eta > 0$  such that the following holds. Let  $T$  be a finite set of finite places away from those dividing  $D$  and  $N$ . Let  $\xi \in \mathfrak{a}^*$ . For any  $k_T \in \mathcal{H}_T^{\leq \kappa}$  and any  $k_\xi \in \mathcal{H}_\infty$  satisfying  $(P_\infty)$  we have*

$$\text{Vol}_N \sum_{i \geq 0} \widehat{k_T \otimes k_\xi}(\psi_i) = \mu_G^{\text{can}}([G]) k_T(1) k_\xi(1) + O(N^{-\delta} q_T^{A\kappa+B} \beta(\xi) (1 + \|\xi\|)^{-\eta} \|k_T\|_\infty),$$

where  $\{\psi_i\}$  runs over an orthonormal basis of Hecke-Maass forms for  $L^2(Y_N)$ . The implied constant depends only on  $G$  and the cardinality of  $T$ .

**5.2. Canonical and Tamagawa measures.** If  $G$  is a general connected reductive group over  $F$ , Gross [9, (1.5)] attaches to  $G$  an Artin-Tate motive

$$M_G = \bigoplus_{d \geq 1} M_{G,d}(1-d)$$

with coefficients in  $\mathbb{Q}$ . Here  $(1-d)$  denotes the Tate twist. We let  $\epsilon(M_G)$  be the  $\epsilon$ -factor of this motive, which is given by

$$\epsilon(M_G) = |d_F|^{\dim G/2} \prod_{d \geq 1} \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{f}(M_{G,d}))^{d-1/2},$$

where  $\mathfrak{f}(M_{G,d})$  denotes the conductor of the Artin motive  $M_{G,d}$  (see [9, (9.8)]). We let  $L(M_{G_v}^\vee(1))$  denote the  $L$ -function of the local motive  $M_{G_v}^\vee(1)$ , and  $L(M_G^\vee(1))$  and  $\Lambda(M_G^\vee(1))$  denote the finite and completed  $L$ -functions of  $M_G^\vee(1)$ . Then  $L(M_{G_v}^\vee(1))$  is a positive real number, and  $L(M_G^\vee(1))$  and  $\Lambda(M_G^\vee(1))$  are finite if  $Z(G)$  does not contain an  $F$ -split torus (see [9, Proposition 9.4]). From now on we shall assume that  $G$  satisfies this condition.

In [9, §11] Gross defines a canonical measure  $|\omega_{G_v}|$  on  $G_v$  at any place of  $F$ . We define  $\mu_{G,v}^{\text{can}} = L(M_{G_v}^\vee(1)) \cdot |\omega_{G_v}|$  as in [9]. When  $v$  is finite and  $G$  is unramified at  $v$ ,  $\mu_{G,v}^{\text{can}}$  assigns volume 1 to a hyperspecial subgroup of  $G(F_v)$ , and so we can define the measure  $\mu_G^{\text{can}} = \prod_v \mu_{G,v}^{\text{can}}$  on  $G(\mathbb{A})$ .

Now let  $\omega$  be a nonzero differential form of top degree on  $G$  defined over  $F$ . For each  $v$ , one may associate to  $\omega$  a Haar measure  $|\omega|_v$  on  $G(F_v)$ . For almost all  $v$ ,  $L(M_{G_v}^\vee(1)) \cdot |\omega|_v$  assigns volume 1 to a hyperspecial subgroup of  $G(F_v)$ . Let  $\mu_G^{\text{Tam}}$  be the Tamagawa measure on  $G(\mathbb{A})$ , which is defined by

$$\mu_G^{\text{Tam}} = \Lambda(M_G^\vee(1))^{-1} |d_F|^{-\dim G/2} \bigotimes_v L(M_{G_v}^\vee(1)) |\omega|_v$$

(see [9, (10.2)]) and satisfies

$$\mu_G^{\text{Tam}}(G(F) \backslash G(\mathbb{A})) = |\pi_0(Z(\widehat{G})^\Gamma) / \ker^1(F, Z(\widehat{G}))|^{-1}.$$

The comparison between  $\mu_G^{\text{can}}$  and  $\mu_G^{\text{Tam}}$  is given by [9, Theorem 11.5],

$$(7) \quad \frac{\mu_G^{\text{can}}}{\mu_G^{\text{Tam}}} = \epsilon(M_G) \Lambda(M_G^\vee(1)).$$

**5.3. The trace formula.** The trace formula is a distributional identity

$$I_{\text{spec}}(\phi, \mu_G^{\text{can}}) = I_{\text{geom}}(\phi, \mu_G^{\text{can}}),$$

for  $\phi \in C_c^\infty(G(\mathbb{A}))$ . More precisely,

$$I_{\text{spec}}(\phi, \mu_G^{\text{can}}) = \sum_{\pi} m(\pi) \text{tr}(\pi(\phi)),$$

where  $\pi$  runs over all irreducible subrepresentations of  $L^2(G(F) \backslash G(\mathbb{A}))$  occurring with multiplicity  $m(\pi)$ , and

$$I_{\text{geom}}(\phi, \mu_G^{\text{can}}) = \sum_{\{\gamma\}} \frac{\mu_{I_\gamma}^{\text{can}}(I_\gamma(F) \backslash I_\gamma(\mathbb{A}))}{|G_\gamma : I_\gamma|} O_\gamma(\phi),$$

where  $\{\gamma\}$  runs over all  $G(F)$ -conjugacy classes,  $G_\gamma$  is the centraliser of  $\gamma$  in  $G$ ,  $I_\gamma$  is the connected component of  $G_\gamma$ , and

$$O_\gamma(\phi) = \int_{I_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} \phi(x^{-1} \gamma x) d\mu_\gamma(x).$$

The measure  $\mu_\gamma$  above denotes the quotient measure  $d\mu_G^{\text{can}} / d\mu_{I_\gamma}^{\text{can}}$ .

We shall bound the terms in  $I_{\text{geom}}(\phi, \mu_G^{\text{can}})$  using the Weyl discriminant. For any  $v$  and  $\gamma \in G_v$ , this is defined by

$$D_v(\gamma) = |\det(1 - \text{Ad}(\gamma)|_{\mathfrak{g}_v / \mathfrak{g}_{v,\gamma}})|_v,$$

where  $\mathfrak{g}_{v,\gamma}$  denotes the centraliser of  $\gamma$  in  $\mathfrak{g}_v$ . If  $S$  is any set of places and  $\gamma \in G(F)$ , we define  $D_S(\gamma) = \prod_{v \in S} D_v(\gamma)$  and  $D^S(\gamma) = \prod_{v \notin S} D_v(\gamma)$ .

**5.4. Bounding volumes.** We again let  $G$  denote a group satisfying the conditions of Theorem 1.1. Throughout this rest of this section,  $A, B$  and  $C$  will denote sufficiently large positive constants that may vary from line to line, and will never depend on a choice of place of  $F$ .

In preparation for the following result, we introduce some additional notation. Given  $\kappa \geq 0$  and a finite set of finite places  $T$ , we write  $U_T^{\leq \kappa}$  for the open compact subset  $\text{supp } \mathcal{H}_T^{\leq \kappa}$ . Furthermore, we denote by  $\mathcal{C}_T^{\leq \kappa}$  the set of  $G(F)$ -conjugacy classes of elements in  $G(F) - Z(F)$  whose  $G(\mathbb{A})$ -conjugacy classes have non empty intersection with  $K^T \cdot U_T^{\leq \kappa} \cdot G(F_\infty)$ .

**Proposition 5.2.** *There exist  $A, B > 0$  such that for any  $\kappa \geq 0$ , any finite set of finite places  $T$  away  $D$ , and any  $\{\gamma\} \in \mathcal{C}_T^{\leq \kappa}$ , we have*

$$\mu_{I_\gamma}^{\text{can}}(I_\gamma(F) \backslash I_\gamma(\mathbb{A})) \ll q_T^{A\kappa+B}.$$

*The implied constant depends only on  $G$ .*

*Proof.* Let  $S_D$  denote the set of places dividing  $D$ . Put  $S_\gamma = \{v \notin S_D \cup \infty : D_v(\gamma) \neq 1\}$ . We begin by noting that for any  $\gamma \in G(F)$  we have

$$(8) \quad \mu_{I_\gamma}^{\text{can}}(I_\gamma(F) \backslash I_\gamma(\mathbb{A})) \ll q_{S_\gamma}^B,$$

where the implied constant depends only on  $G$ . Indeed, from the proof of [31, Corollary 6.16] we have

$$\epsilon(M_{I_\gamma})L(M_{I_\gamma}^\vee(1)) \ll \prod_{v \in \text{Ram}(I_\gamma)} q_v^B \ll q_{S_\gamma}^B,$$

where  $\text{Ram}(I_\gamma)$  is the set of finite places where  $I_\gamma$  is ramified. Note that the last bound follows from the inclusion  $\text{Ram}(I_\gamma) \subset S_D \cup S_\gamma$ . Moreover, from the definition of the local archimedean factors in (7.1) and (7.2) of [9], combined with [31, Proposition 6.3], we find that  $L_\infty(M_{I_\gamma}^\vee(1)) \ll 1$ , the implied constant depending only on  $G$ . Finally, Corollary 8.12 and Lemma 8.13 of [31] imply  $\mu_{I_\gamma}^{\text{Tam}}(I_\gamma(F) \backslash I_\gamma(\mathbb{A})) \ll 1$ . By combining these estimates with (7) we obtain (8).

Now for  $\{\gamma\} \in \mathcal{C}_T^{\leq \kappa}$  we have

$$D_v(\gamma) \leq \begin{cases} q_v^{A\kappa+B}, & \text{for } v \in T, \\ C, & \text{for } v \mid \infty, \\ 1, & \text{for } v \notin T \cup \infty. \end{cases}$$

From this and the product formula we deduce that

$$(9) \quad 1 = \prod_{v \in T} D_v(\gamma) \prod_{v \in S_D \cup \infty} D_v(\gamma) \prod_{v \in S_\gamma, v \notin T} D_v(\gamma) \ll q_T^{A\kappa+B} q_{S_\gamma}^{-1},$$

since  $D_v(\gamma) \leq q_v^{-1}$  for every  $v \in S_\gamma$ . Inserting this into (8) gives the proposition.  $\square$

**5.5. Bounding adelic orbital integrals.** Let  $T$  be a finite set of places away from those dividing  $N$ ,  $D$ , and  $\infty$ . Recall the definition of  $\mathbf{1}_N^T$  from (6). Let  $\xi \in \mathfrak{a}^*$ . For  $k_T \in \mathcal{H}_T^{\leq \kappa}$  and  $k_\xi \in \mathcal{H}_\infty$  satisfying  $(P_\infty)$  we put

$$\phi = \mathbf{1}_N^T \otimes k_T \otimes k_\xi.$$

We now look at  $O_\gamma(\phi)$ .

**Proposition 5.3.** *There are constants  $A, B, \delta, \eta > 0$  such that*

$$O_\gamma(\phi) \ll N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\eta} q_T^{A\kappa+B} \|k_T\|_\infty$$

for every  $\gamma \in G(F) - Z(F)$ . The implied constant depends only on  $G$ .

*Proof.* Note that we may write  $k_T$  as a sum of at most  $q_T^{A\kappa+B}$  pure tensors in  $\mathcal{H}_T^{\leq \kappa}$ , whose sup norms are all bounded by  $\|k_T\|_\infty$ . We shall therefore assume that  $k_T$ , and hence  $\phi$ , is a pure tensor product. This assumption implies that the orbital integral factorizes as  $O_\gamma(\phi) = \prod_v O_\gamma(\phi_v)$ , where for any  $\gamma_v \in G(F_v)$  we have

$$O_{\gamma_v}(\phi_v) = \int_{I_{\gamma_v}(F_v) \backslash G(F_v)} \phi_v(x_v^{-1} \gamma_v x_v) d\mu_{\gamma_v}(x_v)$$

and  $\mu_{\gamma,v} = \mu_{G,v}^{\text{can}} / \mu_{I_{\gamma_v},v}^{\text{can}}$ . It therefore suffices to work place by place.

In [31, Theorems 7.3 and B.1] it is shown that

$$O_\gamma(k_T) \ll q_T^{A\kappa+B} D_T(\gamma)^{-C} \|k_T\|_\infty.$$

We may prove the following bound for the integral at infinity using the results of Section 7.

**Lemma 5.4.** *We have the bound*

$$O_\gamma(k_\xi) \ll \beta(\xi) (1 + \|\xi\|)^{-\eta} D_\infty(\gamma)^{-C}.$$

*Proof.* Let  $G_{\text{cpt}}$  and  $G_{\text{cpt},v_0}$  be the groups associated to  $G_\infty$  and  $G_{v_0}$  in Section 7.2. We begin by showing that, as a consequence of Proposition 7.4, the following statement holds. Let  $0 < \eta < \min(\eta_0, 1/2)$ , and let  $k_\xi \in \mathcal{H}_\infty$  satisfy property (P $_\infty$ ). Then

$$(10) \quad O_\gamma(k_\xi) \ll \beta(\xi) (1 + \|\xi\|)^{-\eta} D_\infty(\gamma)^{-A}$$

for every semisimple  $\gamma \in G_\infty - G_{\text{cpt}}$ .

To see how (10) follows from Proposition 7.4, first note that for any non-negative  $f \in C_c^\infty(G_\infty)$  such that  $f(1) = 1$  on the support of  $k_\xi$ , we have

$$\begin{aligned} k_\xi(g) &\ll \beta(\xi) (1 + \|\xi\| d(g, H_{v_0}))^{-1/2} f(g) \\ &\leq \beta(\xi) (1 + \|\xi\| d(g, H_{v_0}))^{-\eta} f(g) \\ &\ll \beta(\xi) (1 + \|\xi\|)^{-\eta} d(g, H_{v_0})^{-\eta} f(g). \end{aligned}$$

Thus

$$O_\gamma(k_\xi) \ll \beta(\xi) (1 + \|\xi\|)^{-\eta} O_\gamma(f d(g, H_{v_0})^{-\eta}),$$

to which we may apply (22). Indeed, since  $Z$  is finite, the function  $\|X(g)\|$  used there satisfies  $\|X(g)\| \ll d(g, H_{v_0})$ .

The proof of Lemma 5.4 then follows from (10) once we have verified that a non-central element  $\gamma \in G(F)$  cannot lie in  $G_{\text{cpt}}$ . Because  $G_\infty = \prod_{v|\infty} G_v$ , we have  $G_{\text{cpt}} = G_{\text{cpt},v_0} \times \prod_{v \neq v_0} G_v$ , and so it suffices to verify that  $\gamma_{v_0} \notin G_{\text{cpt},v_0}$ .

In the case at hand,  $G_{\text{cpt},v_0}$  is a compact normal subgroup of  $G_{v_0}$ ; this is the content of Lemma 7.2. If we let  $H^+$  be the fixed point set of  $\theta$  in  $G$  (we write  $H^+$  to distinguish it from its identity component  $H$ ), then  $H_{v_0}^+$  is a maximal compact subgroup of  $G_{v_0}$ , and so we

have  $G_{\text{cpt},v_0} \subset gH_{v_0}^+g^{-1}$  for all  $g \in G_{v_0}$ . Thus, if  $\gamma_{v_0} \in G_{\text{cpt},v_0}$ , we have  $\gamma \in gH^+(F)g^{-1}$  for all  $g \in G(F)$ . The group

$$\bigcap_{g \in G(F)} gH^+(F)g^{-1}$$

is a proper normal  $F$ -subgroup of  $G$ , and so (using the assumption that  $G$  is  $F$ -almost simple) it must be contained in  $Z(G)$ . We therefore have  $\gamma \in Z(F)$ , a contradiction.  $\square$

It remains then to address the size of the orbital integral at finite places away from  $T$ . We claim that

$$(11) \quad O_\gamma(\mathbf{1}_N^T) \ll N^{-\delta} q_T^{A\kappa+B} D^{T\infty}(\gamma)^{-C}.$$

Taken together (and using the product rule for the product of Weyl discriminants), the above estimates imply the proposition.

Recall the sets  $S_D$  and  $S_\gamma$  from the proof of Proposition 5.2. Let  $S_N$  denote the set of places dividing  $N$ . We are free to take  $\gamma \in \mathcal{C}_T^{\leq \kappa}$ , for otherwise the orbital integral vanishes.

- If  $v \notin S_D \cup S_N \cup S_\gamma \cup T \cup \infty$ , then  $K_v(N)K_{H_v} = K_v$  and we have  $O_\gamma(\mathbf{1}_{K_v}) = 1$ ; see, for example, [16, Corollary 7.3].
- If  $v \in S_D$ , then  $K_v(N)K_{H_v} = K_vK_{H_v}$  and a general bound of Kottwitz [31, Theorem A.1] establishes that  $O_\gamma(\mathbf{1}_{K_vK_{H_v}}) \ll_v D_v(\gamma)^{-1/2}$ .
- If  $v \in S_N \cup S_\gamma$ ,  $v \notin T$ , we argue as follows.

We begin by estimating the orbital integrals at places  $v \in S_N$ , for which we will use as a critical input the work of Finis-Lapid [7]. As the setting of [7] is that of  $\mathbb{Q}$ -groups, we shall need to restrict scalars from  $F$  to  $\mathbb{Q}$  to properly invoke their results. We thus let  $p$  denote the rational prime over which  $v$  lies, and we note that all places lying over  $p$  belong to  $S_N$ . We set  $K_p = \prod_{v|p} K_v$ , and  $K_p(N)K_{H,p} = \prod_{v|p} K_v(N)K_{H,v}$ . Factorize  $N = \prod_{p|N} p^{n_p}$  and put  $N_p = p^{n_p}$ .

Let  $\mu_{G,p}^{\text{can}}$  (resp.,  $\mu_{I_\gamma,p}^{\text{can}}$ ) be the product measure on  $G_p = \prod_{v|p} G_v$  (resp.,  $I_{\gamma,p} = \prod_{v|p} I_{\gamma,v}$ ). Write  $C_{\gamma,G_p}$  for the  $G_p$ -conjugacy class of  $\gamma$ . Letting  $\mu_{\gamma,p} = \mu_{G,p}^{\text{can}}/\mu_{I_\gamma,p}^{\text{can}}$  be the natural measure on  $C_{\gamma,G_p}$ , we have

$$O_\gamma(\mathbf{1}_{K_p(N)K_{H,p}}) = \mu_{\gamma,p}(C_{\gamma,G_p} \cap K_p(N)K_{H,p}).$$

Now  $C_{\gamma,G_p}$  is closed since  $\gamma$  is semi-simple. We may therefore break up the compact intersection  $C_{\gamma,G_p} \cap K_p(N)K_{H,p}$  into a finite number of (open)  $K_p$ -conjugacy classes meeting  $K_p(N)K_{H,p}$ . Choose representatives  $x_i \in K_p$  for these, and let  $C_{x_i,K_p}$  denote the corresponding  $K_p$ -conjugacy class. Then  $C_{\gamma,G_p} \cap K_p(N)K_{H,p} = \coprod C_{x_i,K_p}$  and we get

$$O_\gamma(\mathbf{1}_{K_p(N)K_{H,p}}) = \sum_i \mu_{\gamma,p}(C_{x_i,K_p} \cap K_p(N)K_{H,p}).$$

Note that for any open compact subgroup  $K'_p$  of  $G_p$ , and any  $x \in K_p$ , we have

$$\mu_{\gamma,p}(C_{x,K_p} \cap K'_p) = \frac{\mu_{G,p}^{\text{can}}(k \in K_p : k^{-1}xk \in K'_p)}{\mu_{I_\gamma,p}^{\text{can}}(I_{x,p} \cap K_p)}.$$

We deduce from this, and the fact that  $\mu_{G,v}^{\text{can}}(K_v) = 1$  for all  $v \notin S_D \cup \infty$ , that

$$\frac{\mu_{\gamma,p}(C_{x,K_p} \cap K'_p)}{\mu_{\gamma,p}(C_{x,K_p})} = \mu_{G,p}^{\text{can}}(k \in K_p : k^{-1}xk \in K'_p).$$

One can deduce from Propositions 5.10 and 5.11 in [7] (see Remark 5.5 below) that there are constants  $\epsilon, \delta > 0$  (independent of  $p$ ) such that for  $x \in K_p$  with  $D_p(x) > N_p^{-\epsilon}$  one has

$$(12) \quad \mu_{G,p}^{\text{can}}(k \in K_p : k^{-1}xk \in K_p(N)K_{H,p}) \ll_{G,\rho} N_p^{-\delta}.$$

(Here  $\ll_{G,\rho}$  means that, in particular, the implied constant is independent of  $p$ .) On the other hand  $\sum_i \mu_{\gamma,p}(C_{x_i, K_p}) = \mu_{\gamma,p}(C_{\gamma, G_p} \cap K_p) = O_\gamma(\mathbf{1}_{K_p})$ . Noting that  $D_p(\gamma) = D_p(x_i)$  for all  $i$ , we deduce that for  $D_p(\gamma) > N_p^{-\epsilon}$  we have

$$(13) \quad O_\gamma(\mathbf{1}_{K_p(N)K_{H_p}}) \ll_{G,\rho} N_p^{-\delta} \sum_i \mu_{\gamma,p}(C_{x_i, K_p}) = N_p^{-\delta} O_\gamma(\mathbf{1}_{K_p}).$$

In the remaining range  $D_p(\gamma) \leq N_p^{-\epsilon}$ , we have

$$O_\gamma(\mathbf{1}_{K_p(N)K_{H_p}}) \leq N_p^{-\delta} D_p(\gamma)^{-\delta/\epsilon} O_\gamma(\mathbf{1}_{K_p}).$$

We now return to the product of orbital integrals over all  $v \in S_N \cup S_\gamma$ ,  $v \notin T$ . Recalling that  $O_\gamma(\mathbf{1}_{K_v}) = 1$  for  $v \notin S_\gamma$ , we have just shown

$$\prod_{v \in S_N} O_\gamma(\mathbf{1}_{K_v(N)K_{H_v}}) \prod_{\substack{v \in S_\gamma \\ v \notin S_N \cup T}} O_\gamma(\mathbf{1}_{K_v}) \ll N^{-\delta} D_{S_N}(\gamma)^{-C} \prod_{\substack{v \in S_\gamma \\ v \notin T}} O_\gamma(\mathbf{1}_{K_v}),$$

since  $D_v(\gamma) \leq 1$  for all  $v \notin T \cup \infty$  (and we shrink  $\delta$  to absorb the implied constant in (13)). For  $v \in S_\gamma$  we again apply [31, Theorems 7.3 and B.1] to get  $O_\gamma(\mathbf{1}_{K_v}) \leq q_v^B D_v(\gamma)^{-C}$ . Since  $\{\gamma\} \in \mathcal{C}_T^{\leq \kappa}$  we may invoke (9) to obtain

$$\prod_{\substack{v \in S_\gamma \\ v \notin T}} O_\gamma(\mathbf{1}_{K_v}) \ll \prod_{\substack{v \in S_\gamma \\ v \notin T}} q_v^B D_v(\gamma)^{-C} \ll D_{S_\gamma - T}(\gamma)^{-C} \prod_{v \in S_\gamma} q_v^B \ll q_T^{A\kappa + B} D_{S_\gamma - T}(\gamma)^{-C}.$$

Putting these estimates together completes the proof of (11) and hence the proposition.  $\square$

*Remark 5.5.* We make a few remarks on the various bounds we have imported into the above proof.

As the authors point out in [31, Remark 7.4], the bound [31, Theorems 7.3 and B.1] is uniform in the place  $v \notin S_D$  whereas the bound [31, Theorem A.1] of Kottwitz applies to  $v \in S_D$  but it not uniform in  $v$ . As we allow the implied constant in Proposition 5.3 to depend on the group, this non-uniformity is not an issue.

We now explain how to extract from Propositions 5.10 and 5.11 of [7] the bound we stated in (12). In what follows, we simplify notation by writing  $K'_p$  for  $K_p(N)K_{H,p}$ . We recall that  $N = \prod_{p|N} p^{n_p}$  and that  $x$  is taken to lie in  $K_p$ .

• We first remark that we may assume that  $x$  lies in  $K'_p$ , for if there is no such representative then the left-hand side of (12) is zero. We then have

$$\mu_{G,p}^{\text{can}}(k \in K_p : k^{-1}xk \in K'_p) = \mu_{G,p}^{\text{can}}(k \in K_p : [k, x] \in K'_p).$$

If we let  $\phi_{K'_p}(x)$  be as in [7, Definition 5.1], then

$$\mu_{G,p}^{\text{can}}(k \in K_p : [k, x] \in K'_p) \ll_{G,\rho} \phi_{K'_p}(x).$$

(Note that the two expressions are not necessarily equal because of the passage to  $G^{\text{ad}}$  in [7]).

• Define  $\lambda_p(x)$  as in [7, Definition 5.2]. We claim that one can deduce from Propositions 5.10 and 5.11 of [7] that for every  $\epsilon > 0$  small enough there is  $\delta > 0$  such that if  $\lambda_p(x) < \epsilon n_p$

then  $\phi_{K'_p}(x) \ll_{G,\rho} p^{-\delta n_p}$ . The argument is already present in [7, §5.2] in the deduction of the global result [7, Theorem 5.3] from these two local results.

From [7, Proposition 5.11] it follows that there are positive constants  $a, b > 0$  and  $c \geq 0$  (depending on  $G$  and  $\rho$ ) such that  $\phi_{K'_p}(x) \leq p^{a(c+\lambda_p(x)-bn_p)}$ . The presence of the constant  $c$  renders this bound useless for small  $n_p$ . Taking  $\epsilon$  small enough to satisfy  $0 < \epsilon < (c+1)^{-1}b$ , we shall apply this bound only in the range  $n_p > \epsilon^{-1}$ ; we obtain  $\phi_{K'_p}(x) \leq p^{-a(b-\epsilon(c+1))n_p}$ . In the remaining range  $n_p \leq \epsilon^{-1}$  we see that  $\lambda_p(x) < 1$  so that in fact  $\lambda_p(x) = 0$ . In this case [7, Proposition 5.10] ensures<sup>1</sup> that  $\phi_{K'_p}(x) \ll_{G,\rho} p^{-1} \leq p^{-\epsilon n_p}$ . Taking  $\delta = \min\{\epsilon, a(b-\epsilon(c+1))\}$ , we establish the claim.

• It remains to show that if  $D_p(x) > p^{-\epsilon n_p}$  then  $\lambda_p(x) < \epsilon n_p$ . To see this, we may assume that  $x$  is semisimple. Write  $x = (x_v)_{v|p}$ . As a point of reference, note that for  $x \in K_p$  we have  $\lambda_p(x) \geq 0$ , and  $D_v(x_v) \leq 1$  for all  $x_v \in K_v$ . Now if  $\lambda_p(x) \geq \epsilon n_p$  then there is some  $v \mid p$  and some eigenvalue of  $1 - \text{Ad}(x_v)|_{\mathfrak{g}/\mathfrak{g}_{x_v}}$  with  $v$ -adic valuation at least  $e_v \epsilon n_p$ , where  $e_v$  is the ramification index of  $F_v$  over  $\mathbb{Q}_p$ . (As these eigenvalues may not lie in  $F_v$ , we extend the valuation on  $F_v$  to the field containing the eigenvalue in such a way that restricting to  $F_v$  gives the original valuation.) This then implies that  $D_v(x_v) \leq |\varpi_v|_v^{e_v \epsilon n_p} = p^{-\epsilon n_p}$ , so that  $D_p(x) \leq p^{-\epsilon n_p}$ .

**5.6. Proof of Theorem 5.1.** We retain the notation for the test function  $\phi$  from §5.5 and for the set  $\mathcal{C}_T^{\leq \kappa}$  from §5.4. Then

$$I_{\text{spec}}(\phi, \mu_G^{\text{can}}) = \text{Vol}_N \sum_{i \geq 0} \widehat{k_T \otimes k_\xi(\psi_i)},$$

the sum ranging over an orthonormal basis of Hecke-Maass forms for  $Y_N$ , and

$$I_{\text{geom}}(\phi, \mu_G^{\text{can}}) = \mu_G^{\text{can}}([G])k_T(1)k_\xi(1) + \sum_{\{\gamma\} \in \mathcal{C}_T^{\leq \kappa}} \frac{\mu_{I_\gamma}^{\text{can}}(I_\gamma(F) \backslash I_\gamma(\mathbb{A}))}{|G_\gamma : I_\gamma|} O_\gamma(\phi).$$

Here we have used the hypothesis on  $K_f$  from §3.5 that  $Z(F) \cap K_f = \{e\}$ . Now by [31, Corollary 8.10] we have  $|\mathcal{C}_T^{\leq \kappa}| \ll q_T^{A\kappa+B}$ . From this and Propositions 5.2 and 5.3 we find

$$I_{\text{geom}}(\phi, \mu_G^{\text{can}}) = \mu_G^{\text{can}}([G])k_T(1)k_\xi(1) + O(q_T^{A\kappa+B} N^{-\delta} \beta(\xi)(1 + \|\xi\|)^{-\eta} \|k_T\|_\infty),$$

as desired. This completes the proof of Theorem 5.1.

## 6. COMPARISON OF TRACE FORMULAE AND THE PROOF OF THEOREM 1.2

In this section we prove our main result, Theorem 1.2. The argument is based on a comparison of the trace formulae described in the preceding two sections. For this, we must choose test functions  $k_\xi \in \mathcal{H}_\infty$  and  $k_S \in \mathcal{H}_S$  to insert into Proposition 4.1 and Theorem 5.1 and explicate the error terms in those results.

<sup>1</sup>Note that [7, Proposition 5.10] assumes that  $G$  is simply connected. As was pointed out to us by Finis and Lapid, this assumption can be dropped for those subgroups  $K'_p$  not containing the intersection of  $K_p$  with  $G_p^+$ . (The statement of [7, Proposition 5.10] provides bounds on  $\phi_{K'_p}(x)$  for every proper subgroup  $K'_p$  of  $K_p$ .) Here,  $G_p^+$  denotes the image in  $G_p$  of the  $\mathbb{Q}_p$ -points of the simply connected cover of  $G_p$ . That our subgroups  $K'_p = K_p(N)K_{H,p}$  satisfy this condition (for  $p$  large enough with respect to  $G$  and  $F$ ) can be seen from comparing indices. The subgroups  $K_p(N)K_{H,p}$  have indices growing like a power of  $p$ , whereas those containing  $K_p \cap G_p^+$  are of index bounded in terms of  $G$  and  $F$ .

**6.1. The archimedean test function.** Let  $G_\infty^0$  denote the connected component of the identity in  $G_\infty$  in the real topology. If  $\mu \in \mathfrak{a}_\mathbb{C}^*$ , we define  $\varphi_\mu$  to be the corresponding spherical function on  $G_\infty^0$ . If  $k_\infty \in C_c^\infty(G_\infty^0)$ , we define its Harish-Chandra transform by

$$\widehat{k}_\infty(\mu) = \int_{G_\infty^0} k_\infty(g) \varphi_{-\mu}(g) d\mu_{G_\infty^0}^{\text{can}}(g).$$

We shall choose  $k_\xi$  so that its spherical transform concentrates around  $\xi \in \mathfrak{a}^*$ . For this we first take a function  $h_0 \in C^\infty(\mathfrak{a}^*)$  of Paley-Wiener type that is real, nonnegative, and satisfies  $h_0(0) = 1$ . Let

$$h_\xi(\nu) = \sum_{w \in W_\mathbb{R}} h_0(w\nu - \xi),$$

and let  $k_\xi$  be the bi- $K_\infty$ -invariant function on  $G_\infty^0$  satisfying  $\widehat{k}_\xi = h_\xi$ . Note that  $\widehat{k}_\xi(\xi) = h_\xi(\xi) \geq 1$ .

**Lemma 6.1.** *The function  $k_\xi$  satisfies property  $(P_\infty)$ .*

*Proof.* Note that  $k_\xi$  is of compact support independent of  $\xi$  from the Paley-Wiener theorem of [8]; we may thus take  $h_0$  so that the support of  $k_\xi$  lies in  $\{g \in G(F_\infty) : d(g, H_{v_0}) < 1\}$  for all  $\xi$ .

If  $k_\xi$  is bi-invariant under  $K_\infty$ , we have the inversion formula

$$k_\xi(g) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \widehat{k}_\xi(\mu) \varphi_\mu(g) |c(\mu)|^{-2} d\mu,$$

where  $c(\mu)$  is Harish-Chandra's  $c$ -function; see [11, Ch. II §3.3]. We can now quote either Theorem 2 of [2] or Proposition 7.2 of [21], and apply our conditions on  $G_\infty$ , to find

$$(14) \quad \varphi_\mu(x) \ll_C (1 + \|\mu\|d(x, K_\infty))^{-1/2},$$

for  $\mu \in \mathfrak{a}^*$  and  $x$  in a compact set  $C \subset G_\infty$ . By inverting the Harish-Chandra transform and applying (14) as in Lemma 2.7 of [20], it follows that

$$k_\xi(x) \ll \beta(\xi)(1 + \|\xi\|d(x, K_\infty))^{-1/2},$$

whence the claim. □

**6.2. The  $S$ -adic test function.** Recall from Section 3.2 that  $\mathcal{P}$  denotes the set of finite places of  $F$  that do not divide  $D$  and at which  $T$  splits. For  $v \in \mathcal{P}$  and  $\mu \in X_*(T)$ , we define  $\tau(v, \mu) \in \mathcal{H}_v$  to be the function supported on  $G(\mathcal{O}_v)\mu(\pi_v)G(\mathcal{O}_v)$  and taking the value  $q_v^{-\|\mu\|}$  there. (Note that this is compatible with the conventions of the first paragraphs of Section 5 if we choose  $A_v = T_v$  there.) Let  $P$  be a positive integer and put

$$(15) \quad S = \{v \in \mathcal{P} : P/2 \leq q_v < P\}.$$

To define a test function at places in  $S$ , choose any non-zero  $\nu \in X_*(T)$ , and let

$$(16) \quad \omega_S = \sum_{v \in S} \tau(v, \nu),$$

where, as in §3.6, we are identifying  $\mathcal{H}_v$  with a subalgebra of  $\mathcal{H}_S$  in the natural way. The parameter  $P$  and the cocharacter  $\nu$  will be chosen later in §6.3. Then define  $k_S = \omega_S \omega_S^* \in \mathcal{H}_S$ .

**Lemma 6.2.** *The following properties hold for the above function  $k_S$ .*



- (a) There is  $B > 0$  such that  $\|g\|_S \ll P^B$  for all  $g \in \text{supp}(k_S)$ .  
(b) We have  $\|k_S\|_\infty \ll P$ .  
(c) There is  $C > 0$  such that  $\#(\text{supp } k_S/K_S) \ll P^C$ .

The exponents  $B$  and  $C$  depend on the underlying choice of  $\nu$  in the definition of  $k_S$ . All implied constants depend on  $G$  and  $\nu$ .

*Proof.* (a) It may be shown as in [20, Lemma 2.4] that there is  $B > 0$  such that for all  $v \in \mathcal{P}$  and all  $g$  in the support of  $\tau(v, \nu)$  one has  $\|g\|_v \ll q_v^B$ . From this one easily deduces that  $\omega_S$  satisfies the first property. As supports add under convolution, the same holds for  $k_S$ .

(b) We note that

$$(17) \quad k_S = \sum_{v \in S} \tau(v, \nu) \tau(v, \nu)^* + \sum_{\substack{v, w \in S \\ v \neq w}} \tau(v, \nu) \tau(w, \nu)^*.$$

The bound is clear for the second sum, because the terms satisfy  $\|\tau(v, \nu) \tau(w, \nu)^*\|_\infty \leq 1$  and the supports of the terms are disjoint. For the first sum, we have

$$\|\tau(v, \nu) \tau(v, \nu)^*\|_\infty \leq \|\tau(v, \nu)\|_2^2,$$

and  $\|\tau(v, \nu)\|_2 \ll 1$  follows from  $\#(K_v \nu(\pi_v) K_v)/K_v \sim q_v^{2\|\nu\|}$ .

(c) We may write the first sum in (17) as a linear combination of  $\tau(v, \mu)$  with  $\mu$  lying in a finite set depending on  $\nu$ . The bound now follows from the asymptotic  $\#(K_v \mu(\pi_v) K_v)/K_v \sim q_v^{2\|\mu\|}$ . The second sum may be treated similarly.  $\square$

**6.3. Proof of Theorem 1.2.** We are now in a position to prove Theorem 1.2.

We first note that  $k_\xi(1) \asymp \beta(\xi)$ , where  $k_\xi$  is defined as in §6.1. Moreover,  $\beta(\xi)$  is bounded above (and below) by a power of  $(1 + \|\xi\|)$ .

We borrow the constructions from §6.2. Namely, we take  $S$  as in (15) (for a parameter  $P$  to be chosen later) and  $\omega_S \in \mathcal{H}_S$  as in (16) (relative to a cocharacter  $\nu \in X_*(T)$  to be chosen in Lemma 6.4).

We now apply Proposition 4.1 with test functions  $k_S k_\xi$ , where  $k_S = \omega_S \omega_S^*$ . Moreover, for every pair  $v, w \in S$  we put  $T = \{v, w\}$  and apply Theorem 5.1 with test function  $k_T k_\xi$  where  $k_T = \tau(v, \nu) \tau(w, \nu)^*$ ; we then sum over such pairs  $v, w$ . As a result, we deduce the existence of constants  $A > 2$  (depending on  $\nu$ ) and  $\delta > 0$  such that

$$(18) \quad \text{Vol}_N \sum_{i \geq 0} |\widehat{\omega_S}(\psi_i)|^2 h_\xi(\xi_i) \asymp \omega_S \omega_S^*(1) \beta(\xi) + O(P^A N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\delta})$$

and

$$(19) \quad \text{Vol}_N \sum_{i \geq 0} |\widehat{\omega_S}(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \asymp \Pi_H \omega_S \omega_S^*(1) \beta(\xi) + O(P^A N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\delta}).$$

The error term in (18) was obtained by observing that  $\tau(v, \nu) \tau(w, \nu)^* \in \mathcal{H}_T^{\leq \kappa}$ , for some  $\kappa$  depending only on  $\nu$ , and  $q_T = q_v q_w < P^2$ , so that  $q_T^{A\kappa+B}$  is bounded by a power of  $P$ ; we must also insert the  $L^\infty$  norm estimate for  $k_T$  coming from the proof of Lemma 6.2. The error term in (19) was obtained by taking  $B$  as in Lemma 6.2 and setting  $R = P^B$  in condition  $(P_S)$ , and inserting the  $L^\infty$  norm estimate of Lemma 6.2. It remains then to explicate the size of  $\Pi_H \omega_S \omega_S^*(1)$  and  $\omega_S \omega_S^*(1)$  (upon taking an appropriate  $\nu$ ), truncate the spectral sums,

and choose the length of the amplifier  $P$  in terms of  $N$  and  $\|\xi\|$ . We continue to use the convention that the values of the exponents  $A, \delta > 0$  can vary from line to line.

We first examine (19). We begin by truncating the spectral sum about  $\xi$ .

**Lemma 6.3.** *For any  $Q > 1$  the left-hand side of (19) can be written as*

$$\begin{aligned} \text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \\ + O_M(\Pi_H \omega_S \omega_S^*(1) \beta(\xi) Q^{-M}) + O(P^A N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\delta}). \end{aligned}$$

*Proof.* Break the region in the positive chamber  $\mathfrak{a}_+^*$  defined by  $\|\mu - \xi\| > Q$  into an overlapping union of  $O(1)$ -balls  $B(\mu_n)$  centered at points  $\mu_n \in \mathfrak{a}^*$ . On each ball we apply the rapid decay estimate  $h_\xi(\mu) \ll_M \|\mu - \xi\|^{-M}$  to obtain

$$\sum_{\xi_i \in B(\mu_n)} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \ll_M \|\mu_n - \xi\|^{-M} \sum_{\xi_i \in B(\mu_n)} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2.$$

We may use (19) with the test function  $h_{\mu_n}$  together with the positivity of the spectral terms to show that

$$\sum_{\xi_i \in B(\mu_n)} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 \ll \Pi_H \omega_S \omega_S^*(1) \beta(\mu_n) + P^A \beta(\mu_n) (1 + \|\mu_n\|)^{-\delta} N^{-\delta}.$$

Summing over  $n$  we obtain

$$\begin{aligned} \sum_{\|\xi_i - \xi\| > Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \ll_M \Pi_H \omega_S \omega_S^*(1) \sum_n \|\mu_n - \xi\|^{-M} \beta(\mu_n) \\ + P^A N^{-\delta} \sum_n \|\mu_n - \xi\|^{-M} (1 + \|\mu_n\|)^{-\delta} \beta(\mu_n). \end{aligned}$$

From  $\beta(\mu_n) \ll \|\mu_n - \xi\|^k \beta(\xi)$ , where  $k$  is the number of roots of  $G$  counted with multiplicity, we may simplify this to

$$\begin{aligned} \sum_{\|\xi_i - \xi\| > Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \ll_M \Pi_H \omega_S \omega_S^*(1) \beta(\xi) \sum_n \|\mu_n - \xi\|^{-M} \\ + \beta(\xi) P^A N^{-\delta} \sum_n \|\mu_n - \xi\|^{-M} (1 + \|\mu_n\|)^{-\delta}. \end{aligned}$$

The first sum is  $\ll_M Q^{-M}$ . We bound the second sum by breaking it into  $\|\mu_n\| \leq \|\xi\|/2$  and the complement. The first sum is  $\ll (1 + \|\xi\|)^{-M}$ , and the second is  $\ll Q^{-M} (1 + \|\xi\|)^{-\delta}$ . Both of these are dominated by  $(1 + \|\xi\|)^{-\delta}$ , which gives

$$\sum_{\|\xi_i - \xi\| > Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \ll_M \Pi_H \omega_S \omega_S^*(1) \beta(\xi) Q^{-M} + P^A N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\delta},$$

as desired. □

We combine Lemma 6.3 (taking any  $M > 0$  and large enough  $Q$ ) and (19) to obtain

$$\text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \asymp \Pi_H \omega_S \omega_S^*(1) \beta(\xi) + O(P^A N^{-\delta} \beta(\xi) (1 + \|\xi\|)^{-\delta}).$$

We now make use of the critical assumption that  $G$  is  $H$ -large to bound from below the right hand side.

**Lemma 6.4.** *If  $\nu \in X_*(T_H)$  satisfies (5) then  $\Pi_H \omega_S \omega_S^*(1) \gg_\epsilon P^{2-\epsilon}$ .*

*Proof.* Note that for any  $v \in \mathcal{P}$  and  $\nu \in X_*(T_H)$  we have

$$q_v^{\|\nu\|^*} \int_{H_v} \tau(v, \nu)(x) dx = \text{vol}(H_v \cap K_v \nu(\pi_v) K_v) \geq \text{vol}(K_{H,v} \nu(\pi_v) K_{H,v}) \gg q_v^{2\|\nu\|_H^*},$$

where we have used our assumption that  $T$ , and hence  $T_H$ , are split at  $v$ . If  $\nu \in X_*(T)$  satisfies (5), then so does  $-\nu$ , and applying the above bound with  $\pm\nu$ , we get  $\int_{H(F_T)} \tau(v, \nu) \tau(w, \nu)^* \gg 1$  if  $v \neq w$ . Summing over  $v, w \in S$  yields the claim.  $\square$

We deduce from the above lemma that for such a choice of  $\nu$  we have

$$\text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \gg_\epsilon P^{2-\epsilon} \beta(\xi) (1 + O(P^A N^{-\delta} (1 + \|\xi\|)^{-\delta})).$$

We now treat (18). Lemma 6.2 (b) gives  $\omega_S \omega_S^*(1) \ll P$ . By positivity we may truncate the spectral sum to obtain the upper bound

$$\text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 h_\xi(\xi_i) \ll P \beta(\xi) (1 + P^A N^{-\delta} (1 + \|\xi\|)^{-\delta}).$$

Choosing  $P$  to be a small power of  $(1 + \|\xi\|)N$ , we find  $\delta > 0$  such that

$$(20) \quad \text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 h_\xi(\xi_i) \ll \beta(\xi) (1 + \|\xi\|)^\delta N^\delta$$

and

$$(21) \quad \text{Vol}_N \sum_{\|\xi_i - \xi\| \leq Q} |\widehat{\omega}_S(\psi_i)|^2 |\mathcal{P}_H(\psi_i)|^2 h_\xi(\xi_i) \gg_\epsilon \beta(\xi) (1 + \|\xi\|)^{2\delta - \epsilon} N^{2\delta - \epsilon}.$$

Comparing (20) and (21) we find that there is  $\delta > 0$  and  $Q > 1$  such that for every  $\xi \in \mathfrak{a}^*$  and every  $N$  there is a Hecke-Maass form  $\psi_i$  on  $Y_N$  with spectral parameter  $\|\xi_i - \xi\| \leq Q$  and satisfying  $|\mathcal{P}_H(\psi_i)| \gg (1 + \|\xi_i\|)^\delta N^\delta$ . This implies the same lower bound on  $\|\psi_i\|_\infty$ , and in particular proves Theorem 1.1.

It remains to refine this power growth to obtain the stated lower bounds of Theorem 1.2. For this, we will make a special choice of orthonormal basis of Hecke-Maass forms for  $L^2(Y_N)$ . First recall that we have a Hilbert direct sum decomposition

$$L^2(Y_N) = \bigoplus_{\pi} m(\pi) \pi^{K(N)K_H K_\infty},$$

the sum ranging over irreducible representations of  $G_\infty^0 \times G(\mathbb{A}_f)$  in  $L^2(G(F) \backslash G(\mathbb{A}))$  with  $\pi^{K(N)K_H K_\infty} \neq 0$ , each occurring with multiplicity  $m(\pi)$ . Each  $\pi$  gives rise to a spectral datum  $c_\pi$  (as defined in the paragraph preceding Theorem 1.2), and this assignment  $\pi \mapsto c_\pi$  is finite to one. Given a spectral datum  $c$  for  $(G, N)$ , the space  $V(N, c)$  is the direct sum of  $m(\pi) \pi^{K(N)K_H K_\infty}$  over all  $\pi$  in the fiber over  $c$ .

Let  $\{c_i\}$  be an enumeration of the spectral data, and for each  $c_i$  write  $V_i = V(N, c_i)$  and  $\xi_i = \xi(c_i)$ . The automorphic period  $\mathcal{P}_H$  defines a linear functional on each finite dimensional vector space  $V_i$ , and its kernel is of codimension at most 1. If the kernel is codimension 1 we let  $\phi_i \in V_i$  be a unit normal to it, and otherwise choose  $\phi_i \in V_i$  to be an arbitrary unit vector. We may complete this set of vectors to an orthonormal basis of Hecke-Maass forms for  $L^2(Y_N)$ , to which we apply (20) and (21). Since  $\mathcal{H}$  acts as a scalar on  $V_i$  we may write the left-hand side of (20) as

$$\text{Vol}_N \sum_{\substack{c_i \\ \|\xi_i - \xi\| \leq Q}} |\widehat{\omega}_S(\phi_i)|^2 h_\xi(\xi_i) \dim V_i,$$

and the left-hand side of (21) as

$$\text{Vol}_N \sum_{\substack{c_i \\ \|\xi_i - \xi\| \leq Q}} |\widehat{\omega}_S(\phi_i)|^2 |\mathcal{P}_H(\phi_i)|^2 h_\xi(\xi_i).$$

We obtain Theorem 1.2 by comparing the right-hand sides of (20) and (21) as before.  $\square$

## 7. BOUNDS FOR REAL ORBITAL INTEGRALS

The aim of this section is to establish uniform bounds on real orbital integrals that were used in the proof of the global bounds of Proposition 5.3.

**7.1. Notation.** We adopt the following notation in this section.

- $G$  is a (Zariski-) connected reductive group over  $\mathbb{R}$  with real Lie algebra  $\mathfrak{g}$ .
- $\theta$  is a Cartan involution of  $G$
- $K$  is the fixed point set of  $\theta$ , so that  $K$  is a maximal compact subgroup of  $G$ .
- $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$  is the Cartan decomposition associated to  $\theta$ .
- $A$  and  $A_G$  are maximal split tori in  $G$  and  $Z(G)$  respectively. We assume that  $\theta$  acts by  $-1$  on  $A$  and  $A_G$ .
- $A^0$  is the connected component of  $A$  in the real topology.
- $\mathfrak{a}$  and  $\mathfrak{a}_G$  are the Lie algebras of  $A$  and  $A_G$ .
- $W = N_G(A)/Z_G(A)$  is the Weyl group.
- $\mathfrak{a}^+$  is a choice of open Weyl chamber in  $\mathfrak{a}$ .

For a semisimple element  $\gamma \in G$  let  $G_\gamma$  be its centralizer. The Lie algebra of  $G_\gamma$  is denoted by  $\mathfrak{g}_\gamma$ . By [17, Thm 7.39],  $G$  has a Cartan decomposition  $G = KA^0K$ , and any  $g \in G$  may be written as  $g = k_1 e^H k_2$  for a unique  $H \in \mathfrak{a}^+$ . We use this to define a map  $X : G \rightarrow \mathfrak{a}^+$  by  $g \in Ke^{X(g)}K$ . Let  $\|\cdot\|$  be the norm on  $\mathfrak{a}/\mathfrak{a}_G$  obtained by restricting the Killing form, which we pull back to a function on  $\mathfrak{a}$ . We let  $D^G$  (or  $D$  if there is no confusion) denote the Weyl discriminant.

**7.2. The subgroup  $G_{\text{cpt}}$ .** It may be seen that Proposition 7.4 below cannot hold for all semisimple  $\gamma \in G$ . For instance, if  $\gamma$  lies in a compact normal subgroup of  $G$  then the conjugacy class of  $\gamma$  will be contained in  $K$ , and the function  $f\|X(\cdot)\|^{-\eta}$  will be singular everywhere on the conjugacy class. We now define a subgroup of elements of  $G$  that we will exclude, denoted  $G_{\text{cpt}}$ . Note that in the following discussion we take care to distinguish between an algebraic group and its real points.

Let  $G^{\text{ad}}$  be the adjoint form of  $G$ , so that there is an exact sequence  $Z(G) \rightarrow G \xrightarrow{\pi} G^{\text{ad}}$  of algebraic  $\mathbb{R}$ -groups. The group  $G^{\text{ad}}$  breaks up into a product of  $\mathbb{R}$ -almost simple factors. We let  $G_{\text{cpt}}^{\text{ad}}$  and  $G_{\text{nc}}^{\text{ad}}$  be the product of the compact and noncompact factors respectively. It may be seen that  $G_{\text{cpt}}^{\text{ad}}(\mathbb{R})$  is the maximal compact normal subgroup of  $G^{\text{ad}}(\mathbb{R})$ . We define  $G_{\text{cpt}} = \pi^{-1}(G_{\text{cpt}}^{\text{ad}}(\mathbb{R})) \subset G(\mathbb{R})$ . We record two properties of  $G_{\text{cpt}}$  for later use.

**Lemma 7.1.** *Let  $\mathfrak{g}_{\text{nc}}$  be the product of the  $\mathbb{R}$ -simple factors of  $\mathfrak{g}$  of noncompact type. Then if  $g \in G(\mathbb{R})$ ,  $\text{Ad}(g)$  is trivial on  $\mathfrak{g}_{\text{nc}}$  if and only if  $g \in G_{\text{cpt}}$ .*

Indeed, it is clear that if  $g \in G^{\text{ad}}(\mathbb{R})$ , then  $\text{Ad}(g)$  is trivial on  $\mathfrak{g}_{\text{nc}}$  if and only if  $g \in G_c(\mathbb{R})$ . The lemma follows because  $\pi$  commutes with the adjoint action on  $\mathfrak{g}_{\text{nc}}$ . The following is also clear.

**Lemma 7.2.** *If  $Z(G)$  is anisotropic, then  $G_{\text{cpt}}$  is a compact normal subgroup of  $G(\mathbb{R})$ .*

**7.3. Statement of results and sketch of proof.** The aim of this section is to prove the following two propositions bounding  $O_\gamma(f)$  for semisimple  $\gamma \in G$ . We shall prove them together, by induction on the semisimple rank of  $G$ .

**Proposition 7.3.** *There is  $A > 0$  depending only on  $G$  with the following property. Let  $f \in C(G)$  be bounded and compactly supported modulo center. Then we have  $O_\gamma(f) \ll_f D(\gamma)^{-A}$  for every semisimple  $\gamma \in G$ .*

**Proposition 7.4.** *Suppose that the semisimple rank of  $G$  is at least 1. There exists  $A > 0$  depending only on  $G$  with the following property. Let  $0 < \eta < 1/2$  and let  $f \in C(G)$  be bounded and compactly supported modulo center. Then there is a constant  $c(\eta, f) > 0$  such that*

$$(22) \quad O_\gamma(f \|X(\cdot)\|^{-\eta}) < c(\eta, f) D(\gamma)^{-A}$$

for every semisimple  $\gamma \in G - G_{\text{cpt}}$ .

We sketch the proof of Propositions 7.3 and 7.4; complete details will be given in the subsequent paragraphs. We use an induction argument with three steps, based on the general approach of [31, §7].

*Step 1:* We reduce to the case where  $Z(G)$  is anisotropic. This is simple; if  $A_G$  is the maximal split torus in  $Z(G)$ , we simply push the orbital integrals forward to  $G/A_G$ .

To describe the remaining steps, we begin with a definition.

**Definition 7.5.** *We say that  $\gamma \in G$  is elliptic if  $Z(G_\gamma)$  is anisotropic.*

*Step 2:* If  $\gamma$  is not elliptic, we may choose a nontrivial split torus  $S \subset Z(G_\gamma)$  and define  $M$  to be the centraliser of  $S$  in  $G$ . Because we have assumed that  $Z(G)$  is anisotropic,  $M$  is a proper Levi subgroup satisfying  $G_\gamma \subset M$ , and we may apply parabolic descent to  $O_\gamma(\phi)$ .

*Step 3:* If  $\gamma$  is elliptic, we may assume without loss of generality that  $\gamma \in K$ . For  $\epsilon > 0$ , define  $K(\epsilon) = \{g \in G : \|X(g)\| < \epsilon\}$ . Proving the bound (22) is roughly equivalent to controlling the volume of the conjugacy class of  $\gamma$  that lies inside  $K(\epsilon)$ , uniformly for  $\epsilon$  and  $D(\gamma)$  small. This turns out to be equivalent to bounding the set of points in  $G/K$  that are

moved distance at most  $\epsilon$  by the rotation  $\gamma$ . Note that this set will be noncompact if  $I_\gamma$  is noncompact. We solve this problem by writing the metric on  $G/K$  in polar co-ordinates. A key point is that the component of the metric in the angular variables grows exponentially.

**7.4. Reduction to the case of  $Z(G)$  anisotropic.** Let  $A_G$  denote the maximal split torus in  $Z(G)$ , and let  $\overline{G} = G/A_G$ . We wish to show that if Propositions 7.3 and 7.4 hold for  $\overline{G}$  then they hold for  $G$ . We have an exact sequence  $1 \rightarrow A_G \rightarrow G \rightarrow \overline{G} \rightarrow 1$  of algebraic groups over  $\mathbb{R}$ , and by Hilbert 90 this gives an exact sequence on points. If  $\gamma \in G$ , we denote its image in  $\overline{G}$  by  $\overline{\gamma}$ . If  $\overline{G}_{\text{cpt}}$  is the subgroup of  $\overline{G}$  defined as in Section 7.2, we have  $\overline{C}_c = G_{\text{cpt}}/A_G$  and so  $\overline{\gamma}$  still satisfies hypotheses of Proposition 7.4. We denote the connected centraliser of  $\overline{\gamma}$  by  $\overline{I}_{\overline{\gamma}}$ . There is an exact sequence  $1 \rightarrow A_G \rightarrow I_\gamma \rightarrow \overline{I}_{\overline{\gamma}} \rightarrow 1$ , and so  $G \rightarrow \overline{G}$  induces a bijection  $I_\gamma \backslash G \simeq \overline{I}_{\overline{\gamma}} \backslash \overline{G}$ . Moreover, this bijection carries  $\mu_G^{\text{can}}/\mu_{I_\gamma}^{\text{can}}$  to  $\mu_{\overline{G}}^{\text{can}}/\mu_{\overline{I}_{\overline{\gamma}}}^{\text{can}}$ . It follows that if  $f \in C(G)$  is invariant under  $A_G$ , and we let  $\overline{f}$  be its image in  $C(\overline{G})$ , we have  $O_\gamma^G(f) = O_{\overline{\gamma}}^{\overline{G}}(\overline{f})$ .

Let  $f \in C(G)$  be bounded and compactly supported modulo center. We may assume without loss of generality that  $f \geq 0$ , by replacing  $f$  by its absolute value. We may choose  $h \in C(G)$  that is invariant under  $A_G$  and satisfies  $h \geq f$ , and let  $\overline{h}$  be the image of  $h$  in  $C(\overline{G})$ . We have  $O_\gamma^G(f) \leq O_\gamma^G(h) = O_{\overline{\gamma}}^{\overline{G}}(\overline{h})$ , and likewise for  $f\|X(\cdot)\|^{-\eta}$ . Because  $D(\gamma) = D(\overline{\gamma})$ , if Propositions 7.3 and 7.4 hold for  $\overline{G}$  then they hold for  $G$ .

We may henceforth assume that  $Z(G)$  is isotropic. We may then take the function  $f$  of Propositions 7.3 and 7.4 to lie in  $C_c(G)$ . As in Section 5, we write  $\mu_\gamma = \mu_G^{\text{can}}/\mu_{I_\gamma}^{\text{can}}$ .

*Remark 7.6.* From this point on, we will only ever invoke Proposition 7.3 in our induction. As a result, there is no need to worry about the extra hypotheses of Proposition 7.4 holding after reduction to a parabolic subgroup.

**7.5. Parabolic descent.** We handle the case when  $\gamma$  is not elliptic by the process of parabolic descent, which we now recall. Let  $P = MN$  be a semi-standard parabolic subgroup of  $G$ , so that  $A \subset M$ . Choose Haar measures on  $N$  and  $K$  so that  $dk$  gives  $K$  measure 1, and we have  $d\mu_G^{\text{can}} = d\mu_M^{\text{can}} dndk$  in Langlands  $MNK$  co-ordinates. The parabolic descent along  $P$  is defined by

$$f \in L^1(G) \mapsto f^P \in L^1(M),$$

where

$$f^P(m) = \delta_P^{1/2}(m) \int_N \int_K f(k^{-1}mnk) dndk.$$

If  $\gamma \in M$ , we define  $D_M^G(\gamma)$  by choosing a maximal torus  $T \subset M$ , letting  $\Delta$  and  $\Delta_M$  be the roots of  $T$  in  $G$  and  $M$ , and setting  $D_M^G(\gamma) = \prod_{\alpha \in \Delta - \Delta_M} |\alpha(\gamma) - 1|$ . It may be seen that this is independent of the choice of  $T$ . We say that  $\gamma \in M$  is  $(G, M)$ -regular if  $D_M^G(\gamma) \neq 0$ . We recall the descent relation between the orbital integrals of  $f$  and  $f^P$ .

**Lemma 7.7.** *If  $\gamma \in M$  is  $(G, M)$ -regular and  $f \in C_c(G)$ , we have*

$$D_M^G(\gamma)^{1/2} O_\gamma(f) = O_\gamma^M(f^P).$$

*Proof.* Because  $\gamma$  is  $(G, M)$  regular, we have  $I_\gamma \subset M$ . Let  $\mu_\gamma^M = \mu_M^{\text{can}}/\mu_{I_\gamma}^{\text{can}}$ . We may parametrize  $I_\gamma \backslash G$  in Langlands co-ordinates as  $(I_\gamma \backslash M)NK$ , which allows us to rewrite  $O_\gamma(f)$  as follows.

$$\begin{aligned}
O_\gamma(f) &= \int_{I_\gamma \backslash G} f(x^{-1}\gamma x) d\mu_\gamma(x) \\
&= \int_{I_\gamma \backslash M} \int_N \int_K f(k^{-1}n^{-1}m^{-1}\gamma mnk) d\mu_\gamma^M(m) dndk \\
&= D_M^G(\gamma)^{-1/2} \delta_P(\gamma)^{1/2} \int_{I_\gamma \backslash M} \int_N \int_K f(k^{-1}m^{-1}\gamma mnk) d\mu_\gamma^M(m) dndk \\
&= D_M^G(\gamma)^{-1/2} O_\gamma^M(f^P).
\end{aligned}$$

□

We shall need a version of Lemma 7.7 that can be applied to the singular functions  $f\|X(\cdot)\|^{-\eta}$ , which we now derive. Define the positive definite inner product  $\langle \cdot, \cdot \rangle_+$  on  $[\mathfrak{g}, \mathfrak{g}]$  by  $\langle u, v \rangle_+ = -\langle \theta u, v \rangle$ . If  $A \in GL([\mathfrak{g}, \mathfrak{g}])$ , define  ${}^tA$  to be the transpose with respect to  $\langle \cdot, \cdot \rangle_+$ . Following [21, §3.2], we define the functions  $\mathcal{L}, \mathcal{N} : GL([\mathfrak{g}, \mathfrak{g}]) \rightarrow \mathbb{R}$  by  $\mathcal{L}(A) = \log(\text{tr}(A^t A)/\dim \mathfrak{g})$  and  $\mathcal{N}(A) = \text{tr}(A^t A)/\dim \mathfrak{g}$ .

**Lemma 7.8.** *We have  $\mathcal{L}(\text{Ad}(g)) \leq 2\|X(g)\|$  for all  $g \in G$ .*

*Proof.* We follow [21, Lemma 3.2]. It may be seen that  ${}^t\text{Ad}(g) = \text{Ad}(\theta(g))^{-1}$  for  $g \in G$ . We write  $g \in G$  as  $k_1 e^{X(g)} k_2$ , so that  $\text{Ad}(g) {}^t\text{Ad}(g) = \text{Ad}(g\theta(g)^{-1}) = \text{Ad}(k_2^{-1} e^{2X(g)} k_2)$ . Taking traces gives

$$\mathcal{N}(\text{Ad}(g)) = \text{tr}(\text{Ad}(e^{2X(g)}))/\dim \mathfrak{g} \leq e^{2\|X(g)\|},$$

and taking logs completes the proof.

□

**Lemma 7.9.** *Let  $B_M \subset M$  and  $B_{\mathfrak{n}} \subset \mathfrak{n}$  be compact. There is  $C > 0$  depending on  $B_M$  and  $B_{\mathfrak{n}}$  such that for all  $m \in B_M$ ,  $V \in B_{\mathfrak{n}}$ , we have  $\mathcal{N}(\text{Ad}(me^V)) \geq 1 + C\|V\|^2$ .*

*Proof.* Choose a basis for  $[\mathfrak{g}, \mathfrak{g}]$  subordinate to the root space decomposition  $[\mathfrak{g}, \mathfrak{g}] = (Z_{\mathfrak{g}}(\mathfrak{a}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$  that is orthonormal with respect to  $\langle \cdot, \cdot \rangle_+$ , where  $\Delta$  are the roots of  $\mathfrak{a}$  in  $\mathfrak{g}$ . It may be seen that the nonzero entries of  $\text{Ad}(m)$  and  $\text{Ad}(m)(\text{Ad}(e^V) - 1)$  with respect to this basis are disjoint, so we have

$$\mathcal{N}(\text{Ad}(me^V)) = \mathcal{N}(\text{Ad}(m)) + \mathcal{N}(\text{Ad}(m)(\text{Ad}(e^V) - 1)) \geq 1 + \mathcal{N}(\text{Ad}(m)(\text{Ad}(e^V) - 1)).$$

It follows that if  $\mathcal{N}(\text{Ad}(me^V)) = 1$  then  $\mathcal{N}(\text{Ad}(m)(\text{Ad}(e^V) - 1)) = 0$ , so  $V = 0$ . By a compactness argument, we may therefore assume that  $\|V\| \leq 1$ . By applying the Taylor expansion of  $e^V$  and the fact that  $\mathcal{N}$  is a quadratic form, we have

$$\mathcal{N}(\text{Ad}(m)(\text{Ad}(e^V) - 1)) = \mathcal{N}(\text{Ad}(m)\text{ad}(V)) + O_{B_M}(\|V\|^3).$$

By compactness, we have  $\mathcal{N}(\text{Ad}(m)\text{ad}(V)) \geq C$  for all  $\|V\| = 1$  and  $m \in B_M$ . As  $\mathcal{N}$  is quadratic, we have  $\mathcal{N}(\text{Ad}(m)\text{ad}(V)) \geq C\|V\|^2$  for all  $V$  and  $m \in B_M$ , which completes the proof.

□

**Lemma 7.10.** *Let  $P \subsetneq G$  be semi-standard. Let  $f \in C_c(G)$ , and for  $0 < \eta < 1/2$  define  $F_\eta = f\|X(\cdot)\|^{-\eta}$ . We have*

- (i)  $F_\eta \in L^1(G)$ , and the parabolic descent  $F_\eta^P(m)$  is well defined and finite for any  $m \in M$ .
- (ii) There exists  $f_1 \in C_c(M)$  depending only on  $f$  and  $\eta$  such that  $|F_\eta^P(m)| \leq f_1(m)$  for all  $m \in M$ .

*Proof.* We are free to replace  $f$  by its absolute value, so that  $f \geq 0$ , and assume that  $f$  is bi- $K$ -invariant. The claim that  $F_\eta \in L^1(G)$  follows from the Cartan decomposition. Let  $B \subset G$  be a compact set containing  $\text{supp}(f)$ . There are compact sets  $B_M \subset M$  and  $B_{\mathfrak{n}} \subset \mathfrak{n}$  depending only on  $B$  such that  $me^V \in B$  implies  $m \in B_M$  and  $V \in B_{\mathfrak{n}}$ . It follows that  $\text{supp}(F_\eta^P) \subset B_M$ .

Lemma 7.8 implies

$$F_\eta^P(m)\delta_P^{-1/2}(m) = \int_N f(mn)\|X(mn)\|^{-\eta}dn \ll \int_N f(mn)\mathcal{L}(\text{Ad}(mn))^{-\eta}dn.$$

Writing  $n = e^V$  and applying Lemma 7.9 gives

$$F_\eta^P(m)\delta_P^{-1/2}(m) \ll \int_{\mathfrak{n}} f(me^V)\|V\|^{-2\eta}dV.$$

The right-hand side is bounded by  $C(B, \eta)\|f\|_\infty$ . This proves (i), and (ii) follows by combining this with  $\text{supp}(F_\eta^P) \subset B_M$ .  $\square$

**Lemma 7.11.** *Let  $f \in C_c(G)$ , and for  $0 < \eta < 1/2$  define  $F_\eta = f\|X(\cdot)\|^{-\eta}$ . If  $\gamma \in M$  is  $(G, M)$  regular, then the integral  $O_\gamma(F_\eta)$  converges absolutely, and we have*

$$(23) \quad D_M^G(\gamma)^{1/2}O_\gamma(F_\eta) = O_\gamma^M(F_\eta^P).$$

*Proof.* We first assume that  $f \geq 0$ . Define  $\phi_\epsilon$  by

$$\phi_\epsilon(y) = \begin{cases} f(y)\|X(y)\|^{-\eta}, & \|X(y)\| > \epsilon, \\ f(y)\epsilon^{-\eta}, & \|X(y)\| \leq \epsilon. \end{cases}$$

The monotone convergence theorem implies that  $O_\gamma(\phi_\epsilon) \rightarrow O_\gamma(F_\eta)$  and  $\phi_\epsilon^P \rightarrow F_\eta^P$  monotonically as  $\epsilon \rightarrow 0$ . Another application of monotone convergence gives  $O_\gamma^M(\phi_\epsilon^P) \rightarrow O_\gamma^M(F_\eta^P)$ . Lemma 7.7 then implies that  $O_\gamma(\phi_\epsilon) = D_M^G(\gamma)^{-1/2}O_\gamma^M(\phi_\epsilon^P)$  converges to  $D_M^G(\gamma)^{-1/2}O_\gamma^M(F_\eta^P)$ , which from Lemma 7.10 is finite. Therefore  $O_\gamma(F_\eta)$  converges absolutely and satisfies (23).

In the general case, define  $\tilde{f} = |f|$ , and let  $\tilde{F}_\eta$  and  $\tilde{F}_\eta^P$  be the corresponding functions on  $G$  and  $M$ . Then  $\tilde{F}_\eta$  and  $\tilde{F}_\eta^P$  dominate  $F_\eta$  and  $F_\eta^P$ , and the integrals  $O_\gamma(\tilde{F}_\eta)$  and  $O_\gamma^M(\tilde{F}_\eta^P)$  are finite. We may then repeat the argument in the case  $f \geq 0$ , with monotone convergence replaced with dominated convergence.  $\square$

**7.6. The case when  $\gamma$  is not elliptic.** We only prove Proposition 7.4, as Proposition 7.3 follows by a similar argument. Let  $0 < \eta < 1/2$ , and let  $f \in C_c(G)$  and  $F_\eta = f\|X(\cdot)\|^{-\eta}$ . We assume that  $f \geq 0$ . Assume that  $\gamma \in G$  is not elliptic, and let  $M$  be a proper Levi subgroup with  $G_\gamma \subset M$ . By conjugation, we may assume that  $M$  is standard. Because  $G_\gamma \subset M$ ,



$\text{Ad}(\gamma) - 1$  must be invertible on  $\mathfrak{g}/\text{Lie}(M)$ , so that  $\gamma$  is  $(G, M)$ -regular. Apply Lemma 7.10 to  $F_\eta$  to obtain  $f_1 \in C_c(M)$  such that  $|F_\eta^P(m)| \leq f_1(m)$  for all  $m \in M$ . Lemma 7.11 gives

$$O_\gamma^G(F_\eta) = (D_M^G(\gamma))^{-1/2} O_\gamma^M(F_\eta^P) \leq (D_M^G(\gamma))^{-1/2} O_\gamma^M(f_1).$$

Because the semisimple rank of  $M$  is smaller than that of  $G$ , we may apply Proposition 7.3 on  $M$  to obtain  $O_\gamma^M(f_1) < c(\eta, f) D^M(\gamma)^{-A}$ . Combining these gives

$$O_\gamma^G(F_\eta) < c(\eta, f) (D_M^G(\gamma))^{-1/2} D^M(\gamma)^{-A} < c'(\eta, f) D^G(\gamma)^{-A},$$

which completes the proof.

**7.7. The case of  $\gamma$  elliptic.** We first observe that, if  $\gamma \in G$  is elliptic, then it is conjugate to an element of  $K$ . This is because if  $Z(G_\gamma)$  is compact, then  $\gamma$  lies in the compact group  $Z(G_\gamma)$ , which is conjugate to a subgroup of  $K$ . We shall assume that  $\gamma \in K$  for the rest of Section 7.

We next prove Proposition 7.3 in the case  $\gamma \in G_{\text{cpt}}$ . By Lemma 7.2,  $G_{\text{cpt}}$  is a compact normal subgroup of  $G$ . If  $\gamma \in G_{\text{cpt}}$ , the conjugacy class of  $\gamma$  is a closed subset of  $G_{\text{cpt}}$ , hence compact and of finite volume. We therefore have

$$O_\gamma(f) \leq \|f\|_\infty \mu_\gamma(I_\gamma \backslash G).$$

Because there are only finitely many possibilities for  $I_\gamma$  up to conjugacy, we have  $\mu_\gamma(G/I_\gamma) \ll 1$ .

We shall assume  $\gamma \in K - G_{\text{cpt}}$  for the rest of Section 7. Because  $\theta(\gamma) = \gamma$ ,  $\mathfrak{g}_\gamma$  is  $\theta$ -stable. We may then write  $\mathfrak{g}_\gamma = \mathfrak{p}_\gamma + \mathfrak{k}_\gamma$  where  $\mathfrak{k}_\gamma = \mathfrak{k} \cap \mathfrak{g}_\gamma$  and  $\mathfrak{p}_\gamma = \mathfrak{p} \cap \mathfrak{g}_\gamma$ . Let  $\mathfrak{p}_\gamma^\perp$  be the orthocomplement of  $\mathfrak{p}_\gamma$  in  $\mathfrak{p}$ . We note that Lemma 7.2 implies that  $\mathfrak{p}_\gamma \neq \mathfrak{p}$ . Indeed, if  $\mathfrak{p}_\gamma = \mathfrak{p}$  then  $\text{Ad}(\gamma)$  fixes  $\mathfrak{p}$ , and hence  $[\mathfrak{p}, \mathfrak{p}]$ . However,  $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$  is the product of the  $\mathbb{R}$ -simple factors of  $\mathfrak{g}$  of noncompact type, see [10, Ch. V Thm 1.1] and the subsequent proof.

We next convert the problem to one on the symmetric space  $S = G/K$ . We extend  $\|\cdot\|$  to all of  $\mathfrak{p}$  by setting it equal to the norm induced by the Killing form. Then  $ds$  will denote the associated metric tensor on  $S$ . We note that the distance function  $d_S$  attached to  $ds$  is given by  $d_S(g_1, g_2) = \|X(g_1^{-1}g_2)\|$ . For  $\epsilon > 0$  we put  $K(\epsilon) = \{g \in G : \|X(g)\| < \epsilon\}$  and write  $1_{K(\epsilon)}$  for the characteristic function of  $K(\epsilon)$ .

To set up Proposition 7.3, we let  $f \in C_c(G)$  and take  $\epsilon_f > 0$  to be such that  $\text{supp}(f) \subset K(\epsilon_f)$ ; it follows that

$$(24) \quad |O_\gamma(f)| \leq O_\gamma(1_{K(\epsilon_f)}) \|f\|_\infty.$$

For Proposition 7.4, we may assume that  $f \geq 0$ , and choose  $f_0 \in C_c(G)$  with  $\text{supp}(f_0) \subset K(\epsilon_f)$  such that

$$f \|X(\cdot)\|^{-\eta} \ll_f f_0 + \sum_{k=1}^{\infty} 2^{\eta k} 1_{K(2^{-k})}.$$

We deduce that

$$O_\gamma(f \|X(\cdot)\|^{-\eta}) \ll_f O_\gamma(1_{K(\epsilon_f)}) + \sum_{k=1}^{\infty} 2^{\eta k} O_\gamma(1_{K(2^{-k})}).$$

In either case, it therefore suffices to bound  $O_\gamma(1_{K(\epsilon)})$  uniformly in  $\epsilon > 0$  and  $\gamma \in K - G_{\text{cpt}}$ . We note that for  $\gamma \in K - G_{\text{cpt}}$  we have  $D(\gamma) \ll 1$ , where the implied constant depends only on  $G$ .

**Lemma 7.12.** *There is  $A > 0$  such that for all  $\gamma \in K - G_{\text{cpt}}$ ,  $T \geq 1$ , and  $0 < \epsilon < T$ , we have*

$$O_\gamma(1_{K(\epsilon)}) \ll_T \begin{cases} \epsilon^{d_\gamma} D(\gamma)^{-d_\gamma}, & \text{if } \epsilon \leq 2D(\gamma), \\ \epsilon^A D(\gamma)^{-A}, & \text{if } \epsilon > 2D(\gamma), \end{cases}$$

where  $d_\gamma \geq 1$  is the codimension of  $I_\gamma K$  in  $G$ .

From Lemma 7.12 it follows that  $O_\gamma(1_{K(\epsilon)}) \ll_\epsilon D(\gamma)^{-A}$  for all  $\epsilon$ ; when combined with (24), this proves Proposition 7.3 for elliptic  $\gamma$ .

To complete the proof of Proposition 7.4 we apply Proposition 7.3 and Lemma 7.12 to deduce that

$$O_\gamma(f \|X(\cdot)\|^{-\eta}) \ll_f D(\gamma)^{-A} + \sum_{2^{k+1} \geq D(\gamma)^{-1}} 2^{\eta k - k} D(\gamma)^{-1} + \sum_{2^{k+1} < D(\gamma)^{-1}} 2^{\eta k - Ak} D(\gamma)^{-A}.$$

Our assumption that  $0 < \eta < 1/2$  implies that both geometric series are bounded by  $D(\gamma)^{-A}$ . This gives

$$O_\gamma(f \|X(\cdot)\|^{-\eta}) < c(\eta, f) D(\gamma)^{-A},$$

as desired.

*Proof of Lemma 7.12.* We first observe that  $O_\gamma(1_{K(\epsilon)}) = \mu_\gamma(I_\gamma \backslash G(\epsilon))$ , where  $G(\epsilon) = \{x \in G : x^{-1}\gamma x \in K(\epsilon)\}$ . The set  $G(\epsilon)$  is right  $K$ -invariant, and  $G(\epsilon)/K$  is the set of points  $x \in S$  such that  $d_S(x, \gamma x) < \epsilon$ . Moreover,  $G(\epsilon)$  is left  $I_\gamma$ -invariant, and we will see it is roughly a tube around  $I_\gamma$ . Bounding the volume of  $I_\gamma \backslash G(\epsilon)$  is therefore roughly equivalent to finding the radius of this tube, in a way which we now make precise.

Let  $x \in G(\epsilon)$ . By [10, Chapter VI, Theorem 1.4], we may write  $x = e^{X_\gamma} e^{X^\gamma} k$  with  $X_\gamma \in \mathfrak{p}_\gamma$ ,  $X^\gamma \in \mathfrak{p}_\gamma^\perp$ , and  $k \in K$ . The condition  $x^{-1}\gamma x \in K(\epsilon)$  simplifies to  $e^{-X^\gamma} \gamma e^{X^\gamma} \in K(\epsilon)$ , which (since  $\gamma \in K$ ) is equivalent to  $e^{-X^\gamma} e^{\gamma \cdot X^\gamma} \in K(\epsilon)$ . This implies that  $d_S(e^{X^\gamma}, e^{\gamma \cdot X^\gamma}) < \epsilon$ : the element  $e^{X^\gamma}$  is rotated by  $\gamma$  by distance at most  $\epsilon$ . Proposition 8.1 below then states  $\|X^\gamma\| \leq r_{\gamma, \epsilon}$ , for an expression  $r_{\gamma, \epsilon} > 0$  depending on  $\gamma$  and  $\epsilon$ . If we let  $B_{\mathfrak{p}_\gamma^\perp}(r)$  be the ball of radius  $r$  around 0 in  $\mathfrak{p}_\gamma^\perp$  with respect to  $\|\cdot\|$ , it follows that  $G(\epsilon) \subset I_\gamma \exp(B_{\mathfrak{p}_\gamma^\perp}(r_{\gamma, \epsilon}))K$ .

Let  $I_\gamma^c \subset I_\gamma$  be a compact set such that  $\mu_{I_\gamma^c}^{\text{can}}(I_\gamma^c) = 1$ . Note that we may assume that  $\gamma$  lies in a fixed maximal torus in  $K$ , so that there are only finitely many possibilities for  $I_\gamma$  and we may ignore the dependence of our bounds on  $I_\gamma^c$  in what follows. We have

$$\mu_\gamma(I_\gamma \backslash G(\epsilon)) \leq \mu_G(I_\gamma^c \exp(B_{\mathfrak{p}_\gamma^\perp}(r_{\gamma, \epsilon}))K).$$

We now bound  $\mu_G(I_\gamma^c \exp(B_{\mathfrak{p}_\gamma^\perp}(r_{\gamma, \epsilon}))K)$ , using the radius estimates of Proposition 8.1 below. We first take  $\epsilon \leq 2D(\gamma)$ . Let  $B_{\mathfrak{p}}(r)$  denote the ball in  $\mathfrak{p}$  of radius  $r$  with respect to  $\|\cdot\|$ . We have

$$I_\gamma^c \exp(B_{\mathfrak{p}_\gamma^\perp}(r_{\gamma, \epsilon}))K \subset I_\gamma^c \exp(B_{\mathfrak{p}}(r_{\gamma, \epsilon}))K = I_\gamma^c K \exp(B_{\mathfrak{p}}(r_{\gamma, \epsilon})).$$

Proposition 8.1 gives  $r_{\gamma, \epsilon} = C\epsilon D(\gamma)^{-1}$  for some  $C > 0$ . Since  $\gamma \notin G_{\text{cpt}}$  we have  $\mathfrak{p}_\gamma \neq \mathfrak{p}$ , so that the codimension  $d_\gamma$  of  $I_\gamma K$  in  $G$  is at least 1. Because  $I_\gamma^c K$  is compact and contained in  $I_\gamma K$ , the result follows.

For the remaining range, we will reduce the problem to well-known volume estimates for expanding balls in the symmetric space  $S$ . Let  $B_G(r) = \exp(B_{\mathfrak{p}}(r))K$  be the ball of radius  $r$  around the identity in  $G$  with respect to  $d_S$ . Then

$$I_\gamma^c \exp(B_{\mathfrak{p}_\gamma^\perp}(r))K \subset I_\gamma^c \exp(B_{\mathfrak{p}}(r))K = I_\gamma^c B_G(r).$$

We also have  $I_\gamma^c B_G(r) \subset B_G(r + C_1)$  for some  $C_1 > 0$ . If  $2D(\gamma) < \epsilon < T$ , Proposition 8.1 gives  $r_{\gamma,\epsilon} = C \log(\epsilon D(\gamma)^{-1})$  for some  $C > 0$ . If  $r \gg 1$ , we have  $\mu_G(B_G(r)) \ll e^{C_2 r}$  for some  $C_2 > 0$ . We deduce that  $\mu_G(I_\gamma^c B_G(r_{\gamma,\epsilon}))$  is bounded by  $\epsilon^A D(\gamma)^{-A}$  for some  $A > 0$ , as desired.  $\square$

## 8. RADIUS BOUNDS ON TUBES

This section is devoted to the proof of the following result, used in the proof of Lemma 7.12.

**Proposition 8.1.** *Let  $\gamma \in K - G_{\text{cpt}}$ . If  $V \in \mathfrak{p}_\gamma^\perp$ ,  $T \geq 1$ , and  $d_S(e^V, e^{\gamma \cdot V}) < \epsilon < T$ , then*

$$\|V\| \ll_T \begin{cases} \epsilon D(\gamma)^{-1}, & \text{if } \epsilon D(\gamma)^{-1} \leq 2 \\ \log(\epsilon D(\gamma)^{-1}), & \text{if } \epsilon D(\gamma)^{-1} > 2. \end{cases}$$

We shall prove Proposition 8.1 by expressing  $d_S$  in polar co-ordinates. To deal with the singularities of the Cartan decomposition, we will introduce a system of polar co-ordinates for every conjugacy class of Levi subgroups in  $G$ .

**8.1. Notation.** We let  $\|\cdot\|$  denote the norms on either  $\mathfrak{p}$  or  $\mathfrak{k}$  obtained by restricting the Killing form (resp. minus the Killing form). Let  $\Delta$  be the roots of  $\mathfrak{a}$  in  $\mathfrak{g}$ . Let  $A \subset P_0$  be a minimal parabolic subgroup of  $G$ . Let  $\Delta^+$  be the set of positive roots corresponding to  $P_0$ . We may assume that the positive Weyl chamber  $\mathfrak{a}^+$  chosen in Section 7.1 is the one corresponding to  $\Delta^+$ . Let  $\Phi \subset \Delta^+$  be the simple roots.

We say that a parabolic subgroup  $P$  is standard if  $P_0 \subset P$ , and that a Levi subgroup  $L$  of a parabolic  $P$  is standard if  $P$  is standard and  $A \subset L$ . Let  $\mathcal{L}$  be the set of proper standard Levi subgroups of  $G$ . Then  $\mathcal{L}$  contains exactly one representative of every conjugacy class of proper Levi subgroups of  $G$ .  $\mathcal{L}$  is in bijection with the proper subsets of  $\Phi$ , and we now recall this correspondence. If  $\Phi_L \subset \Phi$ , we define  $A_L \subset A$  to be the neutral component of the kernel of all  $\alpha \in \Phi_L$ . The Levi  $L$  associated to  $\Phi_L$  is  $Z_G(A_L)$ , which is Zariski-connected by [3, Prop 18.4]. We introduce the following notation for  $L$ .

- $\mathfrak{l}$  is the Lie algebra of  $L$ .
- $K_L = K \cap L$ . It is known that  $K_L$  is a maximal compact subgroup of  $L$ , and that  $K_L$  meets every connected component of  $G$  in the real topology (see for instance [17, Prop. 7.25] and [17, Prop. 7.33]).
- $\mathfrak{l} = \mathfrak{k}_L + \mathfrak{p}_L$  is the Cartan decomposition corresponding to  $K_L$ .
- $\mathfrak{k}_L^\perp$  is the Killing-orthogonal complement of  $\mathfrak{k}_L$  in  $\mathfrak{k}$ .
- $\mathfrak{a}_L$  is the Lie algebra of  $A_L$ , which is the center of  $\mathfrak{l}$ .
- $\Delta_L$  is the set of roots vanishing on  $\mathfrak{a}_L$ .
- $\mathfrak{a}_L^+ = \{H \in \mathfrak{a}_L : \alpha(H) > 0 \text{ for } \alpha \in \Delta^+ - \Delta_L\}$ .
- $S_{\mathfrak{a}}$  is the unit sphere in  $\mathfrak{a}$ .
- $S_{\mathfrak{a}}^+ = S_{\mathfrak{a}} \cap \bar{\mathfrak{a}}^+$ .

- Let  $\kappa > 0$  be a constant to be chosen later. We define  $S_L^+ = \{H \in S_{\mathfrak{a}} \cap \mathfrak{a}_L^+ : \alpha(H) > \kappa \text{ for } \alpha \in \Delta^+ - \Delta_L\}$ .

**8.2. A system of approximating open sets.** For each  $L \in \mathcal{L}$ , we choose open sets  $U_L, U'_L \subset S_{\mathfrak{a}}^+$  that approximate  $S_L^+$  using the following lemma.

**Lemma 8.2.** *There exist two collections of open sets  $\{U_L \subset S_{\mathfrak{a}}^+ : L \in \mathcal{L}\}$ ,  $\{U'_L \subset S_{\mathfrak{a}}^+ : L \in \mathcal{L}\}$  (for the relative topology on  $S_{\mathfrak{a}}^+$ ) with the following properties.*

- (a)  $U'_L$  cover  $S_{\mathfrak{a}}^+$ .
- (b)  $\overline{U'_L} \subset U_L$ .
- (c)  $U_L$  lies in the ball of radius  $1/2$  around  $S_L^+$ .
- (d) If  $\alpha \notin \Delta_L$ ,  $\alpha$  does not vanish on  $\overline{U'_L}$ .

*Proof.* We let  $\delta > 0$  to be chosen later. We define  $U_L \subset S_{\mathfrak{a}}^+$  to be the subset of  $H$  satisfying

$$\begin{aligned} \alpha(H) &< \delta 2^{2|\Phi_L|+1}, & \alpha \in \Phi_L, \\ \alpha(H) &> \delta 2^{2|\Phi_L|}, & \alpha \notin \Phi_L, \end{aligned}$$

and  $U'_L \subset S_{\mathfrak{a}}^+$  to be the subset of  $H$  satisfying

$$\begin{aligned} \alpha(H) &< \delta 2^{2|\Phi_L|}, & \alpha \in \Phi_L, \\ \alpha(H) &> \delta 2^{2|\Phi_L|+1}, & \alpha \notin \Phi_L. \end{aligned}$$

It is immediate from the definition that  $\overline{U'_L} \subset U_L$ . Because  $S_L^+ \subset S_{\mathfrak{a}}^+$  is the subset of  $H$  satisfying

$$\begin{aligned} \alpha(H) &= 0, & \alpha \in \Phi_L, \\ \alpha(H) &> \kappa, & \alpha \notin \Phi_L, \end{aligned}$$

it is clear that (c) holds if  $\delta$  is chosen small enough and we let  $\kappa = \delta$ . For (d), let  $\alpha \notin \Delta_L$ . We may assume without loss of generality that  $\alpha \in \Delta^+$ . We then have  $\alpha = \sum_{\beta \in \Phi} n_{\beta} \beta$  with  $n_{\beta} \geq 0$  and  $n_{\beta} > 0$  for at least one  $\beta \notin \Phi_L$ . It is then clear that  $\alpha$  is nonzero on  $\overline{U'_L}$ .

It remains to prove (a). Let  $H \in S_{\mathfrak{a}}^+$ . Let  $\Phi_L \subset \Phi$  be a set such that

$$(25) \quad \alpha(H) < \delta 2^{2|\Phi_L|} \text{ for all } \alpha \in \Phi_L.$$

Note that at least one such set exists, namely  $\emptyset$ . Furthermore, suppose that  $\Phi_L$  is maximal with this property. Note that we cannot have  $\Phi_L = \Phi$  if  $\delta$  is chosen small enough. If  $\alpha(H) > \delta 2^{2|\Phi_L|+1}$  for all  $\alpha \notin \Phi_L$ , then  $H \in U'_L$ . If there is some  $\beta \notin \Phi_L$  such that  $\beta(H) \leq \delta 2^{2|\Phi_L|+1}$ , then  $\Phi_L \cup \{\beta\}$  also satisfies (25), which contradicts maximality.  $\square$

**8.3. A system of polar co-ordinates on  $\mathfrak{p}$ .** Let  $C_L = \{tH : t > 0, H \in U_L\}$  be the cone over  $U_L$ . We define  $\mathcal{C}_L^0 = \text{Ad}(K_L)C_L \subset \mathfrak{p}_L$  and  $\mathcal{C}_L = \text{Ad}(K)C_L \subset \mathfrak{p}$ , and define  $\mathcal{C}'_L, \mathcal{C}_L^{0'}$ , and  $\mathcal{C}'_L$  in the same way using  $U'_L$ . It follows that  $\mathcal{C}_L^0$  is open in  $\mathfrak{p}_L$ , and  $\mathcal{C}_L$  is open in  $\mathfrak{p}$ , and that  $\mathcal{C}'_L$  cover  $\mathfrak{p} \setminus \{0\}$ .

We recall the definition of the principal bundle  $K \times_{K_L} \mathfrak{p}_L$ , which is the quotient of  $K \times \mathfrak{p}_L$  by the action of  $K_L$  given by

$$k_L \cdot (k, V) = (kk_L^{-1}, k_L.V).$$

Define  $P_L = K \times_{K_L} \mathcal{C}_L^0 \subset K \times_{K_L} \mathfrak{p}_L$ . The natural map  $K \times \mathcal{C}_L^0 \rightarrow \mathcal{C}_L \subset \mathfrak{p}$  given by  $(k, V) \mapsto k.V$  factors to a map  $\Pi_L : P_L \rightarrow \mathcal{C}_L$ . We shall prove that  $\Pi_L$  is a diffeomorphism;

this will provide us with a system of polar coordinates on  $\mathcal{C}_L$  with angular variable in  $K/K_L$  and radial variable in  $\mathcal{C}_L^0$ . We shall then give a formula for the metric on  $S$  in these polar coordinates.

To lift our polar coordinate map to a map of tangent spaces, define  $B_L$  to be the quotient of  $K \times \mathcal{C}_L^0 \times \mathfrak{k}_L^\perp \times \mathfrak{p}_L$  by the action

$$k_L \cdot (k, V, X, Y) = (kk_L^{-1}, k_L.V, k_L.X, k_L.Y).$$

We may naturally view  $B_L$  as a vector bundle over  $P_L$  with fiber  $\mathfrak{k}_L^\perp \times \mathfrak{p}_L$ . There is a map  $d\Pi_L : B_L \rightarrow T\mathcal{C}_L$  given by  $d\Pi_L(k, V, X, Y) = (k.V, k.[X, V] + k.Y)$ .

**Proposition 8.3.** *The map  $\Pi_L : P_L \rightarrow \mathcal{C}_L$  is a diffeomorphism, and  $d\Pi_L : B_L \rightarrow T\mathcal{C}_L$  is an isomorphism of vector bundles.*

*Proof.* We first show that  $\Pi_L$  is a bijection. Surjectivity is clear, to prove injectivity suppose that  $(k_1, V_1), (k_2, V_2) \in K \times \mathcal{C}_L^0$  satisfy  $k_1.V_1 = k_2.V_2$ . By the definition of  $\mathcal{C}_L^0$ , we may write  $V_i = k_{i,L}H_i$  with  $k_{i,L} \in K_L$  and  $H_i \in \mathcal{C}_L \subset \bar{\mathfrak{a}}^+$ . Because  $H_1$  and  $H_2$  are conjugate under  $K$ , we have  $H_1 = H_2 = H$  (see e.g. [17, Lemma 7.38] for the proof of this in our case where the groups may be disconnected in the real topology). We then have  $k_1k_{1,L} = k_2k_{2,L}.H$ , so that  $k_1k_{1,L} \in k_2k_{2,L}K_H$  where  $K_H$  is the stabilizer of  $H$  in  $K$ . Because  $H \in \mathcal{C}_L$ , we have  $\alpha(H) \neq 0$  for all  $\alpha \notin \Delta_L$ , and Lemma 8.4 implies that  $k_2^{-1}k_1 \in K_L$ . It follows that  $(k_2, V_2) = (k_1, (k_1^{-1}k_2).V_2) = (k_1, V_1)$  in  $P_L$ .

Because  $P_L$  and  $\mathcal{C}_L$  are smooth manifolds of the same dimension, to show  $\Pi_L$  is smooth it suffices to show that its differential is surjective on tangent spaces everywhere. In addition, because  $\Pi_L$  is  $K$ -equivariant it suffices to check that this holds at a point  $(e, H)$  with  $H \in \mathcal{C}_L$ . Let  $X \in \mathfrak{k}_L^\perp$  and  $Y \in \mathfrak{p}_L$ , and consider the path  $(\exp(tX), H + tY)$  in  $P_L$ . Applying  $\Pi_L$  and differentiating at  $t = 0$  gives

$$\frac{d}{dt} \exp(tX).(H + tY)|_{t=0} = [X, H] + Y.$$

Because  $\alpha(H) \neq 0$  for all  $\alpha \notin \Delta_L$ ,  $\text{ad}(H)$  is an isomorphism from  $\mathfrak{k}_L^\perp$  to  $\mathfrak{p}_L^\perp$ , so the differential of  $\Pi_L$  is surjective.

To prove the claim about  $d\Pi_L$ , we know that it is a map of vector bundles that lies over a diffeomorphism. Moreover, the argument above implies that it is an isomorphism on each fiber, and so it must be an isomorphism.  $\square$

**Lemma 8.4.** *If  $H \in \mathfrak{a}$  satisfies  $\alpha(H) \neq 0$  for all  $\alpha \notin \Delta_L$ , and  $K_H$  is the stabilizer of  $H$  in  $K$ , we have  $K_H \subset K_L$ .*

*Proof.* Let  $k \in K_H$ . The algebras  $\mathfrak{a}$  and  $k.\mathfrak{a}$  are both contained in  $Z_{\mathfrak{g}}(H)$ . The root space decomposition and our assumption on  $H$  imply that  $Z_{\mathfrak{g}}(H) \subset \mathfrak{l}$ . Therefore  $\mathfrak{a}$  and  $k.\mathfrak{a}$  are two maximal abelian subalgebras in  $\mathfrak{p}_L$ , and so by [17, Lemma 7.29] there is  $k_L \in K_L$  such that  $k_Lk.\mathfrak{a} = \mathfrak{a}$ . We have  $k_L.H = k_Lk.H \in \mathfrak{a}$ , and because  $H, k_L.H \in \mathfrak{a}$  there is an element of the Weyl group of  $L$  that maps  $H$  to  $k_L.H$ . It follows that we may assume  $k_Lk.H = H$  without loss of generality. Because  $k_Lk.\mathfrak{a} = \mathfrak{a}$  and  $k_Lk.H = H$ , the action of  $k_Lk$  on  $\mathfrak{a}$  must be by an element of the subgroup of the Weyl group of  $G$  generated by reflections in the roots with  $\alpha(H) = 0$ . The set of such roots is contained in  $\Delta_L$ , which implies that this Weyl element fixes  $\mathfrak{a}_L$ . Therefore  $k_Lk \in Z_G(\mathfrak{a}_L) = L$ , which completes the proof.  $\square$

**8.4. Lower bounds on angular displacement.** From now on, we shall identify  $B_L$  with the tangent space of  $P_L$ . Define  $\Theta_L : \mathcal{C}_L \rightarrow K/K_L$  to be the composition of the isomorphism  $\mathcal{C}_L \simeq P_L$  with projection to the base. This is the angular variable of our polar coordinate system.

Define a metric tensor  $ds_{K_L}$  on  $K/K_L$  as follows. For  $V \in T_e(K/K_L) \simeq \mathfrak{k}_L^\perp$  we define  $ds_{K_L}(V) = \|V\|$ , and we extend  $ds_{K_L}$  to the whole manifold by  $K$ -invariance. We let  $d_{K_L}$  be the associated distance function. The next lemma gives a lower bound for how much  $\gamma$  rotates any  $V \in \mathfrak{p}_L^\perp$ .

**Lemma 8.5.** *If  $V \in \mathcal{C}_L \cap \mathfrak{p}_\gamma^\perp$ , then  $d_{K_L}(\Theta_L(V), \gamma\Theta_L(V)) \gg D(\gamma)$ .*

*Proof.* We may assume without loss of generality that  $\|V\| = 1$ . If we let  $V = k.H$  with  $H \in U_L$ , we have  $\Theta_L(V) = kK_L$  and  $d_{K_L}(\Theta_L(V), \gamma\Theta_L(V)) = d_{K_L}(k, \gamma k)$ . By property (c) of  $U_L$ , there exists  $H_0 \in S_L^+$  with  $\|H - H_0\| \leq 1/2$ . Let  $V_0 = k.H_0$ , and let  $W$  be the projection of  $V_0$  to  $\mathfrak{p}_\gamma^\perp$ . We have  $\|W\| \geq 1/2$ . Because  $I - \gamma$  acts on  $\mathfrak{p}_\gamma^\perp$  with determinant  $\gg D(\gamma)$  and all its eigenvalues are  $\leq 2$  in absolute value, we have  $\|\gamma.V_0 - V_0\| = \|\gamma.W - W\| \gg D(\gamma)$ . We then have  $\|\gamma k.H_0 - k.H_0\| \gg D(\gamma)$ , and applying Lemma 8.6 to  $H_0$ ,  $k$ , and  $\gamma k$  completes the proof.  $\square$

**Lemma 8.6.** *If  $H \in S_L^+$  and  $k_1, k_2 \in K$ , we have  $d_{K_L}(k_1, k_2) \gg \|k_1.H - k_2.H\|$ .*

*Proof.* Let  $\mathcal{O}_H \subset \mathfrak{p}$  be the orbit of  $H$  under  $K$ . We let  $ds_{\mathcal{O}}$  be the metric tensor on  $\mathcal{O}_H$  obtained by restricting the Killing form, and we let  $d_{\mathcal{O}}$  be the associated distance function. Because  $d_{\mathcal{O}}(k_1.H, k_2.H) \geq \|k_1.H - k_2.H\|$ , it suffices to prove that  $d_{K_L}(k_1, k_2) \gg d_{\mathcal{O}}(k_1.H, k_2.H)$ . Because the map  $\text{Ad} : K/K_L \rightarrow \mathcal{O}_H$  is a diffeomorphism, this would follow from knowing  $ds_{K_L} \gg \text{Ad}^* ds_{\mathcal{O}}$ . Because both metrics are  $K$ -invariant, it suffices to prove this at  $e \in K/K_L$ . If  $V \in \mathfrak{k}_L^\perp \simeq T_e(K/K_L)$ , we have  $ds_{K_L}(V) = \|V\|$  and  $\text{Ad}^* ds_{\mathcal{O}}(V) = \|[H, V]\|$ . Our assumption that  $\alpha(H) > \kappa$  for  $H \in S_L^+$  and  $\alpha \in \Delta^+ - \Delta_L$  implies that  $\text{ad}(H)$  maps  $\mathfrak{k}_L^\perp$  to  $\mathfrak{p}_L^\perp$  with bounded distortion, and the result follows.  $\square$

**8.5. Metrics in polar co-ordinates.** Let  $ds$  be the metric on  $S$ ,  $ds_{\mathfrak{p}}$  the pullback of  $ds$  via the exponential, and  $ds_P$  the pullback of  $ds_{\mathfrak{p}}$  to  $P_L$  under the map  $P_L \rightarrow \mathcal{C}_L^0 \hookrightarrow \mathfrak{p}$ . We shall compare  $ds_P$  to the (degenerate) metric  $ds_0$  on  $P_L$  obtained by pulling back  $ds_{K_L}$  under the natural projection  $P_L \rightarrow K/K_L$ .

**Proposition 8.7.** *There is  $c > 0$  such that*

$$ds_P(k, V, X, Y) \geq \sinh(c\|V\|) ds_0(k, V, X, Y)$$

for all  $(k, V, X, Y) \in TP_L \simeq B_L$ .

*Proof.* First note that  $ds_0(k, V, X, Y) = \|X\|$ . Our immediate goal is then to give a similarly convenient formula for  $ds_P$ , allowing for a comparison with  $ds_0$ . We will do so by expressing  $ds_{\mathfrak{p}}$  in polar coordinates.

We begin by picking a convenient basis of  $\mathfrak{p}$ . For each  $\alpha \in \Delta^+$ , choose a basis  $V_{\alpha,1}, \dots, V_{\alpha,p(\alpha)}$  for  $\mathfrak{g}_\alpha$  such that  $\langle V_{\alpha,i}, \theta(V_{\alpha,j}) \rangle = -\delta_{ij}/2$ . Let  $X_{\alpha,i} = V_{\alpha,i} + \theta(V_{\alpha,i})$  and  $Y_{\alpha,i} = V_{\alpha,i} - \theta(V_{\alpha,i})$ . Let  $M = Z_K(\mathfrak{a})$ , and let  $\mathfrak{k}_M^\perp$  be the complement of  $\text{Lie}(M)$  in  $\mathfrak{k}$ . Then the vectors  $X_{\alpha,i}$  for  $\alpha \in \Delta^+$  (resp.  $\alpha \in \Delta^+ - \Delta^L$ ) form a basis for  $\mathfrak{k}_M^\perp$  (resp.  $\mathfrak{k}_L^\perp$ ) that is orthonormal with respect to minus the Killing form. The vectors  $Y_{\alpha,i}$  are orthonormal in  $\mathfrak{p}$ , and together with a basis of  $\mathfrak{a}$  they form a basis of  $\mathfrak{p}$ . Note that we will omit the index  $i$  from sums of the  $X_\alpha$  and  $Y_\alpha$  from now on.

We identify the tangent space to  $K/M$  at  $kM$  with  $\mathfrak{k}_M^\perp$  via the differential of the map  $\mathfrak{k}_M^\perp \ni X \mapsto k \exp(X)M \in K/M$ . The polar co-ordinate map  $K/M \times \mathfrak{a} \rightarrow \mathfrak{p}$  allows us to identify the tangent space to  $\mathfrak{p}$  at a point  $k.H$  with  $\mathfrak{k}_M^\perp + \mathfrak{a}$ , and under this identification the metric  $ds_{\mathfrak{p}}$  becomes

$$ds_{\mathfrak{p}}^2 = d\mathfrak{a}^2 + \sum_{\alpha \in \Delta^+} \sinh^2 \alpha(H) dX_\alpha^2,$$

where  $d\mathfrak{a}$  is the Killing metric on  $\mathfrak{a}$ , see for instance [5, Proposition 2.3]. If  $H \in \mathfrak{a}^+$  then it follows that the metric  $ds_{\mathfrak{p}}$  at  $H$  with respect to the vectors  $Y_\alpha$  and the subspace  $\mathfrak{a}$  is given by

$$(26) \quad ds_{\mathfrak{p}}^2 = d\mathfrak{a}^2 + \sum_{\alpha \in \Delta^+} \frac{\sinh^2 \alpha(H)}{\alpha(H)^2} dY_\alpha^2.$$

Returning to the proof of the proposition, note that from the left  $K$ -invariance of both metrics, it suffices to check the claimed inequality at  $T_x P_L$  where  $x = (e, H)$  and  $H \in C_L$ . By continuity, we may also assume that  $H \in C_L \cap \mathfrak{a}^+$ . Consequently, condition (d) in the definition of  $U_L$  ensures that

$$(27) \quad \alpha(H) \gg \|H\|$$

for all  $\alpha \in \Delta^+ - \Delta_L$ .

Let  $(e, H, X, Y) \in T_x P_L$ . We have  $d\Pi_L(e, H, X, Y) = (H, [X, H] + Y)$ . If

$$X = \sum_{\alpha \in \Delta^+ - \Delta_L} c_\alpha X_\alpha,$$

then

$$[X, H] = - \sum_{\alpha \in \Delta^+ - \Delta_L} \alpha(H) c_\alpha Y_\alpha \in \mathfrak{p}_L^\perp.$$

Applying (26) we find that

$$ds_{\mathfrak{p}}^2(H, [X, H] + Y) \geq ds_{\mathfrak{p}}^2(H, [X, H]) = \sum_{\alpha \in \Delta^+ - \Delta_L} c_\alpha^2 \sinh^2 \alpha(H).$$

From (27) it follows that  $\sinh^2 \alpha(H) \geq \sinh^2(c\|H\|)$  for some  $c > 0$ . We therefore have

$$ds_{\mathfrak{p}}^2(e, H, X, Y) \geq \sinh^2(c\|H\|) \sum_{\alpha \in \Delta^+ - \Delta_L} c_\alpha^2 = \sinh^2(c\|H\|) \|X\|^2$$

as required. □

**8.6. Proof of Proposition 8.1.** Let  $\gamma, V, \epsilon$ , and  $T$  be as in the statement of Proposition 8.1. Let  $1/2 > \delta > 0$  be such that for all  $L$  the  $\delta$ -neighbourhood of  $U'_L$  in  $\mathfrak{p}$  is contained in  $\mathcal{C}_L$ .

We first consider the case where  $\|V\| \leq \max(T\delta^{-1}, 1)$ . Because  $d_S \circ \exp$  is quasi-isometric to the Killing metric on the ball of radius  $T\delta^{-1}$  in  $\mathfrak{p}$ , we have  $d_S(e^V, e^{\gamma \cdot V}) \gg_T \|\gamma \cdot V - V\|$  where the implied constant depends only on  $T$  and  $G$ . Arguing as in Lemma 8.5 gives  $\|\gamma \cdot V - V\| \gg \|V\| D(\gamma)$ , and combining this with our assumption  $\epsilon > d_S(e^V, e^{\gamma \cdot V})$  gives  $\epsilon D(\gamma)^{-1} \gg \|V\|$  in this case.

We now assume that  $\|V\| > \max(T\delta^{-1}, 1)$ . Choose  $L \in \mathcal{L}$  such that  $V \in \mathcal{C}'_L$ . Let  $p$  be a path of length  $< \epsilon$  from  $e^V$  to  $e^{\gamma \cdot V}$  in  $S$ . Because the map  $X : S \rightarrow \bar{\mathfrak{a}}^+$  is distance

non-increasing, our assumptions that  $\|V\| > T\delta^{-1}$  and  $\epsilon < T$  imply that  $p(t) \in \mathcal{C}_L$  for all  $t$ . We may therefore bound the length of  $p$  using Proposition 8.7. If  $\tilde{p}$  is the pullback of  $p$  to  $P_L$ , we have

$$\epsilon > \int ds(p'(t)) = \int ds_P(\tilde{p}'(t)).$$

Because  $\|V\| > T\delta^{-1} > 2\epsilon$ , we have  $\|\tilde{p}(t)\| \geq \|V\|/2$  for all  $t$ . We may therefore apply Proposition 8.7 (after shrinking  $c$ ) to obtain

$$\epsilon > \int ds_P(\tilde{p}'(t)) \geq \sinh(c\|V\|) \int ds_0(\tilde{p}'(t)).$$

The metric  $ds_0$  computes the length of the projection of  $\tilde{p}$  to  $K/K_L$ , or equivalently the length of the path  $\Theta_L(p(t))$ . Lemma 8.5 therefore gives

$$\int ds_0(\tilde{p}'(t)) \geq d_{K_L}(\Theta_L(V), \Theta_L(\gamma V)) \gg D(\gamma).$$

Combining these gives  $\epsilon D(\gamma)^{-1} \gg \sinh(c\|V\|)$ .

To prove Proposition 8.1 in the case that  $\epsilon D(\gamma)^{-1} \leq 2$  we use  $\sinh(c\|V\|) \geq c\|V\|$  to obtain  $\epsilon D(\gamma)^{-1} \gg \|V\|$  for both ranges  $\|V\| \leq \max(T\delta^{-1}, 1)$  and  $\|V\| > \max(T\delta^{-1}, 1)$ .

To prove Proposition 8.1 for  $\epsilon D(\gamma)^{-1} > 2$ , note that we may assume that  $\|V\| > \max(T\delta^{-1}, 1)$ . The bound then follows from  $\epsilon D(\gamma)^{-1} \gg \sinh(c\|V\|) \gg e^{c\|V\|}$ .

## 9. CONSTRUCTION OF ADMISSIBLE GROUPS $G$

This section is devoted to the proof of the following proposition, which provides examples of groups satisfying the hypotheses of Theorem 1.1.

**Proposition 9.1.** *Let  $G'/\mathbb{R}$  be connected, simply connected, and  $\mathbb{R}$ -almost simple. Let  $F$  be a totally real number field, and let  $v_0$  be a real place of  $F$ . There is a connected semisimple group  $G/F$  with  $G_{v_0} \simeq G'$  that satisfies conditions (2) and (3) of Theorem 1.1.*

It is an interesting question whether condition (3) on the existence of a rational Cartan involution is automatic or not. We believe that it is not when  $G$  is almost simple of type  $A_n$ ,  $D_n$ , or  $E_6$ , but are unsure otherwise.

**9.1. Lie algebra version.** We begin by proving the analogue of Proposition 9.1 for absolutely simple Lie algebras.

**Proposition 9.2.** *Let  $\mathfrak{g}'/\mathbb{R}$  be an absolutely simple Lie algebra. There exists a Lie algebra  $\mathfrak{g}/\mathbb{Q}$  with an involution  $\tau$  defined over  $\mathbb{Q}$  such that  $\mathfrak{g} \otimes \mathbb{R} \simeq \mathfrak{g}'$ , and  $\tau$  is a Cartan involution of  $\mathfrak{g} \otimes \mathbb{R}$ .*

*Proof.* Let  $\mathfrak{g}^*/\mathbb{Q}$  be the  $\mathbb{Q}$ -split Lie algebra satisfying  $\mathfrak{g}^* \otimes \mathbb{C} \simeq \mathfrak{g}' \otimes \mathbb{C}$ . We shall obtain the Lie algebra  $\mathfrak{g}$  of the proposition by appropriately twisting  $\mathfrak{g}^*$ , as we now explain.

There is a canonical  $\mathbb{Q}$ -involution on  $\mathfrak{g}^*$  inducing a Cartan involution of  $\mathfrak{g}^* \otimes \mathbb{R}$ . To define it, first let  $\mathfrak{h}$  be a rational splitting Cartan subalgebra of  $\mathfrak{g}^*$ , i.e. so that the adjoint action of  $\mathfrak{h}$  is diagonalisable over  $\mathbb{Q}$ . Let  $\Delta$  be the roots of  $\mathfrak{h}$  in  $\mathfrak{g}^*$ , and let  $\mathfrak{g}^* = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha^*$  be the root space decomposition. Let  $\{X_\alpha : \alpha \in \Delta\} \cup \{H_i : 1 \leq i \leq n\}$  be a rational Chevalley basis with respect to this root space decomposition. Let  $\theta^*$  be the involution of  $\mathfrak{g}^*$  defined by  $\theta^*(X_\alpha) = X_{-\alpha}$  and  $\theta^*(H_i) = -H_i$ . Then  $\theta^*$  induces a Cartan involution of  $\mathfrak{g}^* \otimes \mathbb{R}$ .



Next let  $\theta'$  be a Cartan involution of  $\mathfrak{g}'$  and write  $\tau_0$  for the involution on  $\mathfrak{g}^* \otimes \mathbb{C}$  obtained by extending  $\theta'$  to  $\mathfrak{g}^* \otimes \mathbb{C} \simeq \mathfrak{g}' \otimes \mathbb{C}$ . Lemma 9.3 below, applied with the pair  $(\mathfrak{g}^*, \tau_0)$ , then produces a  $\mathbb{Q}$ -involution  $\tau$  of  $\mathfrak{g}^*$  commuting with  $\theta^*$  and conjugate to  $\tau_0$  in  $\text{Aut}(\mathfrak{g}^* \otimes \mathbb{C})$ .

We obtain  $\mathfrak{g}$  by twisting  $\mathfrak{g}^*$  by  $\theta^*\tau$ . More precisely, we let  $\mathfrak{g}_+^*$  and  $\mathfrak{g}_-^*$  be the  $\pm 1$  eigenspaces of  $\theta^*\tau$  on  $\mathfrak{g}^*$  and define  $\mathfrak{g} = \mathfrak{g}_+^* + i\mathfrak{g}_-^* \subset \mathfrak{g}^* \otimes \mathbb{Q}(i)$ . It remains to verify the required properties of  $\mathfrak{g}$  and to check that  $\tau$  defines a Cartan involution on  $\mathfrak{g} \otimes \mathbb{R}$ .

Firstly, it is clear that  $\mathfrak{g}$  is a Lie algebra over  $\mathbb{Q}$ , and that  $\tau$  gives a  $\mathbb{Q}$ -involution of  $\mathfrak{g}$ , since  $\tau$  and  $\theta^*$  commute. Let  $c$  denote the complex conjugation of  $\mathfrak{g}^* \otimes \mathbb{C}$  with respect to  $\mathfrak{g}^* \otimes \mathbb{R}$ . Because  $\theta^*$  is a Cartan involution, the fixed-point set of  $\theta^*c$  is a compact real form  $\mathfrak{u}$  of  $\mathfrak{g}^* \otimes \mathbb{C}$ . The complex conjugation of  $\mathfrak{g}^* \otimes \mathbb{C} \simeq \mathfrak{g} \otimes \mathbb{C}$  with respect to  $\mathfrak{g} \otimes \mathbb{R}$  is equal to  $\theta^*\tau c$ . Because  $\mathfrak{u}$  is also the fixed point set of  $\tau \circ (\theta^*\tau c)$ , this implies that  $\tau$  is a Cartan involution of  $\mathfrak{g} \otimes \mathbb{R}$ .

We now show that  $\mathfrak{g} \otimes \mathbb{R} \simeq \mathfrak{g}'$ . Indeed,  $\mathfrak{g} \otimes \mathbb{R}$  and  $\mathfrak{g}'$  are both real forms of  $\mathfrak{g}^* \otimes \mathbb{C}$ . If we let  $\mathfrak{g} \otimes \mathbb{R} = \mathfrak{k} + \mathfrak{p}$  and  $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$  be the Cartan decompositions induced by  $\tau$  and  $\theta'$ , we have  $\mathfrak{k} \otimes \mathbb{C} \simeq \mathfrak{k}' \otimes \mathbb{C}$  because  $\tau$  and  $\theta'$  are conjugate in  $\text{Aut}(\mathfrak{g}^* \otimes \mathbb{C})$ . As  $\mathfrak{k}$  and  $\mathfrak{k}'$  are both of compact type, this implies that  $\mathfrak{k} \simeq \mathfrak{k}'$  and [10, Ch. X, Theorem 6.2] implies that  $\mathfrak{g} \otimes \mathbb{R} \simeq \mathfrak{g}'$ .  $\square$

**Lemma 9.3.** *Let  $\mathfrak{g}^*/\mathbb{Q}$  be a  $\mathbb{Q}$ -split Lie algebra with  $\mathbb{Q}$ -involution  $\theta^*$  as above. Let  $\tau_0$  be an involution of  $\mathfrak{g}^* \otimes \mathbb{C}$ . Then there exists a  $\mathbb{Q}$ -involution  $\tau$  of  $\mathfrak{g}^*$  which commutes with  $\theta^*$  and is conjugate to  $\tau_0$  in  $\text{Aut}(\mathfrak{g}^* \otimes \mathbb{C})$ .*

*Proof.* We shall use the classification of automorphisms of  $\mathfrak{g} \otimes \mathbb{C}$  in [10, Ch. X, Theorem 5.15]. This implies that one of the following holds:

- (1)  $\tau_0$  is conjugate to an involution  $\tau$  that satisfies  $\tau(H) = H$  for  $H \in \mathfrak{h}$  and  $\tau(X_\alpha) = \epsilon_\alpha X_\alpha$  where  $\epsilon_\alpha = \pm 1$ . Because  $\tau$  leaves  $\mathfrak{h}$  invariant, we must have  $\epsilon_\alpha = \epsilon_{-\alpha}$ .
- (2)  $\tau_0$  is conjugate to an involution  $\tau$  induced by an automorphism of the Dynkin diagram. This means that there is a linear map  $\sigma : \mathfrak{h} \rightarrow \mathfrak{h}$  that induces a map on  $\Delta$ , and  $\tau$  satisfies  $\tau(H) = \sigma(H)$  for  $H \in \mathfrak{h}$  and  $\tau(X_\alpha) = X_{\sigma(\alpha)}$  for  $\alpha \in \Delta$ .

In both cases, it may be seen that  $\tau$  commutes with  $\theta^*$ . That  $\tau$  is defined over  $\mathbb{Q}$  is clear in case (1), and in case (2) it follows from the fact that  $\sigma$  preserves  $\Delta$  and hence the  $\mathbb{Q}$ -structure on  $\mathfrak{h}$ .  $\square$

We may immediately extend Proposition 9.2 to allow  $\mathfrak{g}'/\mathbb{R}$  to be  $\mathbb{R}$ -simple. To do this, we note that if  $\mathfrak{g}'$  is  $\mathbb{R}$ -simple but not absolutely simple, then  $\mathfrak{g}' = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathfrak{h}'$  where  $\mathfrak{h}'/\mathbb{C}$  is simple. We treat this case by finding a rational form  $\mathfrak{h}$  of  $\mathfrak{h}'$  such that  $\mathfrak{h} \otimes \mathbb{R}$  is compact, and then setting  $\mathfrak{g} = \mathbb{Q}(i) \otimes \mathfrak{h}$ , considered as a  $\mathbb{Q}$ -algebra.

Finally, we extend the above construction to totally real number fields, as in Lemma 9.1.

**Lemma 9.4.** *Let  $F$  be a totally real number field, and let  $v_0$  be a real place of  $F$ . Let  $\mathfrak{g}'/\mathbb{R}$  be an  $\mathbb{R}$ -simple Lie algebra. There exists  $\mathfrak{g}/F$  with an involution  $\theta$  defined over  $F$  such that  $\mathfrak{g}_{v_0} \simeq \mathfrak{g}'$ ,  $\mathfrak{g}_v$  is of compact type for all real  $v \neq v_0$ , and  $\theta$  induces a Cartan involution of  $\mathfrak{g}_{v_0}$ .*

*Proof.* By Proposition 9.2 and the remark above, there exists  $\mathfrak{g}_0/\mathbb{Q}$  and a  $\mathbb{Q}$ -involution  $\theta$  of  $\mathfrak{g}_0$  such that  $\mathfrak{g}_0 \otimes \mathbb{R} \simeq \mathfrak{g}'$ , and  $\theta$  induces a Cartan involution of  $\mathfrak{g}_0 \otimes \mathbb{R}$ . Let  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$  be the rational Cartan decomposition associated to  $\theta$ . Choose  $\alpha \in F$  such that  $\alpha_{v_0} > 0$ , and

$\alpha_v < 0$  for all other real  $v$ . Define  $\mathfrak{g} = \mathfrak{k}_0 \otimes F + \sqrt{\alpha}\mathfrak{p}_0 \otimes F \subset \mathfrak{g}_0 \otimes \overline{\mathbb{Q}}$ . We extend  $\theta$  to an involution of  $\mathfrak{g}$  defined over  $F$ .

We have  $\mathfrak{g}_{v_0} \simeq \mathfrak{g}_0 \otimes \mathbb{R} \simeq \mathfrak{g}'$ , and it is clear that  $\theta$  induces a Cartan involution of  $\mathfrak{g}_{v_0}$ . If  $v \neq v_0$  is real, we have  $\mathfrak{g}_v = \mathfrak{k}_0 \otimes \mathbb{R} + i\mathfrak{p}_0 \otimes \mathbb{R} \subset \mathfrak{g}_0 \otimes \mathbb{C}$  so that  $\mathfrak{g}_v$  is of compact type.  $\square$

**9.2. Proof of Proposition 9.1.** We now return to the proof of Proposition 9.1.

Let  $G'/\mathbb{R}$  be the Lie algebra of the connected, simply connected, and  $\mathbb{R}$ -almost simple group  $G'/\mathbb{R}$  given in the lemma. Apply Lemma 9.4 to  $\mathfrak{g}'$  and the distinguished real place  $v_0$  of  $F$  to obtain  $\mathfrak{g}/F$  and  $\theta$ . Let  $\text{Int}(\mathfrak{g})$  be the identity component of  $\text{Aut}(\mathfrak{g})$ , and let  $G$  be the simply connected cover of  $\text{Int}(\mathfrak{g})$ . Then  $G$  has Lie algebra  $\mathfrak{g}$ , and  $\theta$  induces an involution of  $G$  which we also denote by  $\theta$  (as the derivative at the identity of the former is equal to the latter). Because  $G'$  and  $G_{v_0}$  are connected, simply connected semisimple groups with isomorphic Lie algebras, they are isomorphic. If  $v \neq v_0$  is real, then  $\mathfrak{g}_v$  is of compact type and  $G(F_v)$  is compact.

By Lemma 9.4,  $\theta$  induces a Cartan involution of  $\mathfrak{g}_{v_0}$ , and it follows that  $\theta$  also induces a Cartan involution of  $G_{v_0}$  (in the sense that the fixed point set  $G_{v_0}^\theta$  is a maximal compact subgroup of  $G_{v_0}$ ). This completes the proof.

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