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10–plectic formulation of gravity and Cartan connections

Dimitri VEY [♭]

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Abstract — We give a Hamiltonian formulation of Weyl–Einstein–Cartan gravity which is covariant from the viewpoint of the geometry of the principal fiber bundle. The connection is represented by a 1-form with values in the Poincaré Lie algebra, which is defined on the total space of the orthonormal frame bundle fibered over the space-time. Within the 10-plectic framework we discover that the local equivariance property of the Cartan connection is a consequence of the Hamilton equations.

One of the guiding ideas rooted in this paper is that the *physical laws* (in particular General Relativity) are independent of the point of the space-time manifold **and** the pseudo-orthonormal (or reference frame) in which they are expressed. This viewpoint is adopted in the approach to field theory based on the space of reference frames, which is developed by Toller [28–30]. This standpoint echoed the work initiated by Lurçat [24] towards a Quantum Field Theory on the Poincaré group (see also [14,33]). The original motivation of Lurçat was to put the dynamical role of spin in the foreground. Accordingly, in gauge theories of gravitation the local model of Minkowski space is replaced by the Poincaré group [1,27]. We present a Hamiltonian formulation of first order gravity which is covariant from the viewpoint of the geometry of the principal fiber bundle, i.e. which does not depend on choices of space-time coordinates **nor** on the trivialization of the principal bundle[1]. We work with the multisymplectic approach, which goes back to the discovery of generalizations of the Hamilton equations for variational problems with several variables by Volterra [34,35]. These ideas were first developped in the thirties [2,8,23,36] and later in the seventies of the last century [11–13,20–22,26]. The multisymplectic theory has many recent developments (see e.g. [3,10,15–17]) The basic concept is the notion of a multisymplectic $(m+1)$-form $\boldsymbol{\omega}$ on a smooth manifold $\mathcal{N}$, where m corresponds to the number of independent variables. The form $\boldsymbol{\omega}$ is always closed and one often assumes that it is non degenerate: $\forall \xi \in \Gamma(\mathcal{N}, T\mathcal{N})$, $\xi \lrcorner \boldsymbol{\omega} = 0 \Rightarrow \xi = 0$. The solutions of the Hamilton–Volterra–De Donder–Weyl (HVDW) equations are given by oriented m-dimensional submanifolds $\boldsymbol{\gamma}$

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[1] For a formulation which is covariant from the viewpoint of the geometry of space–time only, we refer to [31,32] and references therein.

of $\mathcal{N}$ which satisfy the condition that, at any point $\mathsf{m} \in \mathcal{N}$, there exists a basis $(X_1, \cdots, X_m)$ of $T_{\mathsf{m}}\boldsymbol{\gamma}$ such that $X_1 \wedge \cdots \wedge X_m \lrcorner \boldsymbol{\omega} = (-1)^m dH$, where $H : \mathcal{N} \longrightarrow \mathbb{R}$ is a Hamiltonian function. Analogously one can replace $\boldsymbol{\omega}$ by its restriction to the level set $H^{-1}(0)$ and describe the solutions as the submanifolds $\boldsymbol{\gamma}$ of $H^{-1}(0)$ such that $X_1 \wedge \cdots \wedge X_m \lrcorner \boldsymbol{\omega} = 0$ everywhere and $\beta^{(m)}(X_1 \wedge \cdots \wedge X_m) \neq 0$, where $\beta^{(m)}$ is some volume m-form, see [16].

Organization of the paper — In Section 1.1 we describe the space-time dynamical fields (e, A) for Weyl–Einstein–Cartan (WEC) gravity. The multisymplectic formulation of the WEC functional $\mathcal{S}_{\mathsf{WEC}}[e, A]$ is delicate because the dynamical fields (e, A) are the components of a Cartan connection (see [4–6, 25]) and subject to gauge invariance. In Section 1.2 we lift the connection defined on some manifold $\mathcal{X}$ representing the space-time to the principal bundle $\mathcal{P}$ over $\mathcal{X}$ with structure group the Lorentz group. The connection is represented by a 1-form (α, ω) on $\mathcal{P}$ with value in the Poincaré algebra $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{t}$, which satisfies *normalization* and *equivariance* hypotheses, see Equations (2). Although *a priori* mandatory the equivariance condition has the shortcoming of being a non holonomic constraint, i.e. on the first order derivatives of the field.

The geometrical background for the 10-plectic formulation of WEC gravity is presented in Section 2. The covariant configuration space is $\mathcal{Z} := \mathfrak{p} \otimes T^*\mathcal{P}$, the space of $\mathfrak{p}$-valued 1-forms over $\mathcal{P}$. Section 2.1 is dedicated to present the WEC 10-form, the first order jet bundle $J^1\mathcal{Z}$ and the De Donder–Weyl (DW) bundle $\Lambda^{10}_1 T^*\mathcal{Z}$. In the following, we compute the Legendre transform for the WEC action by treating connections as normalized and equivariant $\mathfrak{p}$-valued 1-forms on $\mathcal{P}$ (see Section 2.2). We find that the natural multisymplectic manifold can be built from the vector bundles $\mathfrak{p} \otimes T^*\mathcal{P}$ and $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$ over $\mathcal{P}$, where 10 is the dimension of $\mathcal{P}$, $\mathfrak{p}$ is the structure Lie algebra and $\mathfrak{p}^*$ its dual vector space. These vector bundles are endowed with a canonical $\mathfrak{p}$-valued 1-form η and a canonical $\mathfrak{p}^*$-valued 8-form ψ respectively (see also [19]). Then the multisymplectic manifold corresponds to the total space of the vector bundle $\mathcal{M} := \mathbb{R} \oplus_{\mathcal{P}} (\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P}) \oplus_{\mathcal{P}} (\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P})$, equipped with the 10-form $\theta^{(10)} = \varsigma\beta^{(4)} \wedge \gamma^{(6)} + \psi \wedge (d\eta + \eta \wedge \eta)$, where ς is a coordinate on $\mathbb{R}$, $\beta^{(4)} \wedge \gamma^{(6)}$ is the volume form on $\mathcal{P}$ and $\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P} \subset \mathfrak{p} \otimes T^*\mathcal{P}$ is the subbundle of normalized forms.

Finally, the DW formulation of the Hamilton equations is given in Section 3. Any solution of the Hamilton equations is given by a 10-dimensional submanifold of $\mathcal{N}$, more precisely a section ϕ of $\mathcal{N}$ over $\mathcal{P}$. In Section 3.1, we compute the 11-plectic form $\boldsymbol{\omega} := d\theta^{(10)}$. In Section 3.2, we finally discover that the dynamical equations constrain the $\mathfrak{p}$-valued 1-forms to be equivariant, see Proposition 3.1. In addition, the formalism yields Einstein–Cartan type equations:

$$\left\{ \begin{array}{rcl} G^b{}_a & = & \frac{1}{2}\rho_j \cdot p_a{}^{bj} \\ T^a{}_{cd} & = & -\left(\mathsf{h}_{de}\delta^a_{a'}\delta^{c'}_c + \frac{1}{2}\delta^{c'}_{a'}(\delta^a_d \mathsf{h}_{ce} - \delta^a_c \mathsf{h}_{de})\right) \rho_j \cdot p_{c'}{}^{ea'j}, \end{array} \right.$$

where $G^b{}_a$ is the Einstein tensor and $T^a{}_{cd}$ is the torsion tensor, see Section 4.3 in the Annex. In addition, ρ_j is a left invariant vector field acting on the multimomenta coordinates $p_a{}^{bj}$ and $p_a{}^{bcj}$ which are given in Proposition 3.2.

The approach of curved space-time by crystallization of liquid fiber bundles, which is developed elsewhere (see [19]), is more fundamental and includes the one given in this paper as a peculiar case. Hence, Proposition 3.2 reproduces partially the results obtained in the broader context of *liquid fiber bundles*, where the Hamilton equations contain in addition non homogeneous Maxwell type equations (see Equations (88) in [19]). In the former, no *a priori* hypotheses are given to settle the structure of the principal fiber bundle.

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1 Weyl–Einstein–Cartan

Let $\vec{\mathbb{M}}$ be the Minkowski vector space endowed with the Minkowski metric h. We fix a pseudo-orthonormal basis $(E_a)_{0\leq a\leq 3}$ of $(\vec{\mathbb{M}},\mathsf{h})$. In addition, $\mathfrak{T}$ is the Abelian Lie group of translations on $\vec{\mathbb{M}}$, $\mathfrak{t}$ its trivial Lie algebra, with basis $(\mathfrak{t}_a)_{0\leq a\leq 3}$ and $\mathfrak{t}^*$ the dual of $\mathfrak{t}$ with basis $(\mathfrak{t}^a)_{0\leq a\leq 3}$. We denote by $\mathfrak{G}$ the Lorentz group $SO(3,1)$, $\mathfrak{g}$ its Lie algebra, i.e. $\mathfrak{g}:=\mathbf{so}(3,1)$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. We denote by $(\mathfrak{l}_j)_{1\leq j\leq 6}$ a basis of $\mathfrak{g}$ and $(\mathfrak{l}^j)_{1\leq j\leq 6}$ a basis of and $\mathfrak{g}^*$, respectively. Finally, $\mathfrak{G}\ltimes\mathfrak{T}$ is the Poincaré group $ISO(3,1):=SO(3,1)\ltimes\mathfrak{T}$, $\mathfrak{p}:=\mathfrak{g}\oplus\mathfrak{t}$ its Lie algebra and $\mathfrak{p}^*$ is the dual of $\mathfrak{p}$. We fix some basis $(\mathfrak{l}_A)_{0\leq A\leq 9}=(\mathfrak{t}_0,\cdots,\mathfrak{t}_3,\mathfrak{l}_1,\cdots,\mathfrak{l}_6)$ and $(\mathfrak{l}^A)_{0\leq A\leq 9}=(\mathfrak{t}^0,\cdots,\mathfrak{t}^3,\mathfrak{l}^1,\cdots,\mathfrak{l}^6)$ of $\mathfrak{p}$ and $\mathfrak{p}^*$, respectively.

1.1 Space-time dynamics

In WEC formulation of gravity, dynamical fields can be defined locally as being pairs (e,A), where $e=(e^0,e^1,e^2,e^3)$ is a moving coframe on $\mathcal{X}$ and A is a $\mathfrak{g}$-valued connection 1-form on $\mathcal{X}$. We set a volume 4-form $e^{(4)}:=e^0\wedge e^1\wedge e^2\wedge e^3$ and $e^{(2)}_{ab}:=(\frac{\partial}{\partial e^a}\wedge\frac{\partial}{\partial e^b})\lrcorner\, e^{(4)}$. The WEC action reads

$$\mathcal{S}_{\mathsf{WEC}}[e,A]=\int_{\mathcal{X}}e^{(2)}_{ab}\wedge F^{ab}=\int_{\mathcal{X}}u^{ab}_i e^{(2)}_{ab}\wedge F^i, \tag{1}$$

where $F:=dA+A\wedge A$ is the curvature form and $F^{cd}:=\mathsf{h}^{dd'}F^c{}_{d'}$. Elsewhere, this action is termed the « Palatini » action functional, which is inexact [9].

Torsion and curvature — The torsion and curvature 2-forms, which are denoted

by T^a and F^{ab}, are related to the dynamical field (e, A) by the Cartan structure equations: $T^a = de^a + A^a{}_b \wedge e^b$ and $F^a{}_b = dA^a{}_b + A^a{}_c \wedge A^c{}_b$, respectively. We introduce the torsion tensor $T^a{}_{cd}$ such that $T^a = \frac{1}{2}T^a{}_{cd}e^{cd} = \frac{1}{2}T^a_{\mu\nu}\beta^{\mu\nu}$, where the components $T^a_{\mu\nu} = T^a{}_{cd}e^c_\mu e^d_\nu$ are given by $T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + A^a_{\mu c}e^c_\nu - e^c_\mu A^a_{\nu c}$. In addition, the curvature tensor $F^a{}_{bcd}$ is such that $F^a{}_b = \frac{1}{2}F^a{}_{bcd}e^{cd} = F^{ab}{}_{cd}e^c_\mu e^d_\nu$, where the components are written as $F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^a_{\mu c}A^{cb}_\nu - A^a_{\nu c}A^{cb}_\mu$.

Ricci and Einstein tensors — We denote by $\mathrm{Ric}_{ab} := F^{a'}{}_{aa'b} = \mathsf{h}_{ab'}F^{a'b'}{}_{a'b}$ the Ricci tensor. Then, $G_{ab} = \mathrm{Ric}_{ab} - \frac{1}{2}\mathsf{h}_{ab}\mathrm{S}$ is the Einstein tensor, where S is the scalar curvature, which is given by $\mathrm{S} = \mathsf{h}^{ab}\mathrm{Ric}_{ab} = \mathsf{h}^{ab}\mathsf{h}_{ab'}F^{a'b'}{}_{a'b} = F^{a'b}{}_{a'b}$.

1.2 Bundle dynamics

In addition, the WEC functional is invariant by gauge transformations of the form $(e, A) \longmapsto (g^{-1}e, g^{-1}dg + g^{-1}Ag)$, which are written in indices as $e^a \longmapsto (g^{-1})^a{}_{a'}e^{a'}$ and $A^a{}_b \longmapsto (g^{-1})^a{}_{a'}dg^{a'}{}_b + (g^{-1})^a{}_{a'}A^{a'}{}_{b'}g^{b'}{}_b$, where $g : \mathcal{X} \longrightarrow \mathfrak{G}$. In order to fully consider the gauge invariance, we now lift the theory to the total space of the principal fiber bundle $(\mathcal{P}, \mathcal{X}, \pi, \mathfrak{G})$, where $\mathcal{X}$ is the base space, $\mathcal{P}$ is the total space, $\mathfrak{G}$ is the Lorentz structure group and $\pi_\mathcal{X} : \mathcal{P} \to \mathcal{X}$ is the fibration map. We assume that $\mathfrak{G}$ is acting on the right on $\mathcal{P}$:

$$\begin{array}{rccc} R_g : & \mathcal{P} \times \mathfrak{G} & \longrightarrow & \mathcal{P} \\ & (\mathsf{z}, g) & \longmapsto & \mathsf{z} \cdot g = R_g(\mathsf{z}) \end{array}$$

This induces an infinitesimal action of $\mathfrak{g}$, to any $\xi \in \mathfrak{g}$, we associate the vector field $\rho_\xi(\mathsf{z}) = \mathsf{z} \cdot \xi$ on $\mathcal{P}$ defined by: $\forall\ \mathsf{z} \in \mathcal{P}, \forall \xi \in \mathfrak{g}, \rho_\xi(\mathsf{z}) := d/dt(\mathsf{z} \cdot e^{t\xi})|_{t=0}$. For any $\mathsf{z} \in \mathcal{P}$ the orbit of the $\mathfrak{G}$ action containing z is the fiber $\mathcal{P}_\mathsf{x}$, where $\mathsf{x} = \pi_\mathcal{P}(\mathsf{z})$. The tangent vector subspace to $\mathcal{P}_\mathsf{x}$ at z is the vertical subspace $V_\mathsf{z}\mathcal{P} := \ker d(\pi_\mathcal{X})_\mathsf{z}$ and is isomorphic to the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. By choosing a section $\sigma : \mathcal{X} \to \mathcal{P}$ which induces a trivialization $\mathsf{z} = \sigma(\mathsf{x}) \cdot g \simeq (\mathsf{x}, g)$, we set $\partial_\mu(\mathsf{z}) := d(R_g \circ \sigma)_\mathsf{x}(\partial_\mu(\mathsf{x})) \simeq \partial_\mu(\mathsf{x}) \cdot g$, for $\mu = 0, \cdots, 3$, where $(\partial_\mu)_{0\leq\mu\leq 3}$ is a moving frame on $\mathcal{X}$. We consider also the family of independent tangent vector fields $(\rho_i)_{1\leq i\leq 6}$ on $\mathcal{P}$ induced by the right action of u_i on $\mathcal{P}$, which, at every point $\mathsf{z} \in \mathcal{P}$, spans the vertical subspace $V_\mathsf{z}\mathcal{P}$. Then $(\partial_\mu, \rho_i)_{0\leq\mu\leq 3, 4\leq i\leq 9}$ is a moving frame on $\mathcal{P}$. The dual frame $(dx^\mu, \gamma^i)_{0\leq\mu\leq 3, 4\leq i\leq 9}$ is the family of sections of $T^*\mathcal{P}$ such that $dx^\mu(\partial_\nu) = \delta^\mu_\nu$ and $\gamma^i(\rho_j) = \delta^i_j$.

To picture geometrically the gauge invariance we lift the variational problem on the total space $\mathcal{P}$ of the principal bundle of orthonormal frames. We represent each pair (e, A) by a pair of 1-forms (α, ω) on $\mathcal{P}$ with values in $\mathfrak{p}$, i.e. α takes values in $\mathfrak{t}$ and ω takes values in $\mathfrak{g}$. However, (α, ω) needs to satisfy the following

normalization and equivariance hypotheses:

$$\left\{\begin{array}{rcl}\rho_i \lrcorner \alpha &=& 0\\ \rho_i \lrcorner \omega &=& u_i,\end{array}\right. \qquad \left\{\begin{array}{rcl}L_{\rho_i}\alpha + u_i \cdot \alpha &=& 0\\ L_{\rho_i}\omega + [u_i,\omega] &=& 0,\end{array}\right. \tag{2}$$

where L_{ρ_i} is the Lie derivative with respect to a vector field ρ_i. We can lift the action $\mathcal{S}_{\mathsf{WEC}}[e,A]$ to a functional on the space of $\mathfrak{p}$-valued 1-forms (α,ω) by setting:

$$\widehat{\mathcal{S}}_{\mathsf{WEC}}[\alpha,\omega] = \int_{\mathcal{P}} \alpha^{(2)}_{ab} \wedge \Omega^{ab} \wedge \gamma^{(6)} = \int_{\mathcal{P}} u_i^{ab} \alpha^{(2)}_{ab} \wedge \Omega^i \wedge \gamma^{(6)}. \tag{3}$$

where $\alpha^{ab} = \alpha^a \wedge \alpha^b$. By setting $\alpha^{(2)}_{ab} := \left(\frac{\partial}{\partial \alpha^a} \wedge \frac{\partial}{\partial \alpha^b}\right) \lrcorner \alpha^{(4)}$, $\Omega := d\omega + \omega \wedge \omega$, $\Omega^{ab} := \Omega^a{}_{b'} \mathsf{h}^{bb'}$ and $\gamma^{(6)} := \gamma^1 \wedge \cdots \wedge \gamma^6$. Then critical points of $\mathcal{S}_{\mathsf{WEC}}[e,A]$ correspond to critical points of $\widehat{\mathcal{S}}_{\mathsf{WEC}}[\alpha,\omega]$ under the constraints (2).

For any $\mathfrak{p}$-valued 1-form (α,ω) on $\mathcal{P}$ which satisfies (2) and for any local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, we obtain a pair (e,A) on $\mathcal{X}$ simply by setting $e = \sigma^*\alpha$ and $A = \sigma^*\omega$. Conversely, given a pair (e,A) on $\mathcal{X}$ and a local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, this provides us with a local trivialization $\tau : \mathcal{P} \longrightarrow \mathcal{X} \times \mathfrak{G} : \mathsf{z} \longmapsto (\mathsf{x}, g)$, where (x, g) is such that $\mathsf{z} = \sigma(\mathsf{x}) \cdot g$. We associate to (e,A) a $\mathfrak{p}$-valued 1-form (α,ω) on $\mathcal{P}$ which satisfies (2), given by $\alpha = \tau^*(g^{-1}e)$ and $\omega = \tau^*(g^{-1}Ag + g^{-1}dg)$.

$$(\alpha,\omega) = (g^{-1}e, g^{-1}dg + g^{-1}Ag) \quad \Longleftrightarrow \quad (\alpha,\omega) = (0, g^{-1}dg) + \mathrm{Ad}_{g^{-1}}(e,A), \tag{4}$$

where (e,A) is a $\mathfrak{p}$-valued 1-form whose coefficients depend only on the x variables. In particular, by using the representation $\omega = g^{-1}dg + g^{-1}Ag$ and $\alpha = g^{-1}e$, then we obtain $d\omega + \omega \wedge \omega = g^{-1}(dA + A \wedge A)g$ and $d\alpha + \omega \wedge \alpha = g^{-1}\left(de + A \wedge e\right)$.

2 Towards the 10-plectic formulation

2.1 Geometrical background

The covariant configuration space is the 110-dimensional vector bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ over $\mathcal{P}$, whose fiber at point $\mathsf{z} \in \mathcal{P}$ is the tensor product $\mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$. Note that $\dim(\mathcal{P}) = 10$ and $\dim(\mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}) = \dim(\mathfrak{p}) \cdot \dim(T^*_{\mathsf{z}}\mathcal{P}) = 100$. A point in $\mathfrak{p} \otimes T^*\mathcal{P}$ will be denoted by (z,y), where $\mathsf{z} \in \mathcal{P}$ and $\mathsf{y} \in \mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$.

2.1.1 Weyl–Einstein–Cartan 10-form

We consider the canonical $\mathfrak{p}$-valued 1-form η (a section of $\mathfrak{p} \otimes T^*(\mathfrak{p} \otimes T^*\mathcal{P})$) defined by:

$$\forall (\mathsf{z},\mathsf{y}) \in \mathfrak{p} \otimes T^*\mathcal{P}, \forall v \in T_{(\mathsf{z},\mathsf{y})}(\mathfrak{p} \otimes T^*\mathcal{P}), \quad \eta_{(\mathsf{z},\mathsf{y})}(v) = \mathsf{y}(d\pi_{(\mathsf{z},\mathsf{y})}(v)),$$

where $\pi : \mathfrak{p} \otimes T^*\mathcal{P} \longrightarrow \mathcal{P}$ is the canonical projection map. This $\mathfrak{p}$-valued 1-form can be decomposed as $\eta = \mathfrak{l}_A \eta^A$, where each η^A is a 1-form on $\mathcal{P}$. Any pair (α, ω) as considered in previously is a section of $\mathfrak{p} \otimes T^*\mathcal{P}$ over $\mathcal{P}$. In the following we identify such a pair with a map φ from $\mathcal{P}$ to the total space of $\mathfrak{p} \otimes T^*\mathcal{P}$ such that $\pi \circ \varphi(\mathsf{z}) = \mathsf{z}$, $\forall \mathsf{z} \in \mathcal{P}$, by letting

$$(\alpha, \omega) = \varphi^* \eta. \tag{5}$$

We introduce the following coordinates on $\mathfrak{p} \otimes T^*\mathcal{P}$. $(z^I)_{1 \leq I \leq 10}$ are local coordinates on $\mathcal{P}$; thus they provide us with locally defined functions $z^I \simeq z^I \circ \pi$ on $\mathfrak{p} \otimes T^*\mathcal{P}$. In a given trivialization $\tau : \mathcal{P} \longrightarrow \mathcal{X} \times \mathfrak{G} : \mathsf{z} \longmapsto (\mathsf{x}, g)$, we denote by (x^μ, g) the coordinates functions for a point $\mathsf{z} \in \mathcal{P}$. We denote by $(\eta^A_I)_{0 \leq A \leq 9; 1 \leq I \leq 10}$ the local coordinates on the space $\mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$ in the basis $(\mathfrak{l}_A \otimes dz^I)_{0 \leq A \leq 9; 1 \leq I \leq 10}$. Furthermore, by using the splitting $\eta = \overset{\circ}{\eta} + \overset{\imath}{\eta}$ and the standard representation (see Section 4.1 in the Annex) we write equivalently $(\overset{\circ}{\eta}{}^c_{d\mu}, \overset{\circ}{\eta}{}^c_{dj}, \overset{\imath}{\eta}{}^c_{\mu}, \overset{\imath}{\eta}{}^c_{j})$ for the coordinates on $\mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$ in the basis $(u^d_c \otimes dx^\mu, u^d_c \otimes \gamma^j, t_c \otimes dx^\mu, t_c \otimes \gamma^j)$. The bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ is endowed with local coordinates $(x^\mu, g, \overset{\circ}{\eta}{}^c_{d\mu}, \overset{\circ}{\eta}{}^c_{dj}, \overset{\imath}{\eta}{}^c_{\mu}, \overset{\imath}{\eta}{}^c_{j})$. In these coordinates η reads $(\overset{\circ}{\eta}, \overset{\imath}{\eta}) = (\overset{\circ}{\eta}{}^c_{d\mu} u^d_c \otimes dx^\mu + \overset{\circ}{\eta}{}^c_{dj} u^d_c \otimes \gamma^j, \overset{\imath}{\eta}{}^c_{\mu} t_c \otimes dx^\mu + \overset{\imath}{\eta}{}^c_{j} t_c \otimes \gamma^j) \in \mathfrak{p} \otimes T^*_{\mathsf{z}}\mathcal{P}$.

We define the following 10-form on $\mathfrak{p} \otimes T^*\mathcal{P}$ (a section of $\Lambda^{10} T^*(\mathfrak{p} \otimes T^*\mathcal{P})$):

$$\boldsymbol{\lambda} := u^{ab}_i \overset{\imath}{\eta}{}^{(2)}_{ab} \wedge (d\overset{\circ}{\eta} + \overset{\circ}{\eta} \wedge \overset{\circ}{\eta})^i \wedge \overset{\circ}{\eta}{}^{(6)}, \tag{6}$$

where $\overset{\imath}{\eta}{}^{(2)}_{ab} := (\frac{\partial}{\partial \overset{\imath}{\eta}{}^a} \wedge \frac{\partial}{\partial \overset{\imath}{\eta}{}^b}) \lrcorner \overset{\imath}{\eta}{}^{(4)} = \frac{1}{2} \epsilon_{abcd} \eta^c \wedge \eta^d$. Then, the WEC action (3) is written as $\widehat{\mathcal{S}}_{\mathsf{WEC}}[\alpha, \omega] = \int_{\mathcal{P}} \varphi^* \boldsymbol{\lambda}$, where φ is such that (5) holds.

2.1.2 First jet bundle

We introduce the first jet bundle $J^1\mathcal{Z} := J^1(\mathcal{P}, \mathfrak{p} \otimes T^*\mathcal{P})$. A section φ of the fiber bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ can be seen as a map $\varphi : \mathcal{P} \longrightarrow \mathfrak{p} \otimes T^*\mathcal{P}$ such that $\pi_{\mathfrak{p} \otimes T^*\mathcal{P}} \circ \varphi = \mathrm{Id}_{\mathcal{P}}$. The first jet space $J^1\mathcal{Z}$ is the manifold of triplets $(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}})$, where $(\mathsf{z}, \mathsf{y}) \in \mathfrak{p} \otimes T^*\mathcal{P}$ and $\dot{\mathsf{y}}$ is the equivalence class of local sections φ of $\mathfrak{p} \otimes T^*\mathcal{P}$ over a neighborhood of z such that $\varphi(\mathsf{z}) = \mathsf{y}$, for the equivalence relation: $\varphi_1 \simeq \varphi_2$ iff $d(\eta^A_I \circ \varphi_1)_{\mathsf{z}} = d(\eta^A_I \circ \varphi_2)_{\mathsf{z}}$, $\forall I, A$. We then write $[\varphi]_{\mathsf{z},\mathsf{y}}$ the class of φ. Local coordinates on $J^1(\mathcal{P}, \mathfrak{p} \otimes T^*\mathcal{P})$ are $(z^I, \eta^A_I, \eta^A_{I;J})$, where

$$\eta^A_{I;J}(\dot{\mathsf{y}}) = \frac{\partial(\eta^A_I \circ \varphi)}{\partial z^J}(\mathsf{z}) \quad \text{where} \quad \dot{\mathsf{y}} = [\varphi]_{\mathsf{z},\mathsf{y}},$$

Equivalently, the first jet space $J^1\mathcal{Z}$ is identified (see [18]) with $T^*\mathcal{P} \otimes_{\mathfrak{p} \otimes T^*\mathcal{P}} (T(\mathfrak{p} \otimes T^*\mathcal{P})/T\mathcal{P})$, the bundle whose fiber at $(\mathsf{z}, \mathsf{y}) \in \mathfrak{p} \otimes T^*\mathcal{P}$ is the space of linear maps $\ell : T_{\mathsf{z}}\mathcal{P} \longrightarrow T_{(\mathsf{z},\mathsf{y})}(\mathfrak{p} \otimes T^*\mathcal{P})$ such that $d(\pi_{\mathcal{P}})_{(\mathsf{z},\mathsf{y})} \circ \ell = \mathrm{Id}_{T_{\mathsf{z}}\mathcal{P}}$, which is canonically identified with $T^*\mathcal{P}_{\mathsf{z}} \otimes (T_{(\mathsf{z},\mathsf{y})}(\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P})/T_{\mathsf{z}}\mathcal{P}))$.

2.1.3 De Donder–Weyl bundle

Consider the fiber bundle $\Lambda^{10}T^*\mathcal{Z}$ of 10-forms over $\mathcal{Z} := \mathfrak{p} \otimes T^*\mathcal{P}$. By using the fibration $\pi_{\mathcal{Z}} : \Lambda^{10}T^*\mathcal{Z} \longrightarrow \mathcal{Z}$ we define a canonical 10-form $\theta^{(10)}$ on $\Lambda^{10}T^*\mathcal{Z}$ by $\forall(\mathsf{z},\mathsf{y}) \in \mathcal{Z}$, $\forall \mathsf{p} \in \Lambda^{10}T^*_{(\mathsf{z},\mathsf{y})}\mathcal{Z}$, $\forall X_1, \cdots, X_{10} \in T_{(\mathsf{z},\mathsf{y},\mathsf{p})}(\Lambda^{10}T^*\mathcal{Z})$,

$$\theta^{(10)}_{(\mathsf{z},\mathsf{y},\mathsf{p})}(X_1, \cdots, X_{10}) := \mathsf{p}((\pi_{\mathcal{Z}})_* X_1, \cdots, (\pi_{\mathcal{Z}})_* X_{10}). \tag{7}$$

The 46 897 636 624 091-dimensional universal Lepage–Dedecker manifold $\Lambda^{10}T^*\mathcal{Z}$ is far too big. We define the subbundle of 9-horizontal forms (following the terminology used in [3, 10])

$$\Lambda^{10}_1 T^*\mathcal{Z} := \left\{(\mathsf{z},\mathsf{y},\mathsf{p}) \in \Lambda^{10}T^*\mathcal{Z}; \forall v_1, v_2 \in V_{(\mathsf{z},\mathsf{y})}\mathcal{Z}, v_1 \wedge v_2 \lrcorner \mathsf{p} = 0\right\}.$$

where the projection map $\pi_{\mathcal{Z}} : \mathcal{Z} \longrightarrow \mathcal{P}$ defines in each tangent space $T_Z\mathcal{Z}$ a vertical subspace $V_{(\mathsf{z},\mathsf{y})}\mathcal{Z} := \ker(\pi_{\mathcal{Z}*})$. We denote by $\mathcal{N}_{\mathsf{DW}} := \Lambda^{10}_1 T^*\mathcal{Z}$ the DW bundle, the « multimomentum phase space » of the DW theory. Local coordinates on $\mathcal{N}_{\mathsf{DW}}$ are $(x^\mu, g, \eta^A_\mu, \eta^A_j, \varsigma, \psi^{\mu\nu}_A, \psi^{j\nu}_A, \psi^{\mu j}_A, \psi^{jk}_A)$, where $(\varsigma, \psi^{\mu\nu}_A, \psi^{j\nu}_A, \psi^{\mu j}_A, \psi^{jk}_A)$ are the components of $\psi \in \Lambda^{10}_1 T^*_{(\mathsf{z},\mathsf{y})}(\mathfrak{p} \otimes T^*\mathcal{P})$ in the basis $(\beta^{(4)} \wedge \gamma^{(6)}, d\eta^A_\mu \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)}, d\eta^A_j \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)}, d\eta^A_\mu \wedge \beta^{(4)} \wedge \gamma^{(5)}_j, d\eta^A_j \wedge \beta^{(4)} \wedge \gamma^{(5)}_k)$.
The canonical 10-form (7) restricted to $\mathcal{N}_{\mathsf{DW}}$ is denoted by $\theta^{(10)} \in \Omega^{10}(\Lambda^{10}_1 T^*\mathcal{Z})$ and reads

$$\begin{array}{rcccccc} \theta^{(10)} = & \varsigma\beta^{(4)} \wedge \gamma^{(6)} & + & \psi^{\mu\nu}_A d\eta^A_\mu \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)} & + & \psi^{j\nu}_A d\eta^A_j \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)} \\ & & + & \psi^{\mu j}_A d\eta^A_\mu \wedge \beta^{(4)} \wedge \gamma^{(5)}_j & + & \psi^{jk}_A d\eta^A_j \wedge \beta^{(4)} \wedge \gamma^{(5)}_k. \end{array} \tag{8}$$

Since we are interested in normalized sections of $\mathfrak{p} \otimes T^*\mathcal{P}$, we actually work on the bundle $\Lambda^{10}_1 T^*(\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P})$ over $\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P}$, which is constructed through a reduction of $\Lambda^{10}_1 T^*(\mathfrak{p} \otimes T^*\mathcal{P})$ (see [18]). This amounts to impose $\eta^A_j = \delta^i_j$ if $A = i$ and $\eta^A_j = 0$ otherwise. We note that, when these constraints are assumed, the terms with $\psi^{j\nu}_A$ and ψ^{jk}_A vanish and actually don't play any role in the following, so that we may forget about coordinates $(\psi^{j\nu}_A, \psi^{jk}_A)$. Denoting simply by $\theta^{(10)}$ the restriction to $\Lambda^{10}_1 T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ of $\theta^{(10)}$ given in (8), this leads to the simplification:

$$\theta^{(10)} = \varsigma\beta^{(4)} \wedge \gamma^{(6)} + \psi^{\mu\nu}_A d\eta^A_\mu \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)} + \psi^{\mu j}_A d\eta^A_\mu \wedge \beta^{(4)} \wedge \gamma^{(5)}_j. \tag{9}$$

2.2 Legendre transform

Let $(\mathsf{z},\mathsf{y},\dot{\mathsf{y}}) \in J^1(\mathcal{P}, \mathfrak{p} \otimes T^*\mathcal{P})$ and let φ be a section such that $[\varphi]_{\mathsf{z},\mathsf{y}} = \dot{\mathsf{y}}$. In order to compute the Legendre transform at $(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},\mathsf{p})$ we need to determine the value of the quantity $W(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},\mathsf{p})$ which is defined by $\varphi^*(\theta^{(10)} - \boldsymbol{\lambda}) = W(\mathsf{z},\mathsf{y},\dot{\mathsf{y}},\mathsf{p})\beta^{(4)} \wedge \gamma^{(6)}$,

where we denote $\beta^{(4)} \wedge \gamma^{(6)} := \varphi^*(\beta^{(4)} \wedge \gamma^{(6)})$ (see [17] for details). Using the standard representation of $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{t}$, the canonical form (9) is then given by:

$$\begin{array}{rcl} \theta^{(10)} & = & \varsigma\beta^{(4)} \wedge \gamma^{(6)} \\ & + & \mathring{\psi}_c^{d\mu\nu} d\mathring{\eta}^c_{d\mu} \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)} + \mathring{\psi}_c^{d\mu j} d\mathring{\eta}^c_{d\mu} \wedge \beta^{(4)} \wedge \gamma^{(5)}_j \\ & + & \overset{1}{\psi}_c{}^{\mu\nu} d\overset{1}{\eta}{}^c_\mu \wedge \beta^{(3)}_\nu \wedge \gamma^{(6)} + \overset{1}{\psi}_c{}^{\mu j} d\overset{1}{\eta}{}^c_\mu \wedge \beta^{(4)} \wedge \gamma^{(5)}_j. \end{array} \tag{10}$$

The problem we start with concerns gauge fields on the space-time manifold $\mathcal{X}$ which are not only normalized but also *equivariant* sections, i.e. such that:

$$\mathring{\eta}^c_{d\mu;j} + [\mathfrak{l}_j, \mathring{\eta}_\mu]^c_d = 0, \qquad\qquad \overset{1}{\eta}{}^c_{\mu;j} + \mathfrak{l}_i \cdot \overset{1}{\eta}{}^c_\mu = 0, \tag{11}$$

To simplify the computation we choose the right coframe, as we learned from Cartan. Here given some $(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{p})$, we replace the arbitrary coframe $(dx^\mu, \gamma^i, d\eta^A_\mu)$ by $(dx^\mu, \gamma^i, \delta\eta^A_\mu)$ in the expression of $\theta^{(10)}_{(\mathsf{z},\mathsf{y},\mathsf{p})}$, where $\delta\eta^A_\mu$ is given by

$$\delta\eta^A_\mu = d\eta^A_\mu - \varphi^*(d\eta^A_\mu) = d\eta^A_\mu - \eta^A_{\mu;\nu} dx^\nu - \eta^A_{\mu;j}\gamma^j, \tag{12}$$

since $\eta^A_{I;J}(\dot{\mathsf{y}}) dz^J = d(\eta^A_I \circ \varphi)_{\mathsf{z}} = (\varphi^* d\eta^A_I)_{\mathsf{z}}$. Then,

$$\forall v \in T_{\mathsf{z}}\mathcal{P}, \quad \delta\eta^A_\mu(\ell(v)) = 0. \tag{13}$$

We compute the Legendre correspondence along equivariant sections, i.e. which satisfy the condition (11). Therefore, the change of coframe (12) is given by

$$\left\{ \begin{array}{rcl} \delta\mathring{\eta}^c_{d\mu} & = & d\mathring{\eta}^c_{d\mu} - \mathring{\eta}^c_{d\mu;\nu} dx^\nu + [\mathfrak{l}_j, \mathring{\eta}_\mu]^c_d \gamma^j, \\ \delta\overset{1}{\eta}{}^c_\mu & = & d\overset{1}{\eta}{}^c_\mu - \overset{1}{\eta}{}^c_{\mu;\nu} dx^\nu + \mathfrak{l}_j \overset{1}{\eta}{}^c_\mu \gamma^j, \end{array} \right.$$

Note that, by using the standard representation of the WEC form (6) (and since $\overset{1}{\eta}{}^{d\mu\nu}_c + \overset{1}{\eta}{}^{d\nu\mu}_c = 0$), we obtain

$$\varphi^*\boldsymbol{\lambda} = -\overset{1}{\eta}{}^{d\mu\nu}_c \left(\mathring{\eta}^c_{d\mu;\nu} - \frac{1}{2}[\mathring{\eta}_\mu, \mathring{\eta}_\nu]^c_d \right) \beta^{(4)} \wedge \gamma^{(6)}. \tag{14}$$

It follows that:

$$\begin{array}{rcl} W(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{p}) & = & (\varsigma \circ \varphi) \\ & + & \left(\overset{1}{\psi}_c{}^{\mu\nu} \circ \varphi \right) \overset{1}{\eta}{}^c_{\mu;\nu} + \left(\mathring{\psi}_c^{d\mu\nu} \circ \varphi - \overset{1}{\eta}{}^{d\mu\nu}_c \right) \mathring{\eta}^c_{d\mu;\nu} \\ & - & \left(\mathring{\psi}_c^{d\mu j} \circ \varphi \right) [\mathfrak{l}_j, \mathring{\eta}_\mu]^c_d - \left(\overset{1}{\psi}_c{}^{\mu j} \circ \varphi \right) \mathfrak{l}_j \overset{1}{\eta}{}^c_\mu \\ & - & \frac{1}{2} \overset{1}{\eta}{}^{d\mu\nu}_c [\mathring{\eta}_\mu, \mathring{\eta}_\nu]^c_d. \end{array} \tag{15}$$

The Legendre correspondence holds on the points with coordinates $(\mathsf{z}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{p})$ which are critical points of W with respect to infinitesimal variations of $\dot{\mathsf{y}}$ which

respect the constraints, i.e., such that $\partial W/\partial \mathring{\eta}^c_{d\mu;\nu} = 0$ and $\partial W/\partial \mathring{\eta}^c_{\mu;\nu} = 0$. The Legendre condition yields:

$$\mathring{\psi}^{\mu\nu}_a \circ \varphi = 0 \quad \text{and} \quad \mathring{\psi}^{d\mu\nu}_c \circ \varphi = \mathring{\eta}^{d\mu\nu}_c. \tag{16}$$

The image of the Legendre transform is denoted by

$$\mathcal{N} := \{(\mathsf{z},\mathsf{y},\mathsf{p}) \in \Lambda^{10}_1 T^*(\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P})/\mathring{\psi}^{d\mu\nu}_c = \mathring{\eta}^{d\mu\nu}_c, \mathring{\psi}^{\mu\nu}_a = 0\} \tag{17}$$

Thus, the value of the Hamiltonian function is then the restriction of W at the points where (16) holds, i.e. simply:

$$H(\mathsf{z},\mathsf{y},\mathsf{p}) = \varsigma - \frac{1}{2}\mathring{\eta}^{d\mu\nu}_c[\mathring{\eta}_\mu, \mathring{\eta}_\nu]^c_d - \left(\mathring{\psi}^{d\mu j}_c[\mathfrak{l}_j, \mathring{\eta}_\mu]^c_d + \mathring{\psi}_c{}^{\mu j}\mathfrak{l}_j\mathring{\eta}^c_\mu\right).$$

We change the coordinates on $\mathcal{N}$ in order to simplify the Hamiltonian function and in such a way that $\theta^{(10)}$ depends on η uniquely through the quantity $d\eta+\eta\wedge\eta$. We set $\varkappa := \varsigma - \frac{1}{2}\mathring{\eta}^{d\mu\nu}_c[\mathring{\eta}_\mu, \mathring{\eta}_\nu]^c_d - (\mathring{\psi}^{d\mu j}_c[\mathfrak{l}_j, \mathring{\eta}_\mu]^c_d + \mathring{\psi}_c{}^{\mu j}\mathfrak{l}_j\mathring{\eta}^c_\mu)$, so that $H(\mathsf{z},\mathsf{y},\mathsf{p}) = \varkappa$. The canonical 10-form $\theta^{(10)}$ on $\mathcal{N}$ then reads

$$\begin{aligned}\theta^{(10)} \;=\; \varkappa\beta^{(4)}\wedge\gamma^{(6)} \;+\; &\left(\frac{1}{2}\mathring{\psi}_c{}^{\mu\nu}\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)} + \mathring{\psi}_c{}^{\mu j}\beta^{(3)}_\mu\wedge\gamma^{(5)}_j\right)\wedge(d\mathring{\eta}+\mathring{\eta}\wedge\mathring{\eta})^c\\ +\; &\left(\frac{1}{2}\mathring{\psi}^{d\mu\nu}_c\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)} + \mathring{\psi}^{d\mu j}_c\beta^{(3)}_\mu\wedge\gamma^{(5)}_j\right)\wedge(d\mathring{\eta}+\mathring{\eta}\wedge\mathring{\eta})^c_d.\end{aligned}$$

The multisymplectic manifold $(\mathcal{N},\boldsymbol{\omega})$ has another construction (see also [18, 19]). We choose, as suitable submanifold of $\Lambda^{10}T^*(\mathfrak{p}\otimes T^*\mathcal{P})$, the total space of the fiber bundle $\mathcal{M} := \mathbb{R}\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})$ over $\mathcal{P}$. The base $\mathcal{P}$ is equipped with the volume form $\beta^{(4)}\wedge\gamma^{(6)}$ and ς is a coordinate on $\mathbb{R}$. Denote by $(\psi^{\mu\nu},\psi^{\mu j},\psi^{jk})$ the $\mathfrak{p}^*$-valued coordinates on the fibers of $\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}$ in the basis $(\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)}, \beta^{(3)}_\mu\wedge\gamma^{(5)}_j, \beta^{(4)}\wedge\gamma^{(4)}_{jk})$. The bundle $\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}$ is endowed with the canonical $\mathfrak{p}^*$-valued 8-form ψ defined by (see [19]): $\forall(\mathsf{z},\mathsf{p})\in\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}$, $\forall w_1,\cdots,w_8\in T_{(\mathsf{z},\mathsf{p})}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})$,

$$\psi_{(\mathsf{z},\mathsf{p})}(w_1,\cdots,w_8) = \mathsf{p}(d\pi_{(\mathsf{z},\mathsf{p})}(w_1),\cdots,d\pi_{(\mathsf{z},\mathsf{p})}(w_8)),$$

where $\pi = \pi_{\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}} : \mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}\longrightarrow\mathcal{P}$ is the canonical projection map. This $\mathfrak{p}^*$-valued 8-form decomposes as $\psi = \psi_A\mathfrak{l}^A$ where ψ_A is written as:

$$\psi_A := \frac{1}{2}\psi^{\mu\nu}_A\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)} + \psi^{\mu j}_A\beta^{(3)}_\mu\wedge\gamma^{(5)}_j + \frac{1}{2}\psi^{ij}_A\beta^{(4)}\wedge\gamma^{(4)}_{ij}. \tag{18}$$

The final multisymplectic space under consideration is the submanifold $\mathcal{N}_\circ := \mathcal{N}\cap H^{-1}(0)$ of $\mathcal{M}$ which is the intersection of the image of the Legendre correspondence, defined by the constraints (16), with the level set $H^{-1}(0)$. Thus, the canonical 10-form on $\mathcal{N}_\circ$ has the simple structure: $\theta^{(10)} := \psi_A\wedge(d\eta+\eta\wedge\eta)^A$.

3 The Hamilton equations

We consider a point $(\mathsf{z},\mathsf{m},\dot{\mathsf{m}})\in J^1(\mathcal{P},(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P}))$, where $\mathsf{z}\in\mathcal{P}$, $\mathsf{m}\in(\mathfrak{p}\otimes^{\mathrm{N}}T^*_{\mathsf{z}}\mathcal{P})\oplus(\mathfrak{p}^*\otimes^{\mathrm{N}}\Lambda^8T^*_{\mathsf{z}}\mathcal{P})$ and $\dot{\mathsf{m}}$ represent the tangent space to a section of the first jet bundle $J^1(\mathcal{P},(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes^{\mathrm{N}}\Lambda^8T^*\mathcal{P}))$ at (z,m). Local coordinates on $(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes^{\mathrm{N}}\Lambda^8T^*\mathcal{P})$ are $(z^I,\eta^A_\mu,\psi^{\mu\nu}_A,\psi^{\mu j}_A)$, where $\eta^A=\eta^A_I dz^I$. We identify $\dot{\mathsf{m}}$ with the equivalence class of sections ϕ of $(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes^{\mathrm{N}}\Lambda^8T^*\mathcal{P})$ over $\mathcal{P}$ such that $\phi(\mathsf{z})=\mathsf{m}$, for the equivalence relation:

$$\phi_1\sim\phi_2\quad\text{iff}\quad\left\{\begin{array}{rcl} d(\eta^A_I\circ\phi_1)(\mathsf{z}) &=& d(\eta^A_I\circ\phi_2)(\mathsf{z}),\\ d({\psi_A}^{\mu\nu}\circ\phi_1)(\mathsf{z}) &=& d({\psi_A}^{\mu\nu}\circ\phi_2)(\mathsf{z}),\\ d({\psi_A}^{\mu j}\circ\phi_1)(\mathsf{z}) &=& d({\psi_A}^{\mu j}\circ\phi_2)(\mathsf{z}).\end{array}\right.$$

The HVDW equations in $(\mathcal{N}_\circ,\boldsymbol{\omega})$ consists in a condition on a 10-dimensional oriented submanifold $\boldsymbol{\gamma}$ of $(\mathfrak{p}\otimes T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})$ which says that, for any point m of coordinates $(z^I,\eta^A_\mu,\psi^{\mu\nu}_A,\psi^{\mu j}_A)$ of $\boldsymbol{\gamma}$, if $(X_1,\cdots,X_{10})$ is a basis of the tangent space to $\boldsymbol{\gamma}$ at m such that $\beta^{(4)}\wedge\gamma^{(6)}(X_1,\cdots,X_{10})=1$, then

$$X_1\wedge\cdots\wedge X_{10}\lrcorner\, d\theta^{(10)}=0, \tag{19}$$

(see [17]). The independence condition $\beta^{(4)}\wedge\gamma^{(6)}(X_1,\cdots,X_{10})\neq 0$ means that such sub-manifolds are locally the graph of some section ϕ of the fiber $(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})$ over $\mathcal{P}$. For any section ϕ, we denote by $\phi^{\circledast}d\theta^{(10)}$ the 10-form on $\mathcal{P}$, such that the fiber at $\mathsf{z}\in\mathcal{P}$ is $T^*_{\phi(\mathsf{z})}\left[(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})\right]$, which is defined by: $\forall Z_1,\cdots,Z_{10}\in T_{\mathsf{z}}\mathcal{P}$, $\forall v\in T_{\phi(\mathsf{z})}\left[(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes\Lambda^8T^*\mathcal{P})\right]$,

$$(d\theta^{(10)})_{\phi(\mathsf{z})}(\phi_*Z_1,\cdots,\phi_*Z_{10},v)=\left((\phi^{\circledast}d\theta^{(10)})_{\mathsf{z}}(Z_1,\cdots,Z_{10}),v\right),$$

or equivalently $((\phi^{\circledast}d\theta^{(10)})_{\mathsf{z}},v)=(-1)^{10}\psi^*(v\lrcorner\, d\theta^{(10)})$. The HVDW equations then read $\phi^{\circledast}d\theta^{(10)}=0$. The 11-plectic form $\boldsymbol{\omega}=d\theta^{(10)}$ on $\mathcal{N}_\circ$ is

$$\boldsymbol{\omega}=d\psi_A\wedge(d\eta+\eta\wedge\eta)^A+[d\eta\wedge\eta]^A\wedge\psi_A. \tag{20}$$

where we denote $[d\eta\wedge\eta]:=d\eta\wedge\eta-\eta\wedge d\eta$. Following the same steps as in [18], we evaluate separately the terms in (20) in view of finding the Hamilton equations. Given some point m of $\boldsymbol{\gamma}$ of coordinates $(z^I,\eta^A_\mu,\psi^{\mu\nu}_A,\psi^{\mu j}_A)$, we replace the coframe $(dz^I,d\eta^A_\mu,d\psi^{\mu\nu}_A,d\psi^{\mu j}_A)$ at m by the coframe $(dz^I,\delta\eta^A_\mu,\delta\psi^{\mu\nu}_A,\delta\psi^{\mu j}_A)$, where:

$$\left\{\begin{array}{rcl}\delta\eta^A_\mu &:=& d\eta^A_\mu-\phi^*d\eta^A_\mu\\ \delta\psi^{\mu\nu}_A &:=& d\psi^{\mu\nu}_A-\phi^*d\psi^{\mu\nu}_A\\ \delta\psi^{\mu j}_A &:=& d\psi^{\mu j}_A-\phi^*d\psi^{\mu j}_A\end{array}\right. \tag{21}$$

which is equivalently written as:

$$\left\{\begin{array}{rcl} d\eta^A_\mu &=& \delta\eta^A_\mu + \eta^A_{\nu;\mu}dx^\nu + \eta^A_{\mu;j}\gamma^j \\ d\psi^{\mu\nu}_A &=& \delta\psi^{\mu\nu}_A + \psi^{\mu\nu}_{A;\rho}dx^\rho + \psi^{\mu\nu}_{A;j}\gamma^j \\ d\psi^{\mu j}_A &=& \delta\psi^{\mu j}_A + \psi^{\mu j}_{A;\rho}dx^\rho + \psi^{\mu j}_{A;k}\gamma^k \end{array}\right. \tag{22}$$

Since $\psi^A_{I;J}(\dot{\mathsf{y}})dz^J = d(\eta^A_I \circ \phi)_{\mathsf{z}} = (\phi^* d\eta^A_I)_{\mathsf{z}}$ and $\psi^{IJ}_{A;K}(\dot{\mathsf{y}})dz^K = d(\psi^{IJ}_K \circ \phi)_{\mathsf{z}} = (\phi^* d\psi^{IJ}_{A;K})_{\mathsf{z}}$. Note that in the following, we abuse notations $\eta^A_\mu := \phi^*\eta^A_\mu$, $\psi^{\mu\nu}_A := \phi^*\psi^{\mu\nu}_A$ and $\psi^{\mu j}_A = \phi^*\psi^{\mu j}_A$.

3.1 Computation of the 11-plectic form $\boldsymbol{\omega}$

Let us denote by $\boldsymbol{\omega}_1 := d\psi_A \wedge (d\eta + \eta\wedge\eta)^A$ and $\boldsymbol{\omega}_2 := [d\eta\wedge\eta]^A \wedge \psi_A$ such that $\boldsymbol{\omega} := \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

3.1.1 The computation of $\boldsymbol{\omega}_1$

Recall that we work on normalized sections i.e. on the space $(\mathfrak{p}\otimes^{\mathrm{N}} T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{p}^*\otimes^{\mathrm{N}}\Lambda^8 T^*\mathcal{P})$ over $\mathcal{P}$. Then using the change of coframe (21), we have

$$\begin{array}{rcl}(d\eta+\eta\wedge\eta)^A &=& \delta\eta^A_\mu\wedge dx^\mu + \frac{1}{2}\left(\eta^A_{\nu;\mu} - \eta^A_{\mu;\nu} + [\eta_\mu,\eta_\nu]^A\right)\beta^{\mu\nu} \\ && \qquad - \left(\eta^A_{\mu;j} - [\eta_\mu, \mathfrak{l}_j]^A\right) dx^\mu\wedge\gamma^j,\end{array}$$

which in the standard representation is written as:

$$\left\{\begin{array}{rcl}(d\eta+\eta\wedge\eta)^a &=& \delta\overset{\scriptscriptstyle 1}{\eta}{}^d_\mu\wedge dx^\mu + \frac{1}{2}\left(\overset{\scriptscriptstyle 1}{\eta}{}^a_{\nu;\mu} - \overset{\scriptscriptstyle 1}{\eta}{}^a_{\mu;\nu} + [\overset{\circ}{\eta}_\mu, \overset{\scriptscriptstyle 1}{\eta}_\nu]^a\right)\beta^{\mu\nu} \\ && \qquad - \left(\overset{\scriptscriptstyle 1}{\eta}{}^a_{\mu;j} - [\overset{\circ}{\eta}_\mu, \mathfrak{l}_j]^a\right) dx^\mu\wedge\gamma^j, \\ (d\eta+\eta\wedge\eta)^d_c &=& \delta\overset{\circ}{\eta}{}^d_{c\mu}\wedge dx^\mu + \frac{1}{2}\left(\overset{\circ}{\eta}{}^d_{c\nu;\mu} - \overset{\circ}{\eta}{}^d_{c\mu;\nu} + [\overset{\circ}{\eta}_\mu, \overset{\circ}{\eta}_\nu]^d_c\right)\beta^{\mu\nu} \\ && \qquad - \left(\overset{\circ}{\eta}{}^d_{c\mu;j} - [\overset{\circ}{\eta}_\mu, \mathfrak{l}_j]^d_c\right) dx^\mu\wedge\gamma^j.\end{array}\right.$$

Note also that $d\psi_A = \frac{1}{2}d\psi^{\mu\nu}_A\wedge\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)} + d\psi^{\mu j}_A\wedge\beta^{(3)}_\mu\wedge\gamma^{(5)}_j$. Then, we apply the change of coframe (21) and algebraic relations which are given in Section 4.2 of the Annex:

$$\begin{array}{rcl} d\psi_A &=& \frac{1}{2}\delta\psi^{\mu\nu}_A\wedge\beta^{(2)}_{\mu\nu}\wedge\gamma^{(6)} + \frac{1}{2}\psi^{\mu\nu}_{A;\rho}\left(\delta^\rho_\nu\beta^{(3)}_\mu - \delta^\rho_\mu\beta^{(3)}_\nu\right)\wedge\gamma^{(6)} \\ && + \delta\psi^{\mu j}_A\wedge\beta^{(3)}_\mu\wedge\gamma^{(5)}_j + \psi^{\mu j}_{A;\mu}\beta^{(4)}\wedge\gamma^{(5)}_j - \psi^{\mu j}_{A;j}\beta^{(3)}_\mu\wedge\gamma^{(6)}.\end{array}$$

We translate this expression by using the standard representation and imposing the constraints (16):

$$\left\{\begin{array}{rcl} d\overset{\scriptscriptstyle 1}{\psi}_a &=& \delta\overset{\scriptscriptstyle 1}{\psi}_a{}^{\mu j}\wedge\beta^{(3)}_\mu\wedge\gamma^{(5)}_j + \overset{\scriptscriptstyle 1}{\psi}{}^{\mu j}_{a;\mu}\beta^{(4)}\wedge\gamma^{(5)}_j - \overset{\scriptscriptstyle 1}{\psi}{}^{\mu j}_{a;k}\gamma^k\wedge\beta^{(3)}_\mu\wedge\gamma^{(5)}_j, \\ d\overset{\circ}{\psi}{}^d_c &=& \varepsilon_{abc}{}^d\delta\overset{\scriptscriptstyle 1}{\eta}{}^a_\mu\wedge dx^\mu\wedge\overset{\scriptscriptstyle 1}{\eta}{}^b\wedge\gamma^{(6)} + \varepsilon_{abc}{}^d\overset{\scriptscriptstyle 1}{\eta}{}^a{}_{\mu;\nu}\beta^{\nu\mu}\wedge\overset{\scriptscriptstyle 1}{\eta}{}^b\wedge\gamma^{(6)} \\ && \quad + \delta\overset{\circ}{\psi}{}^{d\mu j}_c\wedge\beta^{(3)}_\mu\wedge\gamma^{(3)}_j + \overset{\circ}{\psi}{}^{d\mu j}_{c;\mu}\beta^{(4)}\wedge\gamma^{(5)}_j - \overset{\circ}{\psi}{}^{d\mu j}_{c;j}\beta^{(3)}_\mu\wedge\gamma^{(6)}.\end{array}\right.$$

Finally, the expression of $\boldsymbol{\omega}_1 := d\psi \wedge (d\eta + \eta \wedge \eta)$, is given by:

$$\begin{aligned}
\boldsymbol{\omega}_1 &= \epsilon_{abc}{}^{d}\delta\overset{1}{\eta}{}^{a}_{\mu} \wedge \overset{1}{\eta}{}^{b} \wedge \delta\overset{\circ}{\eta}{}^{c}_{d\sigma} \wedge \beta^{\mu\sigma} \wedge \gamma^{(6)} \\
&+ \left[\delta\overset{\circ}{\psi}{}^{d\mu j}_{c} \wedge \delta\overset{\circ}{\eta}{}^{c}_{d\mu} + \delta\overset{1}{\psi}{}_{a}{}^{\mu j} \wedge \delta\overset{\circ}{\eta}{}^{a}_{\mu}\right] \wedge \beta^{(4)} \wedge \gamma^{(5)}_{j} \\
&+ \delta\overset{1}{\eta}{}^{a}_{\mu} \wedge \left[\tfrac{1}{2}\epsilon_{abc}{}^{d}\left(\overset{\circ}{\eta}{}^{c}_{d\tau;\sigma} - \overset{\circ}{\eta}{}^{c}_{d\sigma;\tau} + [\overset{\circ}{\eta}_{\sigma}, \overset{\circ}{\eta}_{\tau}]^{c}_{d}\right)\beta^{\sigma\tau\mu} \wedge \overset{1}{\eta}{}^{b} - \overset{1}{\psi}{}_{a;j}^{\mu j}\beta^{(4)}\right] \wedge \gamma^{(6)} \\
&+ \delta\overset{\circ}{\eta}{}^{c}_{d\sigma} \wedge \left[\tfrac{1}{2}\epsilon_{abc}{}^{d}\left(\overset{1}{\eta}{}^{a}_{\mu;\nu} - \overset{1}{\eta}{}^{a}_{\nu;\mu}\right)\beta^{\nu\mu\sigma} \wedge \overset{1}{\eta}{}^{b} - \overset{\circ}{\psi}{}^{d\mu j}_{c;j}\beta^{(4)}\right] \wedge \gamma^{(6)} \\
&+ \delta\overset{\circ}{\psi}{}^{d\mu j}_{c} \wedge \left(\overset{\circ}{\eta}{}^{c}_{d\tau;k} - [\overset{\circ}{\eta}_{\tau}, \mathfrak{l}_{k}]^{c}_{d}\right)\beta^{(4)} \wedge \gamma^{(6)} \\
&+ \delta\overset{1}{\psi}{}_{a}^{\mu j} \wedge \left(\overset{\circ}{\eta}{}^{a}_{\tau;k} - [\overset{\circ}{\eta}_{\tau}, \mathfrak{l}_{k}]^{a}\right)\beta^{(4)} \wedge \gamma^{(6)}.
\end{aligned} \tag{23}$$

3.1.2 The computation of $\boldsymbol{\omega}_2$

Let us compute $\boldsymbol{\omega}_2 := [d\eta \wedge \eta]^A \wedge \psi_A$. First, note that $[d\eta \wedge \eta]^A = [d\eta, \eta_\mu]^A \wedge dx^\mu + [d\eta, \mathfrak{l}_j]^A \wedge \gamma^j$, then

$$\begin{aligned}
\boldsymbol{\omega}_2 &= \tfrac{1}{2}[d\eta, \eta_\nu]^A \wedge \psi^{\mu\rho}_A(\delta^\nu_\rho\beta^{(3)}_\mu - \delta^\nu_\mu\beta^{(3)}_\rho) \wedge \gamma^{(6)} \\
&+ [d\eta, \eta_\nu]^A \wedge \psi^{\mu k}_A\delta^\nu_\mu\beta^{(4)} \wedge \gamma^{(5)}_k - [d\eta, \mathfrak{l}_j]^A \wedge \psi^{\mu k}_A\delta^j_k\beta^{(3)}_\mu \wedge \gamma^{(6)},
\end{aligned}$$

so that:

$$\begin{aligned}
\boldsymbol{\omega}_2 &= [d\eta, \eta_\nu]^A \wedge \psi^{\mu\nu}_A\beta^{(3)}_\mu \wedge \gamma^{(6)} + [d\eta, \eta_a]^A \wedge \psi^{\mu k}_A\beta^{(4)} \wedge \gamma^{(5)}_k \\
&\quad - [d\eta, \mathfrak{l}_j]^A \wedge \psi^{\mu j}_A\beta^{(3)}_\mu \wedge \gamma^{(6)}.
\end{aligned} \tag{24}$$

By using the change of coframe given in Equation (21), we are able to simplify Equation (24) such that:

$$\boldsymbol{\omega}_2 = \left(\psi^{\mu\nu}_A[\eta_b, \delta\eta_\mu]^A + \psi^{\mu j}_A[\mathfrak{l}_j, \delta\eta_\mu]^A\right) \wedge \beta^{(4)} \wedge \gamma^{(6)}. \tag{25}$$

In Equation (25) we have a duality product between $\psi^{\mu\nu}_A$ and $(\mathrm{ad}_{\eta_\nu}(\delta\eta_\mu))^A := [\eta_\nu, \delta\eta_\mu]^A$, which is equivalently seen as the product between $(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu}))_A$ and $\delta\eta^A_\mu$, where $\mathrm{ad}^*_{\eta_\nu}$ is the adjoint of ad_{η_ν}. We have also the duality product between $\psi^{\mu j}_A$ and $(\mathrm{ad}_{\mathfrak{l}_j}(\delta\eta_\mu))^A := [\mathfrak{l}_j, \delta\eta_\mu]^A$, which is equivalent with the product between $(\mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j}))_A$ and $\delta\eta^A_\mu$. We refer to Section 4.1 in the Annex for further details on adjoint and coadjoint actions. Hence (25) reads:

$$\boldsymbol{\omega}_2 = \left(\left(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu})\right)_A + \left(\mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j})\right)_A\right)\delta\eta^A_\mu \wedge \beta^{(4)} \wedge \gamma^{(6)}. \tag{26}$$

which, in the standard representation, is equivalently given by

$$\begin{aligned}
\boldsymbol{\omega}_2 &= \left(\left(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu})\right)^d_c + \left(\mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j})\right)^d_c\right)\delta\overset{\circ}{\eta}{}^{c}_{d\mu} \wedge \beta^{(4)} \wedge \gamma^{(6)} \\
&+ \left(\left(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu})\right)_a + \left(\mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j})\right)_a\right)\delta\overset{1}{\eta}{}^{a}_{\mu} \wedge \beta^{(4)} \wedge \gamma^{(6)}.
\end{aligned} \tag{27}$$

Working on the submanifold of constraints $\mathcal{N}$, see (16), and using Lemma 4.10 (see Section 4.3), Equation (27) yields:

$$\begin{aligned}\boldsymbol{\omega}_2 &= \delta\mathring{\eta}^c_{d\mu}\wedge\Big(dx^\mu\wedge\epsilon_{abc}{}^d\mathring{\eta}^a_{a'}\wedge\overset{1}{\eta}{}^{a'}\wedge\overset{1}{\eta}{}^b+[\mathring{\psi}^{\mu j},\mathfrak{l}_j]^d_c\beta^{(4)}\Big)\wedge\gamma^{(6)}\\ &\quad+\delta\overset{1}{\eta}{}^a_\mu\wedge(\overset{1}{\psi}{}^{\mu j}\mathfrak{l}_j)_a\beta^{(4)}\wedge\gamma^{(6)}.\end{aligned}\tag{28}$$

3.2 HVDW–WEC equations

Collecting (23) and (28), the 11-plectic form $\boldsymbol{\omega}:=d\theta^{(10)}$ is then written as:

$$\begin{aligned}\boldsymbol{\omega} &= \text{ non linear terms in }\delta\eta^A_\mu\text{ and }\delta\psi^{\mu j}_A\\ &\quad+\delta\overset{1}{\eta}{}^a_\sigma\wedge\Big[2\Upsilon_a\wedge dx^\mu-\Big(\overset{1}{\psi}{}_{a;j}^{\sigma j}-\overset{1}{\psi}{}_b{}^{\sigma j}\mathfrak{l}_j{}^b_a\Big)\beta^{(4)}\Big]\wedge\gamma^{(6)}\\ &\quad+\delta\mathring{\eta}^c_{d\sigma}\wedge\Big[2\Sigma_c{}^d\wedge dx^\mu-\Big(\mathring{\psi}{}^{d\sigma j}_{c;j}-[\mathring{\psi}^{\sigma j},\mathfrak{l}_j]^d_c\Big)\beta^{(4)}\Big]\wedge\gamma^{(6)}\\ &\quad+\delta\mathring{\psi}{}^{d\mu j}_c\wedge\Big(\mathring{\eta}^c_{d\mu;k}+[\mathfrak{l}_k,\mathring{\eta}_\mu]^c_d\Big)\beta^{(4)}\wedge\gamma^{(6)}\\ &\quad+\delta\overset{1}{\psi}{}_c{}^{\mu j}\wedge\Big(\overset{1}{\eta}{}^c_{\mu;k}+\overset{1}{\eta}{}^c_\mu\mathfrak{l}_k\Big)\beta^{(4)}\wedge\gamma^{(6)}.\end{aligned}\tag{29}$$

We have recognized the Einstein 3-forms $\Upsilon_a:=\frac{1}{2}\epsilon_{abc}{}^d(d\mathring{\eta}^d_c+\mathring{\eta}^d_{c'}\wedge\mathring{\eta}^{c'}_c)\wedge\overset{1}{\eta}{}^b$ and the Spin 3-forms $\Sigma_c{}^d:=\frac{1}{2}\epsilon_{abc}{}^d(d\overset{1}{\eta}{}^a+\mathring{\eta}^a_{a'}\wedge\mathring{\eta}^{a'})\wedge\overset{1}{\eta}{}^b$, which are given by Definitions (4.1) and (4.2), respectively (see Section 4.3 in the Annex). Since $\Upsilon_a\wedge dx^\rho=\frac{1}{3!}\epsilon^{\rho\lambda\mu\nu}\Upsilon_{a\lambda\mu\nu}\beta^{(4)}$ and $\Sigma_c{}^d\wedge dx^\rho=\frac{1}{3!}\epsilon^{\rho\lambda\mu\nu}\Sigma_c{}^d{}_{\lambda\mu\nu}\beta^{(4)}$, Equation (19) yields:

$$\begin{aligned}X_1\wedge\cdots\wedge X_{10}\lrcorner\,d\theta^{(10)} &= +\delta\overset{1}{\eta}{}^a_\mu\Big(\big(\tfrac{2}{3!}\big)\,\epsilon^{\sigma\lambda\mu\nu}\Upsilon_{a\lambda\mu\nu}-\overset{1}{\psi}{}_{a;j}^{\sigma j}+\overset{1}{\psi}{}_b{}^{\sigma j}\mathfrak{l}_j{}^b_a\Big)\\ &\quad+\delta\mathring{\eta}^c_{d\sigma}\Big(\big(\tfrac{2}{3!}\big)\,\epsilon^{\sigma\lambda\mu\nu}\Sigma_c{}^d{}_{\lambda\mu\nu}-\mathring{\psi}{}^{d\sigma j}_{c;j}+[\mathring{\psi}^{\sigma j},\mathfrak{l}_j]^d_c\Big)\\ &\quad+\delta\mathring{\psi}{}^{d\mu j}_c\Big(\mathring{\eta}^c_{d\mu;k}+[\mathfrak{l}_k,\mathring{\eta}_\mu]^c_d\Big)\\ &\quad+\delta\overset{1}{\psi}{}_c{}^{\mu j}\Big(\mathring{\eta}^c_{\mu;k}+[\mathfrak{l}_k,\mathring{\eta}_\mu]^c\Big),\end{aligned}\tag{30}$$

where the first line in the right hand side of (29) do not contribute because of terms quadratic in $\delta(\cdot)$.

Proposition 3.1. The HVDW–WEC equations $\phi^\circledast d\theta^{(10)}=0$ (see (19)) yields the following system of equations:

$$\text{HVDW–WEC}\left\{\begin{array}{c}\big(\tfrac{2}{3!}\big)\,\epsilon^{\sigma\lambda\mu\nu}\Sigma_a{}^b{}_{\lambda\mu\nu}=\mathring{\Xi}_a{}^{b\sigma}\\ \big(\tfrac{2}{3!}\big)\,\epsilon^{\sigma\lambda\mu\nu}\Upsilon_{a\lambda\mu\nu}=\overset{1}{\Xi}{}_a{}^\sigma\end{array}\right\}\text{Einstein–Cartan}$$

$$\left\{\begin{array}{c}\mathring{\eta}_{\mu;k}+[\mathfrak{l}_k,\mathring{\eta}_\mu]=0\\ \overset{1}{\eta}{}_{\mu;k}+\mathfrak{l}_k\overset{1}{\eta}{}_\mu=0\end{array}\right\}\text{Equivariance}$$

where we use the notations $\overset{\circ}{\Xi}{}^{d\mu}_{c} := \overset{\circ}{\psi}{}^{d}_{c;j}{}^{\mu j} + [\mathfrak{l}_j, \overset{\circ}{\psi}{}^{\mu j}]^{d}_{c}$ and $\overset{1}{\Xi}{}_{a}{}^{\mu} := \overset{1}{\psi}{}_{a;j}{}^{\mu j} - \overset{1}{\psi}{}_{b}{}^{\mu j}\mathfrak{l}_{j}{}^{b}_{a}$.

The Hamilton equations are composed of the Einstein–Cartan system of equations together with the equivariance condition for the for the 1-form η. Note that the latter is not assumed a priori but is obtained by unfolding the dynamics: the fields $\overset{\circ}{\psi}{}^{d\mu j}_{c}$ and $\overset{1}{\psi}{}_{c}{}^{\mu j}$ plays the role of Lagrange multipliers for the constraints given by Equation (11).

Proposition 3.2. We denote $\overset{\circ}{p}{}^{d\mu j}_{c} := g^{d}_{d'}\overset{\circ}{\psi}{}^{d'\mu j}_{c'}(g^{-1})^{c'}_{c}$ and $\overset{1}{p}{}^{\mu j}_{a} := \overset{1}{\psi}{}_{a'}{}^{\mu j}(g^{-1})^{a'}_{a}$. The HVDW–WEC equations $\phi^{\circledast}d\theta^{(10)} = 0$ yields the following Einstein–Cartan system of equations:

$$\left\{\begin{array}{rcl} G^{b}{}_{a} & = & \frac{1}{2}\rho_j \cdot p_{a}{}^{bj} \\ T^{a}{}_{cd} & = & -\left(\mathsf{h}_{de}\delta^{a}_{a'}\delta^{c'}_{c} + \frac{1}{2}\delta^{c'}_{a'}(\delta^{a}_{d}\mathsf{h}_{ce} - \delta^{a}_{c}\mathsf{h}_{de})\right)\rho_j \cdot p_{c'}{}^{ea'j}, \end{array}\right. \tag{31}$$

where $p_{a}{}^{bcj} := \overset{\circ}{p}{}_{a}{}^{b\sigma j}e^{c}_{\sigma}$ and $p_{a}{}^{bj} := \overset{1}{p}{}_{a}{}^{\sigma j}e^{b}_{\sigma}$.

Proof. The equivariance condition found in Proposition 3.1 is equivalent to say that there exists $\mathfrak{g}$-valued functions $A_\mu(x)$ and $\mathfrak{t}$-valued functions $e_\mu(x)$, which depends only on $x \in \mathcal{X}$ such that (see Equation (4)) $\forall x \in \mathcal{X}, \forall g \in \mathfrak{G}$,

$$\overset{\circ}{\eta}{}^{a}_{b\,\mu}(x,g) = (g^{-1})^{a}_{b'}A^{b'}_{\mu a'}(x)g^{a'}_{b}, \qquad \overset{1}{\eta}{}^{a}_{\mu}(x,g) = (g^{-1})^{a}_{a'}e^{a'}_{\mu}(x).$$

Using Lemmata 4.2 and 4.5 of the Annex, $\Upsilon_a := G_{a'}g^{a'}{}_{a}$ and $\Sigma_{c}{}^{d} := H_{c'}{}^{d'}g^{c'}_{c}(g^{-1})^{d}_{d'}$, respectively.

$$\left\{\begin{array}{rcl} \left(\frac{1}{3!}\right)\epsilon^{\sigma\lambda\mu\nu}G_{a\lambda\mu\nu} & = & \frac{1}{2}\overset{1}{p}{}_{a;j}{}^{\sigma j} \\ \left(\frac{1}{3!}\right)\epsilon^{\sigma\lambda\mu\nu}H_{a}{}^{b}{}_{\lambda\mu\nu} & = & \frac{1}{2}\overset{\circ}{p}{}_{a}{}^{b\sigma j}_{\;;j}, \end{array}\right. \tag{32}$$

Then, we use Lemmata 4.4 and 4.7 so that the system of Equations (32) is equivalent to:

$$\left\{\begin{array}{rcl} G^{b}{}_{a}e^{\sigma}_{b} & = & \frac{1}{2}\overset{1}{p}{}_{a;j}{}^{\sigma j} \\ \mathsf{h}^{bb'}\left(T^{c}{}_{b'c}e^{\sigma}_{a} + T^{c}{}_{ca}e^{\sigma}_{b'} + T^{c}{}_{ab'}e^{\sigma}_{c}\right) & = & \overset{\circ}{p}{}_{a}{}^{b\sigma j}_{\;;j}. \end{array}\right. \tag{33}$$

The first line of the system (33) yields $G^{b}{}_{a}e^{\sigma}_{b}e^{b'}_{\sigma} = \frac{1}{2}\overset{1}{p}{}_{a;j}{}^{\sigma j}e^{b'}_{\sigma} \Rightarrow G^{b'}{}_{a} = \frac{1}{2}\overset{1}{p}{}_{a;j}{}^{\sigma j}e^{b'}_{\sigma} = \frac{1}{2}p_{a;j}{}^{b'j}$. Analogously, $\overset{\circ}{p}{}_{a}{}^{b\sigma j}_{\;;j} = \overset{\circ}{p}{}_{a}{}^{bgj}_{\;;j}e^{\sigma}_{g}$. Then, the second line in the system of equations (33) yields:

$$\overset{\circ}{p}{}_{a}{}^{bcj}_{\;;j} = \mathsf{h}^{bb'}\left(T^{d}{}_{b'd}\delta^{c}_{a} + T^{d}{}_{da}\delta^{c}_{b'} + T^{d}{}_{ab'}\delta^{c}_{d}\right) = \mathsf{h}^{be}T^{d}{}_{ed}\delta^{c}_{a} - \mathsf{h}^{bc}T^{d}{}_{ad} + \mathsf{h}^{bd}T^{c}{}_{ad},$$

which is equivalent to $\mathrm{T}^{a}{}_{cd} = -\left(\mathsf{h}_{de}\delta^{a}_{a'}\delta^{c'}_{c} + \frac{1}{2}\delta^{c'}_{a'}(\delta^{a}_{d}\mathsf{h}_{ce} - \delta^{a}_{c}\mathsf{h}_{de})\right)p_{c'}{}^{ea'j}_{\;;j}$ (see [19] for further details). □

Since the Lorentz group $SO(3,1)$ is not compact we cannot conclude that the right hand side of the HVDW–WEC equations (32) vanish in general (as opposed to the case of the Yang–Mills system, see [18]). One way to overcome this difficulty, (see also [19]), is to suppose that the field $\phi^*\psi_a{}^{\mu j}$ and $\phi^*\psi_a{}^{b\mu j}$ have compact support in $\mathcal{P}$ or decay at infinity. Then, the right hand side of (31) vanish and the system of equations reduces to the Einstein–Cartan system in vacuum.

Conclusion

In the context of *liquid fiber bundles*, it is difficult to give any meaning to a predetermined space–time topology. Nonetheless, the theory which is presented in this paper (as in Toller's approach [29]) allows to prescribe the topology of the space–time manifold (e.g. globally hyperbolic space-times, where the topology is given by some foliation of Cauchy hypersurfaces). In addition, the 10-plectic formulation of WEC gravity might shed new light on cosmology and could open a road to the question of dark energy. In particular, the point to address is to see whether the right hand side of Equations (32) could be interpreted as a dark source. This would entails an interpretation of the multimomenta $p_a{}^{bcj}$ and $p_a{}^{cj}$ and of the relevant hypotheses to consider on those fields which would implicate that the right hand side of (32) vanish or, within the perspective of the cosmological constant, is independent of the point of the space-time manifold.

4 Annex

4.1 Lie algebra and representations

We denote by $\mathcal{R} : \mathfrak{G} \to GL(\vec{\mathbb{M}})$ the standard representations of $\mathfrak{G}$. We fix $\mathcal{R}(g)(E_b) = E_a g^a{}_b$, $\forall g \in \mathfrak{G}$, $\forall 0 \leq a, b \leq 3$, where $g^a{}_b$ are the coefficients of $\mathcal{R}(g)$ in the basis $(E_a)_{0\leq a\leq 3}$. We denote also $\mathcal{R} : \mathfrak{g} \to \mathbf{gl}(\vec{\mathbb{M}})$ the standard representations of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. Analogously, $\forall 0 \leq a, b \leq 3$, $\mathcal{R}(\xi)(E_b) = E_a \xi^a{}_b$, where $\xi^a{}_b$ are the coefficients of $\mathcal{R}(\xi)$ in the basis $(E_a)_{0\leq a\leq 3}$. Note that $\xi^{ab} + \xi^{ba} = 0$, where $\xi^{ab} = \xi^a{}_{b'}\mathsf{h}^{b'b}$. In addition, $\forall 1 \leq i \leq 6$, $\mathfrak{l}_i$ is identified with the matrix with coefficients u^a_{ib}; we also note $u_i^{ab} := u^a_{ib'}\mathsf{h}^{b'b}$ and $u_i^{ab} + u_i^{ba} = 0$. Then, $(t_a, u^a_{ib}) := (t_0, \cdots t_3, u^a_{4b}, \cdots, u^a_{9b})$ is a basis of $\mathfrak{p}$ whereas $(t^a, u_a^{ib}) := (t^0, \cdots t^3, u_a^{4b}, \cdots, u_a^{9b})$ is a basis of $\mathfrak{p}^*$.

Let consider the vector subspace $\vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^* := \{t^a{}_b E_a{}^b \in \vec{\mathbb{M}} \otimes \vec{\mathbb{M}}^*; t^a{}_{b'}\mathsf{h}^{b'b} + t^b{}_{a'}\mathsf{h}^{a'a} = 0\}$, where $E_a{}^b := E_a \otimes E^b$. The standard representation $\mathcal{R}$ of $\mathfrak{G}$ induces the map $\mathfrak{G} \longrightarrow \vec{\mathbb{M}} \wedge \vec{\mathbb{M}}^*$, $g \longmapsto g^a{}_b E_a{}^b$.

Adjoint action — The restriction to $\mathfrak{G}$ of the adjoint representation of $\mathfrak{P}$ on $\mathfrak{p}$

reads $\forall \xi \in \mathfrak{p}$,

$$\mathrm{Ad}_g(\xi^a{}_b E_a{}^b, \xi^a E_a) = \Big((g^a{}_{a'} \xi^{a'}{}_{b'} (g^{-1})^{b'}{}_b) E_a{}^b,\ g^a{}_{a'} \xi^{a'} E_a \Big).$$

Coadjoint action — The coadjoint action of $\mathfrak{G}$ on $\mathfrak{p}^*$ is defined by: $\forall g \in \mathfrak{G}$, $\forall \lambda \in \mathfrak{p}^*$, $\mathrm{Ad}_g^* \lambda$ is the vector in $\mathfrak{p}^*$ such that: $\forall \xi \in \mathfrak{p}, (\mathrm{Ad}_g^* \lambda)(\xi) := \lambda(\mathrm{Ad}_g \xi)$. Then

$$(\mathrm{Ad}_g^* \lambda)(\xi) \ = \ \tfrac{1}{2} \left(g^{a'}{}_a \lambda_{a'}{}^{b'} (g^{-1})^b{}_{b'} \right) \xi^a{}_b + \left(g^{a'}{}_a \lambda_{a'} \right) \xi^a.$$

At the level of Lie algebra, the coadjoint action of $\mathfrak{p}$ on $\mathfrak{p}^*$ is defined by: $\forall \xi \in \mathfrak{p}$, $\forall \lambda \in \mathfrak{p}^*$, $\mathrm{ad}_\xi^* \lambda$ is the vector in $\mathfrak{p}^*$ such that: $\forall \zeta \in \mathfrak{p}, (\mathrm{ad}_\xi^* \lambda)(\zeta) := \lambda(\mathrm{ad}_\xi \zeta) = \lambda([\xi, \zeta])$. This gives us:

$$(\mathrm{ad}_\xi^* \lambda)(\zeta) = \frac{1}{2} \left(\xi^c{}_a \lambda_c{}^b - \lambda_a{}^c \xi^b{}_c - 2 \lambda_a \xi^b \right) \zeta^a{}_b + (\xi^a{}_b \lambda_a) \, \zeta^b \tag{34}$$

4.2 Coframe yoga

For any form $\alpha \in \Omega^*(\mathcal{X})$, for any multivector field $v := v_1 \wedge \cdots \wedge v_p \in \mathfrak{X}^p(\mathcal{X})$, we have $v \lrcorner \alpha = (v_1 \wedge \cdots \wedge v_p) \lrcorner \alpha := v_p \lrcorner \cdots \lrcorner v_1 \lrcorner \alpha$. We consider a moving frame $(\partial_\mu, \rho_i) := (\partial_0, \cdots, \partial_3, \rho_1, \cdots, \rho_6)$ and its dual moving coframe $(dx^\mu, \gamma^i) := (dx^0, \cdots, dx^3, \gamma^1, \cdots, \gamma^6)$ defined on the total space of the pseudo-orthonormal frame bundle (see Section 1.2). Consider the volume 4-form $\beta^{(4)} := dx^0 \wedge \cdots \wedge dx^3$. We define the family of basis p-forms $\beta^{(p)}_{\mu_1 \cdots \mu_{4-p}}$ such that:

$$\beta^{(3)}_\mu \ := \ \tfrac{\partial}{\partial x^\mu} \lrcorner \beta^{(4)}; \quad \beta^{(2)}_{\mu\nu} := \tfrac{\partial}{\partial x^\nu} \lrcorner \beta^{(3)}_\mu; \quad \beta^{(1)}_{\mu\nu\rho} := \tfrac{\partial}{\partial x^\rho} \lrcorner \beta^{(2)}_{\mu\nu}; \quad \beta^{(0)}_{\mu\nu\rho\sigma} := \tfrac{\partial}{\partial x^\sigma} \lrcorner \beta^{(1)}_{\mu\nu\rho}.$$

and we denote by $\beta^{\mu_1 \cdots \mu_p} := dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$. We have $\beta^\sigma \wedge \beta^{(3)}_\mu = \delta^\sigma_\mu \beta^{(4)}$ and the following algebraic relations (see also [19]):

$$\left\{ \begin{array}{rcl} \beta^\sigma \wedge \beta^{(2)}_{\mu\nu} & = & \delta^\sigma_\nu \beta^{(3)}_\mu - \delta^\sigma_\mu \beta^{(3)}_\nu, \\ \beta^{\mu\nu} \wedge \beta^{(2)}_{\rho\sigma} & = & \delta^{\mu\nu}_{\rho\sigma} \beta^{(4)}, \\ \beta^\sigma \wedge \beta^{(1)}_{\mu\nu\rho} & = & \delta^\sigma_\mu \beta^{(2)}_{\nu\rho} + \delta^\sigma_\nu \beta^{(2)}_{\rho\mu} + \delta^\sigma_\rho \beta^{(2)}_{\mu\nu}, \\ \beta^{\sigma\kappa} \wedge \beta^{(1)}_{\mu\nu\rho} & = & \delta^{\sigma\kappa}_{\nu\rho} \beta^{(3)}_\mu + \delta^{\sigma\kappa}_{\nu\rho} \beta^{(3)}_\mu + \delta^{\sigma\kappa}_{\nu\rho} \beta^{(3)}_\mu, \end{array} \right.$$

where $\delta^{\mu\nu}_{\rho\sigma} := (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$. Analogously, we consider the volume 4-form $e^{(4)} = e^0 \wedge \cdots \wedge e^3$ and define the family of basis p-forms $e^{(p)}_{a_1 \cdots a_{(4-p)}}$ such that:

$$e^{(3)}_a \ := \ \tfrac{\partial}{\partial e^a} \lrcorner e^{(4)}; \quad e^{(2)}_{ab} := \tfrac{\partial}{\partial e^b} \lrcorner e^{(3)}_a; \quad e^{(1)}_{abc} := \tfrac{\partial}{\partial e^c} \lrcorner e^{(2)}_{ab}; \quad e^{(0)}_{abcd} := \tfrac{\partial}{\partial e^d} \lrcorner e^{(1)}_{abc}.$$

and $e^{a_1 \cdots a_p} := e^{a_1} \wedge \cdots e^{a_p}$. We have $\beta^\sigma \wedge \beta^{(3)}_\mu = \delta^\sigma_\mu \beta^{(4)}$, $e^g \wedge e^{(3)}_a = \delta^g_a e^{(4)}$ and the following algebraic relations (see also [19]):

$$\left\{ \begin{array}{rcl} e^g \wedge e^{(2)}_{ab} & = & \delta^g_b e^{(3)}_a - \delta^g_a e^{(3)}_b, \\ e^{gh} \wedge e^{(2)}_{ab} & = & \delta^{gh}_{ab} e^{(4)}, \\ e^g \wedge e^{(1)}_{abc} & = & \delta^g_a e^{(2)}_{bc} + \delta^g_b e^{(2)}_{ca} + \delta^g_c e^{(2)}_{ab}, \\ e^{gh} \wedge e^{(1)}_{abc} & = & \delta^{gh}_{bc} e^{(3)}_a + \delta^{gh}_{ca} e^{(3)}_b + \delta^{gh}_{ab} e^{(3)}_c. \end{array} \right.$$

Finally, we denote $\gamma^{(6)} := \gamma^1 \wedge \cdots \wedge \gamma^6$, $\gamma^{i_1 \cdots i_q} := \gamma^{i_1} \wedge \cdots \wedge \gamma^{i_q}$ and we define the family of basis q-forms $\gamma^{(q)}_{i_1 \cdots i_{(6-q)}}$ such that:

$$\gamma^{(5)}_i := \rho_i \lrcorner \gamma^{(6)}; \quad \gamma^{(4)}_{ij} := \rho_j \lrcorner \gamma^{(5)}_i; \quad \gamma^{(3)}_{ijk} := \rho_k \lrcorner \gamma^{(4)}_{ij}; \quad \gamma^{(2)}_{ijkl} := \rho_l \lrcorner \gamma^{(3)}_{ijk}.$$

We have also (using $\delta^{ij}_{kl} := (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)$) the useful relations:

$$\left\{ \begin{array}{rcl} \gamma^i \wedge \gamma^{(5)}_j & = & \delta^i_j \gamma^{(6)}, \\ \gamma^i \wedge \gamma^{(4)}_{kl} & = & \delta^i_l \gamma^{(5)}_k - \delta^i_k \gamma^{(5)}_l, \\ \gamma^{ij} \wedge \gamma^{(4)}_{kl} & = & \delta^{ij}_{kl} \gamma^{(6)}, \\ \gamma^{ij} \wedge \gamma^{(3)}_{klm} & = & \delta^{ij}_{lm} \gamma^{(5)} + \delta^{ij}_{mk} \gamma^{(5)} + \delta^{ij}_{kl} \gamma^{(5)}. \end{array} \right.$$

4.3 Einstein and Spin forms

Lemma 4.1. $\forall g \in \mathfrak{G}$, $\varepsilon_{abc}{}^d (g^{-1})^b_{b'} (g^{-1})^c_{c'} g^{d'}_d = g^{a'}_a \varepsilon_{a'b'c'}{}^{d'}$.

Proof. By definition of the Lorentz group we have, $\forall g \in \mathfrak{G}$, $g_a{}^b \mathsf{h}_{a'b'} g^{b'}{}_b = \mathsf{h}_{ab}$ so that $\forall g \in \mathfrak{G}$, $(g^{-1})_a{}^b = \mathsf{h}_{aa'} g^{a'}{}_{b'} \mathsf{h}^{b'b}$. Note that $\forall g \in \mathfrak{G}$, $\det(g) = 1$, then:

$$\begin{array}{rcl@{\quad}rcl} \varepsilon_{abcd} = \varepsilon_{a'b'c'd'} g^{a'}_a g^{b'}_b g^{c'}_c g^{d'}_d & \Longleftrightarrow & \varepsilon_{abc}{}^{d''} = \varepsilon_{a'b'c'}{}^{d'} g^{a'}_a g^{b'}_b g^{c'}_c g^{d'''}_{d''} \mathsf{h}_{d'''d'}, \\ & \Longleftrightarrow & \varepsilon_{abc}{}^{d} = \varepsilon_{a'b'c'}{}^{d'} g^{a'}_a g^{b'}_b g^{c'}_c \mathsf{h}^{dd''} g^{d'''}_{d''} \mathsf{h}_{d'''d'}, \\ & \Longleftrightarrow & \varepsilon_{abc}{}^{d} = \varepsilon_{a'b'c'}{}^{d'} g^{a'}_a g^{b'}_b g^{c'}_c (g^{-1})^d_{d'}. \end{array}$$

□

4.3.1 Einstein 3-forms

Definition 4.1. The Einstein $\mathfrak{t}^*$-valued 3-form is denoted by $\Upsilon = \Upsilon_a \otimes t^a$, where $\forall 0 \leq a \leq 3$, $\Upsilon_a := \frac{1}{2} \epsilon_{abc}{}^d (d\mathring{\eta}^c{}_d + \mathring{\eta}^c{}_{c'} \wedge \mathring{\eta}^{c'}{}_d) \wedge \mathring{\eta}^b$.

Lemma 4.2. $\forall \mathsf{x} \in \mathcal{X}, \forall g \in \mathfrak{G}$, if $\mathring{\eta}^a_{b\mu}(x, g) = (g^{-1})^a_{b'} A^{b'}_{\mu a'}(x) g^{a'}_b$ and $\mathring{\eta}^a_\mu(x, g) = (g^{-1})^a_{a'} e^{a'}_\mu(x)$, then $\Upsilon_a = G_{a'} g^{a'}_a$, where $G_a = \frac{1}{2} \epsilon_{abc}{}^d \left(dA^c{}_d + A^c{}_{d'} \wedge A^{d'}{}_d \right) \wedge e^b$.

Proof. By using Lemma 4.1, we have the straightforward calculation:

$$\begin{array}{rcl} \Upsilon_a & = & \frac{1}{2} \varepsilon_{abc}{}^d (g^{-1})^c_{c'} (g^{-1})^b_{b'} g^{d'}_d (dA + A \wedge A)^{c'}_{d'} \wedge e^{b'} \\ & = & \frac{1}{2} g^{a'}_a \varepsilon_{a'b'c'}{}^{d'} (dA^{c'}{}_{d'} + A^{c'}{}_{d''} \wedge A^{d''}{}_{d'}) \wedge e^{b'}. \end{array}$$

Then, $\Upsilon_a = G_{a'} g^{a'}_a$. □

Lemma 4.3. $G_a = G^{a'}{}_a e^{(3)}_{a'} = G^{a'}{}_a \boldsymbol{e}^\mu_{a'} \beta^{(3)}_\mu$, where $G^a{}_b$ are the components of the Einstein tensor and $\boldsymbol{e}^\mu_a := \det(e) e^\mu_a$.

Proof. Note that $G_a = \frac{1}{2}\varepsilon_{abcd}e^b \wedge F^{cd} = \frac{1}{2}\varepsilon_{abcd}e^d \wedge F^{bc} = \frac{1}{2}F^{bc} \wedge e^{(1)}_{abc} = \frac{1}{4}F^{bc}{}_{b'c'}(e^{b'c'} \wedge e^{(1)}_{abc})$. By using algebraic relations in Section 4.2, we have the straightforward calculation:

$$\begin{aligned} G_a &= \tfrac{1}{2}\left(F^{bc}{}_{bc}e^{(3)}_a + F^{bc}{}_{ca}e^{(3)}_b + F^{bc}{}_{ab}e^{(3)}_c\right) = \tfrac{1}{2}\left(F^{bc}{}_{ca} + F^{cb}{}_{ac}\right)e^{(3)}_b + \tfrac{1}{2}\mathrm{S}e^{(3)}_a, \\ &= -\mathsf{h}^{bb'}F^{c}{}_{b'ca}e^{(3)}_b + \tfrac{1}{2}\mathrm{S}e^{(3)}_a = -\left(\mathrm{Ric}^b{}_a - \tfrac{1}{2}\mathrm{S}\right)e^{(3)}_b \end{aligned}$$

Then $G_a = -G^b{}_a e^{(3)}_b$. □

Lemma 4.4. We have the identity $\left(\frac{1}{3!}\right)\epsilon^{\mu\nu\rho\sigma}G_{a\nu\rho\sigma} = G^b{}_a e^\mu_b$.

Proof. Consider Lemma 4.3 and since $e^{(3)}_a := \boldsymbol{e}^\mu_a \beta^{(3)}_\mu := \left(\left(\frac{1}{3!}\right)\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}e^b_\nu e^c_\rho e^d_\sigma\right)\beta^{(3)}_\mu$, we obtain:

$$\begin{aligned} G_a &= -G^{a'}{}_a e^{(3)}_{a'} = -\left(\tfrac{1}{3!}\right)\epsilon_{a'bcd}\epsilon^{\mu\nu\rho\sigma}G^{a'}{}_a e^b_\nu e^c_\rho e^d_\sigma \beta^{(3)}_\mu \\ &= -\left(\tfrac{1}{3!}\right)\epsilon_{a'bcd}\epsilon^{a''bcd}G^{a'}{}_a e^\mu_{a''}\beta^{(3)}_\mu = \delta^{a''}_{a'}G^{a'}{}_a e^\mu_{a''}\beta^{(3)}_\mu = G^b{}_a e^\mu_b \beta^{(3)}_\mu. \end{aligned}$$

Also, since $2G_a = \epsilon_{abc}{}^d F^c{}_d \wedge e^b = \epsilon_{abcd}e^d \wedge F^{bc} = \frac{1}{2}\epsilon_{abcd}e^d_\rho F^{bc}_{\mu\nu}\beta^{\rho\mu\nu}$, we then obtain $G_a = \frac{1}{3!}G_{a\mu\nu\rho}\beta^{\mu\nu\rho} = \frac{1}{3!}\epsilon^{\sigma\lambda\mu\nu}G_{a\lambda\mu\nu}\beta^{(3)}_\sigma$, where $G_{a\mu\nu\rho} = \left(\frac{3!}{4}\right)\epsilon_{abcd}e^d_\rho F^{bc}_{\mu\nu}$. □

4.3.2 Spin 3-forms

Definition 4.2. The Spin $\mathfrak{g}^*$-valued 3-form is $\Sigma := \Sigma_c{}^d \otimes u^c_{i\,d}\mathfrak{l}^i$, where $\forall 0 \leq a, b \leq 3$, $\Sigma_c{}^d := \frac{1}{2}\epsilon_{abc}{}^d(d\overset{1}{\eta}{}^a + \overset{\circ}{\eta}{}^a{}_{a'} \wedge \overset{1}{\eta}{}^{a'}) \wedge \overset{1}{\eta}{}^b$.

Lemma 4.5. $\forall \mathsf{x} \in \mathcal{X}, \forall g \in \mathfrak{G}$, if $\overset{\circ}{\eta}{}^a_{b\,\mu}(x,g) = (g^{-1})^a_{b'}A^{b'}_{\mu a'}(x)g^{a'}_b$ and $\overset{1}{\eta}{}^a_\mu(x,g) = (g^{-1})^a_{a'}e^{a'}_\mu(x)$, then $\Sigma_c{}^d = H_{c'}{}^{d'}g^{c'}_c(g^{-1})^d_{d'}$, where $H_c{}^d = \frac{1}{2}\epsilon_{abc}{}^d(de^a + A^a{}_{a'} \wedge e^{a'}) \wedge e^b$.

Proof. By using Lemma 4.1, we have the straightforward calculation:

$$\begin{aligned} \Sigma_c{}^d &= \tfrac{1}{2}\varepsilon_{abc}{}^d(g^{-1})^a_{a'}(g^{-1})^b_{b'}\left(de + A \wedge e\right)^{a'} \wedge e^{b'} \\ &= \tfrac{1}{2}\varepsilon_{a'b'c'}{}^{d'}(g)^{c'}_c(g^{-1})^b_{b'}(de^{a'} + A^{a'}{}_{a''} \wedge e^{a''}) \wedge e^{b'} \end{aligned}$$

Then, $\Sigma_c{}^d = H_c{}^d g^{c'}_c(g^{-1})^d_{d'}$. □

Lemma 4.6. $H_a{}^b = \frac{1}{2}\mathsf{h}^{bb'}(T^c{}_{b'c}e^{(3)}_a + T^c{}_{ca}e^{(3)}_{b'} - T^c{}_{b'a}e^{(3)}_c)$, where $T^c{}_{ab}$ are the components of the torsion tensor.

Proof. Note that $H_a{}^b := \frac{1}{2}\varepsilon_{ab'cd}\mathsf{h}^{bb'}e^d \wedge T^c = \frac{1}{2}\mathsf{h}^{bb'}T^c \wedge e^{(1)}_{ab'c} = \frac{1}{4}\mathsf{h}^{bb'}T^c{}_{a'c'}(e^{a'c'} \wedge e^{(1)}_{ab'c})$. By using algebraic relations given in Section 4.2:

$$\begin{aligned} H_a{}^b &= \tfrac{1}{4}\mathsf{h}^{bb'}T^c{}_{a'c'}\left(\delta^{a'c'}_{b'c}e^{(3)}_a + \delta^{a'c'}_{ca}e^{(3)}_b + \delta^{a'c'}_{ab'}e^{(3)}_c\right) \\ &= \tfrac{1}{2}\mathsf{h}^{bb'}\left(T^c{}_{b'c}e^{(3)}_a + T^c{}_{ca}e^{(3)}_{b'} + T^c{}_{ab'}e^{(3)}_c\right). \end{aligned}$$

□

Lemma 4.7. The identity $\left(\frac{1}{3!}\right)\epsilon^{\mu\nu\rho\sigma}H_a{}^b{}_{\nu\rho\sigma} = \frac{1}{2}\mathsf{h}^{bb'}\left(T^c{}_{b'c}\delta^{a'}_a + T^c{}_{ca}\delta^{a'}_{b'} + T^c{}_{ab'}\delta^{a'}_c\right)e^\mu_{a'}$ holds.

Proof. By using Lemma 4.6 and since $e^{(3)}_a := \boldsymbol{e}^\mu_a\beta^{(3)}_\mu := \left(\left(\frac{1}{3!}\right)\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}e^b_\nu e^c_\rho e^d_\sigma\right)\beta^{(3)}_\mu$, we have:

$$\begin{aligned} H_a{}^b &= \tfrac{1}{2}\left[\mathsf{h}^{bb'}\left(T^c{}_{b'c}\boldsymbol{e}^\mu_a + T^c{}_{ca}\boldsymbol{e}^\mu_{b'} + T^c{}_{ab'}\boldsymbol{e}^\mu_c\right)\right]\beta^{(3)}_\mu \\ &= \left(\tfrac{1}{2\cdot 3!}\right)\left[\epsilon^{\mu\nu\rho\sigma}\mathsf{h}^{bb'}\left(\epsilon_{adef}T^c{}_{b'c} + \epsilon_{b'def}T^c{}_{ca} + \epsilon_{cdef}T^c{}_{ab'}\right)e^d_\nu e^e_\rho e^f_\sigma\right]\beta^{(3)}_\mu \\ &= -\tfrac{1}{2}\left(\delta^{a'}_a T^c{}_{b'c}e^\mu_{a'} + T^c{}_{ca}\delta^{b''}_{b'}e^\mu_{b''} + \delta^c_{c'}T^c{}_{ab'}e^\mu_{c'}\right)\beta^{(3)}_\mu \\ &= -\tfrac{1}{2}\mathsf{h}^{bb'}\left(T^c{}_{b'c}\delta^{a'}_a + T^c{}_{ca}\delta^{a'}_{b'} + T^c{}_{ab'}\delta^{a'}_c\right)e^\mu_{a'}\beta^{(3)}_\mu. \end{aligned}$$

where $e^\mu_{a'}$ is such that $e^\mu_{a'}(x)e^a_\mu(x) = \delta^a_{a'}$. Finally, since $2H_a{}^b = \epsilon_{cda}{}^b e^d \wedge T^c = \epsilon_{cdab'}\mathsf{h}^{bb'}e^d \wedge T^c = \frac{1}{2}\epsilon_{ab'cd}e^d_\rho\mathsf{h}^{bb'}T^c_{\mu\nu}\beta^{\rho\mu\nu}{}_{,,}$ then the Spin 3-forms $H_c{}^d$ are given by $H_a{}^b = \frac{1}{3!}H_a{}^b{}_{\mu\nu\rho}\beta^{\mu\nu\rho} = \frac{1}{3!}\epsilon^{\sigma\lambda\mu\nu}H_a{}^b{}_{\lambda\mu\nu}\beta^{(3)}_\sigma$, where $H_a{}^b{}_{\mu\nu\rho} = \left(\frac{3!}{4}\right)\epsilon_{ab'cd}e^d_\rho T^c_{\mu\nu}$. □

4.4 Coadjoint exterior yoga

Let us denote by $\mathcal{E}_{abc}{}^d := \mathring{\eta}^{a'}_a\varepsilon_{a'bc}{}^d + \mathring{\eta}^{b'}_b\varepsilon_{ab'c}{}^d + \mathring{\eta}^{c'}_c\varepsilon_{abc'}{}^d - \mathring{\eta}^d_{d'}\varepsilon_{abc}{}^{d'}$

Lemma 4.8. The identity $\mathcal{E}_{abc}{}^d = 0$ holds. Equivalently, $\mathring{\eta}^d_{d'}\varepsilon_{abc}{}^{d'} - \mathring{\eta}^{c'}_c\varepsilon_{abc'}{}^d = \mathring{\eta}^{a'}_a\varepsilon_{a'bc}{}^d + \mathring{\eta}^{b'}_b\varepsilon_{ab'c}{}^d$.

Proof. $-\mathring{\eta}^d_{d'}\varepsilon_{abc}{}^{d'} = \varepsilon_{abcd''}(-\mathring{\eta}^d_{d'}\mathsf{h}^{d'd''}) = \varepsilon_{abcd''}(\mathsf{h}^{dd'}\mathring{\eta}^{d''}_{d'}) = \varepsilon_{abcd''}\mathring{\eta}^{d''}_{d'}\mathsf{h}^{d'd}$. Therefore: $\mathcal{E}_{abc}{}^d = (\mathring{\eta}^{a'}_a\varepsilon_{a'bcd''} + \mathring{\eta}^{b'}_b\varepsilon_{ab'cd''} + \mathring{\eta}^{c'}_c\varepsilon_{abc'd''} + \mathring{\eta}^{d''}_{d'}\varepsilon_{abcd''})\mathsf{h}^{d'd}$, which is identically vanishing. □

Lemma 4.9. On the submanifold of constraints $\mathcal{N}$, see (17), we have

$$\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu}) = \frac{1}{2}[\overset{1}{\eta}{}^{\mu\nu}, \mathring{\eta}_\nu]^d_c u^c{}_d \qquad \mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j}) = \frac{1}{2}[\mathring{\psi}^{\mu j}, \mathfrak{l}_j]^d_c u^c{}_d + \left(\overset{1}{\psi}{}^{\mu j}_{c'}\mathfrak{l}_j{}^{c'}_c\right)t^c$$

Proof. First, using Equation (34) in Section 4.1,

$$\begin{cases} \mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu}) &= \frac{1}{2}\left(\mathring{\psi}^{d\,\mu\nu}_{c'}\mathring{\eta}^{c'}_{c\ \nu} - \mathring{\eta}^d_{d'\,\nu}\mathring{\psi}^{d'\,\mu\nu}_c - 2\overset{1}{\psi}{}^{\mu\nu}_c\overset{1}{\eta}{}^d_\nu\right)u^c{}_d + \left(\overset{1}{\psi}{}^{\mu\nu}_c\overset{1}{\eta}{}^d_\mu\right)t^c \\ \mathrm{ad}^*_{\mathfrak{l}_j}(\psi^{\mu j}) &= \frac{1}{2}\left(\mathring{\psi}^{d\,\mu j}_{c'}\mathfrak{l}_j{}^{c'}_c - \mathfrak{l}_j{}^d_{d'}\mathring{\psi}^{d'\,\mu j}_c\right)u^c{}_d + \left(\overset{1}{\psi}{}^{\mu j}_{c'}\mathfrak{l}_j{}^{c'}_c\right)t^c \end{cases}$$

We conclude by using relations (16) i.e. $\mathring{\psi}^{d\mu\nu}_c = \overset{1}{\eta}{}^{d\mu\nu}_c$ and $\overset{1}{\psi}{}^{\mu\nu}_a = 0$, where we denote $[\overset{1}{\eta}{}^{\mu\nu}, \mathring{\eta}_\nu]^d_c := \overset{1}{\eta}{}^{d\mu\nu}_{c'}\mathring{\eta}^{c'}_{c\ \nu} - \mathring{\eta}^d_{d'\,\nu}\overset{1}{\eta}{}^{d'\,\mu\nu}_c$ and $[\mathring{\psi}^{\mu j}, \mathfrak{l}_j] := \mathring{\psi}^{d\,\mu j}_{c'}\mathfrak{l}_j{}^{c'}_c - \mathfrak{l}_j{}^d_{d'}\mathring{\psi}^{d'\,\mu j}_c$. □

Lemma 4.10. On $\mathcal{N} \subset \mathcal{N}$ (see (17)), we have $(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu}))^d_c\beta^{(3)}_\mu = \varepsilon_{abc}{}^d\mathring{\eta}^a_{a'} \wedge \overset{1}{\eta}{}^{a'} \wedge \overset{1}{\eta}{}^b$.

Proof. Consider the 2-form $\overset{\circ}{\psi} = \frac{1}{2}\overset{\circ}{\psi}^{\mu\nu}\beta^{(2)}_{\mu\nu}$ and the 1-form $\overset{\circ}{\eta} = \overset{\circ}{\eta}_\rho dx^\rho$, Then:

$$\begin{aligned}
[\overset{\circ}{\psi} \wedge \overset{\circ}{\eta}] &:= \tfrac{1}{2}\left(\overset{\circ}{\psi}^{\mu\nu}\overset{\circ}{\eta}_\rho\beta^{(2)}_{\mu\nu} \wedge dx^\rho - \overset{\circ}{\eta}_\rho\overset{\circ}{\psi}^{\mu\nu}dx^\rho \wedge \beta^{(2)}_{\mu\nu}\right)\\
&= \tfrac{1}{2}\left(\overset{\circ}{\psi}^{\mu\nu}\overset{\circ}{\eta}_\rho\left(\delta^\rho_\nu\beta^{(3)}_\mu - \delta^\rho_\mu\beta^{(3)}_\nu\right) - \overset{\circ}{\eta}_\rho\overset{\circ}{\psi}^{\mu\nu}\left(\delta^\rho_\nu\beta^{(3)}_\mu - \delta^\rho_\mu\beta^{(3)}_\nu\right)\right)\\
&= \overset{\circ}{\psi}^{\mu\nu}\overset{\circ}{\eta}_\nu\beta^{(3)}_\mu - \overset{\circ}{\eta}_\nu\overset{\circ}{\psi}^{\mu\nu}\beta^{(3)}_\mu\\
&= [\overset{\circ}{\psi}^{\mu\nu}, \overset{\circ}{\eta}_\nu]\beta^{(3)}_\mu.
\end{aligned}$$

Finally, on the submanifold of constraints $\mathcal{N}$, we have $\overset{\circ}{\psi}{}^{d\mu\nu}_c = \overset{1}{\eta}{}^{d\mu\nu}_c$ (see (16)), therefore, we obtain $\frac{1}{2}\overset{\circ}{\psi}{}^{d\mu\nu}_c\beta^{(2)}_{\mu\nu} = \frac{1}{2}\varepsilon_{abc}{}^d\overset{1}{\eta}{}^{ab}$, where $\overset{1}{\eta}{}^{ab} := \overset{1}{\eta}{}^a \wedge \overset{1}{\eta}{}^b$.

$$\begin{aligned}
(\mathrm{ad}^*_{\eta_\nu}(\psi^{\mu\nu}))^d_c\beta^{(3)}_\mu &= \tfrac{1}{2}\left(\varepsilon_{abc'}{}^d\overset{1}{\eta}{}^{ab} \wedge \overset{\circ}{\eta}{}^{c'}_c - \varepsilon_{abc}{}^{d'}\overset{1}{\eta}{}^{ab} \wedge \overset{\circ}{\eta}{}^d_{d'}\right)\\
&= \tfrac{1}{2}\left(\overset{\circ}{\eta}{}^{c'}_c\varepsilon_{abc'}{}^d - \overset{\circ}{\eta}{}^d_{d'}\varepsilon_{abc}{}^{d'}\right) \wedge \overset{1}{\eta}{}^{ab}\\
&= \tfrac{1}{2}\left(\varepsilon_{a'bc}{}^d\overset{\circ}{\eta}{}^{a'}_a \wedge \overset{1}{\eta}{}^{ab} + \varepsilon_{b'ac}{}^d\overset{\circ}{\eta}{}^{b'}_b \wedge \overset{1}{\eta}{}^{ba}\right)\\
&= \varepsilon_{abc}{}^d\overset{\circ}{\eta}{}^a_{a'} \wedge \overset{1}{\eta}{}^{a'b},
\end{aligned}$$

where in the second line we have used Lemma 4.8. □

Lemma 4.11. $\forall g \in \mathfrak{G}$, $\overset{\circ}{\Xi}{}^{d\mu}_c$ and $\overset{1}{\Xi}{}_a{}^\mu$ are given by:

$$\overset{\circ}{\Xi}{}^{d\mu}_c = (g^{-1})^d_{d'}\left(\overset{\circ}{p}{}^{d'\mu j}_{c';j}\right)g^{c'}_c, \qquad \overset{1}{\Xi}{}_a{}^\mu = \left(\overset{1}{p}{}^{\mu j}_{a';j}\right)g^{a'}_a, \tag{35}$$

where $\overset{\circ}{p}{}^{d\mu j}_c$ and $\overset{1}{p}{}^{\mu j}_a$ are given in Proposition 3.2.

Proof. Note that $\forall g \in \mathfrak{G}$, $(g^{-1}dg)^i$ is the component of the Maurer–Cartan 1-form in the basis $\mathfrak{l}_i$. Note also that $dg^{-1} = -g^{-1}dgg^{-1}$. Straightforwardly:

$$\left\{\begin{aligned}
\overset{\circ}{\Xi}{}^{d\mu}_c &= \overset{\circ}{\psi}{}^d_{c;j}{}^{\mu j} + [\mathfrak{l}_j, \overset{\circ}{\psi}{}^{\mu j}]^d_c\\
&= \overset{\circ}{\psi}{}^{d\mu j}_{c\ ;j} + \left((g^{-1})^d_{d'}g^{d'}_{d'';j}\right)\overset{\circ}{\psi}{}^{d''\mu j}_c - \overset{\circ}{\psi}{}^d_{c''}{}^{\mu j}\left((g^{-1})^{c''}_b g^b_{c;j}\right)\\
&= (g^{-1})^d_{d'}\left(g^{d'}_{d'';j}\overset{\circ}{\psi}{}^{d''\mu j}_{c''}(g^{-1})^{c''}_{c'} + g^{d'}_{d''}\overset{\circ}{\psi}{}^{d''\mu j}_{c''\ ;j}(g^{-1})^{c''}_{c'}\right.\\
&\qquad\qquad \left. - g^{d'}_{d''}\overset{\circ}{\psi}{}^{d''\mu j}_{c''}(g^{-1})^{c''}_b g^b_{b';j}(g^{-1})^{b'}_{c'}\right)g^{c'}_c\\
&= (g^{-1})^d_{d'}(g^{d'}_{d''}\overset{\circ}{\psi}{}^{d''\mu j}_{c''}(g^{-1})^{c''}_{c'})_{;j}g^{c'}_c,\\
\overset{1}{\Xi}{}_a{}^\sigma &= \overset{1}{\psi}{}^{\mu j}_{a;j} - \overset{1}{\psi}{}_b{}^{\mu j}\mathfrak{l}_{ja}{}^b\\
&= \overset{1}{\psi}{}^{\mu j}_{a;j} - \overset{1}{\psi}{}_{a''}{}^{\mu j}(g^{-1})^{a''}_b g^b_{a;j}\\
&= \left(\overset{1}{\psi}{}^{\mu j}_{a'';j}(g^{-1})^{a''}_{a'} - \overset{1}{\psi}{}_{a''}{}^{\mu j}(g^{-1})^{a''}_b g^b_{b';j}(g^{-1})^{b'}_{a'}\right)g^{a'}_a\\
&= (\overset{1}{\psi}{}_{a''}{}^{\mu j}(g^{-1})^{a''}_{a'})_{;j}g^{a'}_a.
\end{aligned}\right.$$

Then, $\overset{\circ}{\Xi}{}^{d\mu}_c = (g^{-1})^d_{d'}\left(\overset{\circ}{p}{}^{d'\mu j}_{c';j}\right)g^{c'}_c$ and $\overset{1}{\Xi}{}_a{}^\sigma = \left(\overset{1}{p}{}^{\mu j}_{a';j}\right)g^{a'}_a$, respectively. □

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